

# LAYOUT OF RANDOM CIRCULANT GRAPHS

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# LAYOUT OF RANDOM CIRCULANT GRAPHS

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ABSTRACT. A circulant graph  $H$  is defined on the set of vertices  $V = \{1, \dots, n\}$  and edges  $E = \{(i, j) : |i - j| \equiv s \pmod{n}, s \in S\}$ , where  $S \subseteq \{1, \dots, \lceil \frac{n-1}{2} \rceil\}$ . A random circulant graph results from deleting edges of  $H$  with probability  $1 - p$ . We provide a polynomial time algorithm that approximates the solution to the minimum linear arrangement problem for random circulant graphs. We then bound the error of the approximation with high probability.

## 1. INTRODUCTION

A layout on the graph  $G = (V, E)$  is a bijection function  $f : V \rightarrow \{1, \dots, |V|\}$ . Layout problems can be used to formulate several well-known optimization problems on graphs. Also known as linear ordering problems or linear arrangement problems, they consist on the minimization of specific metrics. Such metrics would provide the solution to problems as linear arrangement, bandwidth, modified cut, cut width, sum cut, vertex separation and edge separation. All these problems are NP-hard in the general case.

The Minimum Linear Arrangement (MinLA) problem is to find a function  $f$  that minimizes the sum  $\sum_{uv \in E} |f(u) - f(v)|$ . A layout is also called a labeling, a ordering, or a linear arrangement. The MinLA is one of the most important graph layout problems and was introduced in 1964 by Harper to develop error-correcting codes with minimal average absolute errors. In fact, MinLA appear in a vast domain of problems: VLSI circuit design, network reliability, topology awareness of overlay networks, single machine job scheduling, numerical analysis, computational biology, information retrieval, automatic graph drawing, etc. For instance, layout problems appear in the reconstruction of DNA sequences [6], using overlaps of genes between fragments. Also, MinLA has been used in brain cortex modelling [7]. In [5] it is presented a good survey on graph layout problems and its applications.

The main contribution of this paper is a polynomial time algorithm that approximates the solution to the MinLA problem for a random circulant graph. First, a circulant graph  $H$  is defined on the set of vertices  $V = \{1, \dots, n\}$  and edges  $E = \{(i, j) : |i - j| \equiv s \pmod{n}, s \in S\}$ , where  $S \subseteq \{1, \dots, \lceil \frac{n-1}{2} \rceil\}$ . A random circulant graph results from deleting edges of  $H$  with probability  $1 - p$ . Noticeable, circulant graphs and its random instances carry a nice shape. The MinLA problem for these graphs is the reconstruction of that shape.

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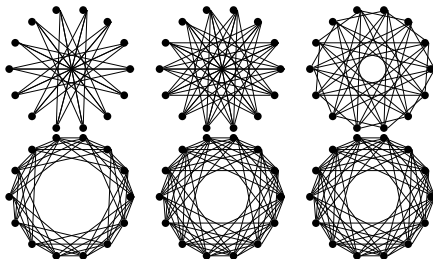


FIGURE 1.1. Circulant graphs

The MinLA problem for circulant graphs is solved in [8], where the authors address the problem of finding an embedding of  $G$  into a graph  $H$ . In that case,  $G$  is a circulant graph and  $H$  is a cycle graph. Certain circulant graphs are of particular interest. In [10] it is presented a polynomial time algorithm solving MinLA of Chord graphs, which is a particular case of circulant graphs. The main motivation of [10] is an application to topology awareness of peer-to-peer overlay networks. The solution of [10] assumes that the Chord graph is complete. However, in real overlay networks nodes can disconnect at any moment, so the remaining network can be regarded as a random circulant graph. Thus, the solution to MinLA for random circulant graphs suits well such applications.

Nevertheless, layout problems for random graphs are significantly more complicated and usually the solution is an approximation of the solution of the model graph. The paper [5] is concerned with the approximability of several layout problems on families of random geometric graphs. It is proven that some of these problems are still NP-complete even for deterministic geometric graphs. The authors present heuristics that turn out to be constant approximation algorithms for layout problems on random geometric graphs, almost surely. The authors of [5] remark that their algorithms use the node coordinates in order to build a layout. That is another feature we do not require in our problem. Even though, the random graph follows a geometric graph model (the circulant structure), we do not have the coordinates of the random graph in advance. The input random graph consists of a set of vertices and edges only and we have to retrieve the circulant layout from that.

Eigenvectors of random matrices are the main tool we use to construct the layout in our problem. We introduce this idea in [9], where one eigenvector would suffice to recover the structure of a random linear graph. Here, as we will see, one eigenvector alone is not enough to encode the whole layout. Fortunately, we can combine two special eigenvectors to find the linear arrangement. Even though, the use of eigenvectors in the same fashion is a common feature of both methods, here we

require some additional technical details that were not present in [9]. Due to the use of angles between subspaces and SVD decomposition, the technique we use here differs significantly from [9]. There, we pointed out the generality of such method and here it turns out we need a more careful analysis. Nevertheless, we have evidence that these methods can be used to implement a general framework for which layout problems can be solved in a broader class of random geometric graphs.

The rest of the paper is organized as follow. In section 2 we define the model matrix, state the algorithm, and the main theorems. In section 3 we describe basic properties of angle between subspaces. Finally, in section 4 we provide the proofs for the results.

## 2. MAIN RESULTS

A circulant matrix  $A$  is a matrix that can be completely specified by only one vector  $a$ , that appears in the first column of  $A$ . The remaining columns are cyclic permutations of  $a$  with offset equal to the column index, i.e., the matrix  $A$  is of the following form

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ \vdots & & \ddots & & \vdots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{bmatrix}.$$

A circulant graph is a graph with circulant adjacency matrix. Let  $H = (V, E_H)$  be a circulant graph with vertex set  $V = \{v_1, \dots, v_n\}$  and adjacency matrix  $A$ , where  $[a_1, \dots, a_n]$  corresponds to the first row of  $A$ . We define the set of indices of non-zero elements in the first half of the row of  $A$  as

$$N := \{k : a_k = 1, 1 \leq k \leq \lceil \frac{n-1}{2} \rceil\}.$$

Equivalently, a circulant graph can be defined as the Cayley graph of a finite cyclic group.

In this paper,  $H$  is referred as the model graph. The random graph we consider is denoted by  $G = (V, E)$  which results from deleting edges of  $H$  with probability  $1 - p$ . The model matrix  $M$  is a circulant matrix that describes the structure of  $H$ , where  $M = pA$ .

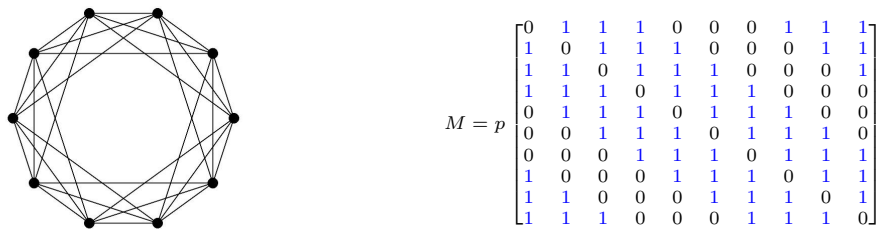


FIGURE 2.1. Model graph and its model matrix

Furthermore, let  $\hat{M}$  be the adjacency matrix of the random graph  $G$ . The entries of  $\hat{M}$  correspond to independent Bernoulli variables, where  $\mathbf{P}(\hat{m}_{ij} = 1) = m_{ij}$ .

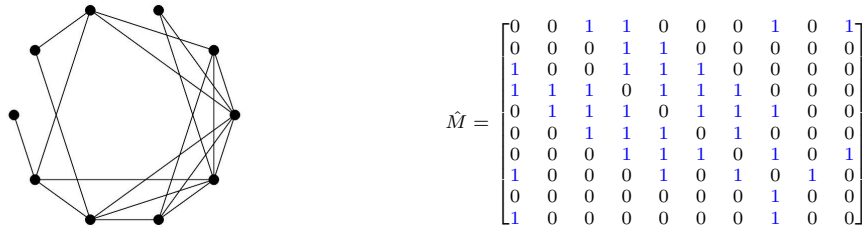


FIGURE 2.2. Random graph and its random matrix

By construction, the labels of vertices in the random graph  $G$  corresponds to the same labels as the graph model in the first figure. However, in the real world we do not have the labels in advance. We are talking about a large amount of disorganized data with additional noise. We only know that this data encodes a circulant structure which is hidden from us. In such situations, finding the labels for the random graph can be rather challenging. See Figure 2.3.

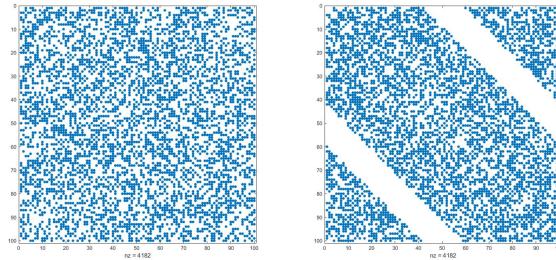


FIGURE 2.3. Data before and after the correct permutation.

That is precisely the problem we address in this paper: given a graph that follows a circulant model, find its circular embedding, or rather, retrieve the correct order of the vertices. We present an algorithm that solves this problem by using eigenvectors corresponding the second and third largest eigenvalues  $\hat{M}$ . The algorithm can be described as follow.

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**Algorithm 1**


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**Require:** Random matrix  $\hat{M}$

- 1: Compute  $\hat{x}$  and  $\hat{y}$ , the eigenvectors for  $\lambda_2(\hat{M})$  and  $\lambda_3(\hat{M})$
  - 2: Compute the angular coordinate  $\varphi_i$  for the point of coordinates  $(\hat{x}_i, \hat{y}_i)$
  - 3: Define a permutation  $\sigma$  such that  $\sigma(i) > \sigma(j)$  iff  $\varphi_i \geq \varphi_j$
  - 4: **return**  $\sigma$
- 

This simple algorithm is shown to return the correct labels with a bounded error. We quantify the error in terms of a rank correlation coefficient we introduce. Before, let us plot the points described in Algorithm 1.

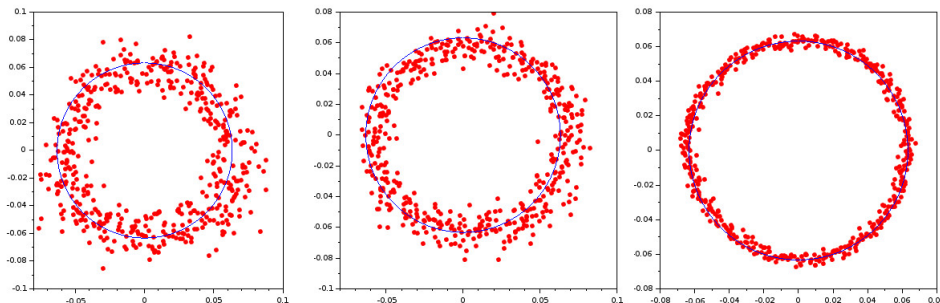


FIGURE 2.4. Points whose coordinates are entries of the eigenvectors. Probabilities: 0.3, 0.5, 0.9.

The circle-like shape follows indeed the layout we are looking to reconstruct. This phenomenon can be explained in terms of angles between spaces which appear in our proofs. Also, notice that the points that are in the wrong position are not a major part. That can be explained in terms of rank correlation coefficients.

A rank correlation coefficient measures the degree of similarity between two lists, and can be used to assess the significance of the relation between them. One can see the rank of one list as a permutation of the rank of the other. Statisticians have used a number of different measures of closeness for permutations. Some popular rank correlation statistics are Kendall's  $\tau$ , Kendall distance, and Spearman's footrule. There are several other metrics, and for different situations some metrics are preferable. For a deeper discussion on metrics on permutations we recommend [4].

To count the total number of inversions in  $\sigma$  one can use

$$D(\sigma) = \sum_{i < j} \mathbf{1}_{\sigma(i) > \sigma(j)} \quad (\text{Kendall Distance})$$

First we define a refined version of the Kendall distance. This version counts inverted pairs whose indices are at least  $k$  positions apart. First note that, for a permutation  $\sigma$ , we can rewrite  $D(\sigma)$  as

$$D(\sigma) = |\{(i, j) : \sigma(j) < \sigma(i) \text{ and } i < j\}|.$$

Given a permutation  $\sigma$  and an index  $k \geq 1$ , let

$$D_k(\sigma) = |\{(i, j) : \sigma(j) < \sigma(i) \text{ and } i + k \leq j \text{ and } i - j \geq k \pmod{n}\}|.$$

Thus,  $D_k$  counts the number of inverted pairs where the vertices have jumped at least  $k$  positions from their original order. In particular,  $D_1(\sigma) = D(\sigma)$ . The module in the definition is used to access the circular structure of the graph we consider.

Consider the eigenvectors  $x$  and  $y$  for  $\lambda_2(M)$  and  $\lambda_3(M)$ , respectively. As we will see, the set of points  $z_i = (x_i, y_i)$  have coordinates on a circle in  $\mathbb{R}^2$ . Let  $\varphi(x)$  be the angular coordinate for  $x \in \mathbb{R}^2$ . A crucial observation is that  $\{\varphi(z_i)\}_{i=1}^n$  is an increasing sequence. That means that the order of  $\varphi(z_i)$  provides the correct order for the vertices in the model graph. Similarly, we can consider the eigenvectors  $\hat{x}$  and  $\hat{y}$  for  $\lambda_2(\widehat{M})$  and  $\lambda_3(\widehat{M})$ , respectively. Here,  $\{\varphi(\hat{z}_i)\}_{i=1}^n$  does not necessarily

form an increasing sequence. Thus, we can construct a permutation  $\sigma$  of indices such that  $\sigma(i) > \sigma(j)$  if and only if  $\varphi(\hat{z}_i) \geq \varphi(\hat{z}_j)$ .

In view of the last observations, the permutation  $\sigma$  has a neat interpretation in terms of  $D_k$ :  $D_k(\sigma)$  counts the pairs in  $\hat{z}$  that disagree with the order induced by  $z$  by at least  $k$  positions. I.e, the permutation  $\sigma$  of Algorithm 1 has  $D_k(\sigma)$  pairs of vertices in the wrong order. Fortunately, the next Theorem bounds the number of such pairs.

**Theorem 1.** *Let  $\sigma$  be the permutation returned by Algorithm 1 for a random circulant graph. Let  $k \in \Omega(n^\beta)$  and  $|N| = cn$ , for a constant  $c > 0$ . Then it holds  $D_k(\sigma) \in \mathcal{O}(n^{5-4\beta})$  with probability  $1 - n^{-3}$ .*

In fact, we prove a more general version of Theorem 1 where we allow the edge density to be variable.

**Theorem 2.** *Let  $\sigma$  be the permutation returned by Algorithm 1 for a random circulant graph with model satisfying  $|N| = cn^\gamma$ , for a constant  $c > 0$ . Let  $k \in \Omega(n^\beta)$ . Then we have  $D_k(\sigma) \in \mathcal{O}(n^{11-6\gamma-4\beta})$  with probability  $1 - n^{-3}$ .*

Furthermore, depending on the parameters  $\gamma$  and  $\beta$  we can improve the bounds of the last Theorems, as shown in the next result.

**Theorem 3.** *Let  $\sigma$  be the permutation returned by Algorithm 1 for a random circulant graph with model satisfying  $|N| = cn^\gamma$ , for a constant  $c > 0$ . Let  $k \in \Omega(n^\beta)$ . Then we have  $D_k(\sigma) \in \mathcal{O}(n^{\frac{13-6\gamma-2\beta}{3}})$  with probability  $1 - n^{-3}$ .*

Notice that the last result shows that there is a trade off between how far vertices can jump and the total number of such incorrectly placed vertices. That is useful for our purpose to establish metrics on the correctness of the rank. For example, consider the worst case of Theorem 3 when all pairs are incorrect. Assuming  $\gamma = 1$ , the number of pairs that drift less than  $k$  positions apart is  $\binom{n}{2} - D_k$ . If we take  $\beta > 1/2$  in Theorem 3, we obtain that  $\binom{n}{2} - D_k$  is asymptotically equivalent to  $n^2$  as  $n \rightarrow \infty$ . That means almost no vertex will drift more than  $n^{1/2}$  slots from its correct position.

Finally, the next theorem shows that the permutation returned by Algorithm 1 is well behaved in terms of the usual Kendall distance.

**Theorem 4.** *Let  $\sigma$  be the permutation returned by Algorithm 1 for a random circulant graph with model satisfying  $|N| = cn^\gamma$  and  $1 \geq \gamma > 0$ . Then  $D(\sigma) \in \mathcal{O}(n^{(15-6\gamma)/5})$  with probability  $1 - n^{-3}$ .*

To prove the results, our technique uses Singular Value Decomposition and angles between subspaces, which require expressions for the eigenvalues and eigenvectors of the model matrix. Fortunately, circulant matrices have known spectrum and, as we will see, there is a specific pair of eigenvectors carrying the desired information about the structure of the graph, providing the correct label of vertices. Moreover, consecutive entries of the eigenvectors differ significantly enough so that a small perturbation will have limited effect on the labels. Further, in Section 3 we show that those eigenvectors are close to the eigenvectors of the random graph. In Section 4 we perform the qualitative analysis of the problem proving the main results.

## 3. SVD AND ANGLES BETWEEN SUBSPACES

The definition of an angle between two vectors can be extended to angles between subspaces.

**Definition 5.** Let  $\mathcal{X} \subset \mathbb{R}^n$  and  $\mathcal{Y} \subset \mathbb{R}^n$  be subspaces with  $\dim(\mathcal{X}) = p$  and  $\dim(\mathcal{Y}) = q$ . Let  $m = \min(p, q)$ . The principal angles

$$\Theta = [\theta_1, \dots, \theta_m], \text{ where } \theta_k \in [0, \pi/2], k = 1, \dots, m,$$

between  $\mathcal{X}$  and  $\mathcal{Y}$  are recursively defined by

$$s_k = \cos(\theta_k) = \max_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} |x^T y| = |x_k^T y_k|,$$

subject to

$$\|x\| = \|y\| = 1, x^T x_i = 0, y^T y_i = 0, \text{ for } i = 1, \dots, k-1.$$

The vectors  $\{x_1, \dots, x_m\}$  and  $\{y_1, \dots, y_m\}$  are called the principal vectors for  $\mathcal{X}$  and  $\mathcal{Y}$ .

The principal angles and principal vectors can be characterized in terms of a Singular Value Decomposition. That provides a constructive form for the principal vectors, which is what we use in the proofs. That is the subject of the next Theorem proved in [1].

**Theorem 6.** *Let the columns of the matrices  $X \in \mathbb{R}^{n \times p}$  and  $Y \in \mathbb{R}^{n \times q}$  form an orthonormal bases for the subspaces  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively. Consider the singular value decomposition*

$$X^T Y = U \Sigma V^T,$$

where  $U$  and  $V$  are unitary matrices and  $\Sigma$  is a  $p \times q$  diagonal matrix with real diagonal entries  $s_1, \dots, s_m$  in nonincreasing order with  $m = \min(p, q)$ . Then

$$\cos \Theta = [s_1, \dots, s_m],$$

where  $\Theta$  denotes the vector of principal angles between  $\mathcal{X}$  and  $\mathcal{Y}$ . Furthermore, the principal vectors for  $\mathcal{X}$  and  $\mathcal{Y}$  are given by the first  $m$  columns of  $XU$  and  $YV$ .

In [12], the authors prove a variant of Davis-Kahan Theorem, which gives an upper bound for the sine of the principal angles between subspaces in terms of eigenvalues of the matrices whose columns are bases for the subspaces. The original version of Davis-Kahan [3] relies on an eigenvalue separation condition for those matrices. However, these conditions are not necessarily met by the eigenvalues of a random matrix. That is the reason we use a different version of Davis-Kahan Theorem. We recast the result here for the eigenvalues of interest of our problem. Here  $\|\cdot\|_F$  denotes the Frobenius norm.

**Theorem 7.** *Let  $M, \widehat{M} \in \mathbb{R}^{n \times n}$  be symmetric matrices, with eigenvalues  $\lambda_1 \geq \dots \geq \lambda_n$  and  $\hat{\lambda}_1 \geq \dots \geq \hat{\lambda}_n$ , respectively. Let  $\lambda_i$  and  $\hat{\lambda}_i$  have corresponding unitary eigenvectors  $v_i$  and  $\hat{v}_i$ . Let  $\min(\lambda_1 - \lambda_2, \lambda_3 - \lambda_4) > 0$ , define  $V = [v_2 \ v_3]$  and  $\widehat{V} = [\hat{v}_2 \ \hat{v}_3]$ . Let  $\Theta$  be a  $2 \times 2$  diagonal matrix whose diagonal contains the principal angles between the subspaces spanned by the columns of  $V$  and  $\widehat{V}$ . Then*

$$\|\sin \Theta\|_F \leq \frac{2 \min \left( \sqrt{2} \|M - \widehat{M}\|, \|M - \widehat{M}\|_F \right)}{\min(\lambda_1 - \lambda_2, \lambda_3 - \lambda_4)}.$$



## 4. BOUNDS AND PROOFS OF THE MAIN RESULTS

To prove the main theorems, we need to bound the differences  $\lambda_1 - \lambda_2$  and  $\lambda_3 - \lambda_4$ . Fortunately, the spectrum of circulant graphs is well known, see for example [2], so we do not need to compute it.

The four largest eigenvalues of  $H$  can be expressed as follows

$$\begin{aligned}\lambda_1 &= \sum_{k \in N} 2a_k, \\ \lambda_2 = \lambda_3 &= \sum_{k \in N} 2a_k \cos\left(\frac{2k\pi}{n}\right), \text{ and} \\ \lambda_4 &= \sum_{k \in N} 2a_k \cos\left(\frac{4k\pi}{n}\right).\end{aligned}$$

Their corresponding unitary eigenvectors are

$$\begin{aligned}v_1 &= \frac{1}{\sqrt{n}}(1, 1, \dots, 1)^T, \\ v_2 &= \frac{2}{\sqrt{2n}}(1, \cos(\frac{2\pi}{n}), \cos(2\frac{2\pi}{n}), \dots, \cos((n-1)\frac{2\pi}{n}))^T, \\ v_3 &= \frac{2}{\sqrt{2n}}(0, \sin(\frac{2\pi}{n}), \sin(2\frac{2\pi}{n}), \dots, \sin((n-1)\frac{2\pi}{n}))^T, \text{ and} \\ v_4 &= \frac{2}{\sqrt{2n}}(1, \cos(\frac{4\pi}{n}), \cos(2\frac{4\pi}{n}), \dots, \cos((n-1)\frac{4\pi}{n}))^T.\end{aligned}$$

Denote by  $v^i$  is the  $i$ -th entry of the vector  $v$ . An important observation is that the set of points with coordinates  $(v_2^i, v_3^i)$  are on a circle in  $\mathbb{R}^2$ . Thus, these points describe the correct structure of the graph, providing the correct label of vertices.

Throughout the paper  $n$  is assumed to be large.

**Lemma 8.** *Let  $H$  be a circulant graph of degree  $d$  and order  $n$  with eigenvalues  $\lambda_1 \geq \lambda_2 = \lambda_3 \geq \lambda_4$ . If  $|N| = cn^\gamma$  for a constant  $c > 0$  and  $1 \geq \gamma > 0$ , there is a constant  $C_1 > 0$  and  $C_2 > 0$  such that  $\lambda_1 - \lambda_2 \geq C_1 n^{3\gamma-2}$  and  $\lambda_3 - \lambda_4 \geq C_2 n^{3\gamma-2}$ .*

*Proof.* We will show the lower bound for  $\lambda_1 - \lambda_2$  first. Using the expression for the eigenvalues as above we have  $\lambda_1 - \lambda_2 = \sum_{k \in N} 2a_k (1 - \cos(\frac{2k\pi}{n}))$ . Note that

$\cos(\theta)$  is a decreasing function in  $\theta$  for  $\theta \in [0, \pi]$  and  $\frac{2k\pi}{n} \leq \pi$  for  $N$  and therefore  $\lambda_1 - \lambda_2 \geq \sum_{k=1}^{|N|} 2(1 - \cos(\frac{2k\pi}{n}))$ . Using the Taylor series of  $\cos(\theta)$  at  $\theta = 0$  we get  $\cos(\theta) \leq 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$ , thus

$$\begin{aligned}\lambda_1 - \lambda_2 &\geq 2 \sum_{k=1}^{|N|} \frac{2k^2\pi^2}{n^2} - \frac{2k^4\pi^4}{3n^4} \\ &= 2 \left( \frac{2\pi^2}{n^2} \frac{2|N|^3 + 3|N|^2 + |N|}{6} - \frac{2\pi^4}{3n^4} \frac{6|N|^5 + 15|N|^4 + 10|N|^3 - |N|}{30} \right) \\ &\geq K_1 \left( \frac{|N|^3}{n^2} - \frac{|N|^5}{n^4} \right),\end{aligned}$$

for a constant  $K_1 > 0$ . Therefore, there is a constant  $C_1 > 0$  such that  $\lambda_1 - \lambda_2 \geq C_1 n^{3\gamma-2}$ .  $\lambda_3 - \lambda_4 = \sum_{k \in N} 2a_k (\cos(\frac{2k\pi}{n}) - \cos(\frac{4k\pi}{n}))$ . Note, that  $f(\theta) = \cos(\theta) - \cos(2\theta)$  is increasing for  $\theta \in [0, \frac{\pi}{4}]$ , decreasing for  $\theta \in [\frac{\pi}{4}, \pi]$  and  $\frac{2k\pi}{n} < \pi$  for  $k \in N$ .

For this reason we will split the sum above using the following partition of  $N$ ,  $N_L := \{k \in N : \frac{2k\pi}{n} \leq \frac{\pi}{4}\}$  and  $N_U := N - N_L$ . Let  $\hat{k} = \max\{k \in N\}$ , then using the Taylor series for  $f(\theta)$ , we have

$$\begin{aligned} \lambda_3 - \lambda_4 &= \sum_{k \in N_L} 2a_k \left( \cos\left(\frac{2k\pi}{n}\right) - \cos\left(\frac{4k\pi}{n}\right) \right) + \sum_{k \in N_U} 2a_k \left( \cos\left(\frac{2k\pi}{n}\right) - \cos\left(\frac{4k\pi}{n}\right) \right) \\ &\geq \sum_{k=1}^{|N_L|} 2a_k \left( \cos\left(\frac{2k\pi}{n}\right) - \cos\left(\frac{4k\pi}{n}\right) \right) + \sum_{k=\hat{k}-|N_U|+1}^{\hat{k}} 2a_k \left( \cos\left(\frac{2k\pi}{n}\right) - \cos\left(\frac{4k\pi}{n}\right) \right) \\ &\geq 2 \sum_{k=1}^{|N_L|} \left( \frac{6\pi^2 k^2}{n^2} - \frac{10\pi^4 k^4}{n^4} \right) + 2 \sum_{k=\hat{k}-|N_U|+1}^{\hat{k}} \left( \frac{6\pi^2 k^2}{n^2} - \frac{10\pi^4 k^4}{n^4} \right) \\ &= K_2 \left( \frac{|N_L|^3}{n^2} - \frac{|N_L|^5}{n^4} \right) + K_3 \left( \frac{\hat{k}^3}{n^2} - \frac{\hat{k}^5}{n^4} \right) - K_4 \left( \frac{(\hat{k}-|N_U|)^3}{n^2} - \frac{(\hat{k}-|N_U|)^5}{n^4} \right), \end{aligned}$$

for nonnegative constants  $K_2, K_3$  and  $K_4$ . Furthermore,  $\hat{k} \geq |N|$  and  $\hat{k} \geq |N_L|$ . The first inequality implies that there is a constant  $K_5$  such that  $\hat{k} = K_5 n^\gamma$ . Therefore, there is a constant  $C_2 > 0$  with  $\lambda_3 - \lambda_4 \geq C_2 n^{3\gamma-2}$   $\square$

Using Lemma 8 we are able to prove an upper bound for the deviations of the eigenvectors corresponding to the second and third eigenvalues of the model matrix and the random matrix, respectively. We will also need the following concentration inequality from [11].

**Lemma 9** (Norm of a random matrix). *There is a constant  $C > 0$  such that the following holds. Let  $E$  be a symmetric matrix whose upper diagonal entries  $e_{ij}$  are independent random variables where  $e_{ij} = 1 - p_{ij}$  or  $-p_{ij}$  with probabilities  $p_{ij}$  and  $1 - p_{ij}$ , respectively, where  $0 \leq p_{ij} \leq 1$ . Let  $\sigma^2 = \max_{ij} p_{ij}(1 - p_{ij})$ . If  $\sigma^2 \geq C \log n/n$ , then*

$$\mathbf{P}(\|E\| \geq C\sigma n^{1/2}) \leq n^{-3}$$

Now we are able to prove the following theorem.

**Theorem 10.** *Let  $M$  be the circulant graph model matrix with constant probability  $p$ , variance  $\sigma^2$ , and  $|N| = cn^\gamma$  for a constant  $1 \geq \gamma > 0$ . Let  $\hat{M}$  the random matrix following the model matrix. Let  $v_2, v_3$  be unitary eigenvectors for  $\lambda_2(M), \lambda_3(M)$  and  $\hat{v}_2, \hat{v}_3$  be unitary eigenvectors for  $\lambda_2(\hat{M}), \lambda_3(\hat{M})$ . Let  $x, y \in \text{Span}\{v_2, v_3\}$  and  $\hat{x}, \hat{y} \in \text{Span}\{\hat{v}_2, \hat{v}_3\}$  be the principal vectors for the principal angles between the spaces  $\text{Span}\{v_2, v_3\}$  and  $\text{Span}\{\hat{v}_2, \hat{v}_3\}$ . Define the matrices  $z = (x, y)$  and  $\hat{z} = (\hat{x}, \hat{y})$ . Then there is an absolute constant  $C_0 > 0$  and such that*

$$\|z - \hat{z}\|_F^2 \leq C_0 \sigma n^{5-6\gamma}$$

with probability at least  $1 - n^{-3}$ .

*Proof.* In view of Theorem 6, consider the singular value decomposition  $[v_2, v_3]^T [\hat{v}_2, \hat{v}_3] = U\Sigma W^T$ . Let  $\theta_2$  and  $\theta_3$  denote the principal angles between the spaces spanned by  $\{v_2, v_3\}$  and  $\{\hat{v}_2, \hat{v}_3\}$ .

Note that  $\min(\lambda_1 - \lambda_2, \lambda_3 - \lambda_4) > 0$ , thus we can apply Theorem 7. We have

$$\begin{aligned} \|z - \hat{z}\|_F^2 &= \|(x, y) - (\hat{x}, \hat{y})\|_F^2 \\ &= (\|x - \hat{x}\|^2 + \|y - \hat{y}\|^2) \\ &\leq 2(\sin^2(\theta_2) + \sin^2(\theta_3)) \\ &= 2\|\sin \Theta(z, \hat{z})\|_F^2 \\ &\leq 2 \left( \frac{\min(\sqrt{2}\|\hat{M}-M\|, \|\hat{M}-M\|_F)}{\min(\lambda_1 - \lambda_2, \lambda_3 - \lambda_4)} \right)^2. \end{aligned}$$

Now, we can view the adjacency matrix  $\hat{M}$  as a perturbation of  $M$ ,  $\hat{M} = M + E$ , where the entries of  $E$  are  $e_{ij} = 1 - p$  with probability  $p$  and  $-p$  with probability  $1 - p$ , thus,  $E$  is as in Lemma 9 and with probability at least  $1 - n^{-3}$  we have  $\|E\| \leq C\sigma\sqrt{n}$ . Furthermore,  $\min(\sqrt{2}\|\hat{M} - M\|, \|\hat{M} - M\|_F) \leq \sqrt{2}\|\hat{M} - M\|_{op} = \|E\|$  and  $\lambda_i(M) = p\lambda_i(H)$ . Together with Lemma 8 we get for some absolute constant  $C_0 > 0$

$$\|z - \hat{z}\|^2 \leq C_0\sigma \left(\frac{\sqrt{n}}{n^{3\gamma-2}}\right)^2 = C_0\sigma n^{5-6\gamma}.$$

That finishes the proof.  $\square$

Now we will provide a lower bound for  $\|z - \hat{z}\|_F$  in terms of  $D_k(\sigma)$  and eventually proof theorem 4.

**Lemma 11.** *Let  $v$  and  $w$  be the unitary eigenvectors for  $\lambda_2(M)$  and  $\lambda_3(M)$ , respectively. Then*

$$\begin{aligned} (v_i - v_{i+k})^2 &= n^{-5} (32\pi^4 k^2 i^2 + 32\pi^4 k^3 i + 8\pi^4 k^4) + \mathcal{O}(n^{-7}) \text{ and,} \\ (w_i - w_{i+k})^2 &= (3n)^{-5} (12\pi^2 k^2 n^2 - 48\pi^4 k^2 i^2 - 48\pi^4 k^3 i - 16\pi^4 k^4) + \mathcal{O}(n^{-7}). \end{aligned}$$

*Proof.* Notice that

$$\begin{aligned} v_i - v_{i+k} &= \frac{2}{\sqrt{2n}} \left( \cos\left(\frac{2\pi i}{n}\right) - \cos\left(\frac{2\pi(i+k)}{n}\right) \right) \text{ and} \\ w_i - w_{i+k} &= \frac{2}{\sqrt{2n}} \left( \sin\left(\frac{2\pi i}{n}\right) - \sin\left(\frac{2\pi(i+k)}{n}\right) \right). \end{aligned}$$

Now the expressions can be obtained from a simple asymptotic expansion as  $n \rightarrow \infty$ .  $\square$

**Theorem 12.** *Let  $M$  be the circulant graph model matrix with constant probability  $p$  and variance  $\sigma^2$ , and  $\hat{M}$  the random matrix following the model matrix. Let  $v_2, v_3$  be unitary eigenvectors for  $\lambda_2(M)$  and  $\lambda_3(M)$ , and  $\hat{v}_2$  and  $\hat{v}_3$  be unitary eigenvectors for  $\lambda_2(\hat{M})$  and  $\lambda_3(\hat{M})$ . Let  $x, y \in \text{Span}\{v_2, v_3\}$  and  $\hat{x}, \hat{y} \in \text{Span}\{\hat{v}_2, \hat{v}_3\}$  be the principal vectors for the principal angles between the spaces  $\text{Span}\{v_2, v_3\}$  and  $\text{Span}\{\hat{v}_2, \hat{v}_3\}$ . Define the matrices  $z = (x, y)$  and  $\hat{z} = (\hat{x}, \hat{y})$ . Then there are constants  $C_1 > 0$  and  $\beta$  such that*

$$\|z - \hat{z}\|^2 > C_0 |R| \frac{n^{4\beta}}{n^6},$$

where  $R = \{(i, j) : \varphi(\hat{z}_j) \geq \varphi(\hat{z}_i), i + k \leq j \text{ and } i - j \geq k \text{ mod } n\}$ .

*Proof.* As in Theorem 6, let  $U\Sigma W^T$  be the singular value decomposition for the matrix  $[v_2, v_3]^T [\hat{v}_2, \hat{v}_3]$ . Thus,  $z = (v_2, v_3)U$  and  $\hat{z} = (\hat{v}_2, \hat{v}_3)W$ . Let  $\varphi(z_i)$  be the angular coordinate of the point  $z_i = (x_i, y_i)$ . Thus,  $\{\varphi(z_i)\}_{i=1}^n$  is an increasing sequence. Fix  $k = k(n) = C(n^\beta)$  and let

$$R = \{(i, j) : \varphi(\hat{z}_j) \leq \varphi(\hat{z}_i) \text{ and } i + k \leq j \text{ and } i - j \geq k \text{ mod } n\}.$$

Then  $R$  is the set of pairs in  $\hat{z}$  that disagree with the order induced by  $z$  by at least  $k$  positions in both directions on the cycle. Now we can write

$$2n\|z - \hat{z}\|^2 = \sum_{i=1}^n \sum_{j=1}^n \|z_i - \hat{z}_i\|^2 + \|z_j - \hat{z}_j\|^2 \geq \sum_{(i,j) \in R} \|z_i - \hat{z}_i\|^2 + \|z_j - \hat{z}_j\|^2.$$

Since  $\varphi(\hat{z}_j) \leq \varphi(\hat{z}_i)$  and  $\varphi(z_j) > \varphi(z_i)$ , the minimum contribution of each term in the sum happens in the median point

$$\hat{z}_i = \hat{z}_j = \frac{z_i + z_j}{2}.$$

Thus, we have

$$\begin{aligned} 2n \|z - \hat{z}\|^2 &> \sum_{(i,j) \in R} \left\| z_i - \frac{z_i + z_j}{2} \right\|^2 + \left\| z_j - \frac{z_i + z_j}{2} \right\|^2 \\ &= \sum_{(i,j) \in R} \frac{\|z_i - z_j\|^2}{2} \end{aligned}$$

Now set  $v = v_2$  and  $w = v_3$  and notice that  $(z_i - z_j)U^T = (v_i - v_j, w_i - w_j)$  and, by the definition of  $R$ ,  $\|z_i - z_j\|^2$  is minimum for  $j = i + k$ . Thus we write

$$\begin{aligned} 2n \|z - \hat{z}\|^2 &> \sum_{(i,j) \in R} \frac{\|z_i - z_{i+k}\|^2}{2} \\ &= \sum_{(i,j) \in R} \frac{\|(z_i - z_{i+k})U^T\|^2}{2} \\ &= \sum_{(i,j) \in R} \frac{(v_i - v_{i+k})^2 + (w_i - w_{i+k})^2}{2}. \end{aligned}$$

By Lemma 11, for  $n$  large enough we obtain

$$2n \|z - \hat{z}\|^2 > C \sum_{(i,j) \in R} \frac{k^4}{n^5},$$

for an absolute constant  $C > 0$ .

Now  $k = k(n) = \Omega(n^\beta)$ , so there exists a constant  $c$  such that for  $n$  large enough  $k \geq cn^\beta$ . We can bound

$$\|z - \hat{z}\|^2 > C_0 |R| \frac{n^{4\beta}}{n^6},$$

for a constant  $C_0 > 0$ . □

Now, the proof of Theorem 2 easily follows from Theorem 12 and Theorem 10. First, we make an observation about the order given by Algorithm 1. Let  $v_2, v_3$  be unitary eigenvectors for  $\lambda_2(M)$  and  $\lambda_3(M)$ , and  $\hat{v}_2$  and  $\hat{v}_3$  be unitary eigenvectors for  $\lambda_2(\hat{M})$  and  $\lambda_3(\hat{M})$ . Let  $U\Sigma W^T$  be the singular value decomposition for the matrix  $[v_2, v_3]^T [\hat{v}_2, \hat{v}_3]$ . Let  $\hat{x} = \hat{v}_2$  and  $\hat{y} = \hat{v}_3$  as in Algorithm 1, and let  $\varphi((\hat{x}_i, \hat{y}_i))$  be the angular coordinate of the point  $(\hat{x}_i, \hat{y}_i)$ . Define the matrices  $z = (v_2 \ v_3)U$  and  $\hat{z} = (\hat{v}_2 \ \hat{v}_3)W$ . Finally, let

$$R = \{(i, j) : \varphi(\hat{z}_j) \leq \varphi(\hat{z}_i) \text{ and } i + k \leq j \text{ and } i - j \geq k \text{ mod } n\}.$$

Notice that since  $W$  is a rotation matrix, it holds

$$\varphi((\hat{x}_i, \hat{y}_i)) - \varphi((\hat{x}_j, \hat{y}_j)) = \varphi(\hat{z}_i) - \varphi(\hat{z}_j).$$

Thus the order induced by the row vectors of  $\hat{z}$  is the same as the order induced by the row vectors of  $[\hat{v}_2, \hat{v}_3]$ . That implies  $D_k(\sigma) = |R|$ , where  $\sigma$  is the permutation returned by the Algorithm 1. Therefore, we proceed bounding  $|R|$ .

*Proof.* (Theorem 2) By theorems 12 and 10 we have

$$C_0 |R| \frac{n^{4\beta}}{n^6} < \|z - \hat{z}\|^2 \leq \bar{C}_0 n^{5-6\gamma},$$

where  $C_0$  and  $\bar{C}_0$  are positive constants and the upper bound holds with probability at least  $1 - n^{-3}$ . Therefore, there is a constant  $C > 0$  such that

$$|R| < C n^{11-6\gamma-4\beta},$$

with probability at least  $1 - n^{-3}$ .  $\square$

The proof of Theorem 3 is similar to the last one but uses a different trick to get another lower bound.

*Proof.* (Theorem 3) As in Theorem 6, let  $U\Sigma W^T$  be the singular value decomposition for the matrix  $[v_2, v_3]^T [\hat{v}_2, \hat{v}_3]$ . Thus,  $z = (v_2, v_3)U$  and  $\hat{z} = (\hat{v}_2, \hat{v}_3)W$ . Let  $\varphi(z_i)$  be the angular coordinate of the point  $z_i = (x_i, y_i)$ . Thus,  $\{\varphi(z_i)\}_{i=1}^n$  is an increasing sequence. Fix  $k = k(n) = C(n^\beta)$  and let

$$R = \{(i, j) : \varphi(\hat{z}_j) \leq \varphi(\hat{z}_i) \text{ and } i + k \leq j \text{ and } i - j \geq k \text{ mod } n\}.$$

Then  $R$  is the set of pairs in  $\hat{z}$  that disagree with the order induced by  $z$  by at least  $k$  positions in both directions on the cycle. Now we can write

$$2n \|z - \hat{z}\|^2 = \sum_{i=1}^n \sum_{j=1}^n \|z_i - \hat{z}_i\|^2 + \|z_j - \hat{z}_j\|^2 \geq \sum_{(i,j) \in R} \|z_i - \hat{z}_i\|^2 + \|z_j - \hat{z}_j\|^2.$$

Since  $\varphi(\hat{z}_j) \leq \varphi(\hat{z}_i)$  and  $\varphi(z_j) > \varphi(z_i)$ , the minimum contribution of each term in the sum happens in the median point

$$\hat{z}_i = \hat{z}_j = \frac{z_i + z_j}{2}.$$

Thus, we have

$$\begin{aligned} 2n \|z - \hat{z}\|^2 &> \sum_{(i,j) \in R} \left\| z_i - \frac{z_i + z_j}{2} \right\|^2 + \left\| z_j - \frac{z_i + z_j}{2} \right\|^2 \\ (4.1) \qquad &= \sum_{(i,j) \in R} \frac{\|z_i - z_j\|^2}{2} \end{aligned}$$

Now, let  $n_i$  denote the number of pairs  $(i, j)$  and label them  $(i, j_{i_1}), \dots, (i, j_{i_{n_i}})$  for  $i = 1, \dots, n$  and therefore  $\sum_{i=1}^n n_i = |R|$ . Furthermore, for a fixed  $i$  we obtain the minimum  $\|z_i - z_j\|^2$  whenever  $j - i$  is minimum. By definition of  $R$  the sum

$$\sum_{t=1}^{n_i} \frac{\|z_i - z_j\|^2}{2}$$

is minimum whenever  $j_{i_1} = i+k$ ,  $j_{i_2} = i+k+1, \dots, j_{i_{n_i}} = i+k+n_i-1$ . Therefore, setting  $v = v_2$  and  $w = v_3$ , inequality 4.1 becomes

$$2n\|z - \hat{z}\|^2 > \sum_{i=1}^n \sum_{t=0}^{n_i-1} \frac{\|z_i - z_{i+k+t}\|^2}{2} = \sum_{i=1}^n \sum_{t=0}^{n_i-1} \frac{(v_i - v_{i+k+t})^2 + (w_i - w_{i+k+t})^2}{2}.$$

By Lemma 11, for  $n$  large enough,

$$2n\|z - \hat{z}\|^2 > C_1 \sum_{i=1}^n \sum_{t=0}^{n_i-1} \frac{(k+t)^4}{n^5} > C_1 \frac{k^2}{n^5} \sum_{i=1}^n \sum_{t=0}^{n_i-1} t^2$$

for a constant  $C_1 > 0$ . Therefore, there is a constant  $C_2 > 0$  such that  $2n\|z - \hat{z}\|^2 > C_2 \frac{k^2}{n^5} \sum_{i=1}^n n_i^3$ . Now, recall that two  $p$ -norms are related by  $\|x\|_p \leq n^{\frac{1}{p}-\frac{1}{q}} \|x\|_q$ . Taking  $p = 1$  and  $q = 3$ , we obtain

$$\sum_{i=1}^n n_i \leq n^{\frac{2}{3}} \left( \sum_{i=1}^n n_i^3 \right)^{\frac{1}{3}}$$

which allows us to rewrite inequality 4.1 as

$$2n\|z - \hat{z}\| > C_1 k^2 |R|^3 n^{-7}.$$

Combining this inequality with the upper bound of Theorem 10 and using  $k = k(n) = \Omega(n^\beta)$ , we obtain a constant  $C_2 > 0$  such that, with probability  $1 - n^{-3}$ ,

$$|R| < C_2 n^{\frac{13-6\gamma-2\beta}{3}}$$

and therefore  $D_k(\sigma) \in \mathcal{O}(n^{\frac{13-6\gamma-2\beta}{3}})$ .  $\square$

Eventually, we give a proof for Theorem 4.

*Proof.* (Theorem 4) Fix  $k = n^{(10-6\gamma)/5}$  and define

$$R = \{(i, j) : \varphi(\hat{z}_j) \leq \varphi(\hat{z}_i) \text{ and } i+k \leq j \text{ and } i-j \geq k \pmod{n}\}$$

and

$$R^C = \{(i, j) : \varphi(\hat{z}_j) \leq \varphi(\hat{z}_i) \text{ and } j < i+k \text{ or } i-j < k \pmod{n}\}.$$

By Theorem 2, taking  $\beta = \frac{10-6\gamma}{5}$ , there is a constant  $C > 0$  so that, for large enough  $n$ ,

$$|R| = D_k(\sigma) \leq C n^{11-6\gamma-4\beta} = C n^{(15-6\gamma)/5}.$$

Furthermore, for each index  $i$  there are at most  $2k$  pairs  $(i, j)$  in  $R^C$ , thus

$$|R^C| \leq 2kn = 2n^{\beta+1} = 2n^{(15-6\gamma)/5}$$

and therefore  $D(\sigma) = |R| + |R^C| \leq (C+2)n^{(15-6\gamma)/5}$ , as required.  $\square$

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