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Duality Results for Extended Multifacility Location Problems*

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Abstract: Duality statements are presented for multifacility location problems as suggested by Drezner (1991), where for each given point the sum of weighted distances to all facilities plus set-up costs is determined and the maximal value of these sums is to be minimized. We develop corresponding dual problems for the cases with and without set-up costs and present associated optimality conditions. In the concluding part of this note we use these optimality conditions for a geometrical characterization of the set of optimal solutions and consider for an illustration corresponding examples.

Key words: Gauges, Continuous Minimax Multifacility Location Problems, Set-up Costs, Conjugate Duality, Optimality Conditions.

AMS subject classification: 49N15, 90C25, 90C46, 90B85.

1 Introduction

In 1991 Drezner developed a location model in [6], which describes the following emergency scenario. A certain number of emergency calls arise and ask for an ambulance. To each of these demand points an ambulance is sent to load and transport the patient to a hospital. The location of the ambulance-station and the hospital must not be necessary on the same site. This assumption may shorten the response time for the patients, especially for the farthest one, in the situation when for example a hospital is completely overcrowded or short of medical supplies. The aim is now to determine the location of the ambulance-station and the hospital such that the maximum time required before the farthest patient arrives at the hospital will be minimized. In this case the maximum time is naturally defined as the sum of the travel time of the ambulance from the ambulance-station to the patient and the travel time to the hospital plus some set-up costs. Set-up costs like the loading time at the emergency and the unloading time at the hospital of the patient are a view examples to cite.
While Drezner suggested a model for the case of the Euclidean norm, Michelot and Plastria [17] work in a higher dimensional space where the distances are measured by a general norm. In this paper we generalize this location model to the situation where the distances are measured by mixed gauges defined on a Fréchet space. The goal is then to describe these type of location problems in the framework of conjugate duality.
To do this we first recall some important elements of Convex Analysis and continue in Section 3

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with the study of location problems with set-up costs. We will construct corresponding conjugate dual problems and prove strong duality from which we derive some optimality conditions. Afterwards, we consider a special case of these location problems where the weights have a multiplicative structure like treated by Michelot and Plastria in [17] and describe the relation to their conjugate dual problems with the Euclidean norm as distance measures. In the end of this note, we study also location problems without set-up costs via conjugate duality. Besides of strong duality assertions and optimality conditions we will give geometrical characterizations of the set of optimal solutions of the conjugate dual problem as well as illustrating examples.

2 Preliminaries

2.1 Elements of convex analysis

Let X be a Fréchet space and X^* its topological dual space endowed with the weak* topology $w(X^*, X)$. For $x \in X$ and $x^* \in X^*$, let $\langle x^*, x\rangle := x^*(x)$ be the value of the linear continuous functional x^* at x. For a subset $A \subseteq X$, its indicator function $\delta_A : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is

$$\delta_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

For a given function $f : X \to \overline{\mathbb{R}}$ we consider its effective domain

$$\operatorname{dom} f := \{x \in X : \ f(x) < +\infty\}$$

and call $f : X \to \overline{\mathbb{R}}$ proper if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. The conjugate function of f is defined by

$$f^* : X^* \to \overline{\mathbb{R}}, \ f^*(x^*) = \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\}.$$

A function $f : X \to \overline{\mathbb{R}}$ is called convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in X$ and all $\lambda \in [0, 1]$. A function $f : X \to \overline{\mathbb{R}}$ is called lower semicontinuous at $\overline{x} \in X$ if $\liminf_{x \to \overline{x}} f(x) \geq f(\overline{x})$ and when this function is lower semicontinuous at all $x \in X$, then we call it lower semicontinuous (l.s.c. for short).

In this paper we do not use the classical differentiability, but we use the notion of subdifferentiability to formulate optimality conditions. If we take an arbitrary $x \in X$ such that $f(x) \in \mathbb{R}$, then we call the set

$$\partial f(x) := \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle \ \forall y \in X\}$$

the (convex) subdifferential of f at x, where the elements are called the subgradients of f at x. Moreover, if $\partial f(x) \neq \emptyset$, then we say that f is subdifferentiable at x and if $f(x) \notin \mathbb{R}$, then we make the convention that $\partial f(x) := \emptyset$. Note, that the subgradients can be characterized by the conjugate function, especially this means

$$x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) = \langle x^*, x\rangle, \ \forall x \in X, \ x^* \in X^*, \tag{1}$$

i.e. the Young-Fenchel inequality is fulfilled with equality.

In the next, we collect some properties of the gauge function. Let $C \subseteq X$, then the gauge function of the subset C, $\gamma_C : X \to \overline{\mathbb{R}}$, is defined by

$$\gamma_C(x) := \begin{cases} +\infty, & \text{if } \{\lambda > 0 : x \in \lambda C\} = \emptyset, \\ \inf\{\lambda > 0 : x \in \lambda C\}, & \text{otherwise.} \end{cases}$$

The following statements with proofs were given in [22].

Theorem 2.1. *Let $C \subseteq X$ be a convex and closed set with $0_X \in C$, then the gauge function γ_C is proper, convex and lower semicontinuous.*

Lemma 2.1. *Let $C \subseteq X$ be a convex and closed set with $0_X \in C$, then the conjugate of the gauge function γ_C is given by*

$$\gamma_C^*(x^*) := \begin{cases} 0, & \text{if } \sigma_C(x^*) \leq 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

where σ_C is the support function of the set C, i.e. $\sigma_C(x^*) = \sup_{x \in C} \langle x^*, x \rangle$.

Definition 2.1. *Let $C \subseteq X$. The polar set of C is defined by*

$$C^0 := \left\{ x^* \in X^* : \sup_{x \in C} \langle x^*, x \rangle \leq 1 \right\} = \{x^* \in X^* : \sigma_C(x^*) \leq 1\}$$

and by means of the polar set the dual gauge is defined by

$$\gamma_{C^0}(x^*) := \sup_{x \in C} \langle x^*, x \rangle = \sigma_C(x^*).$$

Remark 2.1. *Note that C^0 is a convex and closed set containing the origin. Furthermore, by the definition of the dual gauge follows that the conjugate function of γ_C can equivalently be expressed by*

$$\gamma_C^*(x^*) := \begin{cases} 0, & \text{if } \gamma_{C^0}(x^*) \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Moreover, the following theorem and lemma were proven in [22].

Theorem 2.2. *Let $a_i \in \mathbb{R}_+$ be a given point and $h_i : \mathbb{R} \to \overline{\mathbb{R}}$ with $h_i(x) \in \mathbb{R}_+$, if $x \in \mathbb{R}_+$, and $h_i(x) = +\infty$, otherwise, be a proper, lower semicontinuous and convex function, $i = 1, ..., n$. Then the conjugate of the function $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by*

$$g(x_1, ..., x_n) := \begin{cases} \max\{h_1(x_1) + a_1, ..., h_n(x_n) + a_n\}, & \text{if } x_i \in \mathbb{R}_+,\ i = 1, ..., n, \\ +\infty, & \text{otherwise,} \end{cases}$$

is given by $g^* : \mathbb{R}^n \to \mathbb{R}$,

$$g^*(x_1^*, ..., x_n^*) = \min_{\substack{\sum_{i=1}^n z_i^{0*} \leq 1,\ z_i^{0*} \geq 0, \\ i=1,...,n}} \left\{ \sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] \right\}.$$

Lemma 2.2. *Let $a_i \in \mathbb{R}_+$ be a given point and $h_i : \mathbb{R} \to \overline{\mathbb{R}}$ with $h_i(x) \in \mathbb{R}_+$, if $x \in \mathbb{R}_+$, and $h_i(x) = +\infty$, otherwise, be a proper, lower semicontinuous and convex function, $i = 1, ..., n$. Then the function $g : \mathbb{R}^m \to \overline{\mathbb{R}}$,*

$$g(x_1, ..., x_n) = \begin{cases} \max\{h_1(x_1) + a_1, ..., h_n(x_n) + a_n\}, & \text{if } x_i \in \mathbb{R}_+,\ i = 1, ..., n, \\ +\infty, & \text{otherwise,} \end{cases}$$

can equivalently be expressed as

$$g(x_1,...,x_n) = \sup_{\substack{\sum\limits_{i=1}^{n} z_i^{0*}\le 1,\ z_i^{0*}\ge 0,\\ i=1,...,n}} \left\{ \sum_{i=1}^{n} z_i^{0*}[h_i(x_i)+a_i] \right\},\ \forall x_i \ge 0,\ i=1,...,n.$$

2.2 Nonlinear location problems with set-up costs

This section is devoted to recall some basic duality results for nonlinear single minimax location problems with set-up costs stated in [22]. For this purpose, let X be a Fréchet space, S a non-empty, closed and convex subset of X, C_i a non-empty, closed and convex subset of X such that $0_X \in \operatorname{int} C_i$, $\gamma_{C_i} : X \to \mathbb{R}$ a gauge function of the subset C_i and $h_i : \mathbb{R} \to \overline{\mathbb{R}}$ a proper, convex, lower semicontinuous and increasing function on $\mathbb{R}_+$ defined by

$$h_i(x) := \begin{cases} h_i(x) \in \mathbb{R}_+, & \text{if } x \in \mathbb{R}_+,\\ +\infty, & \text{otherwise,} \end{cases}$$

$i = 1,...,n$. For given non-negative set-up costs $a_i \in \mathbb{R}_+$ and distinct points $p_i \in X$ the location problem of interest is then given by

$$(P_{h,a}^S) \qquad \inf_{x\in S} \sup_{1\le i\le n} \{h_i(\gamma_{C_i}(x-p_i)) + a_i\},$$

while its associated conjugate dual problem $(D_{h,a}^S)$ has the form

$$(D_{h,a}^S) \sup_{\substack{I\subseteq R\subseteq\{1,...,n\},\ \lambda_k>0,\ k\in R,\ \lambda_l=0,\ l\notin R,\\ z_i^{0*}>0,\ z_i^{1*}\in X^*,\ \gamma_{C_i^0}(z_i^{1*})\le z_i^{0*},\ i\in I,\\ z_j^{0*}=0,\ z_j^{1*}=0_{X^*},\ j\notin I,\ \sum\limits_{r\in R}\lambda_r\le 1}} \left\{ \inf_{x\in S}\left\{ \sum_{i\in I} \langle z_i^{1*}, x-p_i\rangle \right\} - \sum_{r\in R} \lambda_r \left[h_r^*\left(\frac{z_r^{0*}}{\lambda_r} \right) - a_r \right] \right\}. \quad (2)$$

Under these settings the strong duality statement for $(P_{h,a}^S)$ and its dual problem $(D_{h,a}^S)$ follows.

Theorem 2.3. *(strong duality) Between $(P_{h,a}^S)$ and $(D_{h,a}^S)$ strong duality holds, i.e. $v(P_{h,a}^S) = v(D_{h,a}^S)$ and the conjugate dual problem has an optimal solution.*

Remark 2.2. *Like mentioned in [22], the results in this section hold also for negative set-up costs, with the difference that we have in the constraint set of the dual problem $\sum_{r\in R}\lambda_r = 1$, instead $\sum_{r\in R}\lambda_r \le 1$.*
One can easily observe that this fact holds also in the upcoming remark.

Remark 2.3. *If $h_i : \mathbb{R} \to \overline{\mathbb{R}}$ is defined by*

$$h_i(x) := \begin{cases} x, & \textit{if } x \in \mathbb{R}_+,\\ +\infty, & \textit{otherwise,} \end{cases}$$

then the conjugate function of h_i is

$$h_i^*(x^*) = \sup_{x\in\mathbb{R}_+}\{x^*x - x\} = \sup_{x\in\mathbb{R}_+}\{x(x^*-1)\} = \begin{cases} 0, & \textit{if } x^* \le 1,\\ +\infty, & \textit{otherwise,} \end{cases},\ i=1,...,n,$$

and the conjugate dual problem $(D^S_{h,a})$ *transforms to*

$$(D^S_{h,a}) \quad \sup_{\substack{\widetilde{I}\subseteq R\subseteq\{1,...,n\},\ \lambda_k>0,\ y_k^{0*}\leq\lambda_k,\ k\in R,\ \lambda_l=0,\ l\notin R,\\ y_i^{0*}>0,\ y_i^{1*}\in X^*,\ \gamma_{C_i^0}(y_i^{1*})\leq y_i^{0*},\ i\in\widetilde{I},\\ y_j^{0*}=0,\ y_j^{1*}=0_{X^*},\ j\notin\widetilde{I},\ \sum\limits_{r\in R}\lambda_r\leq 1}} \left\{ \inf_{x\in S}\left\{\sum_{i\in\widetilde{I}}\langle y_i^{1*}, x-p_i\rangle\right\} + \sum_{r\in R}\lambda_r a_r \right\}.$$

This dual problem can be reduced to the following equivalent problem

$$(\widetilde{D}^S_{h,a}) \quad \sup_{\substack{I\subseteq\{1,...,n\},\ z_i^{0*}>0,\ z_i^{1*}\in X^*,\ \gamma_{C_i^0}(z_i^{1*})\leq z_i^{0*},\ i\in I,\\ z_j^{0*}=0,\ z_j^{1*}=0_{X^*},\ j\notin I,\ \sum\limits_{i\in I} z_i^{0*}\leq 1}} \left\{ \inf_{x\in S}\left\{\sum_{i\in I}\langle z_i^{1*}, x-p_i\rangle\right\} + \sum_{i\in I} z_i^{0*} a_i \right\}. \tag{3}$$

To see the equivalence between $(D^S_{h,a})$ *and* $(\widetilde{D}^S_{h,a})$*, take first a feasible element* $(\lambda, y^{0*}, y^{1*}) = (\lambda_1, ..., \lambda_n, y_1^{0*}, ..., y_n^{0*}, y_1^{1*}, ..., y_n^{1*}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times (X^*)^n$ *of the problem* $(D^S_{h,a})$ *and set* $I = R$, $z_i^{0*} = \lambda_i,\ i \in I,\ z_j^{0*} = 0,\ j \notin I$ *and* $z_i^{1*} = y_i^{1*},\ i \in \widetilde{I} \subseteq I,\ z_j^{1*} = 0_{X^*},\ j \notin \widetilde{I}$ *(i.e.* $z_i^{1*} \in X^*,\ i \in I$ *and* $z_j^{1*} = 0_{X^*},\ j \notin I$*), then it follows from the feasibility of* $(\lambda, y^{0*}, y^{1*})$ *that* $\sum_{i\in I}^n z_i^{0*} \leq 1,\ z_i^{0*} > 0,\ z_i^{1*} \in X^*,\ \gamma_{C_i^0}(z_i^{1*}) \leq z_i^{0*},\ i \in I$ *and* $z_j^{0*} = 0,\ z_j^{1*} = 0_{X^*},\ j \notin I$*, i.e.* $(z^{0*}, z^{1*}) = (z_1^{0*}, ..., z_n^{0*}, z_1^{1*}, ..., z_n^{1*}) \in \mathbb{R}^n_+ \times (X^*)^n$ *is feasible to the problem* $(\widetilde{D}^S_{h,a})$*. Hence, it holds*

$$\inf_{x\in S}\left\{\sum_{i=1}^n\langle y_i^{1*}, x-p_i\rangle\right\} + \sum_{i=1}^n \lambda_i a_i = \inf_{x\in S}\left\{\sum_{i=1}^n\langle z_i^{1*}, x-p_i\rangle\right\} + \sum_{i=1}^n z_i^{0*} a_i \leq v(\widetilde{D}^S_{h,a}),$$

for all $(\lambda, y^{0*}, y^{1*})$ *feasible to* $(D^S_{h,a})$*, i.e.* $v(D^S_{h,a}) \leq v(\widetilde{D}^S_{h,a})$ *(where* $v(D^S_{h,a})$ *and* $v(\widetilde{D}^S_{h,a})$ *denote the optimal objective values of the dual problems* $(D^S_{h,a})$ *and* $(\widetilde{D}^S_{h,a})$*, respectively).*
Now, take a feasible element (z^{0*}, z^{1*}) *of the problem* $(\widetilde{D}^S_{h,a})$ *and set* $I = \widetilde{I} = R$, $y_i^{0*} = \lambda_i = z_i^{0*}$ *and* $y_i^{1*} = z_i^{1*}$ *for* $i \in \widetilde{I} = R$ *and* $y_j^{0*} = \lambda_j = 0$ *for* $j \notin \widetilde{I} = R$*, then we have from the feasibility of* (z^{0*}, z^{1*}) *that* $\sum_{r\in R}\lambda_r \leq 1,\ y_k^{0*} = \lambda_k > 0,\ k \in R,\ \lambda_l = 0,\ l \notin R$ *and* $\gamma_{C_i^0}(y_i^{1*}) \leq y_i^{0*},\ i \in \widetilde{I}$*, which means that* $(\lambda, y^{0*}, y^{1*})$ *is a feasible element of* $(D^S_{h,a})$ *and it holds*

$$\inf_{x\in S}\left\{\sum_{i=1}^n\langle z_i^{1*}, x-p_i\rangle\right\} + \sum_{i=1}^n z_i^{0*} a_i = \inf_{x\in S}\left\{\sum_{i=1}^n\langle y_i^{1*}, x-p_i\rangle\right\} + \sum_{i=1}^n \lambda_i a_i \leq v(D^S_{h,a}),$$

for all (z^{0*}, z^{1*}) *feasible to* $(\widetilde{D}^S_{h,a})$*, which implies* $v(\widetilde{D}^S_{h,a}) \leq v(D^S_{h,a})$*. Finally, it follows that* $v(\widetilde{D}^S_{h,a}) = v(D^S_{h,a})$*.*

3 Duality results

3.1 Extended multifacility location problems with set-up costs

The location problem, which we investigate in a more general setting as suggested by Drezner in [6] and studied by Michelot and Plastria in [17], is

$$(EP_a^M) \quad \inf_{(x_1,...,x_m)\in X^m} \max_{1\leq i\leq n} \left\{ \sum_{j=1}^m \tau_{C_{ij}}(x_j - p_i) + a_i \right\},$$

where X is a Fréchet space, $a_i \in \mathbb{R}_+$ are non-negative set-up costs, $p_i \in X$ are distinct points and $\tau_{C_{ij}} : X \to \mathbb{R}$ are gauges defined by closed and convex subsets C_{ij} of X such that $0_X \in \operatorname{int} C_{ij}$,

$$\tau_{C_{ij}}(y) := \inf\{\lambda > 0 : y \in \lambda C_{ij}\},\ i = 1, ..., n,\ j = 1, ..., m.$$

Now, set $\widetilde{X} = X^m$, $x = (x_1, ..., x_m) \in \widetilde{X}$, $\widetilde{p}_i = (p_i, ..., p_i) \in \widetilde{X}$ and define the gauge $\gamma_{C_i} : \widetilde{X} \to \mathbb{R}$ by

$$\gamma_{C_i}(x) := \sum_{j=1}^{m} \tau_{C_{ij}}(x_j),\ x = (x_1, ..., x_m) \in \widetilde{X},$$

where $C_i = \{x \in \widetilde{X} : \gamma_{C_i}(x) \leq 1\},\ i = 1, ..., n$. Then, it is obvious that the location problem (EP_a^M) can also be written in a slightly different form, namely as a single minimax location problem

$$(EP_a^M) \qquad \inf_{x \in \widetilde{X}} \max_{1 \leq i \leq n} \{\gamma_{C_i}(x - \widetilde{p}_i) + a_i\}.$$

Recall that the sum of gauges is itself a gauge. In the following, let $\widetilde{X}^* = (X^*)^m$ be the associated topological dual space of $\widetilde{X}$ where for $x = (x_1, ..., x_m) \in \widetilde{X}$ and $x^* = (x_1^*, ..., x_m^*) \in \widetilde{X}^* = (X^*)^m$ we define $\langle x^*, x\rangle := \sum_{j=1}^{m} \langle x_j^*, x_j\rangle$. Hence, for the associated dual gauge of γ_{C_i} holds

$$\gamma_{C_i^0}(x^*) = \sup_{x \in C_i} \langle x^*, x\rangle,\ i = 1, ..., n.$$

Now, we fix $x^* = (x_1^*, ..., x_m^*) \in \widetilde{X}^*$ and consider the problem

$$(P_i^{\gamma^0}) \quad \inf_{x \in C_i} \langle -x^*, x\rangle = \inf_{x \in \widetilde{X},\ \gamma_{C_i}(x) \leq 1} \langle -x^*, x\rangle,$$

where its associated Lagrange dual problem is

$$\begin{aligned} (D_{iL}^{\gamma^0}) \qquad & \sup_{\lambda \geq 0} \inf_{x \in \widetilde{X}} \{\langle -x^*, x\rangle + \lambda(\gamma_{C_i}(x) - 1)\} = \sup_{\lambda \geq 0} \left\{ -\lambda + \inf_{x \in \widetilde{X}} \{\langle -x^*, x\rangle + \lambda \gamma_{C_i}(x)\} \right\} \\ = \quad & \sup_{\lambda \geq 0} \left\{ -\lambda - \sup_{x \in \widetilde{X}} \{\langle x^*, x\rangle - \lambda \gamma_{C_i}(x)\} \right\} = \sup_{\lambda \geq 0} \left\{ -\lambda - (\lambda \gamma_{C_i})^*(x^*) \right\},\ i = 1, ..., n. \end{aligned}$$

For $\lambda > 0$ it holds (see [2])

$$\begin{aligned} (\lambda \gamma_{C_i})^*(x^*) &= \sup_{x \in \widetilde{X}} \{\langle x^*, x\rangle - \lambda \gamma_{C_i}(x)\} = \sup_{\substack{x_j \in X, \\ j=1,...,m}} \left\{ \sum_{j=1}^{m} \langle x_j^*, x_j\rangle - \lambda \sum_{j=1}^{m} \tau_{C_{ij}}(x_j) \right\} \\ &= \sum_{j=1}^{m} \sup_{x_j \in X} \{\langle x_j^*, x_j\rangle - \lambda \tau_{C_{ij}}(x_j)\} \\ &= \sum_{j=1}^{m} \lambda \sup_{x_j \in X} \left\{ \left\langle \frac{1}{\lambda} x_j^*, x_j \right\rangle - \tau_{C_{ij}}(x_j) \right\} \\ &= \sum_{j=1}^{m} \lambda \tau_{C_{ij}}^* \left(\frac{1}{\lambda} x_j^* \right) = \begin{cases} 0, \text{ if } \sigma_{C_{ij}}(x_j^*) \leq \lambda,\ \forall j = 1, ..., m, \\ +\infty, \text{ otherwise} \end{cases} \\ &= \begin{cases} 0, \text{ if } \tau_{C_{ij}^0}(x_j^*) \leq \lambda,\ \forall j = 1, ..., m, \\ +\infty, \text{ otherwise} \end{cases} \end{aligned} \tag{4}$$

and for $\lambda = 0$ we have

$$\begin{aligned}(0 \cdot \gamma_{C_i})^*(x^*) &= \sup_{x \in \widetilde{X}} \{\langle x^*, x\rangle\} = \sup_{\substack{x_j \in X,\\ j=1,\dots,m}} \left\{ \sum_{j=1}^m \langle x_j^*, x_j\rangle \right\} \\ &= \sum_{j=1}^m \sup_{x_j \in X} \{\langle x_j^*, x_j\rangle\} = \begin{cases} 0, \text{ if } x_j^* = 0_{X^*}, \ \forall j = 1, ..., m, \\ +\infty, \text{ otherwise,} \end{cases} \end{aligned} \tag{5}$$

$i = 1, ..., n$. As $\tau_{C_{ij}^0}(0_{X^*}) = \sup_{x_j \in C_{ij}} \langle 0_{X^*}, x_j\rangle = 0$, one gets by (4) and (5) for the Lagrange dual problem $(D_{iL}^{\gamma^0})$ that

$$(D_{iL}^{\gamma^0}) \qquad \sup_{\lambda \geq 0} \{-\lambda - (\lambda\gamma_{C_i})^*(x^*)\} = \sup_{\lambda \geq 0} \left\{ -\lambda : \tau_{C_{ij}^0}(x_j^*) \leq \lambda, \ \forall j = 1, ..., m \right\}, \ i = 1, ..., n,$$

and since for the primal-dual pair (P^{γ^0})-$(D_{iL}^{\gamma^0})$ the Slater constraint qualification is fulfilled, it holds strong duality. From the last statement we derive an alternative formula for the dual gauge $\gamma_{C_i^0}$,

$$\begin{aligned}\gamma_{C_i^0}(x^*) &= \sup_{x \in C_i} \langle x^*, x\rangle = \min_{\lambda \geq 0} \left\{ \lambda : \tau_{C_{ij}^0}(x_j^*) \leq \lambda, \ \forall j = 1, ..., m \right\} \\ &= \max_{1 \leq j \leq m} \left\{ \min_{\lambda \geq 0} \left\{ \lambda : \tau_{C_{ij}^0}(x_j^*) \leq \lambda \right\} \right\} = \max_{1 \leq j \leq m} \left\{ \tau_{C_{ij}^0}(x_j^*) \right\}, \ i = 1, ..., n. \end{aligned} \tag{6}$$

We use (3) and (6) and get for the dual problem corresponding to (EP_a^M)

$$(ED_a^M) \qquad \sup_{\substack{I \subseteq \{1,\dots,n\}, \ z_i^{0*} > 0, \ z_{ij}^{1*} \in X^*, \ \tau_{C_{ij}^0}(z_{ij}^{1*}) \leq z_i^{0*}, \ i \in I, \\ z_k^{0*} = 0, \ z_{kj}^{1*} = 0_{X^*}, \ k \notin I, \ \sum_{i \in I} z_i^{0*} \leq 1, \ j = 1,\dots,m}} \left\{ \inf_{x \in \widetilde{X}} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - \widetilde{p}_i\rangle \right\} + \sum_{i \in I} z_i^{0*} a_i \right\}.$$

Because

$$\begin{aligned}\inf_{x \in \widetilde{X}} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - \widetilde{p}_i\rangle \right\} &= \inf_{\substack{x_l \in X,\\ l=1,\dots,m}} \left\{ \sum_{i \in I} \sum_{j=1}^m \langle z_{ij}^{1*}, x_j - p_i\rangle \right\} \\ = \sum_{j=1}^m \inf_{x_j \in X} \left\{ \sum_{i \in I} \langle z_{ij}^{1*}, x_j\rangle \right\} &- \sum_{i \in I} \sum_{j=1}^m \langle z_{ij}^{1*}, p_i\rangle,\end{aligned}$$

we obtain finally for the conjugate dual problem of (EP_a^M)

$$(ED_a^M) \qquad \sup_{(z_1^{0*},\dots,z_n^{0*},z_1^{1*},\dots,z_n^{1*}) \in \mathcal{C}} \left\{ -\sum_{i \in I} \left[\left\langle \sum_{j=1}^m z_{ij}^{1*}, p_i \right\rangle - z_i^{0*} a_i \right] \right\},$$

where

$$\mathcal{C} = \Bigg\{ (z_1^{0*}, ..., z_n^{0*}, z_1^{1*}, ..., z_n^{1*}) \in \mathbb{R}^n \times (X^*)^m \times ... \times (X^*)^m : I \subseteq \{1, ..., n\},$$
$$\sum_{i \in I} z_i^{0*} \leq 1, \ \sum_{i \in I} z_{ij}^{1*} = 0_{X^*}, \ z_i^{0*} > 0, \ z_{ij}^{1*} \in X^*, \ \tau_{C_{ij}^0}(z_{ij}^{1*}) \leq z_i^{0*}, \ i \in I,$$
$$z_k^{0*} = 0, \ z_{kj}^{1*} = 0_{X^*}, \ k \notin I, \ j = 1, ..., m \Bigg\}.$$

Remark 3.1. *A similar dual problem was formulated by Michelot and Cornejo in [4] in the situation where X is the Euclidean space, $m = 2$ and the gauges are the Euclidean norm. The authors construct in their paper a Fenchel duality scheme to solve extended minimax location problems by a proximal algorithm.*

Let $v(EP_a^M)$ be the optimal objective value of the location problem (EP_a^M) and $v(ED_a^M)$ be the optimal objective value of the dual problem (ED_a^M), then we obtain the following duality statement as a direct consequence of Theorem 2.3.

Theorem 3.1. *(strong duality) Between (EP_a^M) and (ED_a^M) holds strong duality, i.e. $v(EP_a^M) = v(ED_a^M)$ and the conjugate dual problem has an optimal solution.*

The following necessary and sufficient optimality conditions are a consequence of the previous theorem.

Theorem 3.2. *(optimality conditions) (a) Let $(\overline{x}_1, ..., \overline{x}_m) \in X^m$ be an optimal solution of the problem (EP_a^M). Then there exist $(\overline{z}_1^{0*}, ..., \overline{z}_n^{0*}, \overline{z}_1^{1*}, ..., \overline{z}_n^{1*}) \in \mathbb{R}^n \times (X^*)^m \times ... \times (X^*)^m$ and an index set $\overline{I} \subseteq \{1, ..., n\}$, an optimal solution to (ED_a^M), such that*

(i) $\max\limits_{1\leq u\leq n} \left\{ \sum\limits_{j=1}^{m} \tau_{C_{uj}}(\overline{x}_j - p_u) + a_u \right\} = \sum\limits_{i\in\overline{I}} \overline{z}_i^{0*} \left(\sum\limits_{j=1}^{m} \tau_{C_{ij}}(\overline{x}_j - p_i) + a_i \right),$

(ii) $\overline{z}_i^{0*}\tau_{C_{ij}}(\overline{x}_j - p_i) = \langle \overline{z}_{ij}^{1*}, \overline{x}_j - p_i \rangle,\ i \in \overline{I},\ j = 1, ..., m,$

(iii) $\sum\limits_{i\in\overline{I}} \overline{z}_{ij}^{1*} = 0_{X^*},\ j = 1, ..., m,$

(iv) $\sum\limits_{j\in\overline{I}} \overline{z}_j^{0*} = 1,\ \overline{z}_i^{0*} > 0,\ i \in \overline{I},$ *and* $\overline{z}_k^{0*} = 0,\ k \notin \overline{I},$

(v) $\sum\limits_{j=1}^{m} \tau_{C_{ij}}(\overline{x}_j - p_i) + a_i = \max\limits_{1\leq u\leq n} \left\{ \sum\limits_{j=1}^{m} \tau_{C_{uj}}(\overline{x}_j - p_u) + a_u \right\},\ i \in \overline{I},$

(vi) $\max\limits_{1\leq l\leq m} \left\{ \tau_{C_{il}^0}(\overline{z}_{il}^{1*}) \right\} = \overline{z}_i^{0*},\ \overline{z}_{ij}^{1*} \in X^*,\ i \in \overline{I},$ *and* $\overline{z}_{kj}^{1*} = 0_{X^*},\ k \notin \overline{I},\ j = 1, ..., m.$

(b) If there exists $(\overline{x}_1, ..., \overline{x}_m) \in X^m$ such that for some $(\overline{z}_1^{0*}, ..., \overline{z}_n^{0*}, \overline{z}_1^{1*}, ..., \overline{z}_n^{1*}) \in \mathbb{R}^n \times (X^*)^m \times ... \times (X^*)^m$ and an index set $\overline{I} \subseteq \{1, ..., n\}$ the conditions (i)-(vi) are fulfilled, then $(\overline{x}_1, ..., \overline{x}_m)$ is an optimal solution of (EP_a^M), $(\overline{z}_1^{0*}, ..., \overline{z}_n^{0*}, \overline{z}_1^{1*}, ..., \overline{z}_n^{1*}, \overline{I})$ is an optimal solution for (ED_a^M) and $v(EP_a^M) = v(ED_a^M)$.

Proof. From Theorem 3.1 we have $v(EP_a^M) = v(ED_a^M)$, i.e. for $(\overline{x}_1, ..., \overline{x}_m) \in X^m$ and $(\overline{z}_1^{0*}, ..., \overline{z}_n^{0*}, \overline{z}_1^{1*}, ..., \overline{z}_n^{1*}) \in \mathbb{R}^n \times (X^*)^m \times ... \times (X^*)^m$ and an index set $\overline{I} \subseteq \{1, ..., n\}$ it holds

$$\max_{1\leq u\leq n} \left\{ \sum_{j=1}^{m} \tau_{C_{uj}}(\overline{x}_j - p_u) + a_u \right\} = -\sum_{i\in\overline{I}} \left[\left\langle \sum_{j=1}^{m} \overline{z}_{ij}^{1*}, p_i \right\rangle - \overline{z}_i^{0*} a_i \right]$$

$$\Leftrightarrow \max_{1\leq u\leq n} \left\{ \sum_{j=1}^{m} \tau_{C_{uj}}(\overline{x}_j - p_u) + a_u \right\} + \sum_{i\in\overline{I}} \left[\left\langle \sum_{j=1}^{m} \overline{z}_{ij}^{1*}, p_i \right\rangle - \overline{z}_i^{0*} a_i \right] = 0$$

$$\Leftrightarrow \max_{1\leq u\leq n} \left\{ \sum_{j=1}^{m} \tau_{C_{uj}}(\overline{x}_j - p_u) + a_u \right\} + \sum_{i\in\overline{I}} \left[\left\langle \sum_{j=1}^{m} \overline{z}_{ij}^{1*}, p_i \right\rangle - \overline{z}_i^{0*} a_i \right]$$

$$
\begin{aligned}
&+\sum_{i\in\overline{I}}\sum_{j=1}^{m}\overline{z}_i^{0*}\tau_{C_{ij}}(\overline{x}_j-p_i)-\sum_{i\in\overline{I}}\sum_{j=1}^{m}\overline{z}_i^{0*}\tau_{C_{ij}}(\overline{x}_j-p_i)\\
&+\left\langle\sum_{i\in\overline{I}}\sum_{j=1}^{m}\overline{z}_{ij}^{1*},\overline{x}_j\right\rangle-\left\langle\sum_{i\in\overline{I}}\sum_{j=1}^{m}\overline{z}_{ij}^{1*},\overline{x}_j\right\rangle=0\\
\Leftrightarrow\quad&\left[\max_{1\le u\le n}\left\{\sum_{j=1}^{m}\tau_{C_{uj}}(\overline{x}_j-p_u)+a_u\right\}-\sum_{i\in\overline{I}}\sum_{j=1}^{m}\overline{z}_i^{0*}\tau_{C_{ij}}(\overline{x}_j-p_i)-\sum_{i\in\overline{I}}\overline{z}_i^{0*}a_i\right]\\
&\sum_{i\in\overline{I}}\sum_{j=1}^{m}[\overline{z}_i^{0*}\tau_{C_{ij}}(\overline{x}_j-p_i)-\langle\overline{z}_{ij}^{1*},\overline{x}_j-p_i\rangle]+\sum_{j=1}^{m}\left\langle\sum_{i\in\overline{I}}\overline{z}_{ij}^{1*},\overline{x}_j\right\rangle=0.
\end{aligned}\tag{7}
$$

If we define the function $h_i:\mathbb{R}\to\overline{\mathbb{R}}$ by

$$
h_i(y):=\begin{cases}y, & \text{if } y\in\mathbb{R}_+,\\ +\infty, & \text{otherwise,}\end{cases}\tag{8}
$$

then it follows by Lemma 2.2 that

$$
\begin{aligned}
g\left(\sum_{j=1}^{m}\tau_{C_{1j}}(\overline{x}_j-p_1),...,\sum_{j=1}^{m}\tau_{C_{nj}}(\overline{x}_j-p_n)\right)&=\max_{1\le u\le n}\left\{\sum_{j=1}^{m}\tau_{C_{uj}}(\overline{x}_j-p_u)+a_u\right\}\\
&\ge\sum_{i\in\overline{I}}\overline{z}_i^{0*}\left[\sum_{j=1}^{m}\tau_{C_{ij}}(\overline{x}_j-p_i)+a_i\right],
\end{aligned}
$$

which means that the term in the first bracket of (7) is equal to zero. Moreover, by the Young-Fenchel inequality as well as by the fact that $\sum_{i\in\overline{I}}\overline{z}_{ij}^{1*}=0_{X^*}$, $j=1,...,m$, we get that the terms in the other brackets are also equal to zero. Hence, we derive the optimality conditions (i)-(iii). By the feasibility condition, $\sum_{i\in\overline{I}}\overline{z}_i^{0*}\le 1$, and the equality in the first bracket of (7) it holds

$$
\begin{aligned}
\max_{1\le u\le n}\left\{\sum_{j=1}^{m}\tau_{C_{uj}}(\overline{x}_j-p_u)+a_u\right\}&=\sum_{i\in\overline{I}}\overline{z}_i^{0*}\left(\sum_{j=1}^{m}\tau_{C_{ij}}(\overline{x}_j-p_i)+a_i\right)\\
&\le\sum_{i\in\overline{I}}\overline{z}_i^{0*}\max_{1\le u\le n}\left\{\sum_{j=1}^{m}\tau_{C_{uj}}(\overline{x}_j-p_u)+a_u\right\}\\
&\le\max_{1\le u\le n}\left\{\sum_{j=1}^{m}\tau_{C_{uj}}(\overline{x}_j-p_u)+a_u\right\}
\end{aligned}
$$

and from here follows on the one hand that

$$
\sum_{i\in\overline{I}}\overline{z}_i^{0*}=1,\tag{9}
$$

and on the other hand that

$$
\sum_{j=1}^{m}\tau_{C_{ij}}(\overline{x}_j-p_i)+a_i=\max_{1\le u\le n}\left\{\sum_{j=1}^{m}\tau_{uj}(\overline{x}_j-p_u)+a_u\right\},\ i\in\overline{I}.\tag{10}
$$

Moreover, as $\overline{z}_i^{0*}\tau_{C_{ij}}(\overline{x}_j - p_i) = \langle \overline{z}_{ij}^{1*}, \overline{x}_j - p_i\rangle,\ i \in \overline{I},\ j = 1, ..., m$, one gets by the feasibility condition,

$$\begin{aligned} & \tau_{C_{ij}^0}(\overline{z}_{ij}^{1*}) \leq \overline{z}_i^{0*},\ j = 1, ..., m, \\ \Leftrightarrow\ & \max_{1\leq l\leq m}\left\{\tau_{C_{ij}^0}(\overline{z}_{ij}^{1*})\right\} \leq \overline{z}_i^{0*},\ i \in \overline{I}, \end{aligned} \tag{11}$$

as well as by using the generalized Cauchy-Schwarz inequality that (recall that $\gamma_{C_i}(\overline{x} - \widetilde{p}_i) = \sum_{j=1}^m \tau_{C_{ij}}(\overline{x}_j - p_i)$, $\gamma_{C_i^0}(\overline{z}_i^{1*}) = \max_{1\leq j\leq m}\{\tau_{C_{ij}^0}(\overline{z}_{ij}^{1*})\},\ \widetilde{p}_i = (p_i, ..., p_i) \in X^m$ and $\overline{z}_i^{1*} = (\overline{z}_{i1}^{1*}, ..., \overline{z}_{im}^{1*}) \in (X^*)^m,\ i \in \overline{I}$)

$$\begin{aligned} \overline{z}_i^{0*}\gamma_{C_i}(\overline{x} - \widetilde{p}_i) &= \overline{z}_i^{0*}\sum_{j=1}^m \tau_{C_{ij}}(\overline{x}_j - p_i) = \sum_{j=1}^m \langle \overline{z}_{ij}^{1*}, \overline{x}_j - p_i\rangle = \langle \overline{z}_i^{1*}, \overline{x} - \widetilde{p}_i\rangle \\ &\leq \gamma_{C_i^0}(\overline{z}_i^{1*})\gamma_{C_i}(\overline{x} - \widetilde{p}_i) = \max_{1\leq l\leq m}\left\{\tau_{C_{ij}^0}(\overline{z}_{ij}^{1*})\right\}\sum_{j=1}^m \tau_{C_{ij}}(\overline{x}_j - p_i) \\ &\leq \overline{z}_i^{0*}\sum_{j=1}^m \tau_{C_{ij}}(\overline{x}_j - p_i), \end{aligned}$$

$i \in \overline{I}$, and thus, the inequality in (11) holds as equality. Taking now (9), (10) and (11) as equality together yields the optimality conditions (iv)-(vi) and completes the proof. □

Remark 3.2. *Let $h_i : \mathbb{R} \to \overline{\mathbb{R}}$ be defined by*

$$h_i(x_i) := \begin{cases} x_i, & \text{if } x_i \in \mathbb{R}_+, \\ +\infty, & \text{otherwise,} \end{cases}$$

then the conjugate function of $\lambda_i h_i$, $\lambda_i \geq 0$, is

$$(\lambda_i h_i)^*(x_i^*) = \begin{cases} 0, & \text{if } x_i^* \leq \lambda_i, \\ +\infty, & \text{otherwise,} \end{cases},\ i = 1, ..., n,$$

In addition, we consider the function $f : \mathbb{R} \to \overline{\mathbb{R}}$,

$$f(y^0) = \begin{cases} \max\limits_{1\leq i\leq n}\{y_i^0 + a_i\},\ \text{if } y^0 = (y_1^0, ..., y_n^0)^T \in \mathbb{R}_+^n,\ i = 1, ..., n, \\ +\infty,\ \text{otherwise,} \end{cases}$$

and get by Theorem 2.2 that

$$f^*(z_1^{0*}, ..., z_n^{0*}) = \min_{\substack{\sum_{i=1}^n \lambda_i \leq 1,\ \lambda_i \geq 0,\ z_i^{0*}\leq\lambda_i, \\ i=1,...,n}} \left\{-\sum_{i=1}^n \lambda_i a_i\right\} \leq -\sum_{i=1}^n z_i^{0*}a_i,$$

for all $z_i^{0*} \leq \lambda_i$ with $\lambda_i \geq 0,\ i = 1, ..., n,\ \sum_{i=1}^n \lambda_i \leq 1$. Hence, we have by the Young-Fenchel inequality and the optimal condition (i) of Theorem 3.2 that

$$\begin{aligned} &\sum_{i\in\overline{I}} \overline{z}_i^{0*}\sum_{j=1}^m \tau_{C_{ij}}(\overline{x}_j - p_i) \leq f\left(\sum_{j=1}^m \tau_{C_{1j}}(\overline{x}_j - p_1), ..., \sum_{j=1}^m \tau_{C_{nj}}(\overline{x}_j - p_n))\right) + f^*(\overline{z}_1^{0*}, ..., \overline{z}_n^{0*}) \\ \leq\ & f\left(\sum_{j=1}^m \tau_{C_{1j}}(\overline{x}_j - p_1), ..., \sum_{j=1}^m \tau_{C_{nj}}(\overline{x}_j - p_n))\right) - \sum_{i=1}^n \overline{z}_i^{0*}a_i = \sum_{i\in\overline{I}} \overline{z}_i^{0*}\sum_{j=1}^m \tau_{C_{ij}}(\overline{x}_j - p_i), \end{aligned}$$

i.e.

$$f\left(\sum_{j=1}^{m}\tau_{C_{1j}}(\overline{x}_j-p_1),...,\sum_{j=1}^{m}\tau_{C_{nj}}(\overline{x}_j-p_n))\right)+f^*(\overline{z}_1^{0*},...,\overline{z}_n^{0*})=\sum_{i\in\overline{I}}\overline{z}_i^{0*}\sum_{j=1}^{m}\tau_{C_{ij}}(\overline{x}_j-p_i)$$

and by (1) this equality is equivalent to

$$(z_1^{0*},...,z_n^{0*})\in\partial f\left(\sum_{j=1}^{m}\tau_{C_{1j}}(\overline{x}_j-p_1),...,\sum_{j=1}^{m}\tau_{C_{nj}}(\overline{x}_j-p_n))\right).$$

In other words, the condition (i) *of Theorem 3.2 can be written by means of the subdifferential*

(i) $(\overline{z}_1^{0*},...,\overline{z}_n^{0*})\in\partial\left(\max\limits_{1\leq j\leq n}\{\cdot+a_j\}\right)\left(\sum\limits_{j=1}^{m}\tau_{C_{1j}}(\overline{x}_j-p_1),...,\sum\limits_{j=1}^{m}\tau_{C_{nj}}(\overline{x}_j-p_n))\right),$

Similarly, we can rewrite the condition (ii) *of Theorem 3.2*

(ii) $\overline{z}_{ij}^{1*}\in\partial(\overline{z}_i^{0*}\tau_{C_{ij}})(\overline{x}_j-p_i),\ i\in\overline{I},\ j=1,...,m.$

Moreover, combining this condition with the optimality condition (iii) *of Theorem 3.2 yields that*

$$0_{X^*}\in\sum_{i\in\overline{I}}\partial(\overline{z}_i^{0*}\tau_{C_{ij}})(\overline{x}_j-p_i),\ j=1,...,m.$$

Notice also that the optimality conditions (ii) *and* (vi) *of Theorem 3.2 give a detailed characterization of the subdifferential of* $\overline{z}_i^{0*}\tau_{C_{ij}}$ *at* $\overline{x}_j-p_i$ *such that*

$$\partial(\overline{z}_i^{0*}\tau_{C_{ij}})(\overline{x}_j-p_i)=\left\{\overline{z}_{ij}^{1*}\in X^*:\overline{z}_i^{0*}\tau_{C_{ij}}(\overline{x}_j-p_i)=\langle\overline{z}_{ij}^{1*},\overline{x}_j-p_i\rangle,\ \max_{1\leq l\leq m}\left\{\tau_{C_{il}^0}(\overline{z}_{il}^{1*})\right\}=\overline{z}_i^{0*}\right\},$$

for all $i\in\overline{I},\ j=1,...,m.$

Let us now consider the extended location problem (EP_a^M) in the following framework. We set $X=\mathcal{H}$, where $\mathcal{H}$ is a real Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$ defined by $\|x\|:=\sqrt{\langle x,x\rangle}$. In addition, let $\tau_{C_{ij}}:\mathcal{H}\to\mathbb{R},\ \tau_{C_{ij}}(x):=w_{ij}\|x\|$, where $w_{ij}>0$ for $j=1,...,m,\ i=1,...,n$. Hence, the location problem looks like

$$(EP_{N,a}^M)\quad \inf_{(x_1,...,x_m)\in\mathcal{H}\times...\times\mathcal{H}}\ \max_{1\leq i\leq n}\left\{\sum_{j=1}^{m}w_{ij}\|x_j-p_i\|+a_i\right\}.$$

For this situation, where the gauges are all identical and the distances are measured by a round norm, Michelot and Plastria examined in [17] under which conditions an optimal solution of coincidence type exists. The authors showed that if the weights have a multiplicative structure, i.e. $w_{ij}=\lambda_i\mu_j$ with $\lambda_i,\ \mu_j>0,\ i=1,...,n,\ j=1,...,m$, and $\sum_{j=1}^{m}\mu_j=1$, then there exists an optimal solution of $(EP_{N,a}^M)$ such that all new facilities coincide. Moreover, they described when the optimal solution of coincidence type is unique and presented a full characterization of the set of optimal solutions for extended multifacility location problems where the weights have a multiplicative structure.
The next statement is based on the idea of weights with a multiplicative structure and illustrates in this situation the relation between the extended location problem $(EP_{N,a}^M)$ and its corresponding conjugate dual problem.

Theorem 3.3. *Let $X=\mathcal{H}$, $\tau_{C_{ij}}:\mathcal{H}\to\mathbb{R}$ be defined by $\tau_{C_{ij}}(x):=w_{ij}\|x\|$, $i=1,...,n,\ j=1,...,m$, and $w_{ij}=\lambda_i\mu_j$ with $\lambda_i,\ \mu_j>0$, $i=1,...,n,\ j=1,...,m$, and $\sum_{j=1}^m\mu_j=1$. Assume that $\Delta_{\overline{x}}=(\overline{x},...,\overline{x})\in\underbrace{\mathcal{H}\times...\times\mathcal{H}}_{m-times}$ is an optimal solution of coincidence type of*

$$(EP_{N,a}^M)\quad \inf_{(x_1,...,x_m)\in\mathcal{H}\times...\times\mathcal{H}}\max_{1\le i\le n}\left\{\sum_{j=1}^m w_{ij}\|x_j-p_i\|+a_i\right\}.$$

and $(\overline{z}_1^{0*},...,\overline{z}_n^{0*},\overline{z}_1^{1*},...,\overline{z}_n^{1*},\overline{I})$ and $\overline{I}\subseteq\{1,...,n\}$ is an optimal solution of the corresponding conjugate dual problem

$$(ED_{N,a}^M)\quad \sup_{(z_1^{0*},...,z_n^{0*},z_1^{1*},...,z_n^{1*})\in\mathcal{C}}\left\{-\sum_{i\in I}\left[\left\langle\sum_{j=1}^m z_{ij}^{1*},p_i\right\rangle-z_i^{0*}a_i\right]\right\},$$

where

$$\mathcal{C}=\Bigg\{(z_1^{0*},...,z_n^{0*},z_1^{1*},...,z_n^{1*})\in\mathbb{R}^n\times\underbrace{\mathcal{H}\times...\times\mathcal{H}}_{m-times}\times...\times\underbrace{\mathcal{H}\times...\times\mathcal{H}}_{m-times}:I\subseteq\{1,...,n\},$$
$$\sum_{i\in I}z_i^{0*}\le 1,\ \sum_{i\in I}z_{ij}^{1*}=0_{\mathcal{H}},\ z_i^{0*}>0,\ z_{ij}^{1*}\in\mathcal{H},\ \|z_{ij}^{1*}\|\le z_i^{0*}w_{ij},\ i\in I,$$
$$z_k^{0*}=0,\ z_{kj}^{1*}=0_{\mathcal{H}},\ k\notin I,\ j=1,...,m\Bigg\}.$$

Then, it holds

$$\overline{x}=\frac{1}{\sum\limits_{i\in\overline{I}}\frac{\lambda_i\|\overline{z}_{ij}^{1*}\|}{v(ED_{N,a}^M)-a_i}}\sum_{i\in\overline{I}}\frac{\lambda_i\|\overline{z}_{ij}^{1*}\|}{v(ED_{N,a}^M)-a_i}p_i,\ \forall j\in\overline{J},$$

where

$$\overline{J}:=\left\{j\in\{1,...,m\}:\frac{1}{w_{ij}}\|\overline{z}_{ij}^{1*}\|=\max_{1\le l\le m}\left\{\frac{1}{w_{il}}\|\overline{z}_{il}^{1*}\|\right\}\right\},\ i\in\overline{I}.$$

Proof. First, let us remark that the dual norm of the weighted norm $\tau_{C_{ij}}=w_{ij}\|\cdot\|$ is given by $\tau_{C_{ij}^0}=(1/w_{ij})\|\cdot\|$.
Now, let $\Delta_{\overline{x}}=(\overline{x},...,\overline{x})$ be an optimal solution of coincidence type, then the optimality conditions (ii), (iii), (v) and (vi) of Theorem 3.2 can be written as

(ii) $\overline{z}_i^{0*}w_{ij}\|\overline{x}-p_i\|=\langle\overline{z}_{ij}^{1*},\overline{x}-p_i\rangle,\ i\in\overline{I},\ j=1,...,m,$

(iii) $\sum\limits_{i\in\overline{I}}\overline{z}_{ij}^{1*}=0_{\mathcal{H}},\ j=1,...,m,$

(v) $\sum\limits_{j=1}^m w_{ij}\|\overline{x}-p_i\|+a_i=\max\limits_{1\le u\le n}\left\{\sum\limits_{j=1}^m w_{uj}\|\overline{x}-p_u\|+a_u\right\},\ i\in\overline{I},$

(vi) $\max\limits_{1\le l\le m}\left\{\frac{1}{w_{il}}\|\overline{z}_{il}^{1*}\|\right\}=\overline{z}_i^{0*},\ \overline{z}_{ij}^{1*}\in\mathcal{H},\ i\in\overline{I}$ and $\overline{z}_{kj}^{1*}=0_{\mathcal{H}},\ k\notin\overline{I},\ j=1,...,m.$

By combining the conditions (ii) and (vi), we get

$$\|\overline{z}_{ij}^{1*}\|\|\overline{x}-p_i\| = \langle \overline{z}_{ij}^{1*}, \overline{x}-p_i\rangle,\ i\in\overline{I},\ j\in\overline{J}. \tag{12}$$

Moreover, by Fact 2.10 in [1] there exists $\alpha_{ij}>0$ such that

$$\overline{z}_{ij}^{1*} = \alpha_{ij}\left(\overline{x}-p_i\right) \tag{13}$$

and from here one gets that

$$\|\overline{z}_{ij}^{1*}\| \quad = \quad \alpha_{ij}\|\overline{x}-p_i\|, \tag{14}$$

$i\in\overline{I}, j\in\overline{J}$. By condition (v) follows

$$\begin{aligned}
&\sum_{j=1}^{m} w_{ij}\|\overline{x}-p_i\| + a_i = \max_{1\le u\le n}\left\{\sum_{j=1}^{m} w_{uj}\|\overline{x}-p_u\| + a_u\right\}\\
\Leftrightarrow\quad & \lambda_i\sum_{j=1}^{m}\mu_j\|\overline{x}-p_i\| + a_i = \max_{1\le u\le n}\left\{\lambda_u\sum_{j=1}^{m}\mu_j\|\overline{x}-p_u\| + a_u\right\}\\
\Leftrightarrow\quad & \lambda_i\|\overline{x}-p_i\| + a_i = \max_{1\le u\le n}\left\{\lambda_u\|\overline{x}-p_u\| + a_u\right\},\ i\in\overline{I}.
\end{aligned} \tag{15}$$

Bringing (14) and (15) together yields

$$\begin{aligned}
&\frac{\lambda_i}{\alpha_{ij}}\|\overline{z}_{ij}^{1*}\| + a_i = \max_{1\le u\le n}\left\{\lambda_u\|\overline{x}-p_u\| + a_u\right\}\\
\Leftrightarrow\quad & \alpha_{ij} = \frac{\lambda_i}{\max\limits_{1\le u\le n}\left\{\lambda_u\|\overline{x}-p_u\| + a_u\right\} - a_i}\|\overline{z}_{ij}^{1*}\|,\ i\in\overline{I},\ j\in\overline{J}.
\end{aligned} \tag{16}$$

Taking the sum overall $i\in\overline{I}$ in (16) gives

$$\sum_{i\in\overline{I}}\alpha_{ij} = \sum_{i\in\overline{I}}\frac{\lambda_i\|\overline{z}_{ij}^{1*}\|}{\max\limits_{1\le u\le n}\left\{\lambda_u\|\overline{x}-p_u\| + a_u\right\} - a_i},\ j\in\overline{J}. \tag{17}$$

Now, consider condition (iii), by (13) follows

$$0_{\mathcal{H}} = \sum_{i\in\overline{I}}\overline{z}_{ij}^{1*} = \sum_{i\in\overline{I}}\alpha_{ij}\left(\overline{x}-p_i\right) \Leftrightarrow \overline{x} = \frac{1}{\sum\limits_{i\in\overline{I}}\alpha_{ij}}\sum_{i\in\overline{I}}\alpha_{ij}p_i,\ j\in\overline{J}. \tag{18}$$

Putting (17) and (18) together reveals

$$\begin{aligned}
\overline{x} \quad &= \quad \frac{1}{\sum\limits_{i\in\overline{I}}\frac{\lambda_i\|\overline{z}_{ij}^{1*}\|}{\max\limits_{1\le u\le n}\{\lambda_u\|\overline{x}-p_u\|+a_u\}-a_i}}\sum_{i\in\overline{I}}\frac{\lambda_i\|\overline{z}_{ij}^{1*}\|}{\max\limits_{1\le u\le n}\left\{\lambda_u\|\overline{x}-p_u\| + a_u\right\} - a_i}p_i\\
&= \quad \frac{1}{\sum\limits_{i\in\overline{I}}\frac{\lambda_i\|\overline{z}_{ij}^{1*}\|}{v(ED_{N,a}^{M})-a_i}}\sum_{i\in\overline{I}}\frac{\lambda_i\|\overline{z}_{ij}^{1*}\|}{v(ED_{N,a}^{M})-a_i}p_i,\ \forall j\in\overline{J},
\end{aligned}$$

and the proof is finished. □

Remark 3.3. *In the context of Theorem 3.3, it holds that $\overline{x}-p_i$ and $\overline{z}_{ij}^{1*}$ are multiples of each other and so the vectors $(1/w_{ij})\overline{z}_{ij}^{1*}$, $j\in\overline{J}$, are all multiples of each other. In other words, the vectors $(1/w_{ij})\overline{z}_{ij}^{1*}$, $j\in\overline{J}$, are identical. In this sense, one can understand the optimal solution of the conjugate dual problem also as a solution of coincidence type.*

The next statement holds for any weights, not necessary of multiplicative structure.

Lemma 3.1. *Let $w_{sj}:=\max_{1\leq u\leq n}\{w_{uj}\}$, $X=\mathcal{H}$, $\tau_{C_{ij}}:\mathcal{H}\to\mathbb{R}$ be defined by $\tau_{C_{ij}}(x):=w_{ij}\|x\|$, $i=1,...,n,\ j=1,...,m,$ and $(z_1^{0*},...,z_n^{0*},z_1^{1*},...,z_n^{1*})$ a feasible solution of the conjugate dual problem $(ED_{N,a}^M)$, then it holds*

$$\|z_{ij}^{1*}\|\leq\frac{w_{sj}w_{ij}}{w_{sj}+w_{ij}},\ i\in I,\ j=1,...,m.$$

Proof. Let

$$(z_1^{0*},...,z_n^{0*},z_1^{1*},...,z_n^{1*})\in\mathbb{R}^n\times\underbrace{\mathcal{H}\times...\times\mathcal{H}}_{m-\text{times}}\times...\times\underbrace{\mathcal{H}\times...\times\mathcal{H}}_{m-\text{times}}$$

be a feasible solution of the conjugate dual problem $(ED_{N,a}^M)$, then we have

(i) $\sum\limits_{i\in I}z_i^{0*}\leq 1,$

(ii) $\|z_{ij}^{1*}\|\leq z_i^{0*}w_{ij},\ j=1,...,m,\ i\in I,$

(iii) $\sum\limits_{i\in I}z_{ij}^{1*}=0_{\mathcal{H}}.$

The inequalities (i) and (ii) imply the inequality

$$\sum_{i\in I}\frac{1}{w_{ij}}\|z_{ij}^{1*}\|\leq 1,\ j=1,...,m. \tag{19}$$

Furthermore, by (iii) we have

$$\sum_{i\in I}z_{ij}^{1*}=0_{\mathcal{H}}\Leftrightarrow z_{kj}^{1*}=-\sum_{\substack{i\in I\\ i\neq k}}z_{ij}^{1*},\ k\in I,\ j=1,...,m, \tag{20}$$

and hence,

$$\|z_{kj}^{1*}\|=\|\sum_{\substack{i\in I\\ i\neq k}}z_{ij}^{1*}\|\leq\sum_{\substack{i\in I\\ i\neq k}}\|z_{ij}^{1*}\|,\ k\in I,\ j=1,...,m. \tag{21}$$

By (21) we get in (19)

$$\begin{aligned}1\ &\geq\ \frac{1}{w_{kj}}\|z_{kj}^{1*}\|+\sum_{\substack{i\in I\\ i\neq k}}\frac{1}{w_{ij}}\|z_{ij}^{1*}\|\geq\frac{1}{w_{kj}}\|z_{kj}^{1*}\|+\frac{1}{w_{sj}}\sum_{\substack{i\in I\\ i\neq k}}\|z_{ij}^{1*}\|\\ &\geq\ \frac{1}{w_{kj}}\|z_{kj}^{1*}\|+\frac{1}{w_{sj}}\|z_{kj}^{1*}\|=\frac{w_{sj}+w_{kj}}{w_{kj}w_{ij}}\|z_{kj}^{1*}\|,\ j=1,...,m,\end{aligned}$$

and finally,

$$\|z_{kj}^{1*}\| \leq \frac{w_{sj}w_{kj}}{w_{sj}+w_{kj}},\ k \in I,\ j = 1,...,m.$$

$\square$

Remark 3.4. *If we allow also negative set-up costs, then we have in the constraint set, as stated in Remark 2.2, $\sum_{i\in I} z_i^{0*} = 1$ instead $\sum_{i\in I} z_i^{0*} \leq 1$. One can easy verify that the results we presented above also holds in this case.*

3.2 Extended multifacility location problems without set-up costs

In the next, we study the case where $a_i = 0$ for all $i = 1,...,n$. With this assumption the extended multifacility location problem (EP_a^M) can be stated as

$$(EP^M) \qquad \inf_{(x_1,...,x_m)\in X^m} \max_{1\leq i\leq n} \left\{ \sum_{j=1}^{m} \tau_{C_{ij}}(x_j - p_i) \right\}.$$

In this situation its corresponding conjugated dual problem (ED_a^M) transforms into

$$(ED^M) \qquad \sup_{(z_1^{0*},...,z_n^{0*},z_1^{1*},...,z_n^{1*})\in\mathcal{C}} \left\{ -\sum_{i\in I} \left\langle \sum_{j=1}^{m} z_{ij}^{1*}, p_i \right\rangle \right\}.$$

Additionally, let us consider the following dual problem

$$(E\widetilde{D}^M) \qquad \sup_{(z_1^*,...,z_n^*)\in\widetilde{\mathcal{C}}} \left\{ -\sum_{i\in I} \left\langle \sum_{j=1}^{m} z_{ij}^*, p_i \right\rangle \right\}$$

where

$$\widetilde{\mathcal{C}} = \Bigg\{ (z_1^*,...,z_n^*) \in (X^*)^m \times ... \times (X^*)^m : I \subseteq \{1,...,n\},\ z_{ij}^* \in X^*,\ i \in I,\ z_{kj}^* = 0_{X^*},\ k \notin I,$$
$$\sum_{i\in I} z_{ij}^* = 0_{X^*},\ \sum_{i\in I} \max_{1\leq l\leq m} \left\{ \tau_{C_{ij}^0}(z_{ij}^*) \right\} \leq 1,\ j = 1,...,m \Bigg\}.$$

Let us denote by $v(ED_a^M)$ and $v(E\widetilde{D}^M)$ the optimal objective values of the dual problems (ED_a^M) and $(E\widetilde{D}^M)$, respectively, then we can state.

Theorem 3.4. *It holds $v(ED^M) = v(E\widetilde{D}^M)$.*

Proof. The statement follows immediately by Theorem 4.1 in [22] and by (6). $\square$

The next duality statements follow as a direct consequence of Theorem 3.1 and Theorem 3.4.

Theorem 3.5. *(strong duality) Between (EP^M) and $(E\widetilde{D}^M)$ strong duality holds, i.e. $v(EP^M) = v(E\widetilde{D}^M)$ and the dual problem $v(E\widetilde{D}^M)$ has an optimal solution.*

We define

$$J_\tau := \left\{ j \in \{1,...,m\} : \tau_{C^0_{ij}}(z^*_{ij}) > 0 \right\},\ i \in I,$$

and obtain by using the previous theorem optimality conditions of the following form.

Theorem 3.6. *(optimality conditions) (a) Let $(\overline{x}_1, ..., \overline{x}_m) \in X^m$ be an optimal solution of the problem (EP^M). Then there exists $(\overline{z}^*_1, ..., \overline{z}^*_n) \in (X^*)^m \times ... \times (X^*)^m$ and an index set $\overline{I} \subseteq \{1, ..., n\}$, an optimal solution to $(E\widetilde{D}^M)$, such that*

(i) $\max\limits_{1\leq u\leq n} \left\{ \sum\limits_{j=1}^m \tau_{C_{uj}}(\overline{x}_j - p_u) \right\} = \sum\limits_{i\in\overline{I}} \sum\limits_{j=1}^m \tau_{C^0_{ij}}(\overline{z}^*_{ij}) \tau_{C_{ij}}(\overline{x}_j - p_i),$

(ii) $\sum\limits_{i\in\overline{I}} \overline{z}^*_{ij} = 0_{X^*},\ j = 1, ..., m,$

(iii) $\tau_{C^0_{ij}}(\overline{z}^*_{ij}) \tau_{C_{ij}}(\overline{x}_j - p_i) = \langle \overline{z}^*_{ij}, \overline{x}_j - p_i \rangle,\ i \in \overline{I},\ j = 1, ..., m,$

(iv) $\sum\limits_{i\in\overline{I}} \max\limits_{1\leq l\leq m} \left\{ \tau_{C^0_{il}}(\overline{z}^*_{il}) \right\} = 1,$

(v) $\max\limits_{1\leq u\leq n} \left\{ \sum\limits_{j=1}^m \tau_{C_{uj}}(\overline{x}_j - p_u) \right\} = \sum\limits_{j=1}^m \tau_{C_{ij}}(\overline{x}_j - p_i),\ i \in \overline{I},$

(vi) $\max\limits_{1\leq l\leq m} \left\{ \tau_{C^0_{il}}(\overline{z}^*_{il}) \right\} = \tau_{C^0_{ij}}(\overline{z}^*_{ij}) > 0,\ i \in \overline{I},\ j \in \overline{J}_\tau,$ *and* $\tau_{C^0_{ks}}(\overline{z}^*_{ks}) = 0,\ k \notin \overline{I},\ s = 1, ..., m.$

(b) If there exists $(\overline{x}_1, ..., \overline{x}_m) \in X^m$ such that for some $(\overline{z}^*_1, ..., \overline{z}^*_n) \in (X^*)^m \times ... \times (X^*)^m$ and an index set $\overline{I}$ the conditions (i)-(vi) are fulfilled, then $(\overline{x}_1, ..., \overline{x}_m)$ is an optimal solution of (EP^M), $(\overline{z}^*_1, ..., \overline{z}^*_n, \overline{I})$ is an optimal solution for $(E\widetilde{D}^M)$ and $v(EP^M) = v(E\widetilde{D}^M)$.

Proof. Let $(\overline{x}_1, ..., \overline{x}_m) \in X^m$ be an optimal solution of (EP^M), then by Theorem 3.5 there exists $(\overline{z}^*_1, ..., \overline{z}^*_n) \in (X^*)^m \times ... \times (X^*)^m$ and an index set $\overline{I} \subseteq \{1, ..., n\}$, an optimal solution to $(E\widetilde{D}^M)$, such that $v(EP^M) = v(E\widetilde{D}^M)$. Therefore, we have

$$\begin{aligned}
&\max_{1\leq j\leq n} \left\{ \sum_{j=1}^m \tau_{C_{uj}}(\overline{x}_j - p_u) \right\} = -\sum_{i\in\overline{I}} \left\langle \sum_{j=1}^m \overline{z}^*_{ij}, p_i \right\rangle \\
\Leftrightarrow\ &\max_{1\leq j\leq n} \left\{ \sum_{j=1}^m \tau_{C_{uj}}(\overline{x}_j - p_u) \right\} + \sum_{i\in\overline{I}} \left\langle \sum_{j=1}^m \overline{z}^*_{ij}, p_i \right\rangle = 0 \\
\Leftrightarrow\ &\max_{1\leq j\leq n} \left\{ \sum_{j=1}^m \tau_{C_{uj}}(\overline{x}_j - p_u) \right\} + \sum_{i\in\overline{I}} \left\langle \sum_{j=1}^m \overline{z}^*_{ij}, p_i \right\rangle - \sum_{j=1}^m \left\langle \underbrace{\sum_{i\in\overline{I}} \overline{z}^*_{ij}}_{=0_{X^*}}, \overline{x}_j \right\rangle \\
&+ \sum_{i\in\overline{I}} \sum_{j=1}^m \tau_{C^0_{ij}}(\overline{z}^*_{ij}) \tau_{C_{ij}}(\overline{x}_j - p_i) - \sum_{i\in\overline{I}} \sum_{j=1}^m \tau_{C^0_{ij}}(\overline{z}^*_{ij}) \tau_{C_{ij}}(\overline{x}_j - p_i) = 0
\end{aligned}$$

$$\Leftrightarrow \quad \left[\max_{1\le j\le n}\left\{\sum_{j=1}^{m}\tau_{C_{uj}}(\overline{x}_j - p_u)\right\} - \sum_{i\in\overline{I}}\sum_{j=1}^{m}\tau_{C^0_{ij}}(\overline{z}^*_{ij})\tau_{C_{ij}}(\overline{x}_j - p_i)\right]$$
$$+\sum_{i\in\overline{I}}\sum_{j=1}^{m}\left[\tau_{C^0_{ij}}(\overline{z}^*_{ij})\tau_{C_{ij}}(\overline{x}_j - p_i) - \langle \overline{z}^*_{ij}, \overline{x}_j - p_i\rangle\right] = 0.$$

From Lemma 2.2 follows with (8) that

$$g\left(\sum_{j=1}^{m}\tau_{C_{1j}}(\overline{x}_j - p_1), ..., \sum_{j=1}^{m}\tau_{C_{nj}}(\overline{x}_j - p_n)\right) = \max_{1\le u\le n}\left\{\sum_{j=1}^{m}\tau_{C_{uj}}(\overline{x}_j - p_u)\right\}$$
$$\geq \sum_{i\in\overline{I}}\max_{1\le l\le m}\left\{\tau_{C^0_{il}}(\overline{z}^*_{il})\right\}\sum_{j=1}^{m}\tau_{C_{ij}}(\overline{x}_j - p_i) \geq \sum_{i\in\overline{I}}\sum_{j=1}^{m}\tau_{C^0_{ij}}(\overline{z}^*_{ij})\tau_{C_{ij}}(\overline{x}_j - p_i),$$

i.e. the term within the first bracket is non-negative and by the Young-Fenchel inequality we derive that the terms within the other brackets are also non-negative. Thus, the cases (i)-(iii) are verified.
Furthermore, the first bracket reveals that

$$\max_{1\le j\le n}\left\{\sum_{j=1}^{m}\tau_{C_{uj}}(\overline{x}_j - p_u)\right\} = \sum_{i\in\overline{I}}\sum_{j=1}^{m}\tau_{C^0_{ij}}(\overline{z}^*_{ij})\tau_{C_{ij}}(\overline{x}_j - p_i)$$
$$\leq \sum_{i\in\overline{I}}\max_{1\le l\le m}\left\{\tau_{C^0_{il}}(\overline{z}^*_{ij})\right\}\sum_{j=1}^{m}\tau_{C_{ij}}(\overline{x}_j - p_i)$$
$$\leq \sum_{i\in\overline{I}}\max_{1\le l\le m}\left\{\tau_{C^0_{il}}(\overline{z}^*_{ij})\right\}\max_{1\le j\le n}\left\{\sum_{j=1}^{m}\tau_{C_{uj}}(\overline{x}_j - p_u)\right\} \leq \max_{1\le j\le n}\left\{\sum_{j=1}^{m}\tau_{C_{uj}}(\overline{x}_j - p_u)\right\} \tag{22}$$

and so, we have that $\sum_{i\in\overline{I}}\max_{1\le l\le m}\{\tau_{C^0_{il}}(\overline{z}^*_{ij})\} = 1$ as well as $\max\limits_{1\le l\le m}\left\{\tau_{C^0_{il}}(\overline{z}^*_{il})\right\} = \tau_{C^0_{ij}}(\overline{z}^*_{ij}) > 0$, $j \in \overline{J}_\tau$, $i \in \overline{I}$, which verifies conditions (iv) and (vi). From (22) follows also that

$$\sum_{i\in\overline{I}}\max_{1\le l\le m}\left\{\tau_{C^0_{il}}(\overline{z}^*_{ij})\right\}\left[\max_{1\le j\le n}\left\{\sum_{j=1}^{m}\tau_{C_{uj}}(\overline{x}_j - p_u)\right\} - \sum_{j=1}^{m}\tau_{C_{ij}}(\overline{x}_j - p_i)\right] = 0 \tag{23}$$

As the brackets in (23) are non-negative and $\max_{1\le l\le m}\{\tau_{C^0_{il}}(\overline{z}^*_{ij})\} > 0$, $i \in \overline{I}$, we get that

$$\max_{1\le j\le n}\left\{\sum_{j=1}^{m}\tau_{C_{uj}}(\overline{x}_j - p_u)\right\} = \sum_{j=1}^{m}\tau_{C_{ij}}(\overline{x}_j - p_i),\ i \in \overline{I}.$$

which yields the condition (v) and completes the proof. □

Now, our aim is to investigate the location problem (EP^M) from the geometrical point of view. For this purpose let $X = \mathbb{R}^d$ and the distances are measured by the Euclidean norm. Then, the

(EP^M) turns into

$$(EP_N^M) \qquad \inf_{(x_1,...,x_m)\in\mathbb{R}^d\times...\times\mathbb{R}^d} \max_{1\le i\le n} \left\{ \sum_{j=1}^m w_{ij}\|x_j - p_i\| \right\},$$

while its conjugate dual problem transforms into

$$(E\widetilde{D}_N^M) \qquad \sup_{(z_1^*,...,z_n^*)\in\widetilde{C}} \left\{ -\sum_{i\in I} \left\langle \sum_{j=1}^m z_{ij}^*, p_i \right\rangle \right\}$$

with

$$\widetilde{C} = \Bigg\{ (z_1^*,...,z_n^*) \in \underbrace{\mathbb{R}^d\times...\times\mathbb{R}^d}_{m-\text{times}} \times ... \times \underbrace{\mathbb{R}^d\times...\times\mathbb{R}^d}_{m-\text{times}} : I\subseteq\{1,...,n\},\ z_{ij}^*\in\mathbb{R}^d,\ i\in I$$

$$z_{ij}^* = 0_{\mathbb{R}^d},\ i\notin I,\ \sum_{i\in I} z_{ij}^* = 0_{\mathbb{R}^d},\ \sum_{i\in I}\max_{1\le l\le m}\left\{\frac{1}{w_{il}}\|z_{il}^*\|\right\} \le 1,\ j=1,...,m \Bigg\}.$$

Theorem 3.7. *(strong duality) Between* (EP_M^M) *and* $(E\widetilde{D}_N^M)$ *holds strong duality, i.e.* $v(EP_M^M) = v(E\widetilde{D}_N^M)$ *and the dual problem has an optimal solution.*

Theorem 3.8. *(optimality conditions) (a) Let* $(\overline{x}_1,...,\overline{x}_m)\in\mathbb{R}^d\times...\times\mathbb{R}^d$ *be an optimal solution of the problem* (EP_N^M)*. Then there exists*

$$(\overline{z}_1^*,...,\overline{z}_n^*) \in \underbrace{\mathbb{R}^d\times...\times\mathbb{R}^d}_{m-times} \times ... \times \underbrace{\mathbb{R}^d\times...\times\mathbb{R}^d}_{m-times}$$

and an optimal index set $\overline{I}$*, an optimal solution to* $(E\widetilde{D}_N^M)$*, such that*

(i) $\max\limits_{1\le u\le n}\left\{\sum\limits_{j=1}^m w_{uj}\|\overline{x}_j - p_u\|\right\} = \sum\limits_{i\in\overline{I}}\sum\limits_{j=1}^m \|\overline{z}_{ij}^*\|\|\overline{x}_j - p_i\|,$

(ii) $\sum\limits_{i\in I}\overline{z}_{ij}^* = 0_{\mathbb{R}^d},\ j=1,...,m,$

(iii) $\|\overline{z}_{ij}^*\|\|\overline{x}_j - p_i\| = \langle\overline{z}_{ij}^*, \overline{x}_j - p_i\rangle,\ i\in\overline{I},\ j=1,...,m,$

(iv) $\sum\limits_{i\in\overline{I}}\max\limits_{1\le l\le m}\left\{\frac{1}{w_{il}}\|\overline{z}_{il}^*\|\right\} = 1,$

(v) $\max\limits_{1\le u\le n}\left\{\sum\limits_{j=1}^m w_{uj}\|\overline{x}_j - p_u\|\right\} = \sum\limits_{j=1}^m w_{ij}\|\overline{x}_j - p_i\|,\ i\in\overline{I},$

(vi) $\max\limits_{1\le l\le m}\left\{\frac{1}{w_{il}}\|\overline{z}_{il}^*\|\right\} = \frac{1}{w_{ij}}\|\overline{z}_{ij}^*\|,\ \overline{z}_{ij^*}\in\mathbb{R}^d\setminus\{0_{\mathbb{R}^d}\},\ i\in\overline{I},$ *and* $\overline{z}_{kj}^* = 0_{\mathbb{R}^d},\ k\notin\overline{I},\ j=1,...,m.$

(b) If there exists $(\overline{x}_1,...,\overline{x}_m)\in\mathbb{R}^d\times...\times\mathbb{R}^d$ *such that for some*

$$(\overline{z}_1^*,...,\overline{z}_n^*) \in \underbrace{\mathbb{R}^d\times...\times\mathbb{R}^d}_{m-times} \times ... \times \underbrace{\mathbb{R}^d\times...\times\mathbb{R}^d}_{m-times}$$

and an index set $\overline{I}$ *the conditions (i)-(vi) are fulfilled, then* $(\overline{x}_1,...,\overline{x}_m)$ *is an optimal solution of* (EP_N^M)*,* $(\overline{z}_1^*,...,\overline{z}_n^*,\overline{I})$ *is an optimal solution for* $(E\widetilde{D}_N^M)$ *and* $v(EP_N^M) = v(E\widetilde{D}_N^M)$*.*

We want now, in the concluding part of this paper, to illustrate the results we presented above and describe the set of optimal solutions of the conjugate dual problem. For that end, let us first take a closer look at the optimality conditions stated in Theorem 3.8.
By the condition (iii) follows that the vectors $\overline{z}_{ij}^*$ and $\overline{x}_j - p_i$ are parallel and moreover, these vectors have the same direction, $i \in \overline{I}$, $j = 1, ..., m$. From the optimality condition (vi) we additionally deduce that the vectors $\overline{z}_{ij}^*$, $j = 1, ..., m$, are all unequal to the zero vector if $i \in \overline{I}$, which is the situation when the sum of the weighted distances in condition (v) is equal to the optimal objective value. In the reverse case, when $i \notin \overline{I}$, i.e. the sum of the weighted distances in condition (v) is less than the optimal objective value, the vectors $\overline{z}_{ij}^*$, $j = 1, ..., m$, are all equal to the zero vector.
Therefore, it is appropriate to interpret for $i \in \overline{I}$ the vectors $\overline{z}_{ij}^*$ fulfilling $\sum_{i \in \overline{I}} \overline{z}_{ij}^* = 0_{X^*}$ and $\sum_{i \in \overline{I}} \max_{1 \leq l \leq m} \left\{ \frac{1}{w_{il}} \|\overline{z}_{il}^*\| \right\} = 1$ as force vectors pulling the given point p_i in direction to the associated gravity points $\overline{x}_j$, $j = 1, ..., m$. As an illustration of the nature of the optimal solutions of the conjugate dual problem, let us consider the following example in the plane and especially Figure 1.

Example 3.1. *Let us consider the points* $p_1 = (0,0)^T$, $p_2 = (8,0)^T$ *and* $p_3 = (5,6)^T$ *in the plane* $(d = 2)$. *For the given weights* $w_{11} = 2,\ w_{12} = 3,\ w_{21} = 3,\ w_{22} = 3,\ w_{31} = 2$ *and* $w_{32} = 2$ *we want to determine* $m = 2$ *new points minimizing the objective function of the location problem*

$$(EP_N^M) \quad \inf_{(x_1,x_2)^T \in \mathbb{R}^2 \times \mathbb{R}^2} \max\{2\|x_1 - p_1\| + 3\|x_2 - p1\|, 3\|x_1 - p_2\| + 3\|x_2 - p_2\|, \\ 2\|x_1 - p_3\| + 2\|x_2 - p_3\|\}.$$

To solve this problem, we used the Matlab Optimization Toolbox and obtained as optimal solution $\overline{x}_1 = (6.062, 0.858)^T$, $\overline{x}_2 = (2.997, 0.837)^T$ *and as optimal objective value* $(EP_N^M) = 21.578$. *The corresponding conjugate dual problem becomes to*

$$(E\widetilde{D}_N^M) \quad \sup_{(z_1^*, z_2^*, z_3^*) \in \widetilde{\mathcal{C}}} \left\{ -\langle z_{11}^* + z_{12}^*, p_1 \rangle - \langle z_{21}^* + z_{22}^*, p_2 \rangle - \langle z_{31}^* + z_{32}^*, p_3 \rangle \right\},$$

where

$$\widetilde{\mathcal{C}} = \Big\{ (z_1^*, z_2^*, z_3^*) \in (\mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \mathbb{R}^2) : \\ z_{11}^* + z_{21}^* + z_{31}^* = 0_{\mathbb{R}^2},\ z_{12}^* + z_{22}^* + z_{32}^* = 0_{\mathbb{R}^2}, \\ \max\left\{ \tfrac{1}{2}\|z_{11}^*\|, \tfrac{1}{3}\|z_{12}^*\| \right\} + \max\left\{ \tfrac{1}{3}\|z_{21}^*\|, \tfrac{1}{3}\|z_{22}^*\| \right\} + \max\left\{ \tfrac{1}{2}\|z_{31}^*\|, \tfrac{1}{2}\|z_{32}^*\| \right\} \leq 1 \Big\}.$$

The dual problem $(E\widetilde{D}_N^M)$ *was also solved with the Matlab Optimization Toolbox. The optimal solution was*

$$\overline{z}_{11}^* = (0.803, 0.114)^T,\ \overline{z}_{12}^* = (1.171, 0.327)^T, \\ \overline{z}_{21}^* = (-0.909, 0.402)^T,\ \overline{z}_{22}^* = (-0.98, 0.164)^T, \\ \overline{z}_{31}^* = (0.106, -0.516)^T,\ \overline{z}_{32}^* = (-0.191, -0.491)^T$$

and the optimal objective function value $v(E\widetilde{D}_N^M) = 21.578 = v(EP_N^M)$. *See Figure 1 for an illustration of the relation of the optimal solutions of the primal and its conjugate dual problem.*

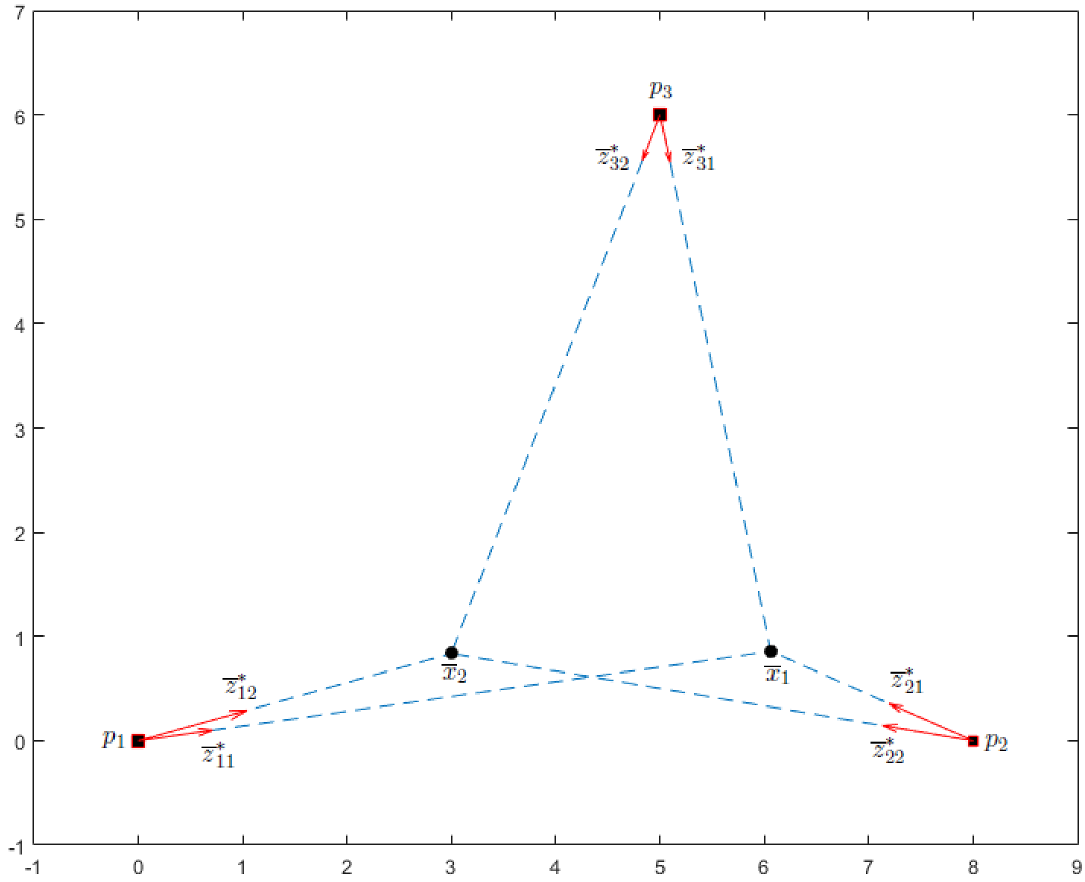

Figure 1: Illustration of the example 3.1.

An alternative geometrical interpretation of the set of optimal solutions of the conjugate dual problem is based on the fact that the extended multifacility location problem (EP^M) can be reduced to a single minimax location problem as seen in the beginning of Section 3. This means precisely that the sum of distances in the objective function of the location problem (EP_N^M) can be understood as the finding the minimum value for n norms d_i defined by the weighted sum of Euclidean norms, i.e. $d_i(y_1, ..., y_m) := \sum_{j=1}^m w_{ij}\|y_j\|$ with $y_j \in \mathbb{R}^d$, $w_{ij} > 0$, $j = 1, ..., m$, such that the associated norm balls centered at the points $\widetilde{p}_i = (p_i, ..., p_i)$ with $p_i \in \mathbb{R}^d$, $i = 1, ..., n$, have a non-empty intersection. In this case, it is possible to interpret the optimal solution of the corresponding conjugate dual problem as force vectors fulfilling the conditions in point (a) of Theorem 3.8 and increasing the norm balls until their intersection is non-empty. Notice that the optimality conditions (v) and (vi) imply that the vectors $\overline{z}_{ij}^*$, $j = 1, ..., m$, are equal to the zero vector if $i \notin \overline{I}$, which is exactly the case when $\overline{x}$ is an element of the interior of the ball associated to the norm d_i. But this also means that the vectors $\overline{z}_{ij}^*$, $j = 1, ..., m$, are all unequal to the zero vector if $i \in \overline{I}$, which exactly holds if $\overline{x}$ is lying on the border of the ball associated to the norm d_i.
For a better geometrical illustration of this interpretation, let us consider an example, where $d = 1$. In this case the Euclidean norm reduces to the absolute value.

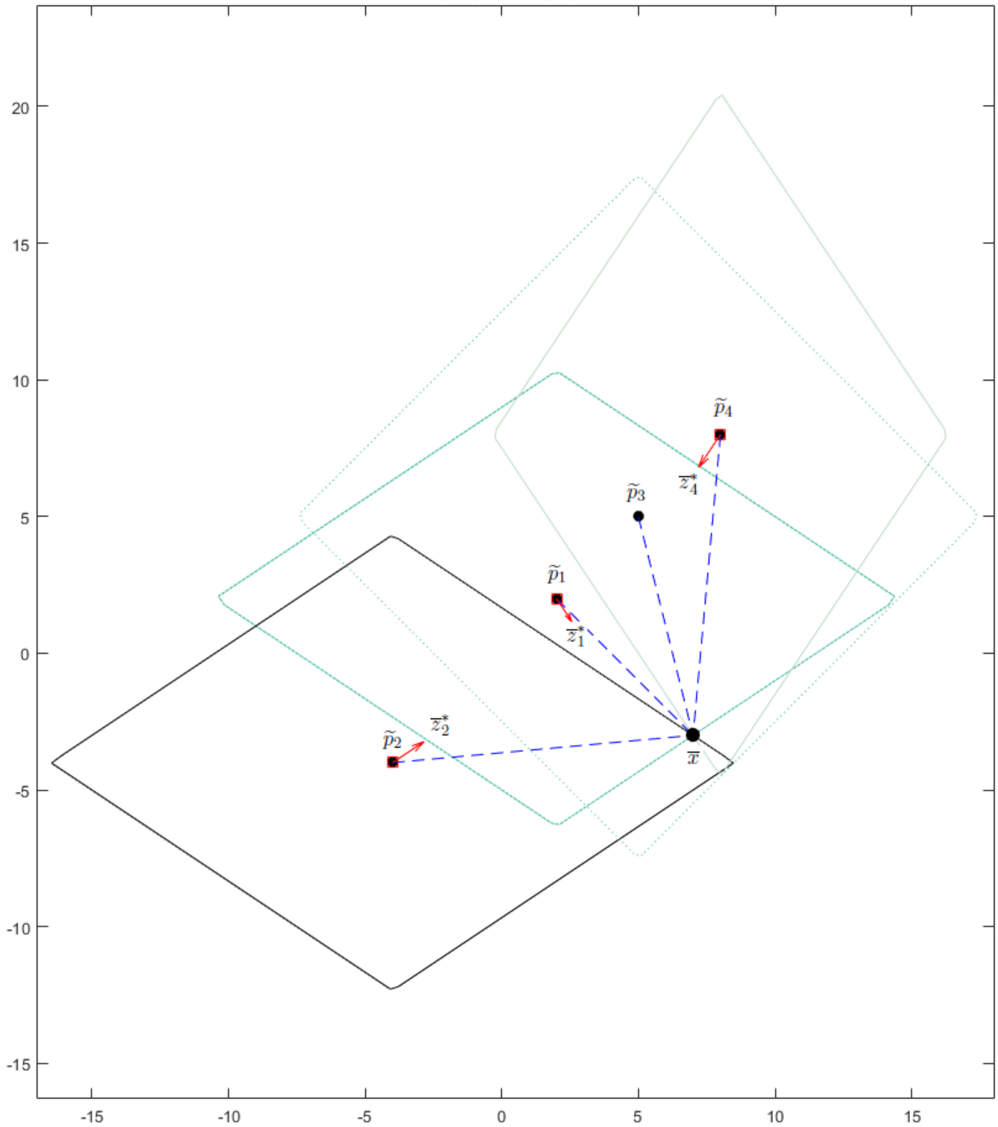

Figure 2: Illustration of the example 3.2.

Example 3.2. *For the given points* $\widetilde{p}_1 = (p_1, p_1) = (2,2)^T$, $\widetilde{p}_2 = (p_2, p_2) = (-4,-4)^T$, $\widetilde{p}_3 = (p_3, p_3) = (5,5)^T$, $\widetilde{p}_4 = (p_4, p_4) = (8,8)^T$ *and the weights* $w_{11} = 2$, $w_{12} = 3$, $w_{21} = 2$, $w_{22} = 3$, $w_{31} = 2$, $w_{32} = 2$, $w_{41} = 3$, $w_{42} = 2$ *we want to locate an optimal solution* $x = (x_1, x_2)^T \in \mathbb{R}^2$ *of the problem*

$$(EP_T^M) \quad \inf_{(x_1,x_2)^T \in \mathbb{R}^2} \max\{2|x_1 - 2| + 3|x_2 - 2|, 2|x_1 + 4| + 3|x_2 + 4|,$$
$$2|x_1 - 5| + 2|x_2 - 5|, 3|x_1 - 8| + 2|x_2 - 8|\}.$$

We solved the problem (EP^M) *with the Matlab Optimization Toolbox and obtain as optimal solution* $\overline{x} = (\overline{x}_1, \overline{x}_2)^T = (7,-3)^T$ *and as optimal objective value* $v(EP^M) = 25$.

For the corresponding conjugate dual problem

$$(E\widetilde{D}^M) \qquad \sup_{(z_1^*,z_2^*,z_3^*,z_4^*)\in\widetilde{\mathcal{C}}} \left\{-2(z_{11}^*+z_{12}^*)+4(z_{21}^*+z_{22}^*)-5(z_{31}^*+z_{32}^*)-8(z_{41}^*+z_{42}^*)\right\},$$

where

$$\widetilde{\mathcal{C}} = \Big\{(z_1^*,z_2^*,z_3^*,z_4^*)\in\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2 : \\ z_{11}^*+z_{21}^*+z_{31}^*+z_{41}^*=0,\ z_{12}^*+z_{22}^*+z_{32}^*+z_{42}^*=0, \\ \max\left\{\tfrac{1}{2}|z_{11}^*|,\tfrac{1}{3}|z_{12}^*|\right\}+\max\left\{\tfrac{1}{2}|z_{21}^*|,\tfrac{1}{3}|z_{22}^*|\right\}+\max\left\{\tfrac{1}{2}|z_{31}^*|,\tfrac{1}{2}|z_{32}^*|\right\}+\max\left\{\tfrac{1}{3}|z_{41}^*|,\tfrac{1}{2}|z_{42}^*|\right\}\leq 1\Big\}$$

we obtain by using again the Matlab Optimization Toolbox the associated optimal solution

$$\overline{z}_1^* = (\overline{z}_{11}^*,\overline{z}_{12}^*)^T = (0.333,-0.5)^T,\ \overline{z}_2^* = (\overline{z}_{21}^*,\overline{z}_{22}^*)^T = (0.867,1.3)^T, \\ \overline{z}_3^* = (\overline{z}_{31}^*,\overline{z}_{32}^*)^T = (0,0)^T,\ \overline{z}_4^* = (\overline{z}_{41}^*,\overline{z}_{42}^*)^T = (-1.2,-0.8)^T$$

and the optimal objective value $v(E\widetilde{D}^M) = 25 = v(EP^M)$. The numerical results are illustrated in Figure 2. Take note that $\overline{x}$ is lying inside the norm ball centered at the point $\widetilde{p}_3$ and that for this reason $\overline{z}_3^*$ is equal to the zero vector.

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