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Duality Results for Nonlinear Single Minimax Location Problems via Multi-Composed Optimization

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Abstract: In the framework of conjugate duality we discuss nonlinear and linear single minimax location problems with geometric constraints, where the gauges are defined by convex sets of a Fréchet space. The version of the nonlinear location problem is additionally considered with set-up costs. Associated dual problems for this kind of location problems will be formulated as well as corresponding duality statements. As conclusion of this paper, we give a geometrical interpretation of the optimal solutions of the dual problem of an unconstraint linear single minimax location problem when the gauges are a norm. For an illustration, an example in the Euclidean space will follow.

Key words: Conjugate Duality, Composed Functions, Gauges, Nonlinear Minimax Location Problems, Set-up Costs, Optimality Conditions.

1 Introduction

In the recent years, location problems attracted enormous attention in the scientific community and a large number of papers studying minisum and minimax location problems have been published (see [3]-[20]). This is due to the fact that location problems cover many practical situations occurring for example in urban area models, computer science, telecommunication and also in emergency facilities location programming.

In this paper minimax location problems form the focal point of our approach. In particular, we are interested to give a detailed duality study for nonlinear and linear single minimax location problems with geometric constraints, where the version of the nonlinear location problem will additionally be equipped with set-up costs. To be more precise, we will formulate to this kind of location problems its corresponding conjugate dual problems and derive necessary and sufficient optimality conditions. Notice that we work in a very general setting, where the underlying space is a Fréchet space and the distances are measured by gauges of convex sets.

But this is not all, we will formulate a new dual problem to the case of a linear single minimax location problem reducing the number of constraints and dual variables compared with the first formulated dual problem. Moreover, just as in the previous consideration we will establish also associated duality results. Besides, we consider to this new dual problem the case where the distances are measured by a norm defined on a Hilbert space and investigate from the optimality conditions additional statements. A geometrical interpretation of the optimal solutions of the

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dual problem and a discussion of an example will complete the paper.
The organization of this article is the following. Section 2 gives some elements of convex analysis and recalls basic statements of the duality approach done in [23] for geometrically and cone constrained multi-composed optimization problems. Then, in Section 3, we apply the previous approach to nonlinear single minimax location problems and give necessary and sufficient optimality conditions. Finally, in Section 4 we consider linear single minimax location problems. In Section 3 as well in Section 4 the location problems will be studied in a Fréchet space followed by a characterization to a Hilbert space endowed with a norm.

2 Preliminaries

2.1 Elements of convex analysis

Let X be a Fréchet space and X^* its topological dual space endowed with the weak* topology $w(X^*, X)$. For $x \in X$ and $x^* \in X^*$, let $\langle x^*, x\rangle := x^*(x)$ be the value of the linear continuous functional x^* at x. For a subset $A \subseteq X$, its indicator function $\delta_A : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is

$$\delta_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

For a given function $f : X \to \overline{\mathbb{R}}$ we consider its effective domain

$$\operatorname{dom} f := \{x \in X : \ f(x) < +\infty\}$$

and call $f : X \to \overline{\mathbb{R}}$ proper if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. The conjugate function of f with respect to the non-empty subset $S \subseteq X$ is defined by

$$f_S^* : X^* \to \overline{\mathbb{R}},\ f_S^*(x^*) = (f + \delta_S)^*(x^*) = \sup_{x \in S}\{\langle x^*, x\rangle - f(x)\}.$$

In the case $S = X$, it is clear that f_S^* turns into the classical Fenchel-Moreau conjugate function of f denoted by f^*. Let us mention that it holds $f^*(x^*) = \sup_{x \in \operatorname{dom} f}\{\langle x^*, x\rangle - f(x)\}$ as well as $f(x) + f(x^*) \geq \langle x^*, x\rangle$ for all $x \in X,\ x^* \in X^*$, which is the so-called Young-Fenchel inequality.
Additionally, we consider a non-empty convex cone $K \subseteq X$, which induces on X a partial ordering relation "$\leqq_K$", defined by

$$\leqq_K := \{(x, y) \in X \times X : y - x \in K\},$$

i.e. for $x, y \in X$ it holds $x \leqq_K y \Leftrightarrow y - x \in K$. Note that we assume that all cones we consider contain the origin. Further, we attach to X a greatest element with respect to "$\leqq_K$", denoted by $+\infty_K$, which does not belong to X and denote $\overline{X} = X \cup \{+\infty_K\}$. Then it holds $x \leqq_K +\infty_K$ for all $x \in \overline{X}$. We also define $x \leq_K y$ if and only if $x \leqq_K y$ and $x \neq y$. Further, we define $\leqq_{\mathbb{R}_+} =: \leq$ and $\leq_{\mathbb{R}_+} =: <$.
On $\overline{X}$ we consider the following operations and conventions: $x + (+\infty_K) = (+\infty_K) + x := +\infty_K\ \forall x \in X \cup \{+\infty_K\}$ and $\lambda \cdot (+\infty_K) := +\infty_K\ \forall \lambda \in [0, +\infty]$. Further, if $K^* := \{x^* \in X^* :\ \langle x^*, x\rangle \geq 0,\ \forall x \in K\}$ is the dual cone of K, then we define $\langle x^*, +\infty_K\rangle := +\infty$ for all $x^* \in K^*$. On the extended real space $\overline{\mathbb{R}}$ we add the following operations and conventions: $\lambda + (+\infty) = (+\infty) + \lambda := +\infty\ \forall \lambda \in (-\infty, +\infty]$, $\lambda + (-\infty) = (-\infty) + \lambda := -\infty\ \forall \lambda \in [-\infty, +\infty)$, $\lambda \cdot (+\infty) := +\infty\ \forall \lambda \in [0, +\infty]$, $\lambda \cdot (+\infty) := -\infty\ \forall \lambda \in [-\infty, 0)$, $\lambda \cdot (-\infty) :=$

$-\infty \ \forall \lambda \in (0, +\infty]$, $\lambda \cdot (-\infty) := +\infty \ \forall \lambda \in [-\infty, 0)$, $(+\infty) + (-\infty) = (-\infty) + (+\infty) := +\infty$, $0(+\infty) := +\infty$ and $0(-\infty) := 0$.
Let Z be another Fréchet space ordered by the convex cone $Q \subseteq Z$, then for a vector function $F : X \to \overline{Z} = Z \cup \{+\infty_Q\}$ the domain is the set $\operatorname{dom} F := \{x \in X : F(x) \neq +\infty_Q\}$. When $F(\lambda x + (1-\lambda)y) \leqq_Q \lambda F(x) + (1-\lambda)F(y)$ holds for all $x, y \in X$ and all $\lambda \in [0,1]$ the function F is said to be Q-convex. A function $f : X \to \overline{\mathbb{R}}$ is called convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in X$ and all $\lambda \in [0,1]$.
Further, we consider the epigraph of a function f defined by $\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$. The Q-epigraph of a vector function F is $\operatorname{epi}_Q F = \{(x, z) \in X \times Z : F(x) \leqq_Q z\}$ and we say that F is Q-epi closed if $\operatorname{epi}_Q F$ is a closed set.
If $Q^* := \{x^* \in X^* : \langle x^*, x\rangle \geq 0, \ \forall x \in Q\}$ is the dual cone of Q, then we define for $z^* \in Q^*$ the function $(z^*F) : X \to \overline{\mathbb{R}}$ by $(z^*F)(x) := \langle z^*, F(x)\rangle$, where it is not hard to see that $\operatorname{dom}(z^*F) = \operatorname{dom} F$. Moreover, it is easy to see that if F is Q-convex, then (z^*F) is convex for all $z^* \in Q^*$.
A function $f : X \to \overline{\mathbb{R}}$ is called lower semicontinuous at $\overline{x} \in X$ if $\liminf_{x \to \overline{x}} f(x) \geq f(\overline{x})$ and when this function is lower semicontinuous at all $x \in X$, then we call it lower semicontinuous (l.s.c. for short). The vector function F is called star Q-lower semicontinuous at $x \in X$ if (z^*F) is lower semicontinuous at x for all $z^* \in Q^*$. The function F is called star Q-lower semicontinuous if it is star Q-lower semicontinuous at every $x \in X$. Note that if F is star Q-lower semicontinuous, then it is also Q-epi closed, while the inverse statement is not true in general (see: Proposition 2.2.19 in [2]). Let us mention that in the case $Z = \mathbb{R}$ and $Q = \mathbb{R}_+$, the notion of Q-epi closedness falls into the classical notion of lower semicontinuity.
A function $f : X \to \overline{\mathbb{R}}$ is called K-increasing, if from $x \leqq_K y$ follows $f(x) \leq f(y)$ for all $x, y \in X$.

Definition 2.1. *The vector function $F : X \to \overline{Z}$ is called K-Q-increasing, if from $x \leqq_K y$ follows $F(x) \leqq_Q F(y)$ for all $x, y \in X$.*

For a set $S \subseteq X$ the conic hull is defined by $\operatorname{cone}(S) := \{\lambda x : x \in S, \ \lambda \geq 0\}$ and sqri is used to denote the strong quasi relative interior, where in the case of having a convex set $S \subseteq X$ it holds

$$\operatorname{sqri}(S) \quad = \quad \{x \in S : \operatorname{cone}(S - x) \text{ is a closed linear subspace}\}.$$

In this paper we do not use the classical differentiability, but we use the notion of subdifferentiability to formulate optimality conditions. If we take an arbitrary $x \in X$ such that $f(x) \in \mathbb{R}$, then we call the set

$$\partial f(x) := \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle \ \forall y \in X\}$$

the (convex) subdifferential of f at x, where the elements are called the subgradients of f at x. Moreover, if $\partial f(x) \neq \emptyset$, then we say that f is subdifferentiable at x and if $f(x) \notin \mathbb{R}$, then we make the convention that $\partial f(x) := \emptyset$. Note, that the subgradients can be characterized by the conjugate function, especially this means

$$x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) = \langle x^*, x\rangle, \ \forall x \in X, \ x^* \in X^*, \tag{1}$$

i.e. the Young-Fenchel inequality is fulfilled with equality.
Let $C \subseteq X$. As conclusion of this section we collect some properties of the gauge function of

the subset C, $\gamma_C : X \to \overline{\mathbb{R}}$ defined by

$$\gamma_C(x) := \begin{cases} +\infty, & \text{if } \{\lambda > 0 : x \in \lambda C\} = \emptyset, \\ \inf\{\lambda > 0 : x \in \lambda C\}, & \text{otherwise.} \end{cases}$$

Let us start with the following theorem.

Theorem 2.1. *Let $C \subseteq X$ be a convex and closed set with $0_X \in C$, then the gauge function γ_C is proper, convex and lower semicontinuous.*

Proof. Let us define the function $g : X^* \to \overline{\mathbb{R}}$ by

$$g(x^*) := \begin{cases} 0, & \text{if } \sigma_C(x^*) \leq 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

where σ_C is the support function of the set C, i.e. $\sigma_C(x^*) = \sup_{x \in C} \langle x^*, x \rangle$. It is obvious that g is proper, convex and lower semicontinuous. For the corresponding conjugate function of g one has

$$g^*(x) = \sup_{x^* \in X^*} \{\langle x^*, x \rangle - g(x^*)\} = \sup_{\substack{x^* \in X^*, \\ \sigma_C(x^*) \leq 1}} \langle x^*, x \rangle.$$

There is $g^*(x) = \sup_{x^* \in X^*} \{\langle x^*, x \rangle - g(x^*)\} \geq \langle 0_{X^*}, x \rangle - g(0_{X^*}) = 0$ since $g(0_{X^*}) = 0,\ \forall x \in X$, and $g^*(0_X) = \sup_{x^* \in X^*} \{-g(x^*)\} = 0$, i.e. g^* is proper. At this point it is important to say that from $0_X \in C$ follows that $\gamma_C(0_X) = 0$, i.e. $g^*(0_X) = \gamma_C(0_X)$.
Let us now assume that $x \neq 0_X$ and consider for fixed $x \in X$ the following convex optimization problem

$$(P^\gamma) \quad \inf_{\substack{x^* \in X^*, \\ \sigma_C(x^*) \leq 1}} \langle -x^*, x \rangle.$$

As $\sigma_C(0_{X^*}) = 0 < 1$, the Slater condition is fulfilled and hence, it holds strong duality between the problem (P^γ) and its corresponding Lagrange dual problem

$$(D_L^\gamma) \quad \sup_{\lambda \geq 0} \inf_{x^* \in X^*} \{\langle -x^*, x \rangle + \lambda(\sigma_C(x^*) - 1)\}.$$

Therefore, the conjugate function of g can be represented for $x \neq 0_X$ as

$$\begin{aligned} g^*(x) &= \sup_{\substack{x^* \in X^*, \\ \sigma_C(x^*) \leq 1}} \langle x^*, x \rangle = -\sup_{\lambda \geq 0} \inf_{x^* \in X^*} \{\langle -x^*, x \rangle + \lambda(\sigma_C(x^*) - 1)\} \\ &= \inf_{\lambda \geq 0} \left\{ \lambda + \sup_{x^* \in X^*} \{\langle x^*, x \rangle - \lambda \sigma_C(x^*)\} \right\} \qquad (2) \end{aligned}$$

For $\lambda = 0$ we verify two conceivable cases.
(a) If $\sigma_C(x^*) < +\infty$, then $0 \cdot \sigma_C(x^*) = 0$ and therefore,

$$\sup_{x^* \in X^*} \{\langle x^*, x \rangle - 0 \cdot \sigma_C(x^*)\} = \sup_{x^* \in X^*} \langle x^*, x \rangle = \begin{cases} 0, & \text{if } x = 0_X, \\ +\infty, & \text{if } x \neq 0_X. \end{cases}$$

As by assumption $x \neq 0_X$, we have $\sup_{x^* \in X^*} = +\infty$, but this has no effect on the infimum in (2).
(b) If $\sigma_C(x^*) = +\infty$, then one has by convention that $\lambda \cdot \sigma_C(x^*) = 0 \cdot (+\infty) = +\infty$ and hence,

$$\langle x^*, x\rangle - \lambda\sigma_C(x^*) = \langle x^*, x\rangle - \infty = -\infty,$$

which has no effect on $\sup_{x^* \in X^*}\{\langle x^*, x\rangle - \lambda\sigma_C(x)\}$.
Hence, as the cases (a) and (b) are not relevant for g^*, we can omit the situation when $\lambda = 0$ and can write

$$g^*(x) = \inf_{\lambda > 0} \left\{ \lambda + \lambda \sup_{x^* \in X^*} \left\{ \left\langle x^*, \frac{1}{\lambda} x \right\rangle - \sigma_C(x^*) \right\} \right\}.$$

Moreover, as C is a non-empty, closed and convex subset of X, the conjugate of the support function σ_C is the indicator function δ_C, i.e.

$$g^*(x) = \inf_{\lambda > 0} \left\{ \lambda + \lambda \delta_C \left(\frac{1}{\lambda} x \right) \right\} = \inf_{\lambda > 0,\ \frac{1}{\lambda} x \in C} \lambda = \inf\{\lambda > 0 : x \in \lambda C\}.$$

Taking the situations where $x = 0_X$ and $x \neq 0_X$ together implies that $g^*(x) = \gamma_C(x),\ \forall x \in X$. Hence, γ_C is the conjugate function of g and by the definition of the conjugate function it follows that γ_C is convex and lower semicontinuous. This completes the proof. □

Remark 2.1. *Note that the gauge function γ_C is not only convex but also sublinear. Moreover, if $0_X \in \operatorname{int} C$, then γ_C is well-defined, which means that $\operatorname{dom} \gamma_C = X$.*

Lemma 2.1. *Let $C \subseteq X$ be a convex and closed set with $0_X \in C$, then the conjugate of the gauge function γ is given by*

$$\gamma_C^*(x^*) := \begin{cases} 0, & \text{if } \sigma_C(x^*) \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Proof. In the proof of Theorem 2.1 we have shown that γ_C is the conjugate function of g, i.e. $\gamma_C = g^*$, and as g is proper, convex and lower semicontinuous we have $g = g^{**}$. As g^{**} is also the conjugate function of γ_C, it holds $\gamma_C^* = g$. □

Definition 2.2. *Let $C \subseteq X$. The polar set of C is defined by*

$$C^0 := \left\{ x^* \in X^* : \sup_{x \in C} \langle x^*, x\rangle \leq 1 \right\} = \{x^* \in X^* : \sigma_C(x^*) \leq 1\}$$

and by means of the polar set the dual gauge is defined by

$$\gamma_{C^0}(x^*) := \sup_{x \in C} \langle x^*, x\rangle = \sigma_C(x^*).$$

Remark 2.2. *Note that C^0 is a convex and closed set containing the origin. Furthermore, by the definition of the dual gauge follows that the conjugate function of γ_C can equivalently be expressed by*

$$\gamma_C^*(x^*) := \begin{cases} 0, & \text{if } \gamma_{C^0}(x^*) \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

2.2 Lagrange duality approach for multi-composed optimization problems

The purpose of this section is to recall some important results done in [23] by studying multi-composed optimization problems. Let us consider an optimization problem with geometric and cone constraints having as objective function the composition of $n+1$ functions:

$$(P^C) \qquad \inf_{x\in\mathcal{A}} (f\circ F^1\circ ...\circ F^n)(x),$$
$$\mathcal{A} = \{x \in S : g(x) \in -Q\},$$

where X_i is a Fréchet space partially ordered by the non-empty convex cone $K_i \subseteq X_i$ for $i = 0, ..., n-1$. Moreover,

- $S \subseteq X_n$ is a non-empty convex set,
- $f : X_0 \to \overline{\mathbb{R}}$ is proper, convex and K_0-increasing on $F^1(\operatorname{dom} F^1) + K_0 \subseteq \operatorname{dom} f$,
- $F^i : X_i \to \overline{X}_{i-1} = X_{i-1} \cup \{+\infty_{K_{i-1}}\}$ is proper, K_{i-1}-convex and K_i-K_{i-1}-increasing on $F^{i+1}(\operatorname{dom} F^{i+1}) + K_i \subseteq \operatorname{dom} F^i$ for $i = 1, ..., n-2$,
- $F^{n-1} : X_{n-1} \to \overline{X}_{n-2} = X_{n-1} \cup \{+\infty_{K_{n-1}}\}$ is proper and K_{n-1}-K_{n-2}-increasing on $F^n(\operatorname{dom} F^n \cap \mathcal{A}) + K_{n-1} \subseteq \operatorname{dom} F^{n-1}$,
- $F^n : X_n \to \overline{X}_{n-1} = X_{n-1} \cup \{+\infty_{K_{n-1}}\}$ is a proper and K_{n-1}-convex function and
- $g : X_n \to \overline{Z}$ is a proper function fulfilling $S \cap g^{-1}(-Q) \cap ((F^n)^{-1} \circ ... \circ (F^1)^{-1})(\operatorname{dom} f) \neq \emptyset$.

Additionally, we make the convention that $f(+\infty_{K_0}) = +\infty$ and $F^i(+\infty_{K_i}) = +\infty_{K_{i-1}}$, i.e. $f : \overline{X}_0 \to \overline{\mathbb{R}}$ and $F^i : \overline{X}_i \to \overline{X}_{i-1}$, $i = 1, ..., n-1$.

Remark 2.3. *Let us point out that for the convexity of $(f\circ F^1\circ ...\circ F^n)$ we ask that the function f be convex and K_0-increasing on $F^1(\operatorname{dom} F^1) + K_0$ and the function F^i be K_{i-1}-convex and fulfills also the property of monotonicity for $i = 1, ..., n-1$, while the function F^n need just be K_{n-1}-convex. This means that if F^n is an affine function, we do not need the monotonicity of F^{n-1}, since the composition of an affine function and a function, which fulfills the property of convexity, fulfills also the property of convexity. In this context one can choose $K_{n-1} = \{0_{X_{n-1}}\}$ (for more details see Remark 3.1 and 4.1 in [23]).*

The corresponding conjugate dual problem to the problem (P^C) looks like (see [23])

$$(D^C) \qquad \sup_{\substack{z^{n*}\in Q^*,\ z^{i*}\in K_i^*,\\ i=0,...,n-1}} \left\{ \inf_{x\in S}\{\langle z^{(n-1)*}, F^n(x)\rangle + \langle z^{n*}, g(x)\rangle\} - f^*(z^{0*}) - \sum_{i=1}^{n-1}(z^{(i-1)*}F^i)^*(z^{i*}) \right\},$$

where $\widetilde{z}^* := (z^{0*}, ..., z^{(n-1)*}, z^{n*}) \in \widetilde{K}^* := K_0^* \times ... \times K_{n-1}^* \times Q^*$ are the dual variables.
We denote by $v(P^C)$ and $v(D^C)$ the optimal objective values of the optimization problems (P^C) and (D^C), respectively. To guarantee strong duality, i.e. the situation where $v(P^C) = v(D^C)$ and the conjugate dual problem has an optimal solution, we consider the following generalized interior point regularity condition introduced in [23]:

$$
(RC) \quad \left| \begin{array}{l} f \text{ is l.s.c., } S \text{ is closed, } g \text{ is } Q\text{-epi closed, } K_{i-1} \text{ is closed,} \\ \operatorname{int} K_{i-1} \neq \emptyset,\ F^i \text{ is } K_{i-1}\text{-epi closed, } i = 1, ..., n, \\ 0_{X_0} \in \operatorname{sqri}(F^1(\operatorname{dom} F^1) - \operatorname{dom} f + K_0), \\ 0_{X_{i-1}} \in \operatorname{sqri}(F^i(\operatorname{dom} F^i) - \operatorname{dom} F^{i-1} + K_{i-1}), i = 2, ..., n-1, \\ 0_{X_{n-1}} \in \operatorname{sqri}(F^n(\operatorname{dom} F^n \cap \operatorname{dom} g \cap S) - \operatorname{dom} F^{n-1} + K_{n-1}) \text{ and} \\ 0_Z \in \operatorname{sqri}(g(\operatorname{dom} F^n \cap \operatorname{dom} g \cap S) + Q). \end{array} \right.
$$

In [23] the following theorems have been stated.

Theorem 2.2. *(strong duality) If the condition (RC) is fulfilled, then between (P^C) and (D^C) strong duality holds, i.e. $v(P^C) = v(D^C)$ and the conjugate dual problem has an optimal solution.*

Theorem 2.3. *(optimality conditions) (a) Suppose that the regularity condition (RC) is fulfilled and let $\overline{x} \in \mathcal{A}$ be an optimal solution of the problem (P^C). Then there exists $(\overline{z}^{0*}, ..., \overline{z}^{(n-1)*}, \overline{z}^{n*}) \in K_0^* \times ... \times K_{n-1}^* \times Q^*$, an optimal solution to (D^C), such that*

(i) $f((F^1 \circ ... \circ F^n)(\overline{x})) + f^*(\overline{z}^{0*}) = \langle \overline{z}^{0*}, (F^1 \circ ... \circ F^n)(\overline{x}) \rangle$,

(ii) $(\overline{z}^{(i-1)*} F^i)((F^{i+1} \circ ... \circ F^n)(\overline{x})) + (\overline{z}^{(i-1)*} F^i)^*(\overline{z}^{i*}) = \langle \overline{z}^{i*}, (F^{i+1} \circ ... \circ F^n)(\overline{x}) \rangle,\ i = 1, ..., n-1$,

(iii) $(\overline{z}^{(n-1)*} F^n)(\overline{x}) + (\overline{z}^{n*} g)(\overline{x}) + ((\overline{z}^{(n-1)*} F^n) + (\overline{z}^{n*} g))_S^*(0_{X_n^*}) = 0$,

(iv) $\langle \overline{z}^{n*}, g(\overline{x}) \rangle = 0$,

(b) If there exists $\overline{x} \in \mathcal{A}$ such that for some $(\overline{z}^{0*}, ..., \overline{z}^{(n-1)*}, \overline{z}^{n*}) \in K_0^* \times ... \times K_{n-1}^* \times Q^*$ the conditions (i)-(iv) are fulfilled, then $\overline{x}$ is an optimal solution of (P^C), $(\overline{z}^{0*}, ..., \overline{z}^{n*})$ is an optimal solution for (D^C) and $v(P^C) = v(D^C)$.

Remark 2.4. *If for some $i \in \{1, ..., n\}$ the function F^i is star K_{i-1}-lower semicontinuous, then we can omit asking that K_{i-1} is closed, $int(K_{i-1}) \neq \emptyset$ and F^i is K_{i-1}-epi closed in the regularity conditions (RC) (for more details see Remark 4.2 in [23]).*

Theorem 2.4. *Let $a_i \in \mathbb{R}_+$ be a given point and $h_i : \mathbb{R} \to \overline{\mathbb{R}}$ with $h_i(x) \in \mathbb{R}_+$, if $x \in \mathbb{R}_+$, and $h_i(x) = +\infty$, otherwise, be a proper, lower semicontinuous and convex function, $i = 1, ..., n$. Then the conjugate of the function $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by*

$$
g(x_1, ..., x_n) := \begin{cases} \max\{h_1(x_1) + a_1, ..., h_n(x_n) + a_n\}, & \text{if } x_i \in \mathbb{R}_+,\ i = 1, ..., n, \\ +\infty, & \text{otherwise,} \end{cases}
$$

is given by $g^* : \mathbb{R}^n \to \mathbb{R}$,

$$
g^*(x_1^*, ..., x_n^*) = \min_{\substack{\sum_{i=1}^n z_i^{0*} \leq 1,\ z_i^{0*} \geq 0, \\ i=1,...,n}} \left\{ \sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] \right\}.
$$

Proof. We set $X_0 = X_1 = \mathbb{R}^n$ and $K_0 = \mathbb{R}^n_+$. Further, we define the function $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}}$ by

$$
f(y_1^0, ..., y_n^0) := \begin{cases} \max\{y_1^0 + a_1, ..., y_n^0 + a_n\}, & \text{if } y_i^0 \in \mathbb{R}_+,\ i = 1, ..., n, \\ +\infty, & \text{otherwise,} \end{cases}
$$

and the function $F^1 : \mathbb{R}^n \to \overline{\mathbb{R}^n}$ by

$$F^1(x_1,...,x_n) := \begin{cases} (h_1(x_1),...,h_n(x_n))^T, & \text{if } x_i \in \mathbb{R}_+,\ i=1,...,n, \\ +\infty_{\mathbb{R}^n_+}, & \text{otherwise.} \end{cases}$$

Hence, the function g can be written as

$$g(x_1,...,x_n) = (f \circ F^1)(x_1,...,x_n).$$

It can easy be verified that the function f is proper, convex, lower semicontinuous and $\mathbb{R}^n_+$-increasing on $F^1(\operatorname{dom} F^1) + K_0 \subseteq \mathbb{R}^n_+$ (as f is the pointwise supremum of proper, convex and lower semicontinuous functions) and the function F^1 is proper, $\mathbb{R}^n_+$-epi closed and $\mathbb{R}^n_+$-convex. Therefore, it follows by Theorem 5.2 in [23] (note also that $0_{\mathbb{R}^n} \in \operatorname{sqri}(F^1(\operatorname{dom} F^1) - \operatorname{dom} f + K_0) = \operatorname{sqri}(F^1(\operatorname{dom} F^1) - \mathbb{R}^n_+ + \mathbb{R}^n_+) = \mathbb{R}^n$) that

$$g^*(x_1^*,...,x_n^*) = \min_{y_i^{0*}\in\mathbb{R}_+,\ i=1,...,n} \{f^*(y_1^{0*},...,y_n^{0*}) + ((y_1^{0*},...,y_n^{0*})^T F^1)^*(x_1^*,...,x_n^*)\}.$$

For the conjugate of the function f we have

$$\begin{aligned} f^*(y^{0*}) &= \sup_{y_i^0\in\mathbb{R},\ i=1,...,n} \left\{ \sum_{i=1}^n y_i^{0*} y_i^0 - f(y_1^0,...,y_n^0) \right\} \\ &= \sup_{y_i^0\in\mathbb{R}_+,\ i=1,...,n} \left\{ \sum_{i=1}^n y_i^{0*} y_i^0 - \max_{1\le i\le n} \{y_i^0 + a_i\} \right\} \\ &= \sup_{y_i^0\in\mathbb{R}_+,\ i=1,...,n} \left\{ \sum_{i=1}^n y_i^{0*} y_i^0 - \min_{\substack{t\in\mathbb{R}_+, y_i^0+a_i\le t,\\ i=1,...,n}} t \right\} = \sup_{\substack{y_i^0\in\mathbb{R}_+,\ t\in\mathbb{R}_+,\\ y_i^0+a_i\le t,\ i=1,...,n}} \left\{ \sum_{i=1}^n y_i^{0*} y_i^0 - t \right\}. \end{aligned} \tag{3}$$

Now, let us consider for any $y^{0*} \in \mathbb{R}^n_+$ the following primal optimization problem

$$(P^{max}) \quad \inf_{\substack{y_i^0\in\mathbb{R}_+,\ t\in\mathbb{R}_+,\\ y_i^0+a_i\le t,\ i=1,...,n}} \left\{ t - \sum_{i=1}^n y_i^{0*} y_i^0 \right\}. \tag{4}$$

and its corresponding Lagrange dual problem

$$\begin{aligned} (D^{max}) \quad & \sup_{\lambda_i\ge 0,\ i=1,...,n} \inf_{\substack{y_i^0\in\mathbb{R}_+,\ t\in\mathbb{R}_+,\\ i=1,...,n}} \left\{ t - \sum_{i=1}^n y_i^{0*} y_i^0 + \sum_{i=1}^n \lambda_i (y_i^0 + a_i - t) \right\} \\ &= \sup_{\substack{\lambda_i\ge 0,\\ i=1,...,n}} \left\{ -\sup_{t\in\mathbb{R}_+} \left\{ \left(\sum_{i=1}^n \lambda_i - 1 \right) t \right\} - \sup_{\substack{y_i^0\in\mathbb{R}_+,\\ i=1,...,n}} \left\{ \sum_{i=1}^n (y_i^{0*} - \lambda_i) y_i^0 \right\} + \sum_{i=1}^n \lambda_i a_i \right\} \\ &= \sup_{\substack{\sum_{i=1}^n \lambda_i \le 1,\ \lambda_i\ge 0,\\ y_i^{0*}\le\lambda_i,\ i=1,...,n}} \left\{ \sum_{i=1}^n \lambda_i a_i \right\}. \end{aligned}$$

As the Slater constraint qualification is fulfilled, i.e. it holds $v(P^{max}) = v(D^{max})$ and the dual has an optimal solution, one gets for the conjugate function of f

$$f^*(y^{0*}) = \min_{\substack{\sum_{i=1}^n \lambda_i \leq 1,\ \lambda_i \geq 0, \\ y_i^{0*} \leq \lambda_i,\ i=1,...,n}} \left\{ -\sum_{i=1}^n \lambda_i a_i \right\}. \tag{5}$$

Furthermore, one has

$$\begin{aligned} ((y_1^{0*},...,y_n^{0*})^T F^1)^*(x_1^*,...,x_n^*) &= \sup_{x_i \in \mathbb{R},\ i=1,...,n} \left\{ \sum_{i=1}^n x_i^* x_i - (y_1^{0*},...,y_n^{0*})^T F^1(x_1,...,x_n) \right\} \\ &= \sup_{x_i \in \mathbb{R}_+,\ i=1,...,n} \left\{ \sum_{i=1}^n x_i^* x_i - \sum_{i=1}^n y_i^{0*} h_i(x_i) \right\} \\ &= \sum_{i=1}^n \sup_{x_i \in \mathbb{R}_+} \{ x_i^* x_i - y_i^{0*} h_i(x_i) \} \\ &= \sum_{i=1}^n (y_i^{0*} h_i)^*(x_i^*), \end{aligned} \tag{6}$$

and so, the conjugate function of g turns into

$$\begin{aligned} g^*(x_1^*,...,x_n^*) &= \min_{y_i^{0*} \geq 0,\ i=1,...,n} \left\{ \min_{\substack{\sum_{i=1}^n \lambda_i \leq 1,\ \lambda_i \geq 0, \\ y_i^{0*} \leq \lambda_i,\ i=1,...,n}} \left\{ -\sum_{i=1}^n \lambda_i a_i \right\} + \sum_{i=1}^n (y_i^{0*} h_i)^*(x_i^*) \right\} \\ &= \min_{\substack{\sum_{i=1}^n \lambda_i \leq 1,\ \lambda_i \geq 0, \\ 0 \leq y_i^{0*} \leq \lambda_i,\ i=1,...,n}} \left\{ \sum_{i=1}^n [(y_i^{0*} h_i)^*(x_i^*) - \lambda_i a_i] \right\}. \end{aligned} \tag{7}$$

We fix $x_i^* \in \mathbb{R}^n$, $i = 1,...,n$, and emphasize that the problem

$$(P^g) \qquad \min_{\substack{\sum_{i=1}^n \lambda_i \leq 1,\ \lambda_i \geq 0, \\ 0 \leq y_i^{0*} \leq \lambda_i,\ i=1,...,n}} \left\{ \sum_{i=1}^n [(y_i^{0*} h_i)^*(x_i^*) - \lambda_i a_i] \right\} \tag{8}$$

is equivalent to

$$(\widetilde{P}^g) \qquad \min_{\substack{\sum_{i=1}^n z_i^{0*} \leq 1,\ z_i^{0*} \geq 0, \\ i=1,...,n}} \left\{ \sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] \right\} \tag{9}$$

in the sense that $v(P^g) = v(\widetilde{P}^g)$ (where $v(P^g)$ and $v(\widetilde{P}^g)$ denote the optimal objective values of the problems (P^g) and $(\widetilde{P}^g)$, respectively).
To see this, take first a feasible element $(\lambda_1,...,\lambda_n, y_1^{0*},...,y_n^{0*}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ of the problem (P^g) and set $z_i^{0*} = \lambda_i$, $i = 1,...,n$, then it follows from $\sum_{i=1}^n \lambda_i \leq 1,\ \lambda_i, y_i^{0*} \geq 0,\ y_i^{0*} \leq \lambda_i,\ i = 1,...,n,$

that $\sum_{i=1}^n z_i^{0*} \leq 1,\ z_i^{0*} \geq 0,\ i = 1, ..., n$, i.e. $(z_1^{0*}, ..., z_n^{0*})$ is feasible to the problem $(\widetilde{P}^g)$. Hence it holds

$$\sum_{i=1}^n [(y_i^{0*} h_i)^*(x_i^*) - \lambda_i a_i] \geq \sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] \geq v(\widetilde{P}^g) \tag{10}$$

for all $(\lambda_1, ..., \lambda_n, y_1^{0*}, ..., y_n^{0*})$ feasible to (P^g), i.e. $v(P^g) \geq v(\widetilde{P}^g)$.
Now, take a feasible element $(z_1^{0*}, ..., z_n^{0*}) \in \mathbb{R}_+^n$ of the problem $(\widetilde{P}^g)$ and set $y_i^{0*} = \lambda_i = z_i^{0*}$ for all $i = 1, ..., n$, then we have from $\sum_{i=1}^n z_i^{0*} \leq 1,\ z_i^{0*} \geq 0,\ i = 1, ..., n$, that $\sum_{i=1}^n \lambda_i \leq 1$, $\lambda_i,\ y_i^{0*} \geq 0$, $y_i^{0*} = \lambda_i$, $i = 1, ..., n$, which means that $(\lambda_1, ..., \lambda_n, y_1^{0*}, ..., y_n^{0*})$ is a feasible element of (P^g) and it holds

$$\sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] = \sum_{i=1}^n [(y_i^{0*} h_i)^*(x_i^*) - \lambda_i a_i] \geq v(P^g) \tag{11}$$

for all $(z_1^{0*}, ..., z_n^{0*})$ feasible to $v(\widetilde{P}^g)$, which implies $v(P^g) \leq v(\widetilde{P}^g)$. Finally, it follows that $v(P^g) = v(\widetilde{P}^g)$ and thus, the conjugate function of g is given by

$$g^*(x_1^*, ..., x_n^*) = \min_{\substack{\sum_{i=1}^n z_i^{0*} \leq 1,\ z_i^{0*} \geq 0, \\ i=1,...,n}} \left\{ \sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] \right\} \tag{12}$$

and takes only finite values. □

Lemma 2.2. *Let $a_i \in \mathbb{R}_+$ be a given point and $h_i : \mathbb{R} \to \overline{\mathbb{R}}$ with $h_i(x) \in \mathbb{R}_+$, if $x \in \mathbb{R}_+$, and $h_i(x) = +\infty$, otherwise, be a proper, lower semicontinuous and convex function, $i = 1, ..., n$. Then the function $g : \mathbb{R}^m \to \overline{\mathbb{R}}$,*

$$g(x_1, ..., x_n) = \begin{cases} \max\{h_1(x_1) + a_1, ..., h_n(x_n) + a_n\}, & \text{if } x_i \in \mathbb{R}_+,\ i = 1, ..., n, \\ +\infty, & \text{otherwise,} \end{cases}$$

can equivalently be expressed as

$$g(x_1, ..., x_n) = \sup_{\substack{\sum_{i=1}^n z_i^{0*} \leq 1,\ z_i^{0*} \geq 0, \\ i=1,...,n}} \left\{ \sum_{i=1}^n z_i^{0*} [h_i(x_i) + a_i] \right\},\ \forall x_i \geq 0,\ i = 1, ..., n.$$

Proof. By Theorem 2.4 and the definition of the conjugate function we have for the biconjugate function of g

$$\begin{aligned} g^{**}(x_1, ..., x_n) &= \sup_{x_i^* \in \mathbb{R},\ i=1,...,n} \left\{ \sum_{i=1}^n x_i^* x_i - \min_{\substack{\sum_{i=1}^n z_i^{0*} \leq 1,\ z_i^{0*} \geq 0, \\ i=1,...,n}} \left\{ \sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] \right\} \right\} \\ &= \sup_{\substack{x_i^* \in \mathbb{R},\ z_i^{0*} \in \mathbb{R}_+, \\ i=1,...,n,\ \sum_{i=1}^n z_i^{0*} \leq 1}} \left\{ \sum_{i=1}^n x_i^* x_i - \sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] \right\} \end{aligned}$$

$$= \sup_{\substack{z_i^{0*} \in \mathbb{R}_+,\ i=1,\dots,n, \\ \sum_{i=1}^n z_i^{0*} \leq 1}} \left\{ \sum_{i=1}^n [\sup_{x_i^* \in \mathbb{R}} \{x_i^* x_i - (z_i^{0*} h_i)^*(x_i^*)\} + z_i^{0*} a_i] \right\}$$

$$= \sup_{\substack{z_i^{0*} \in \mathbb{R}_+,\ i=1,\dots,n, \\ \sum_{i=1}^n z_i^{0*} \leq 1}} \left\{ \sum_{i=1}^n [(z_i^{0*} h_i)^{**}(x_i) + z_i^{0*} a_i] \right\}. \quad (13)$$

As h_i, $i = 1, ..., n$, are proper, convex and lower semicontinuous functions it follows by the Fenchel-Moreau Theorem that

$$g^{**}(x_1, ..., x_n) = \sup_{\substack{z_i^{0*} \in \mathbb{R}_+,\ i=1,\dots,n, \\ \sum_{i=1}^n z_i^{0*} \leq 1}} \left\{ \sum_{i=1}^n [z_i^{0*} h_i(x_i) + z_i^{0*} a_i] \right\},\ \forall x_i \in \mathbb{R}_+,\ i = 1, ..., n, \quad (14)$$

and moreover, as g is also a proper, convex and lower semicontinuous function it follows by using again the Fenchel-Moreau Theorem that $g = g^{**}$, i.e.

$$g(x_1, ..., x_n) = \sup_{\substack{z_i^{0*} \in \mathbb{R}_+,\ i=1,\dots,n, \\ \sum_{i=1}^n z_i^{0*} \leq 1}} \left\{ \sum_{i=1}^n [z_i^{0*} h_i(x_i) + z_i^{0*} a_i] \right\},\ \forall x_i \in \mathbb{R}_+,\ i = 1, ..., n. \quad (15)$$

□

We close this section by the following remark.

Remark 2.5. *If we consider the situation when the given points a_i, $i = 1, ..., n$, are arbitrary, i.e. $a_i \in \mathbb{R}$, then it can easy be verified that the conjugate function of f in (3) looks like*

$$f^*(y^{0*}) = \sup_{\substack{y_i^0 \in \mathbb{R}_+,\ t \in \mathbb{R}, \\ y_i^0 + a_i \leq t,\ i=1,\dots,n}} \left\{ \sum_{i=1}^n y_i^{0*} y_i^0 - t \right\} \quad (16)$$

(notice that here $t \in \mathbb{R}$ instead of $t \in \mathbb{R}_+$).
If we now construct to the conjugate function in (16) a primal problem in the sense of (P^{max}) in (4), then the corresponding Lagrange dual problem (D^{max}) has the form

$$(D^{max}) \qquad \sup_{\substack{\sum_{i=1}^n \lambda_i = 1,\ \lambda_i \geq 0, \\ y_i^{0*} \leq \lambda_i,\ i=1,\dots,n}} \left\{ \sum_{i=1}^n \lambda_i a_i \right\}.$$

Analogously to the calculations done above in (5) - (15) one derives for the conjugate function of g,

$$g^*(x_1^*, ..., x_n^*) = \min_{\substack{\sum_{i=1}^n z_i^{0*} = 1,\ z_i^{0*} \geq 0, \\ i=1,\dots,n}} \left\{ \sum_{i=1}^n [(z_i^{0*} h_i)^*(x_i^*) - z_i^{0*} a_i] \right\},$$

while its biconjugate is then given by

$$g^{**}(x_1,...,x_n) = g(x_1,...,x_n) = \sup_{\substack{\sum_{i=1}^{n} z_i^{0*}=1,\ z_i^{0*}\geq 0,\\ i=1,...,n}} \left\{ \sum_{i=1}^{n} z_i^{0*}[h_i(x_i)+a_i] \right\},\ \forall x_i \geq 0,\ i=1,...,n.$$

3 Duality results for nonlinear location problems with set-up costs

3.1 Geometrically constrained location problems with gauges in Fréchet spaces

Let us now focus our discussion for given non-negative set-up costs $a_i \in \mathbb{R}_+$ and distinct points $p_i \in X$, $i = 1,...,n$, (where $n \geq 2$) on the following geometrically constrained minimax location problem

$$(P_{h,a}^S) \qquad \inf_{x\in S} \sup_{1\leq i\leq n} \{h_i(\gamma_i(x-p_i)) + a_i\},$$

where

- S is a non-empty, closed and convex subset of a Fréchet space X,
- C_i is a non-empty, closed and convex subset of X such that $0_X \in \operatorname{int} C_i$,
- $\gamma_{C_i} : X \to \mathbb{R}$ is a gauge function of the subset C_i and
- $h_i : \mathbb{R} \to \overline{\mathbb{R}}$, defined by

$$h_i(x) := \begin{cases} h_i(x) \in \mathbb{R}_+, & \text{if } x \in \mathbb{R}_+, \\ +\infty, & \text{otherwise}, \end{cases}$$

  is a proper, convex, lower semicontinuous and increasing function on $\mathbb{R}_+$,

$i = 1,...,n$. Hence, it is clear that the defined gauges are proper, lower semicontinuous and convex functions by Theorem 2.1, which implies that the problem $(P_{h,a}^S)$ is a convex optimization problem. The case where the set-up costs are arbitrary, i.e. $a_i \in \mathbb{R}$, will be discussed in Remark 3.2.

For applying the developed Lagrange dual concept for multi-composed optimization problems, we set $X_0 = \mathbb{R}^n$ ordered by $K_0 = \mathbb{R}^n_+$, $X_1 = X^n$ ordered by the trivial cone $K_1 = \{0_{X^n}\}$ and $X_2 = X$ and introduce the following functions:

- $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$f(y^0) := \begin{cases} \sup\limits_{1\leq i\leq n} \{h_i(y_i^0) + a_i\},\ \text{if } y^0 = (y_1^0,...,y_n^0)^T \in \mathbb{R}^n_+,\ i = 1,...,n, \\ +\infty_{\mathbb{R}^n_+},\ \text{otherwise}, \end{cases}$$

- $F^1 : X^n \to \mathbb{R}^n$ defined by $F^1(y^1) := (\gamma_{C_1}(y^1),...,\gamma_{C_n}(y^1))^T$ with $y^1 = (y_1^1,...,y_n^1) \in X^n$ and

- $F^2 : X \to X^n$ defined by $F^2(x) := (x - p_1, ..., x - p_n)$.

These definitions yield the following equivalent representation for the considered problem

$$(P_{h,a}^S) \qquad \inf_{x \in S} (f \circ F^1 \circ F^2)(x).$$

The function f is proper, convex, $\mathbb{R}_+^n$-increasing on $F^1(\operatorname{dom} F^1) + K_0 = \operatorname{dom} f = \mathbb{R}_+^n$ and lower semicontinous. Additionally, one can verify that the function F^1 is proper, $\mathbb{R}_+^n$-convex and $\mathbb{R}_+^n$-epi closed. Furthermore, since the function F^2 is affine, it follows that the function F^1 does not need to be monotone (see Remark 2.3).
By setting $Z = X$ ordered by the trivial cone $Q = X$ and defining the function $g : X \to X$ by $g(x) := x$, we have that $Q^* = \{0_{X^*}\}$, i.e. $z^{2*} = 0_{X^*}$, and thus, the conjugate dual problem corresponding to $(P_{h,a}^S)$, in accordance with the concept from the previous section, looks like

$$(D_{h,a}^S) \quad \sup_{\substack{z_i^{0*} \in \mathbb{R}_+,\ z_i^{1*} \in X^*, \\ i=1,...,n}} \left\{ \inf_{x \in S} \left\{ \sum_{i=1}^n \langle z_i^{1*}, x - p_i \rangle \right\} - f^*(z^{0*}) - (z^{0*} F^1)^*(z^{1*}) \right\},$$

where $z^{0*} = (z_1^{0*}, ..., z_n^{0*})^T \in \mathbb{R}_+^n$ and $z^{1*} = (z_1^{1*}, ..., z_n^{1*}) \in (X^*)^n$. It remains to determine the conjugate functions of f and $(z^{0*} F^1)$. For the conjugate function of f one gets by Theorem 2.4

$$f^*(z_1^{0*}, ..., z_n^{0*}) = \min_{\substack{\sum_{i=1}^n \lambda_i \le 1,\ \lambda_i \ge 0, \\ i=1,...,n}} \left\{ \sum_{i=1}^n [(\lambda_i h_i)^*(z_i^{0*}) - \lambda_i a_i] \right\},$$

while for the conjugate function of $(z^{0*} F^1)$ we have

$$\begin{aligned} (z^{0*} F^1)^*(z^{1*}) &= \sup_{z_i^1 \in X,\ i=1,...,n} \left\{ \sum_{i=1}^n \langle z_i^{1*}, z_i^1 \rangle - \sum_{i=1}^n z_i^{0*} \gamma_{C_i}(z_i^1) \right\} \\ &= \sum_{i=1}^n \sup_{z_i^1 \in X} \left\{ \langle z_i^{1*}, z_i^1 \rangle - z_i^{0*} \gamma_{C_i}(z_i^1) \right\} = \sum_{i=1}^n (z_i^{0*} \gamma_{C_i})^*(z_i^{1*}). \end{aligned} \tag{17}$$

Therefore, the conjugate dual problem $(D_{h,a}^S)$ turns into

$$(D_{h,a}^S) \quad \sup_{\substack{\sum_{i=1}^n \lambda_i \le 1,\ \lambda_i, z_i^{0*} \ge 0, \\ z_i^{1*} \in X^*,\ i=1,...,n}} \left\{ \inf_{x \in S} \left\{ \sum_{i=1}^n \langle z_i^{1*}, x - p_i \rangle \right\} - \sum_{i=1}^n [(\lambda_i h_i)^*(z_i^{0*}) - \lambda_i a_i] - \sum_{i=1}^n (z_i^{0*} \gamma_{C_i})^*(z_i^{1*}) \right\}.$$

By separating the sum $\sum_{i=1}^n (\lambda_i h_i)^*$ into the terms with $\lambda_i > 0$ and the terms with $\lambda_i = 0$ as well as $\sum_{i=1}^n (z_i^{0*} \gamma_{C_i})^*$ into the terms with $z_i^{0*} > 0$ and the terms with $z_i^{0*} = 0$ in $(D_{h,a}^S)$, the dual problem turns into

$$\begin{aligned} (D_{h,a}^S) \quad \sup_{\substack{R \subseteq \{1,...,n\},\ \lambda_k > 0,\ k \in R,\ \lambda_l = 0,\ l \notin R,\ \sum_{r \in R} \lambda_r \le 1 \\ I \subseteq \{1,...,n\},\ z_i^{0*} > 0,\ i \in I,\ z_j^{0*} = 0,\ j \notin I \\ z_i^{1*} \in X^*,\ i=1,...,n}} & \left\{ \inf_{x \in S} \left\{ \sum_{i=1}^n \langle z_i^{1*}, x - p_i \rangle \right\} - \sum_{r \notin R} (0 \cdot h_r)^*(z_r^{0*}) \right. \\ & \left. - \sum_{r \in R} [(\lambda_r h_r)^*(z_r^{0*}) - \lambda_r a_r] - \sum_{i \notin I} (0 \cdot \gamma_{C_i})^*(z_i^{1*}) - \sum_{i \in I} (z_i^{0*} \gamma_{C_i})^*(z_i^{1*}) \right\}. \end{aligned}$$

If $i \in I$, then we have (see [2])

$$
\begin{aligned}
(z_i^{0*}\gamma_{C_i})^*(z_i^{1*}) &= z_i^{0*}\gamma_{C_i}^*\left(\frac{z_i^{1*}}{z_i^{0*}}\right) = \begin{cases} 0, \text{ if } \sigma_{C_i}\left(\frac{z_i^{1*}}{z_i^{0*}}\right) \leq 1, \\ +\infty, \text{ otherwise,} \end{cases} \\
&= \begin{cases} 0, \text{ if } \sigma_{C_i}(z_i^{1*}) \leq z_i^{0*}, \\ +\infty, \text{ otherwise,} \end{cases} = \begin{cases} 0, \text{ if } \gamma_{C_i^0}(z_i^{1*}) \leq z_i^{0*}, \\ +\infty, \text{ otherwise,} \end{cases}
\end{aligned} \tag{18}
$$

(see Remark 2.2 for the last equality) and if $i \notin I$, then it holds

$$
(0 \cdot \gamma_{C_i})^*(z_i^{1*}) = \sup_{y_i^1 \in X} \{\langle z_i^{1*}, y_i^1 \rangle\} = \begin{cases} 0, \text{ if } z_i^{1*} = 0_{X^*}, \\ +\infty, \text{ otherwise.} \end{cases} \tag{19}
$$

Further, let us consider the case $r \notin R$, i.e. $\lambda_r = 0$, then one has for $z_r^{0*} \geq 0$,

$$
(0 \cdot h_r)^*(z_r^{0*}) = \sup_{y_r^0 \geq 0} \{z_r^{0*} y_r^0\} = \begin{cases} 0, \text{ if } z_r^{0*} = 0, \\ +\infty, \text{ otherwise.} \end{cases} \tag{20}
$$

For $r \in R$, i.e. $\lambda_r > 0$, follows

$$
(\lambda_r h_r)^*(z_r^{0*}) = \lambda_r h_r^*\left(\frac{z_r^{0*}}{\lambda_r}\right).
$$

Hence, the equation in (20) implies that if $r \notin R$, then $z_r^{0*} = 0$, which means that $I \subseteq R$. In summary, the conjugate dual problem $(D_{h,a}^S)$ becomes to

$$
(D_{h,a}^S) \quad \sup_{\substack{I \subseteq R \subseteq \{1,\dots,n\},\ \lambda_k > 0,\ k \in R,\ \lambda_l = 0,\ l \notin R, \\ z_i^{0*} > 0,\ z_i^{1*} \in X^*,\ \gamma_{C_i^0}(z_i^{1*}) \leq z_i^{0*},\ i \in I, \\ z_j^{0*} = 0,\ z_j^{1*} = 0_{X^*},\ j \notin I,\ \sum\limits_{r \in R} \lambda_r \leq 1}} \left\{ \inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - p_i \rangle \right\} - \sum_{r \in R} \lambda_r \left[h_r^*\left(\frac{z_r^{0*}}{\lambda_r}\right) - a_r \right] \right\}. \tag{21}
$$

Let us denote by $v(P_{h,a}^S)$ the optimal objective value of the problem $(P_{h,a}^S)$, then the weak duality between the primal-dual pair $(P_{h,a}^S)$-$(D_{h,a}^S)$ always holds, i.e. $v(P_{h,a}^S) \geq v(D_{h,a}^S)$.
Our aim is now to verify whether strong duality holds. In other words we will give an answer to the question whether under the given settings the situation holds where $v(P_{h,a}^S) = v(D_{h,a}^S)$ and the dual problem $(D_{h,a}^S)$ has an optimal solution.
For this purpose, we use the generalized interior point regularity condition (RC), which was imposed in the previous section. In mind of this regularity condition, let us recall that f is lower semicontinuous, $K_0 = \mathbb{R}_+^n$ is closed, S is closed, $\operatorname{int} \mathbb{R}_+^n \neq \emptyset$ and F^1 is $\mathbb{R}_+^n$-epi closed. Moreover, it holds

$$
0_{\mathbb{R}^n} \in \operatorname{sqri}(F^1(\operatorname{dom} F^1) - \operatorname{dom} f + K_0) = \operatorname{sqri}(F^1(\operatorname{dom} F^1) - \mathbb{R}_+^n + \mathbb{R}_+^n) = \mathbb{R}^n
$$

and

$$
0_{X^n} \in \operatorname{sqri}(F^2(\operatorname{dom} F^2 \cap \operatorname{dom} g \cap S) - \operatorname{dom} F^1 + K_1) = \operatorname{sqri}(F^2(S) - X^n + \{0_{X^n}\}) = X^n.
$$

As the function $g : X \to X$ is defined by $g(x) := x$ it follows that g is Q-epi closed and

$$
0_X \in \operatorname{sqri}(g(X \cap S) + Q) = \operatorname{sqri}(S + X) = X.
$$

Finally, as F^2 is star $\{0_{X^n}\}$-lower semicontinuous the regularity condition is obviously fulfilled (see Remark 2.4) and we can state the following theorem as a consequence of Theorem 2.2.

Theorem 3.1. *(strong duality) Between* $(P_{h,a}^S)$ *and* $(D_{h,a}^S)$ *strong duality holds, i.e.* $v(P_{h,a}^S) = v(D_{h,a}^S)$ *and the conjugate dual problem has an optimal solution.*

The following necessary and sufficient optimality conditions are a consequence of the previous theorem.

Theorem 3.2. *(optimality conditions) (a) Let* $\overline{x} \in S$ *be an optimal solution of the problem* $(P_{h,a}^S)$*. Then there exist* $(\overline{\lambda}_1, ..., \overline{\lambda}_n, \overline{z}_1^{0*}, ..., \overline{z}_n^{0*}, \overline{z}_1^{1*}, ..., \overline{z}_n^{1*}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times (X^*)^n$ *and the index sets* $\overline{I} \subseteq \overline{R} \subseteq \{1, ..., n\}$*, an optimal solution to* $(D_{h,a}^S)$*, such that*

(i) $\max\limits_{1 \le j \le n} \{h_j(\gamma_{C_j}(\overline{x} - p_j)) + a_j\} = \sum\limits_{i \in \overline{I}} \overline{z}_i^{0*} \gamma_{C_i}(\overline{x} - p_i) - \sum\limits_{r \in \overline{R}} \overline{\lambda}_r \left[h_r^* \left(\frac{\overline{z}_r^{0*}}{\overline{\lambda}_r} \right) - a_r \right]$
$= \sum\limits_{r \in \overline{R}} \overline{\lambda}_r [h_r(\gamma_{C_r}(\overline{x} - p_r)) + a_r]$,

(ii) $\overline{\lambda}_r h_r^* \left(\frac{\overline{z}_r^{0*}}{\overline{\lambda}_r} \right) + \overline{\lambda}_r h_r(\gamma_{C_r}(\overline{x} - p_r)) = \overline{z}_r^{0*} \gamma_{C_r}(\overline{x} - p_r), \ \forall r \in \overline{R}$,

(iii) $\overline{z}_i^{0*} \gamma_{C_i}(\overline{x} - p_i) = \langle \overline{z}_i^{1*}, \overline{x} - p_i \rangle, \ \forall i \in \overline{I}$,

(iv) $\sum\limits_{i \in \overline{I}} \langle \overline{z}_i^{1*}, \overline{x} \rangle = -\sigma_S \left(- \sum\limits_{i \in \overline{I}} \overline{z}_i^{1*} \right)$,

(v) $\max\limits_{1 \le j \le n} \{h_j(\gamma_{C_j}(\overline{x} - p_j)) + a_j\} = h_r(\gamma_{C_r}(\overline{x} - p_r)) + a_r, \ \forall r \in \overline{R}$,

(vi) $\sum\limits_{r \in \overline{R}} \overline{\lambda}_r = 1, \ \overline{\lambda}_k > 0, \ k \in \overline{R}, \ \lambda_l = 0, \ l \notin \overline{R}, \ \overline{z}_i^{0*} > 0, \ i \in \overline{I}$, *and* $\overline{z}_j^{0*} = 0, \ j \notin \overline{I}$,

(vii) $\gamma_{C_i^0}(\overline{z}_i^{1*}) = \overline{z}_i^{0*}, \ \overline{z}_i^{1*} \in X^*, \ i \in \overline{I}$ *and* $\overline{z}_j^{1*} = 0_{X^*}, \ j \notin \overline{I}$.

(b) If there exists $\overline{x} \in S$ *such that for some* $(\overline{\lambda}_1, ..., \overline{\lambda}_n, \overline{z}_1^{0*}, ..., \overline{z}_n^{0*}, \overline{z}_1^{1*}, ..., \overline{z}_n^{1*}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times (X^*)^n$ *and the index sets* $\overline{I} \subseteq \overline{R} \subseteq \{1, ..., n\}$ *the conditions (i)-(vii) are fulfilled, then* $\overline{x}$ *is an optimal solution of* $(P_{h,a}^S)$*,* $(\overline{\lambda}_1, ..., \overline{\lambda}_n, \overline{z}_1^{0*}, ..., \overline{z}_n^{0*}, \overline{z}_1^{1*}, ..., \overline{z}_n^{1*}, \overline{I}, \overline{R})$ *is an optimal solution for* $(D_{h,a}^S)$ *and* $v(P_{h,a}^S) = v(D_{h,a}^S)$*.*

Proof. (a) By using Theorem 2.3 we derive the following necessary and sufficient optimality conditions

(i) $\max\limits_{1 \le j \le n} \{h_j(\gamma_{C_j}(\overline{x} - p_j)) + a_j\} + \sum\limits_{r \in \overline{R}} \overline{\lambda}_r \left[h_r^* \left(\frac{\overline{z}_r^{0*}}{\overline{\lambda}_r} \right) - a_r \right] = \sum\limits_{i \in \overline{I}} \overline{z}_i^{0*} \gamma_{C_i}(\overline{x} - p_i)$,

(ii) $\sum\limits_{i \in \overline{I}} \overline{z}_i^{0*} \gamma_{C_i}(\overline{x} - p_i) = \sum\limits_{i \in \overline{I}} \langle \overline{z}_i^{1*}, \overline{x} - p_i \rangle$,

(iii) $\sum\limits_{i \in \overline{I}} \langle \overline{z}_i^{1*}, \overline{x} \rangle + \sigma_S \left(- \sum\limits_{i \in \overline{I}} \overline{z}_i^{1*} \right) = 0$,

(iv) $\sum\limits_{r \in \overline{R}} \overline{\lambda}_r \le 1, \ \overline{\lambda}_k > 0, \ k \in \overline{R}, \ \lambda_l = 0, \ l \notin \overline{R}, \ \overline{z}_i^{0*} > 0, \ i \in \overline{I}$, and $\overline{z}_j^{0*} = 0, \ j \notin \overline{I}$,

(v) $\gamma_{C_i^0}(\overline{z}_i^{1*}) \le \overline{z}_i^{0*}, \ \overline{z}_i^{1*} \in X^*, \ i \in \overline{I}$ and $\overline{z}_j^{1*} = 0_{X^*}, \ j \notin \overline{I}$, where case (iii) arises from condition (*iii*) of Theorem 2.3 by the following observation (note that $\overline{z}^{2*} = 0_{X^*}$)

$$\begin{aligned}
&\sum_{i\in\overline{I}}\langle\overline{z}_i^{1*},\overline{x}-p_i\rangle + (\overline{z}^{2*}g)(\overline{x}) + \sup_{x\in S}\left\{-\sum_{i\in\overline{I}}\langle\overline{z}_i^{1*},x-p_i\rangle - \langle\overline{z}^{2*},g(x)\rangle\right\} = 0\\
\Leftrightarrow\quad &\sum_{i\in\overline{I}}\langle\overline{z}_i^{1*},\overline{x}\rangle - \sum_{i\in\overline{I}}\langle\overline{z}_i^{1*},p_i\rangle + \sup_{x\in S}\left\{-\sum_{i\in\overline{I}}\langle\overline{z}_i^{1*},x\rangle\right\} + \sum_{i\in\overline{I}}\langle\overline{z}_i^{1*},p_i\rangle = 0\\
\Leftrightarrow\quad &\sum_{i\in\overline{I}}\langle\overline{z}_i^{1*},\overline{x}\rangle + \sup_{x\in S}\left\{-\sum_{i\in\overline{I}}\langle\overline{z}_i^{1*},x\rangle\right\} = 0.
\end{aligned}$$

Additionally, one has by Theorem 3.1 that $v(P_{h,a}^S) = v(D_{h,a}^S)$, i.e.

$$\begin{aligned}
&\max_{1\le j\le n}\{h_j(\gamma_{C_j}(\overline{x}-p_j))+a_j\} = \inf_{x\in S}\left\{\sum_{i\in\overline{I}}\langle\overline{z}_i^{1*},x-p_i\rangle\right\} - \sum_{r\in\overline{R}}\overline{\lambda}_r\left[h_r^*\left(\frac{\overline{z}_r^{0*}}{\overline{\lambda}_r}\right)-a_r\right]\\
\Leftrightarrow\quad &\max_{1\le j\le n}\{h_j(\gamma_{C_j}(\overline{x}-p_j))+a_j\} + \sigma_S\left(-\sum_{i\in\overline{I}}\overline{z}_i^{1*}\right) + \sum_{i\in\overline{I}}\langle\overline{z}_i^{1*},p_i\rangle\\
&+\sum_{r\in\overline{R}}\overline{\lambda}_r\left[h_r^*\left(\frac{\overline{z}_r^{0*}}{\overline{\lambda}_r}\right)-a_r\right] = 0\\
\Leftrightarrow\quad &\max_{1\le j\le n}\{h_j(\gamma_{C_j}(\overline{x}-p_j))+a_j\} + \sigma_S\left(-\sum_{i\in\overline{I}}\overline{z}_i^{1*}\right) + \sum_{i\in\overline{I}}\langle\overline{z}_i^{1*},p_i\rangle + \sum_{r\in\overline{R}}\overline{\lambda}_r\left[h_r^*\left(\frac{\overline{z}_r^{0*}}{\overline{\lambda}_r}\right)-a_r\right]\\
&+\sum_{r\in\overline{R}}\overline{\lambda}_r h_r(\gamma_{C_r}(\overline{x}-p_r)) - \sum_{r\in\overline{R}}\overline{\lambda}_r h_r(\gamma_{C_r}(\overline{x}-p_r))\\
&+\sum_{i\in\overline{I}}\overline{z}_i^{0*}\gamma_{C_i}(\overline{x}-p_i) - \sum_{i\in\overline{I}}\overline{z}_i^{0*}\gamma_{C_i}(\overline{x}-p_i) + \sum_{i\in\overline{I}}\langle\overline{z}_i^{1*},\overline{x}\rangle - \sum_{i\in\overline{I}}\langle\overline{z}_i^{1*},\overline{x}\rangle = 0\\
\Leftrightarrow\quad &\left[\max_{1\le j\le n}\{h_j(\gamma_{C_j}(\overline{x}-p_j))+a_j\} - \sum_{r\in\overline{R}}(\overline{\lambda}_r h_r(\gamma_{C_r}(\overline{x}-p_r)) + \overline{\lambda}_r a_r)\right]\\
&+\sum_{i\in\overline{I}}[\overline{z}_i^{0*}\gamma_{C_i}(\overline{x}-p_i) - \langle\overline{z}_i^{1*},\overline{x}-p_i\rangle] + \left[\sigma_S\left(-\sum_{i\in\overline{I}}\overline{z}_i^{1*}\right) + \sum_{i\in\overline{I}}\langle\overline{z}_i^{1*},\overline{x}\rangle\right]\\
&+\sum_{i\in\overline{I}}\left[\overline{\lambda}_i h_i^*\left(\frac{\overline{z}_i^{0*}}{\overline{\lambda}_i}\right) + \overline{\lambda}_i h_i(\gamma_{C_i}(\overline{x}-p_i)) - \overline{z}_i^{0*}\gamma_{C_i}(\overline{x}-p_i)\right]\\
&+\sum_{r\in\overline{R}\setminus\overline{I}}\left[\overline{\lambda}_r h_r^*(0) + \overline{\lambda}_r h_r(\gamma_{C_r}(\overline{x}-p_r)) - 0\cdot\gamma_{C_r}(\overline{x}-p_r)\right] = 0,
\end{aligned}$$

where the last two sums arise from the fact that $\overline{I}\subseteq\overline{R}$. By Lemma 2.2 holds that the term within the first bracket is non-negative. Moreover, by the Young-Fenchel inequality we have that the terms within the other brackets are also non-negative and hence, it follows that all the terms within the brackets must be equal to zero. Combining the last statement with the optimality conditions (i)-(v) yields

(i) $\max\limits_{1\leq j\leq n}\{h_j(\gamma_{C_j}(\overline{x}-p_j))+a_j\} = \sum\limits_{i\in\overline{I}} \overline{z}_i^{0*}\gamma_{C_i}(\overline{x}-p_i) - \sum\limits_{r\in\overline{R}} \overline{\lambda}_r\left[h_r^*\left(\frac{\overline{z}_r^{0*}}{\overline{\lambda}_r}\right)-a_r\right]$
$= \sum\limits_{r\in\overline{R}} \overline{\lambda}_r[h_r(\gamma_{C_r}(\overline{x}-p_r))+a_r]$,

(ii) $\overline{\lambda}_r h_r^*\left(\frac{\overline{z}_r^{0*}}{\overline{\lambda}_r}\right)+\overline{\lambda}_r h_r(\gamma_{C_r}(\overline{x}-p_r)) = \overline{z}_r^{0*}\gamma_{C_r}(\overline{x}-p_r),\ \forall r\in\overline{R}$,

(iii) $\overline{z}_i^{0*}\gamma_{C_i}(\overline{x}-p_i) = \langle\overline{z}_i^{1*},\overline{x}-p_i\rangle,\ \forall i\in\overline{I}$,

(iv) $\sum\limits_{i\in\overline{I}}\langle\overline{z}_i^{1*},\overline{x}\rangle = -\sigma_S\left(-\sum\limits_{i\in\overline{I}}\overline{z}_i^{1*}\right)$,

(v) $\sum\limits_{r\in\overline{R}}\overline{\lambda}_r\leq 1,\ \overline{\lambda}_k>0,\ k\in\overline{R},\ \lambda_l=0,\ l\notin\overline{R},\ \overline{z}_i^{0*}>0,\ i\in\overline{I}$, and $\overline{z}_j^{0*}=0,\ j\notin\overline{I}$,

(vi) $\gamma_{C_i^0}(\overline{z}_i^{1*})\leq\overline{z}_i^{0*},\ \overline{z}_i^{1*}\in X^*,\ i\in\overline{I}$ and $\overline{z}_j^{1*}=0_{X^*},\ j\notin\overline{I}$.

From conditions (i) and (v) we obtain that

$$\begin{aligned}\max_{1\leq j\leq n}\{h_j(\gamma_{C_j}(\overline{x}-p_j))+a_j\} &= \sum_{r\in\overline{R}}(\overline{\lambda}_r h_r(\gamma_{C_r}(\overline{x}-p_r))+\overline{\lambda}_r a_r)\\ &\leq \sum_{r\in\overline{R}}\overline{\lambda}_r\max_{1\leq j\leq n}\{h_j(\gamma_{C_j}(\overline{x}-p_j))+a_j\}\\ &\leq \max_{1\leq j\leq n}\{h_j(\gamma_{C_j}(\overline{x}-p_j))+a_j\},\end{aligned}$$

which means on the one hand that

$$\sum_{r\in\overline{R}}\overline{\lambda}_r\max_{1\leq j\leq n}\{h_j(\gamma_{C_j}(\overline{x}-p_j))+a_j\} = \max_{1\leq j\leq n}\{h_j(\gamma_{C_j}(\overline{x}-p_j))+a_j\},$$

i.e. condition (v) can be written as

$$\sum_{r\in\overline{R}}\overline{\lambda}_r=1,\ \overline{\lambda}_k>0,\ k\in\overline{R},\ \lambda_l=0,\ l\notin\overline{R},\ \overline{z}_i^{0*}>0,\ i\in\overline{I},\ \text{and}\ \overline{z}_j^{0*}=0,\ j\notin\overline{I}, \tag{22}$$

and on the other hand that

$$\sum_{r\in\overline{R}}(\overline{\lambda}_r h_r(\gamma_{C_r}(\overline{x}-p_r))+\overline{\lambda}_r a_r) = \sum_{r\in\overline{R}}\overline{\lambda}_r\max_{1\leq j\leq n}\{h_j(\gamma_{C_j}(\overline{x}-p_j))+a_j\} \tag{23}$$

or, equivalently,

$$\sum_{r\in\overline{R}}\overline{\lambda}_r\left[\max_{1\leq j\leq n}\{h_j(\gamma_{C_j}(\overline{x}-p_j))+a_j\}-(h_r(\gamma_{C_r}(\overline{x}-p_r))+a_r)\right]=0. \tag{24}$$

As the brackets in the sum of (24) are non-negative and $\overline{\lambda}_r>0$ for $r\in\overline{R}$, it follows that the terms inside the brackets must be equal to zero, more precisely

$$\max_{1\leq j\leq n}\{h_j(\gamma_{C_j}(\overline{x}-p_j))+a_j\} = h_r(\gamma_{C_r}(\overline{x}-p_r))+a_r,\ \forall r\in\overline{R}. \tag{25}$$

Further, we obtain by the generalized Cauchy-Schwarz inequality and the conditions (iii) and (vi) that

$$\overline{z}_i^{0*}\gamma_{C_i}(\overline{x}-p_i) = \langle \overline{z}_i^{1*}, \overline{x}-p_i\rangle \leq \gamma_{C_i^0}(\overline{z}_i^{1*})\gamma_{C_i}(\overline{x}-p_i) \leq \overline{z}_i^{0*}\gamma_{C_i}(\overline{x}-p_i),$$

which means that condition (vi) can be expressed as

$$\gamma_{C_i^0}(\overline{z}_i^{1*}) = \overline{z}_i^{0*},\ \overline{z}_i^{1*} \in X^*,\ i \in \overline{I} \text{ and } \overline{z}_j^{1*} = 0_{X^*},\ j \notin \overline{I}. \tag{26}$$

Taking now the optimality conditions (i)-(vi), (22), (25) and (26) together gives the desired statement.
(b) All the calculations done in (a), can also be made in the reverse order. □

Remark 3.1. *We want to state that the optimality conditions* (i)*-*(iv) *of the previous theorem can also be expressed by using the subdifferential. As*

$$f(y^0) = \begin{cases} \sup\limits_{1\leq i\leq n}\{h_i(y_i^0)+a_i\}, & \text{if } y^0 = (y_1^0,...,y_n^0)^T \in \mathbb{R}_+^n,\ i=1,...,n, \\ +\infty_{\mathbb{R}_+^n}, & \text{otherwise,} \end{cases}$$

and

$$f^*(z_1^{0*},...,z_n^{0*}) = \min_{\substack{\sum_{i=1}^n \lambda_i \leq 1,\ \lambda_i\geq 0,\\ i=1,...,n}} \left\{ \sum_{i=1}^n [(\lambda_i h_i)^*(z_i^{0*}) - \lambda_i a_i] \right\},$$

we have by the optimal condition (i) *of Theorem 3.2 that*

$$f(\gamma_{C_1}(\overline{x}-p_1),...,\gamma_{C_n}(\overline{x}-p_n)) + f^*(z_1^{0*},...,z_n^{0*}) = \sum_{i\in\overline{I}} \overline{z}_i^{0*}\gamma_{C_i}(\overline{x}-p_i).$$

By (1) the last equality is equivalent to

$$(z_1^{0*},...,z_n^{0*}) \in \partial f(\gamma_{C_1}(\overline{x}-p_1),...,\gamma_{C_n}(\overline{x}-p_n)).$$

Therefore, the condition (i) *of Theorem 3.2 can equivalently be written as*

(i) $(\overline{z}_1^{0*},...,\overline{z}_n^{0*}) \in \partial\left(\max\limits_{1\leq j\leq n}\{\cdot + a_j\}\right)(\gamma_1(\overline{x}-p_1),...,\gamma_n(\overline{x}-p_n))$,

In the same way, we can rewrite the conditions (ii)*-*(iv)

(ii) $\overline{z}_r^{0*} \in \partial(\overline{\lambda}_r h_r)(\gamma_{C_r}(\overline{x}-p_r)),\ r \in \overline{R}$,

(iii) $\overline{z}_i^{1*} \in \partial(\overline{z}_i^{0*}\gamma_{C_i})(\overline{x}-p_i),\ i \in \overline{I}$,

(iv) $-\sum\limits_{i\in\overline{I}} \overline{z}_i^{1*} \in \partial\delta_S(\overline{x}) = N_S(\overline{x})$,

where $N_S(\overline{x}) := \{x^* \in X^* : \langle x^*, y - \overline{x}\rangle \leq 0,\ \forall y \in S\}$ *is the normal cone of the set* S *at* $\overline{x} \in X$. *Bringing the optimality conditions* (i) *and* (ii) *together yields*

$$\begin{aligned}(\overline{z}_1^{0*}, ..., \overline{z}_n^{0*}) \ \in\ & \partial\left(\max_{1\leq j\leq n}\{\cdot + a_j\}\right)(\gamma_1(\overline{x}-p_1), ..., \gamma_n(\overline{x}-p_n)) \\ & \cap \left(\partial(\overline{\lambda}_1 h_1)(\gamma_1(\overline{x}-p_1)) \times ... \times \partial(\overline{\lambda}_n h_n)(\gamma_n(\overline{x}-p_n))\right).\end{aligned}$$

Moreover, summarizing the optimality conditions (iii) *and* (iv) *reveals that*

$$\sum_{i\in\overline{I}} \overline{z}_i^{1*} \in \sum_{i\in\overline{I}} \partial(\overline{z}_i^{0*}\gamma_{C_i})(\overline{x}-p_i) \cap (-N_S(\overline{x})).$$

Finally, take also note that the optimality conditions (iii) *and* (vii) *of Theorem 3.2 give a detailed characterization of the subdifferential of* $\overline{z}_i^{0*}\gamma_{C_i}$ *at* $\overline{x} - p_i$, $i = 1, ..., n$. *More precisely,*

$$\partial(\overline{z}_i^{0*}\gamma_{C_i})(\overline{x}-p_i) = \left\{\overline{z}_i^{1*} \in X^* : \overline{z}_i^{0*}\gamma_{C_i}(\overline{x}-p_i) = \langle \overline{z}_i^{1*}, \overline{x}-p_i\rangle \text{ and } \gamma_{C_i^0}(\overline{z}_i^{1*}) = \overline{z}_i^{0*}\right\},\ i \in \overline{I}.$$

Remark 3.2. *If we consider the situation when the set-up costs are arbitrary, i.e.* a_i *can also be negative,* $i = 1, ..., n$, *then the conjugate function of* f *looks like (see Remark 2.5)*

$$f^*(z_1^{0*}, ..., z_n^{0*}) = \min_{\substack{\sum_{i=1}^n \lambda_i = 1,\ \lambda_i \geq 0,\\ i=1,...,n}} \left\{\sum_{i=1}^n [(\lambda_i h_i)^*(z_i^{0*}) - \lambda_i a_i]\right\}.$$

As a consequence, we derive the following corresponding dual problem

$$(D_{h,a}^S) \sup_{\substack{I\subseteq R\subseteq\{1,...,n\},\ \lambda_k>0,\ k\in R,\ \lambda_l=0,\ l\notin R,\\ z_i^{0*}>0,\ z_i^{1*}\in X^*,\ \gamma_{C_i^0}(z_i^{1*})\leq z_i^{0*},\ i\in I,\\ z_j^{0*}=0,\ z_j^{1*}=0_{X^*},\ j\notin I,\ \sum_{r\in R}\lambda_r=1}} \left\{\inf_{x\in S}\left\{\sum_{i\in I}\langle z_i^{1*}, x-p_i\rangle\right\} - \sum_{r\in R}\lambda_r\left[h_r^*\left(\frac{z_r^{0*}}{\lambda_r}\right) - a_r\right]\right\}.$$

Therefore, all the statements given in this subsection are also true in the case of arbitrary set-up costs with the difference that $\sum_{r\in R}\lambda_r = 1$ *in the constraint set.*
Minimax location problems with arbitrary set-up costs were considered for example in [6] and [19]. For readers who are also interested in minimax location problems with nonlinear set-up costs, we refer to [4] and [10].

3.2 Unconstrained location problems with norms in Hilbert spaces

Let $\mathcal{H}$ be a real Hilbert space with scalar product $\langle\cdot,\cdot\rangle$, where the associated norm is denoted as usual by $\|\cdot\|$ and defined by $\|x\| := \langle x, x\rangle$. This subsection is devoted to the case where $S = X = \mathcal{H}$, $a_i \geq 0$ and $\gamma_{C_i} : \mathcal{H} \to \mathbb{R}$ is defined by $\gamma_{C_i}(x) := \|x\|$, $i = 1, ..., n$, such that the minimax location problem $(P_{h,a}^S)$ turns into

$$(P_{h,a}^{S,N}) \ \inf_{x\in\mathcal{H}} \max_{1\leq i\leq n} \{h_i(\|x-p_i\|) + a_i\}.$$

Its corresponding dual problem $(D_{h,a}^{S,N})$ transforms to

$$\sup_{\substack{I\subseteq R\subseteq\{1,...,n\},\ \lambda_k>0,\ k\in R,\ \lambda_l=0,\ l\notin R,\\ z_i^{0*}>0,\ z_i^{1*}\in\mathcal{H},\ \|z_i^{1*}\|\leq z_i^{0*},\ i\in I,\\ z_j^{0*}=0,\ z_j^{1*}=0_{\mathcal{H}},\ j\notin I,\ \sum\limits_{r\in R}\lambda_r\leq 1}} \left\{\inf_{x\in\mathcal{H}}\left\{\sum_{i\in I}\langle z_i^{1*}, x-p_i\rangle\right\} - \sum_{r\in R}\lambda_r\left[h_r^*\left(\frac{z_r^{0*}}{\lambda_r}\right)-a_r\right]\right\}$$

$$= \sup_{\substack{I\subseteq R\subseteq\{1,...,n\},\ \lambda_k>0,\ k\in R,\ \lambda_l=0,\ l\notin R,\\ z_i^{0*}>0,\ z_i^{1*}\in\mathcal{H},\ \|z_i^{1*}\|\leq z_i^{0*},\ i\in I,\\ z_j^{0*}=0,\ z_j^{1*}=0_{\mathcal{H}},\ j\notin I,\ \sum\limits_{r\in R}\lambda_r\leq 1}} \left\{-\sup_{x\in\mathcal{H}}\left\{\left\langle -\sum_{i\in I} z_i^{1*}, x\right\rangle\right\}\right.$$

$$\left. - \sum_{i\in I}\langle z_i^{1*}, p_i\rangle - \sum_{r\in R}\lambda_r\left[h_r^*\left(\frac{z_r^{0*}}{\lambda_r}\right)-a_r\right]\right\}$$

$$= \sup_{\substack{I\subseteq R\subseteq\{1,...,n\},\ \lambda_k>0,\ k\in R,\ \lambda_l=0,\ l\notin R,\\ z_i^{0*}>0,\ z_i^{1*}\in\mathcal{H},\ \|z_i^{1*}\|\leq z_i^{0*},\ i\in I,\\ z_j^{0*}=0,\ z_j^{1*}=0_{\mathcal{H}},\ j\notin I,\ \sum\limits_{r\in R}\lambda_r\leq 1,\ \sum\limits_{i\in I} z_i^{1*}=0_{\mathcal{H}}}} \left\{-\sum_{i\in I}\langle z_i^{1*}, p_i\rangle - \sum_{r\in R}\lambda_r\left[h_r^*\left(\frac{z_r^{0*}}{\lambda_r}\right)-a_r\right]\right\}.$$

Obviously, in this setting the regularity condition (RC) is fulfilled and the following duality statements are direct consequences of Theorem 3.1 and 3.2.

Theorem 3.3. *(strong duality) Between* $(P_{h,a}^{S,N})$ *and* $(D_{h,a}^{S,N}))$ *holds strong duality, i.e.* $v(P_{h,a}^{S,N}) = v(D_{h,a}^{S,N})$ *and the dual problem has an optimal solution.*

Theorem 3.4. *(optimality conditions) (a) Let* $\overline{x}\in\mathcal{H}$ *be an optimal solution of the problem* $(P_{h,a}^{S,N})$. *Then there exist* $(\overline{\lambda}_1,...,\overline{\lambda}_n,\overline{z}^{0*},\overline{z}^{1*})\in\mathbb{R}_+^n\times\mathbb{R}_+^n\times\mathcal{H}^n$ *and the index sets* $\overline{I}\subseteq\overline{R}\subseteq\{1,...,n\}$, *an optimal solution to* $(D_{h,a}^{S,N})$, *such that*

(i) $\max\limits_{1\leq j\leq n}\{h_j(\|\overline{x}-p_j\|)+a_j\} = \sum\limits_{i\in\overline{I}}\overline{z}_i^{0*}\|\overline{x}-p_i\| - \sum\limits_{r\in\overline{R}}\overline{\lambda}_r\left[h_r^*\left(\frac{\overline{z}_r^{0*}}{\overline{\lambda}_r}\right)-a_r\right]$
$= \sum\limits_{r\in\overline{R}}\overline{\lambda}_r[h_r(\|\overline{x}-p_r\|)+a_r]$,

(ii) $\overline{\lambda}_r h_r^*\left(\frac{\overline{z}_r^{0*}}{\overline{\lambda}_r}\right)+\overline{\lambda}_r h_r(\|\overline{x}-p_r\|) = \overline{z}_r^{0*}\|\overline{x}-p_r\|,\ \forall r\in\overline{R}$,

(iii) $\overline{z}_i^{0*}\|\overline{x}-p_i\| = \langle\overline{z}_i^{1*},\overline{x}-p_i\rangle,\ \forall i\in\overline{I}$,

(iv) $\sum\limits_{i\in\overline{I}}\overline{z}_i^{1*} = 0_{\mathcal{H}}$,

(v) $\max\limits_{1\leq j\leq n}\{h_j(\|\overline{x}-p_j\|)+a_j\} = h_r(\|\overline{x}-p_r\|)+a_r,\ \forall r\in\overline{R}$,

(vi) $\sum\limits_{r\in\overline{R}}\overline{\lambda}_r = 1,\ \overline{\lambda}_k>0,\ k\in\overline{R},\ \lambda_l=0,\ l\notin\overline{R},\ \overline{z}_i^{0*}>0,\ i\in\overline{I}$, *and* $\overline{z}_j^{0*}=0,\ j\notin\overline{I}$,

(vii) $\|\overline{z}_i^{1*}\| = \overline{z}_i^{0*},\ \overline{z}_i^{1*}\in\mathcal{H}\setminus\{0_{\mathcal{H}}\},\ i\in\overline{I}$ *and* $\overline{z}_j^{1*}=0_{\mathcal{H}},\ j\notin\overline{I}$.

(b) If there exists $\overline{x}\in\mathcal{H}$ *such that for some* $(\overline{\lambda}_1,...,\overline{\lambda}_n,\overline{z}^{0*},\overline{z}^{1*})\in\mathbb{R}_+^n\times\mathbb{R}_+^n\times\mathcal{H}^n$ *and the index sets* $\overline{I}\subseteq\overline{R}$ *the conditions (i)-(vii) are fulfilled, then* $\overline{x}$ *is an optimal solution of* $(P_{h,a}^{S,N})$, $(\overline{\lambda}_1,...,\overline{\lambda}_n,\overline{z}^{0*},\overline{z}^{1*},\overline{I},\overline{R})$ *is an optimal solution for* $(D_{h,a}^{S,N})$ *and* $v(P_{h,a}^{S,N}) = v(D_{h,a}^{S,N})$.

Regarding the relation of the optimal solutions of the primal and its the dual problem the following corollary can be given under the additional assumption that the function h_i is continuous and strictly increasing for all $i = 1, ..., n$.

Corollary 3.1. *Let the function*

$$h_i : \mathbb{R} \to \overline{\mathbb{R}},\ h_i(x) := \begin{cases} h_i(x) \in \mathbb{R}_+, & \text{if } x \in \mathbb{R}_+, \\ +\infty, & \text{otherwise}, \end{cases}$$

be convex, continuous and strictly increasing for all $i = 1, ..., n$, and $\overline{x} \in \mathcal{H}$ an optimal solution of the problem $(P^{S,N}_{h,a})$. If $(\overline{\lambda}_1, ..., \overline{\lambda}_n, \overline{z}^{0*}, \overline{z}^{1*}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathcal{H}^n$ and $\overline{I} \subseteq \overline{R} \subseteq \{1, ..., n\}$ are optimal solutions of the dual problem $(D^{S,N}_{h,a})$, then it holds

$$\overline{x} \ = \ \frac{1}{\sum\limits_{i \in \overline{I}} \frac{\|\overline{z}_i^{1*}\|}{h_i^{-1}\left(v(D^{S,N}_{h,a}) - a_i\right)}} \sum_{i \in \overline{I}} \frac{\|\overline{z}_i^{1*}\| p_i}{h_i^{-1}\left(v(D^{S,N}_{h,a}) - a_i\right)}.$$

Proof. The optimality conditions (iii) and (vii) of Theorem 3.4 imply that

$$\|\overline{z}_i^{1*}\| \|\overline{x} - p_i\| = \langle \overline{z}_i^{1*}, \overline{x} - p_i \rangle,\ i \in \overline{I},$$

By Fact 2.10 in [1] there exists $\alpha_i > 0$ such that

$$\overline{z}_i^{1*} = \alpha_i \left(\overline{x} - p_i\right),\ i \in \overline{I} \tag{27}$$

and so, $\|\overline{z}_i^{1*}\| = \alpha_i \|\overline{x} - p_i\|$, $i \in \overline{I}$. Therefore, it follows from the optimality condition (v) of Theorem 3.4 that (note that $\overline{I} \subseteq \overline{R}$)

$$\begin{aligned} & \max_{1 \le j \le n} \{h_j(\|\overline{x} - p_j\|) + a_j\} = h_i\left(\frac{1}{\alpha_i}\|\overline{z}_i^{1*}\|\right) + a_i \\ \Leftrightarrow\ & h_i^{-1}\left(\max_{1 \le j \le n} \{h_j(\|\overline{x} - p_j\|) + a_j\} - a_i\right) = \frac{1}{\alpha_i}\|\overline{z}_i^{1*}\| \\ \Leftrightarrow\ & \alpha_i = \frac{\|\overline{z}_i^{1*}\|}{h_i^{-1}\left(\max\limits_{1 \le j \le n} \{h_j(\|\overline{x} - p_j\|) + a_j\} - a_i\right)} = \frac{\|\overline{z}_i^{1*}\|}{h_i^{-1}\left(v(D^{S,N}_{h,a}) - a_i\right)},\ i \in \overline{I}. \end{aligned} \tag{28}$$

Now, we take in (27) the sum over all $i \in \overline{I}$, which yields by condition (iv) of Theorem 3.4

$$0_{\mathcal{H}} = \sum_{i \in \overline{I}} \overline{z}_i^{1*} = \sum_{i \in \overline{I}} \alpha_i \left(\overline{x} - p_i\right) \Leftrightarrow \overline{x} = \frac{1}{\sum\limits_{i \in \overline{I}} \alpha_i} \sum_{i \in \overline{I}} \alpha_i p_i. \tag{29}$$

Finally, bringing (28) and (29) together implies

$$\overline{x} \ = \ \frac{1}{\sum\limits_{i \in \overline{I}} \frac{\|\overline{z}_i^{1*}\|}{h_i^{-1}\left(v(D^{S,N}_{h,a}) - a_i\right)}} \sum_{i \in \overline{I}} \frac{\|\overline{z}_i^{1*}\| p_i}{h_i^{-1}\left(v(D^{S,N}_{h,a}) - a_i\right)}.$$

□

Example 3.1. *(a) Let $\alpha_{is},\ \beta_{is} \geq 0,\ s = 1, ..., v,$ and $h_i : \mathbb{R} \to \overline{\mathbb{R}}$ be defined by*

$$h_i(x) := \begin{cases} \max\limits_{1\leq s\leq v} \{\alpha_{is}x + \beta_{is}\}, & \text{if } x \in \mathbb{R}_+, \\ +\infty, & \text{otherwise}, \end{cases}$$

$i = 1, ..., n,$ then the corresponding location problem looks like

$$(P_{h,a}^{S,N}) \qquad \inf_{x\in\mathcal{H}} \max_{1\leq i\leq n} \left\{ \max_{1\leq s\leq v} \{\alpha_{is}\|x - p_i\| + \beta_{is}\} + a_i \right\} = \inf_{x\in\mathcal{H}} \max_{\substack{1\leq i\leq n,\\ 1\leq s\leq v}} \{\alpha_{is}\|x - p_i\| + \beta_{is} + a_i\}.$$

Moreover, we define the function

$$f_s : \mathbb{R} \to \overline{\mathbb{R}},\ f_s(x) := \begin{cases} \alpha_{is}x + \beta_{is}, & \text{if } x \in \mathbb{R}_+, \\ +\infty, & \text{otherwise}, \end{cases}$$

then we derive by Theorem 3.2 in [21]

$$h_i^*(x^*) = \left(\max_{1\leq s\leq v} \{f_s\} \right)^* (x^*) = \inf_{\substack{\sum\limits_{s=1}^{v} x_s^* = x^*,\ \sum\limits_{s=1}^{v} \tau_s = 1,\\ \tau_s\geq 0,\ s=1,...,v}} \left\{ \sum_{s=1}^{v} (\tau_s f_s)^*(x_s^*) \right\}.$$

As the conjugate of the function $\tau_s f_s$ is

$$\begin{aligned} (\tau_s f_s)^*(x^*) &= \sup_{x\in\mathbb{R}} \{x_s^* x - \tau_s f_s(x)\} = \sup_{x\geq 0} \{x_s^* x - \tau_s \alpha_{is} x - \tau_s \beta_{is}\} \\ &= -\tau_s\beta_{is} + \sup_{x\geq 0} \{(x_s^* - \tau_s\alpha_{is})x\} = \begin{cases} -\tau_s\beta_{is}, & \text{if } x_s^* \leq \tau_s\alpha_{is}, \\ +\infty, & \text{otherwise}, \end{cases} \end{aligned}$$

$s = 1, ..., v,$ we have

$$h_i^*(x^*) = \inf_{\substack{\sum\limits_{s=1}^{v} x_s^* = x^*,\ \sum\limits_{s=1}^{v} \tau_s = 1,\\ \tau_s\geq 0,\ x_s\leq \tau_s\alpha_{is},\ s=1,...,v}} \left\{ -\sum_{s=1}^{n} \tau_s \alpha_{is} \right\},\ i = 1, ..., n,$$

and hence, the dual problem is given by

$$(D_{h,a}^{S,N}) \qquad \sup_{\substack{I\subseteq R\subseteq\{1,...,n\},\ \lambda_k>0,\ k\in R,\ \lambda_l=0,\ l\notin R,\\ z_i^{0*}>0,\ z_i^{1*}\in\mathcal{H},\ \|z_i^{1*}\|\leq z_i^{0*},\ i\in I,\\ z_j^{0*}=0,\ z_j^{1*}=0_{\mathcal{H}},\ j\notin I,\ \sum\limits_{r\in R}\lambda_r\leq 1,\ \sum\limits_{i\in I} z_i^{1*}=0_{\mathcal{H}}\\ \sum\limits_{s=1}^{v} x_s^*=\frac{z_r^{0*}}{\lambda_r},\ \sum\limits_{s=1}^{v}\tau_s=1,\ \tau_s\geq 0,\ x_s\leq\tau_s\alpha_{rs},\ s=1,...,v}} \left\{ -\sum_{i\in I} \langle z_i^{1*}, p_i\rangle + \sum_{r\in R} \lambda_r \left[\sum_{s=1}^{n} \tau_s\alpha_{rs} - a_r \right] \right\}.$$

Furthermore, $h_i^{-1}(y) = \min_{1\leq s\leq v} \left\{ \frac{1}{\alpha_{is}}(y - \beta_{is}) \right\}$ for all $i = 1, ..., n,$ and thus, we have by Corollary 3.1

$$\overline{x} = \frac{1}{\sum\limits_{i\in\overline{I}} \frac{\|\overline{z}_i^{1*}\|}{\min\limits_{1\leq s\leq v}\left\{\frac{1}{\alpha_{is}}(v(D_{h,a}^{S,N}) - a_i - \beta_{is})\right\}}} \sum_{i\in\overline{I}} \frac{\|\overline{z}_i^{1*}\| p_i}{\min\limits_{1\leq s\leq v}\left\{\frac{1}{\alpha_{is}}(v(D_{h,a}^{S,N}) - a_i - \beta_{is})\right\}}.$$

(b) Let $h_i : \mathbb{R} \to \overline{\mathbb{R}}$ be defined by

$$h_i(x) := \begin{cases} w_i x^{\beta_i}, & \text{if } x \in \mathbb{R}_+, \\ +\infty, & \text{otherwise,} \end{cases}$$

with $w_i > 0$, $\beta_i > 1$, $i = 1, ..., n$, then

$$(P_{h,a}^{S,N}) \qquad \inf_{x \in \mathcal{H}} \max_{1 \le i \le n} \left\{ w_i \|x - p_i\|^{\beta_i} + a_i \right\}$$

and since the conjugate function of h_i is given by (see Example 13.2 (i) in [1])

$$h_i^*(x^*) = w_i \frac{\beta_i - 1}{\beta_i} \left(\frac{1}{w_i} x^* \right)^{\frac{\beta_i}{\beta_i - 1}} = \frac{\beta_i - 1}{\beta_i w_i^{\frac{1}{\beta_i - 1}}} (x^*)^{\frac{\beta_i}{\beta_i - 1}}, \ i = 1, ..., n,$$

the associated dual problem $(D_{h,a}^{S,N})$ is

$$\sup_{\substack{I \subseteq R \subseteq \{1,...,n\},\ \lambda_k > 0,\ k \in R,\ \lambda_l = 0,\ l \notin R, \\ z_i^{0*} > 0,\ z_i^{1*} \in \mathcal{H},\ \|z_i^{1*}\| \le z_i^{0*},\ i \in I, \\ z_j^{0*} = 0,\ z_j^{1*} = 0_{\mathcal{H}},\ j \notin I,\ \sum\limits_{r \in R} \lambda_r \le 1,\ \sum\limits_{i \in I} z_i^{1*} = 0_{\mathcal{H}}}} \left\{ -\sum_{i \in I} \langle z_i^{1*}, p_i \rangle - \sum_{r \in R} \lambda_r \left[\frac{\beta_r - 1}{\beta_r (\lambda_r w_r)^{\frac{1}{\beta_r - 1}}} (z_r^{0*})^{\frac{\beta_r}{\beta_r - 1}} - a_r \right] \right\}.$$

In addition, as $h_i^{-1}(y) = (y/w_i)^{\frac{1}{\beta_i}}$ for all $i = 1, ..., n$, it holds

$$\overline{x} = \frac{1}{\sum\limits_{i \in \overline{I}} \frac{w_i^{\frac{1}{\beta_i}} \|\overline{z}_i^{1*}\|}{\left(v(D_{h,a}^{S,N}) - a_i \right)^{\frac{1}{\beta_i}}}} \sum_{i \in \overline{I}} \frac{w_i^{\frac{1}{\beta_i}} \|\overline{z}_i^{1*}\| p_i}{\left(v(D_{h,a}^{S,N}) - a_i \right)^{\frac{1}{\beta_i}}}.$$

(c) Let $h_i : \mathbb{R} \to \overline{\mathbb{R}}$ be defined by

$$h_i(x) := \begin{cases} w_i x, & \text{if } x \in \mathbb{R}_+, \\ +\infty, & \text{otherwise,} \end{cases}$$

$w_i > 0$, then $h_i^{-1}(y) = \frac{1}{w_i} y$ for all $i = 1, ..., n$, and hence,

$$\overline{x} = \frac{1}{\sum\limits_{i \in \overline{I}} \frac{w_i \|\overline{z}_i^{1*}\|}{v(D_{h,a}^{S,N}) - a_i}} \sum_{i \in \overline{I}} \frac{w_i \|\overline{z}_i^{1*}\| p_i}{v(D_{h,a}^{S,N}) - a_i}. \tag{30}$$

If $a_i = 0$, $i = 1, ..., n$, then formula in (30) reduces to

$$\overline{x} = \frac{1}{\sum\limits_{i \in \overline{I}} w_i \|\overline{z}_i^{1*}\|} \sum_{i \in \overline{I}} w_i \|\overline{z}_i^{1*}\| p_i. \tag{31}$$

Remark 3.3. *Let us note that all the results in this section holds also for negative set-up costs. Like already mentioned in Remark 3.2, we have in this case in the constraint set of the dual problem $\sum_{r \in R} \lambda_r = 1$.*

4 Duality results for linear location problems without set-up costs

4.1 Geometrically constrained location problems with gauges in Fréchet spaces

In this section we will discuss single minimax location problems without set-up costs ($a_i = 0,\ i = 1, ..., n$), where the function $h_i : \mathbb{R} \to \overline{\mathbb{R}}$ be defined by

$$h_i(x) := \begin{cases} x, & \text{if } x \in \mathbb{R}_+, \\ +\infty, & \text{otherwise,} \end{cases}$$

$i = 1, ..., n$. Hence, the location problem $(P^S_{h,a})$ turns into

$$(P^S) \qquad \inf_{x \in S} \max_{1 \le i \le n} \{\gamma_{C_i}(x - p_i)\}.$$

Since the conjugate function of h_i is given by

$$h_i^*(x^*) = \sup_{x \in \mathbb{R}_+} \{(x^* - 1)x\} = \begin{cases} 0, \text{ if } x^* \le 1, \\ +\infty, \text{ otherwise,} \end{cases}$$

$i = 1, ..., n$, the corresponding conjugate dual problem to (P^S) becomes by (21) to

$$(D^S) \qquad \sup_{\substack{I \subseteq R \subseteq \{1,...,n\},\ \lambda_k > 0,\ z_k^{0*} \le \lambda_k,\ k \in R,\ \lambda_l = 0,\ l \notin R, \\ z_i^{0*} > 0,\ z_i^{1*} \in X^*,\ \gamma_{C_i^0}(z_i^{1*}) \le z_i^{0*},\ i \in I, \\ z_j^{0*} = 0,\ z_j^{1*} = 0_{X^*},\ j \notin I,\ \sum\limits_{r \in R} \lambda_r \le 1}} \left\{ \inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - p_i \rangle \right\} \right\}.$$

The fact $I \subseteq R$ implies that if $\lambda_r = 0$, then $z_r^{0*} = 0$ for some $r \in R$ and so, it holds that $z_i^{0*} \le \lambda_i$ for all $i = 1, ..., n$. Now, if we define the function $\theta : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ by

$$\theta(z_1^{0*}, ..., z_n^{0*}) := \begin{cases} 0, \text{ if } z_i^{0*},\ \lambda_i \ge 0,\ z_i^{0*} \le \lambda_i,\ i = 1, ..., n,\ \sum\limits_{i=1}^{n} \lambda_i \le 1 \\ +\infty, \text{ otherwise,} \end{cases}$$

then it is obvious that

$$\theta(z_1^{0*}, ..., z_n^{0*}) = \begin{cases} 0, \text{ if } z_i^{0*} \ge 0,\ \sum\limits_{i=1}^{n} z_i^{0*} \le 1 \\ +\infty, \text{ otherwise} \end{cases}$$

and therefore, we can write (D^S) as

$$(D^S) \qquad \sup_{\substack{I \subseteq \{1,...,n\},\ z_j^{0*} = 0,\ z_j^{1*} = 0_{X^*},\ j \notin I, \\ z_i^{0*} > 0,\ z_i^{1*} \in X^*,\ \gamma_{C_i^0}(z_i^{1*}) \le z_i^{0*},\ i \in I,\ \sum\limits_{i \in I} z_i^{0*} \le 1}} \left\{ \inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - p_i \rangle \right\} \right\}.$$

Let us now introduce the following optimization problem

$$(\widetilde{D}^S) \qquad \sup_{\substack{I \subseteq \{1,...,n\},\ z_i^* \in X_i^*,\ i \in I, \\ z_j^* = 0_{X^*},\ j \notin I,\ \sum\limits_{i \in I} \gamma_{C_i^0}(z_i^*) \le 1}} \left\{ \inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^*, x - p_i \rangle \right\} \right\}. \tag{32}$$

If we denote by $v(D^S)$ the optimal objective value of the problem (D^S) and by $v(\widetilde{D}^S)$ the optimal objective value of the problem $(\widetilde{D}^S)$, then the following theorem can be formulated.

Theorem 4.1. *It holds $v(D^S) = v(\widetilde{D}^S)$.*

Proof. Let z_i^*, $i = 1, ..., n$, be a feasible element to $(\widetilde{D}^S)$ and set $z_i^{1*} = z_i^*$, $z_i^{0*} = \gamma_{C_i^0}(z_i^*)$ for $i \in I$ and $z_i^{0*} = 0$, $z_1^{1*} = 0_{X^*}$ for $i \notin I$. Then, it is obvious that z_i^{0*} and z_i^{1*}, $i = 1, ..., n$ are feasible elements to (D^S) and it holds

$$\inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^*, x - p_i \rangle \right\} = \inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - p_i \rangle \right\} \leq v(D^S), \tag{33}$$

for all z_i^*, $i = 1, ..., n$, feasible to $(\widetilde{D}^S)$, which implies $v(\widetilde{D}^S) \leq v(D^S)$.
Vice versa, let z_i^{0*} and z_i^{1*} be feasible elements to (D^S) for $i = 1, ..., n$, then we have $\gamma_{C_i^0}(z_i^{1*}) \leq z_i^{0*}$ for $i \in I$, $\sum_{i \in I} z_i^{0*} \leq 1$ and $z_i^{0*} = 0$, $z_i^{1*} = 0_{X^*}$ for $i \notin I$, from which follows by setting $z_i^* = z_i^{1*}$ for $i \in I$ and $z_i^* = 0_{X_i^*}$ for $i \notin I$ that

$$\sum_{i \in I} \gamma_{C_i^0}(z_i^*) \leq 1,$$

in other words z_i^* is a feasible solution to $(\widetilde{D}^S)$ for all $i = 1, ..., n$. Furthermore, we have that

$$\inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^{1*}, x - p_i \rangle \right\} = \inf_{x \in S} \left\{ \sum_{i \in I} \langle z_i^*, x - p_i \rangle \right\} \leq v(\widetilde{D}^S), \tag{34}$$

for all z_i^{0*} and z_i^{1*}, $i = 1, ..., n$, feasible to (D^S), which implies that $v(D^S) \leq v(\widetilde{D}^S)$. Bringing the statements (33) and (34) together reveals that it must hold $v(\widetilde{D}^S) = v(D^S)$. □

Motivated by Theorem 4.1 it follows immediately the following one.

Theorem 4.2. *(strong duality) Between (P^S) and $(\widetilde{D}^S)$ holds strong duality, i.e. $v(P^S) = v(\widetilde{D}^S)$ and the dual problem $v(\widetilde{D}^S)$ has an optimal solution.*

Now, it is possible to formulate the following optimality conditions for the primal-dual pair (P^S)-$(\widetilde{D}^S)$ (note that $a_i = 0$, $i = 1, ..., n$).

Theorem 4.3. *(optimality conditions) (a) Let $\overline{x} \in S$ be an optimal solution of the problem (P^S). Then there exists $\overline{z}^* \in (X^*)^n$ and an index set $\overline{I} \subseteq \{1, ..., n\}$, an optimal solution to $(\widetilde{D}^S)$, such that*

(i) $\max\limits_{1 \leq j \leq n} \{\gamma_{C_j}(\overline{x} - p_j)\} = \sum\limits_{i \in \overline{I}} \gamma_{C_i^0}(\overline{z}_i^*) \gamma_{C_i}(\overline{x} - p_i)$,

(ii) $\sum\limits_{i \in \overline{I}} \langle \overline{z}_i^*, \overline{x} \rangle = -\sigma_S \left(-\sum\limits_{i \in \overline{I}} \overline{z}_i^* \right)$,

(iii) $\gamma_{C_i^0}(\overline{z}_i^*) \gamma_{C_i}(\overline{x} - p_i) = \langle \overline{z}_i^*, \overline{x} - p_i \rangle, \ i \in \overline{I}$,

(iv) $\sum\limits_{j \in \overline{I}} \gamma_{C_j^0}(\overline{z}_j^*) = 1, \ \gamma_{C_i^0}(\overline{z}_i^*) > 0, \ i \in \overline{I}$, *and* $\gamma_{C_i^0}(\overline{z}_i^*) = 0, \ i \notin \overline{I}$,

(v) $\gamma_{C_i}(\overline{x} - p_i) = \max\limits_{1 \leq j \leq n} \{\gamma_{C_j}(\overline{x} - p_j)\}, \ i \in \overline{I}$.

(b) If there exists $\overline{x} \in S$ such that for some $\overline{z}^* \in (X^*)^n$ and an index set $\overline{I}$ the conditions (i)-(v) are fulfilled, then $\overline{x}$ is an optimal solution of (P^S), $(\overline{z}^*, \overline{I})$ is an optimal solution for $(\widetilde{D}^S)$ and $v(P^S) = v(\widetilde{D}^S)$.

Proof. Let $\overline{x} \in S$ be an optimal solution of (P^S), then by Theorem 4.2 there exists $\overline{z}^* \in (X^*)^n$ and an index set $\overline{I} \subseteq \{1, ..., n\}$ such that $v(P^S) = v(\widetilde{D}^S)$, i.e.

$$\begin{aligned}
\Leftrightarrow \quad & \max_{1\leq j\leq n} \{\gamma_{C_j}(\overline{x} - p_j)\} = \inf_{x\in S} \left\{ \sum_{i\in\overline{I}} \langle \overline{z}_i^*, x - p_i \rangle \right\} \\
\Leftrightarrow \quad & \max_{1\leq j\leq n} \{\gamma_{C_j}(\overline{x} - p_j)\} + \sigma_S \left(-\sum_{i\in\overline{I}} \overline{z}_i^* \right) + \sum_{i\in\overline{I}} \langle \overline{z}_i^*, p_i \rangle = 0 \\
\Leftrightarrow \quad & \max_{1\leq j\leq n} \{\gamma_{C_j}(\overline{x} - p_j)\} + \sigma_S \left(-\sum_{i\in\overline{I}} \overline{z}_i^* \right) + \sum_{i\in\overline{I}} \langle \overline{z}_i^*, p_i \rangle \\
& + \sum_{i\in\overline{I}} \gamma_{C_i^0}(\overline{z}_i^*)\gamma_{C_i}(\overline{x} - p_i) - \sum_{i\in\overline{I}} \gamma_{C_i^0}(\overline{z}_i^*)\gamma_{C_i}(\overline{x} - p_i) + \sum_{i\in\overline{I}} \langle \overline{z}_i^*, \overline{x} \rangle - \sum_{i\in\overline{I}} \langle \overline{z}_i^*, \overline{x} \rangle = 0 \\
\Leftrightarrow \quad & \left[\max_{1\leq j\leq n} \{\gamma_{C_j}(\overline{x} - p_j)\} - \sum_{i\in\overline{I}} \gamma_{C_i^0}(\overline{z}_i^*)\gamma_{C_i}(\overline{x} - p_i) \right] + \left[\sigma_S \left(-\sum_{i\in\overline{I}} \overline{z}_i^* \right) + \left\langle \sum_{i\in\overline{I}} \overline{z}_i^*, \overline{x} \right\rangle \right] \\
& + \sum_{i\in\overline{I}} [\gamma_{C_i^0}(\overline{z}_i^*)\gamma_{C_i}(\overline{x} - p_i) + \langle \overline{z}_i^*, p_i - \overline{x} \rangle] = 0.
\end{aligned}$$

By Lemma 2.2 holds that the term within the first bracket is non-negative and by the Young-Fenchel inequality we derive that the terms within the other brackets are also non-negative. This implies the cases (i)-(iii). Further, we obtain by the first bracket

$$\begin{aligned}
& \max_{1\leq j\leq n} \{\gamma_{C_j}(\overline{x} - p_j)\} = \sum_{i\in\overline{I}} \gamma_{C_i^0}(\overline{z}_i^*)\gamma_{C_i}(\overline{x} - p_i) \\
\leq \quad & \sum_{i\in\overline{I}} \gamma_{C_i^0}(\overline{z}_i^*) \max_{1\leq j\leq n} \{\gamma_{C_j}(\overline{x} - p_j)\} \leq \max_{1\leq j\leq n} \{\gamma_{C_j}(\overline{x} - p_j)\}
\end{aligned}$$

and from here follows that $\sum_{i\in\overline{I}} \gamma_{C_i^0}(\overline{z}_i^*) = 1$, which yields condition (iv), as well as

$$\begin{aligned}
& \sum_{i\in\overline{I}} \gamma_{C_i^0}(\overline{z}_i^*) \max_{1\leq j\leq n} \{\gamma_{C_j}(\overline{x} - p_j)\} = \sum_{i\in\overline{I}} \gamma_{C_i^0}(\overline{z}_i^*)\gamma_{C_i}(\overline{x} - p_i) \\
\Leftrightarrow \quad & \sum_{i\in\overline{I}} \gamma_{C_i^0}(\overline{z}_i^*) \left[\max_{1\leq j\leq n} \{\gamma_{C_j}(\overline{x} - p_j)\} - \gamma_{C_i}(\overline{x} - p_i) \right] = 0. \qquad (35)
\end{aligned}$$

As the brackets in (35) are non-negative and $\gamma_{C_i^0}(\overline{z}_i^*) > 0$, $i \in \overline{I}$, we get that

$$\max_{1\leq j\leq n} \{\gamma_{C_j}(\overline{x} - p_j)\} = \gamma_{C_i}(\overline{x} - p_i), \; i \in \overline{I}.$$

which yields the condition (v) and completes the proof. □

4.2 Unconstrained location problems with the Euclidean norm

Now we turn our attention to the case where $S = X = \mathbb{R}^d$ and $w_i > 0,\ i = 1, ..., n$. Furthermore, we use as the gauge functions the Euclidean norm, i.e. $\gamma_{C_i} = w_i\|\cdot\|$, $i = 1, ..., n$. By these settings, the minimax location problem (P^S) transforms into the following one

$$(P_N^S) \qquad \inf_{x\in\mathbb{R}^d} \max_{1\leq i\leq n} \{w_i\|x - p_i\|\}.$$

By using (32) we obtain the following dual problem corresponding to (P_N^S),

$$\begin{aligned}
(\widetilde{D}_N^S) \qquad & \sup_{\substack{\sum\limits_{i\in I} \frac{1}{w_i}\|z_i^*\|\leq 1,\ z_i^*\in\mathbb{R}^d,\ i\in I,\\ z_j^*=0_{\mathbb{R}^d},\ j\notin I}} \inf_{x\in\mathbb{R}^d} \left\{ \sum_{i\in I} \langle z_i^*, x - p_i\rangle \right\} \\
= & \sup_{\substack{\sum\limits_{i\in I} \frac{1}{w_i}\|z_i^*\|\leq 1,\ z_i^*\in\mathbb{R}^d,\ i\in I,\\ z_j^*=0_{\mathbb{R}^d},\ j\notin I}} \left\{ -\sigma_{\mathbb{R}^d}\left(-\sum_{i\in I} z_i^* \right) - \sum_{i\in I} \langle z_i^*, p_i\rangle \right\} \\
= & \sup_{\substack{\sum\limits_{i\in I} \frac{1}{w_i}\|z_i^*\|\leq 1,\ \sum\limits_{i\in I} z_i^*=0_{\mathbb{R}^d},\\ z_i^*\in\mathbb{R}^d,\ i\in I,\ z_j^*=0_{\mathbb{R}^d},\ j\notin I}} \left\{ -\sum_{i\in I} \langle z_i^*, p_i\rangle \right\}. \qquad (36)
\end{aligned}$$

Remark 4.1. *Note that for simplicity it is also possible to substitute $z_i^* = -z_i^*$ for all $i = 1, ..., n$, whence it follows*

$$(\widetilde{D}_N^S) \qquad \sup_{\substack{\sum\limits_{i\in I} \frac{1}{w_i}\|z_i^*\|\leq 1,\ \sum\limits_{i\in I} z_i^*=0_{\mathbb{R}^d},\\ z_i^*\in\mathbb{R}^d,\ i\in I,\ z_j^*=0_{\mathbb{R}^d},\ j\notin I}} \left\{ \sum_{i\in I} \langle z_i^*, p_i\rangle \right\}. \qquad (37)$$

Theorem 4.4. *(strong duality) Between (P_N^S) and $(\widetilde{D}_N^S)$ holds strong duality, i.e. $v(P_N^S) = v(\widetilde{D}_N^S)$ and the dual problem has an optimal solution.*

By Theorem 4.3 and 4.4 we derive the following necessary and sufficient optimality conditions.

Theorem 4.5. *(optimality conditions) (a) Let $\overline{x} \in \mathbb{R}^d$ be an optimal solution of the problem (P_N^S). Then there exists $\overline{z}_i^* \in \mathbb{R}^d$, $i = 1, ..., n$, and an optimal index set $\overline{I}$, an optimal solution to $(\widetilde{D}_N^S)$, such that*

(i) $\max\limits_{1\leq j\leq n} \{w_j\|\overline{x} - p_j\|\} = \sum\limits_{i\in\overline{I}} \|\overline{z}_i^*\|\|\overline{x} - p_i\|$,

(ii) $\sum\limits_{i\in\overline{I}} \overline{z}_i^* = 0_{\mathbb{R}^d}$,

(iii) $\|\overline{z}_i^*\|\|\overline{x} - p_i\| = \langle \overline{z}_i^*, \overline{x} - p_i\rangle,\ i \in \overline{I}$,

(iv) $\sum\limits_{j\in\overline{I}} \frac{1}{w_j}\|\overline{z}_j^*\| = 1,\ \overline{z}_i^* \in \mathbb{R}^d \setminus \{0_{\mathbb{R}^d}\}$ *for* $i \in I$ *and* $\overline{z}_i^* = 0_{\mathbb{R}^d}$ *for* $i \notin \overline{I}$,

(v) $w_i\|\overline{x} - p_i\| = \max\limits_{1\leq j\leq n} \{w_j\|\overline{x} - p_j\|\},\ i \in \overline{I}$.

(b) If there exists $\overline{x} \in \mathbb{R}^d$ *such that for some* $\overline{z}_i^* \in \mathbb{R}^d$, $i = 1, ..., n$, *and an index set* $\overline{I}$ *the conditions (i)-(v) are fulfilled, then* $\overline{x}$ *is an optimal solution of* (P_N^S), $(\overline{z}^*, \overline{I})$ *is an optimal solution for* $(\widetilde{D}_N^S)$ *and* $v(P_N^S) = v(\widetilde{D}_N^S)$.

For the length of the vectors z_i^*, $i \in I$, feasible to $(\widetilde{D}_N^S)$ the following estimation from above can be made.

Corollary 4.1. *Let* $w_s := \max_{1 \leq i \leq n}\{w_i\}$ *and* $z_i^* \in \mathbb{R}^d$, $i = 1, ..., n$, *and* $I \subseteq \{1, ..., n\}$ *be a feasible solution to* $(\widetilde{D}_N^S)$, *then it holds*

$$\|z_i^*\| \leq \frac{w_s w_i}{w_s + w_i}, \ i \in I.$$

Proof. Assume that $z_i^* \in \mathbb{R}^d$, $i = 1, ..., n$ and $I \subseteq \{1, ..., n\}$ are feasible elements of the dual problem $(\widetilde{D}_N^S)$, then one has for $j \in I$,

$$\sum_{i \in I} z_i^* = 0_{\mathbb{R}^d} \Leftrightarrow z_j^* = -\sum_{\substack{i \in I \\ i \neq j}} z_i^*$$

and hence,

$$\|z_j^*\| = \|\sum_{\substack{i \in I \\ i \neq j}} z_i^*\| \leq \sum_{\substack{i \in I \\ i \neq j}} \|z_i^*\|, \ j \in I. \tag{38}$$

Moreover, from the feasibility of z_i^*, $i \in I$, to $(\widetilde{D}_N^S)$ and by (38), we have

$$\begin{aligned} 1 &\geq \sum_{i \in I} \frac{1}{w_i}\|z_i^*\| = \frac{1}{w_j}\|z_j^*\| + \sum_{\substack{i \in I \\ i \neq j}} \frac{1}{w_i}\|z_i^*\| \\ &\geq \frac{1}{w_j}\|z_j^*\| + \frac{1}{w_s}\sum_{\substack{i \in I \\ i \neq j}} \|z_i^*\| \geq \frac{1}{w_j}\|z_j^*\| + \frac{1}{w_s}\|z_j^*\| = \frac{w_s + w_j}{w_s w_j}\|z_j^*\|, \ j \in I, \end{aligned}$$

and so,

$$\|z_j^*\| \leq \frac{w_s w_j}{w_s + w_j}, \ j \in I,$$

□

By the next remark we point out the relationship between the minimax and minisum problems.

Remark 4.2. *The optimal solution* $\overline{x}$ *of the problem* (P_N^S) *is also a solution of the following generalized Fermat-Torricelli problem*

$$(P_N^{FT}) \qquad \min_{x \in \mathbb{R}^d} \sum_{i \in \overline{I}} \widetilde{w}_i \|x - p_i\|,$$

where $\widetilde{w}_i = \|\overline{z}_i^*\|$, $i \in \overline{I}$.
This can be seen like follows: It is well known that x is an optimal solution of the problem (P_N^{FT}) with $x \neq p_i,\ i \in I$ if and only if the resultant force R at x, defined by

$$R(x) := \sum_{i\in\overline{I}} \widetilde{w}_i \frac{x - p_i}{\|x - p_i\|},$$

is zero (see [18]). As $\overline{x}$ is an optimal solution of (P_N^S), we have by (27) that

$$\sum_{i\in\overline{I}} \widetilde{w}_i \frac{\overline{x} - p_i}{\|\overline{x} - p_i\|} = \sum_{i\in\overline{I}} \|\overline{z}_i^*\| \frac{\overline{x} - p_i}{\|\overline{x} - p_i\|} = \sum_{i\in\overline{I}} \alpha_i(\overline{x} - p_i) = \sum_{i\in\overline{I}} \overline{z}_i^* = 0_{\mathbb{R}^d},$$

which implies that $\overline{x}$ is also an optimal solution of the problem (P_N^{FT}). In this context, pay attention also to the fact that for the optimal solution $\overline{x}$ of the problem (P_N^S) it holds $\overline{x} \neq p_i$, $i \in \overline{I}$. Because if there exists $j \in \overline{I}$ such that $\overline{x} = p_j$, then $\overline{x} = p_i$ for all $i \in \overline{I}$, which contradicts the assumption that the given points are distinct.

Geometrical Interpretation.
For simplicity let us suppose that $w_1 = ... = w_n = 1$, then it is well-known that the problem (P_S^N) can be interpreted as the finding a ball with center $\overline{x}$ and minimal radius such that all given points p_i, $i = 1, ..., n$ are covered by this ball. This problem is also known as the minimum covering ball problem.
Our plan is now to give a geometrical interpretation of the set of optimal solutions of the dual problem $(\widetilde{D}_N^S)$ by using Theorem 4.5. By condition (iii) we see that for $i \in \overline{I}$ the dual problem can geometrically be understood as the finding of vectors $\overline{z}_i^*$, which are parallel to the vectors $\overline{x} - p_i$ and directed to $\overline{x}$ fulfilling $\sum_{i\in\overline{I}} \overline{z}_i^* = 0_{\mathbb{R}^d}$ and $\sum_{i\in\overline{I}} \|\overline{z}_i^*\| = 1$. Especially, conditions (iv) and (v) are telling us that for $i \in \overline{I}$, i.e. $\overline{z}_i^* \neq 0_{\mathbb{R}^d}$, the corresponding point p_i is lying on the border of the minimal covering ball and for $i \notin \overline{I}$, i.e. $\overline{z}_i^{1*} = 0_{\mathbb{R}^d}$, the corresponding point p_i is lying inside the mentioned ball. Therefore, for $i \in \overline{I}$ the elements $\overline{z}_i^*$ can be interpreted as force vectors, which pulling the points p_i lying on the border of the minimum covering ball inside of this ball in direction to the center, the gravity point $\overline{x}$, where the resultant force of the sum of these force vectors is zero. For illustration see Example 1 and Figure 1.
Another well-known geometrical characterization of the location problem (P_S^N) is to find the minimum radius of balls centered at the points $p_i,\ i = 1, ..., n$, such that their intersection is non-empty. In this situation the set of optimal solutions of the dual problem can be described as force vectors fulfilling the optimality conditions of Theorem 4.5 and increasing these balls until their intersection is non-empty and the radius of the largest ball is minimal. From the conditions (iv) and (v) we obtain that a force vector $\overline{z}_i^*$ is equal to the zero vector if $\overline{x}$ is an element of the interior of the ball centered at point p_i with radius $v(P_N^S)$, which is exactly the case when $i \in \overline{I}$. If $i \notin \overline{I}$, which is exactly the case when $\overline{x}$ is lying on the border of the ball centered at point p_i with radius $v(P_N^S)$, then the corresponding force vector $\overline{z}_i^*$ is unequal to the zero vector and moreover, by the optimality condition (iii) follows that $\overline{z}_i^*$ is parallel to the vector $\overline{x} - p_i$ and has the the same direction.
To demonstrate the statements we made above, let us discuss the following example.

Example 4.1. *Consider the unconstrained single minimax location problem in $\mathbb{R}^2$ defined by the given points:*

$$p_1 = (-5, -2.5)^T;\ p_2 = (-2, 1)^T;\ p_3 = (2.5, 3)^T;\ p_4 = (3.5, -2)^T \textit{ and } p_5 = (0, -3)^T.$$

The primal problem looks in this case like follows

$$(\overline{P}_N^S) \qquad \inf_{x\in\mathbb{R}^2} \max_{1\le i\le 5}\{\|x-p_i\|\}$$

and by using the Matlab Optimization Toolbox we get the solution $\overline{x} = (-0.866, -0.273)^T$ *with the objective function value* $\max_{1\le i\le 5}\{\|\overline{x}-p_i\|\} = 4.695$.
For the dual problem we have the formulation

$$(\widetilde{D}_N^S) \qquad \sup_{\substack{\sum\limits_{i\in I}\|z_i^*\|\le 1,\ \sum\limits_{i\in I} z_i^*=0_{\mathbb{R}^2},\\ z_i^*\in\mathbb{R}^2,\ i\in I,\ z_i^*=0_{\mathbb{R}^2},\ i\notin I}} \left\{-\sum_{i\in I}\langle z_i^*, p_i\rangle\right\}. \tag{39}$$

with the solution

$$\overline{z}_1^* = (0.412, 0.222)^T;\ \overline{z}_2^* = (0,0)^T;\ \overline{z}_3^* = (-0.281, -0.273)^T;$$
$$\overline{z}_4^* = (-0.131, 0.052)^T;\ \overline{z}_5^* = (0,0)^T.$$

The dual problem was also solved by using the Matlab Optimization Toolbox. In fact, it holds $\overline{I} = \{1,3,4\}$, $\langle\overline{z}_1^*, p_1\rangle + \langle\overline{z}_3^*, p_3\rangle + \langle\overline{z}_4^*, p_4\rangle = 4.695$, $\overline{x} = \|\overline{z}_1^*\|p_1 + \|\overline{z}_3^*\|p_3 + \|\overline{z}_4^*\|p_4 = 0.468 \cdot (-5, -2.5)^T + 0.392 \cdot (2.5, 3)^T + 0.14 \cdot (3.5, -2)^T = (-0.866, -0.273)^T$ *(see (31)) and the points* p_1, p_3 *and* p_4 *are lying on the border of the minimum covering circle as Figure 1 demonstrates.*

Remark 4.3. *Let* $w_i = 1$, *i=1,...,n. Then, for the case* $n = 2$ *it follows immediately by condition (iv) of Theorem 4.5 and Corollary 4.1 the well-known fact that* $\overline{x} = \frac{1}{2}(p_1 + p_2)$.

Remark 4.4. *Let* $w_i = 1$, *i=1,...,n. If we consider the case* $d = 1$, *we can write the dual problem* $(\widetilde{D}_N^S)$ *as*

$$(\widetilde{D}_N^S) \qquad \sup_{\substack{\sum\limits_{i\in I}|z_i^*|\le 1,\ \sum\limits_{i\in I} z_i^*=0,\\ z_i^*\in\mathbb{R},\ i\in I,\ z_i^*=0,\ i\notin I}} \left\{-\sum_{i\in I} z_i^* p_i\right\}$$
$$= \sup_{\substack{z^*\in\mathbb{R}^n,\ \langle z^*, \mathbf{1}\rangle = 0,\\ \|z^*\|_1\le 1}} \{-\langle z^*, p\rangle\},$$

where $z^* = (z_1^*, ..., z_n^*)^T \in \mathbb{R}^n$, $p = (p_1, ..., p_n)^T \in \mathbb{R}^n$, $\mathbf{1} = (1, ..., 1)^T \in \mathbb{R}^n$ *and* $\|\cdot\|_1$ *is the Manhattan norm. From the second formulation of the problem* $(\widetilde{D}_N^S)$ *it is clear that the set of the feasible elements is the intersection of a hyperplane orthogonal to the vector* **1** *and a cross-polytope (or hyperoctahedron), i.e. a convex polytope. Further, it is clear that the optimal solution of this problem can get immediately by the following consideration. Let us assume that* $p_1 < ... < p_n$, *then it holds* $p_1 < \overline{x} < p_n$ *and by condition (v) of Theorem 4.5 one gets*

$$\max_{1\le j\le n}\{|\overline{x}-p_j|\} = |\overline{x}-p_1| = |\overline{x}-p_n|,$$

i.e. $\overline{I} = \{1, n\}$. *By Remark 4.3 this means* $\overline{x} = \frac{1}{2}(p_1 + p_n)$. *Moreover, by Corollary 4.1 we have that* $|\overline{z}_1^*| = |\overline{z}_n^*| = 0.5$ *and by condition (iv) of Theorem 4.5 finally follows that* $\overline{z}_1^* = 0.5$ *and* $\overline{z}_n^* = -0.5$. *A more detailed analysis of location problems using rectilinear distances was given in [5].*

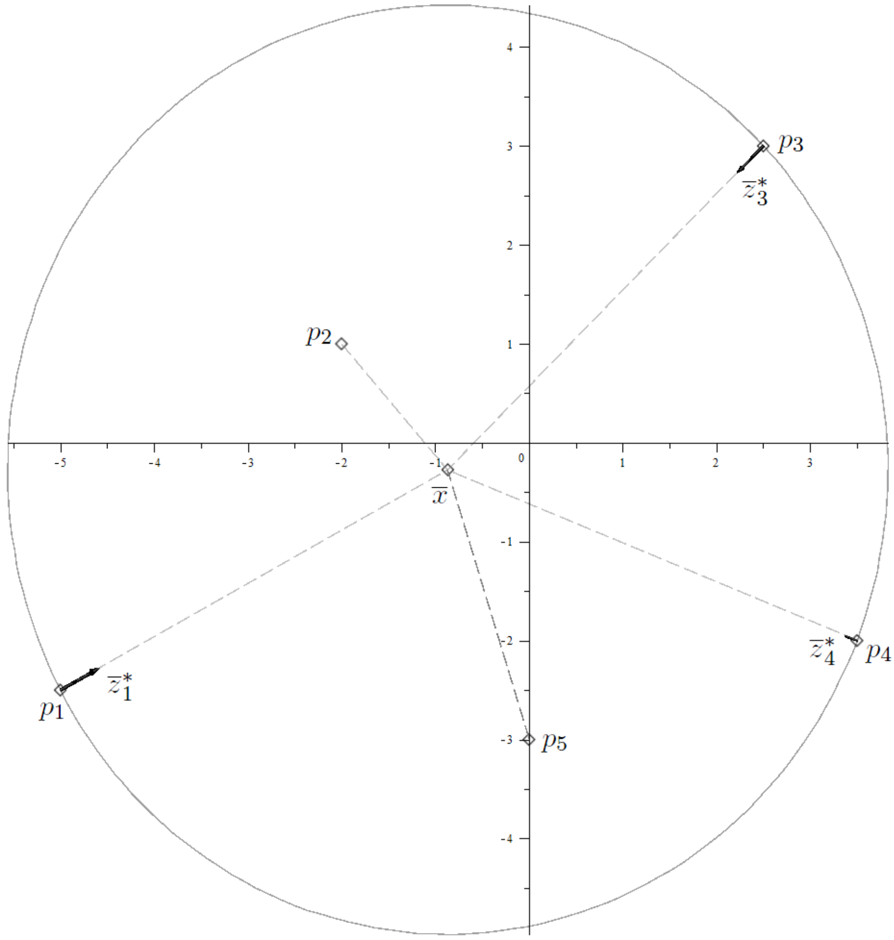

Figure 1: Geometrical illustration of the Example 4.1.

By the next remark, we discover that the Lagrange multiplier associated with the linear equation constraint of the dual problem $(\widetilde{D}_N^S)$ is the optimal solution of the primal problem (P_N^S) and moreover, the Lagrange multiplier associated with the inequality constraint of the dual $(\widetilde{D}_N^S)$ is the optimal objective value. A similar result was shown in [16] for minisum location problems.

Remark 4.5. *First, let us notice that the dual problem $(\widetilde{D}_N^S)$ can be written as*

$$(\widetilde{D}_N^S) \qquad \sup_{\substack{z_i^* \in \mathbb{R}^d,\ i=1,\ldots,n \\ \sum\limits_{i=1}^n \frac{1}{w_i}\|z_i^*\| \leq 1,\ \sum\limits_{i=1}^n z_i^* = 0_{\mathbb{R}^d}}} \left\{ -\sum_{i=1}^n \langle z_i^*, p_i \rangle \right\},$$

then the Lagrange dual of the dual $(\widetilde{D}_N^S)$ *looks like*

$$(D\widetilde{D}_N^S) \qquad \inf_{\lambda \geq 0,\ x \in \mathbb{R}^d} \sup_{z_i^* \in \mathbb{R}^d,\ i=1,\ldots,n} \left\{ -\sum_{i=1}^n \langle z_i^*, p_i \rangle + \left\langle x, \sum_{i=1}^n z_i^* \right\rangle - \lambda \left(\sum_{i=1}^n \frac{1}{w_i} \|z_i^*\| - 1 \right) \right\}$$

$$= \inf_{\lambda \geq 0,\ x \in \mathbb{R}^d} \left\{ \lambda + \sum_{i=1}^n \sup_{z_i^* \in \mathbb{R}^d} \left\{ -\langle z_i^*, p_i \rangle + \langle x, z_i^* \rangle - \frac{\lambda}{w_i} \|z_i^*\| \right\} \right\}$$

$$= \inf_{\lambda \geq 0,\ x \in \mathbb{R}^d} \left\{ \lambda + \sum_{i=1}^n \sup_{z_i^* \in \mathbb{R}^d} \left\{ \langle x - p_i, z_i^* \rangle - \frac{\lambda}{w_i} \|z_i^*\| \right\} \right\}. \tag{40}$$

If $\lambda = 0$*, then we get*

$$\sup_{z_i^* \in \mathbb{R}^d} \langle x - p_i, z_i^* \rangle = \begin{cases} 0, & \text{if } x = p_i, \\ +\infty, & \text{otherwise,} \end{cases}$$

$i = 1, ..., n$*, which contradicts the assumption from the beginning that the given points* p_i*,* $i = 1, ..., n$ *are distinct. Therefore, we can write for (40)*

$$(D\widetilde{D}_N^S) \qquad \inf_{\lambda > 0,\ x \in \mathbb{R}^d} \left\{ \lambda + \lambda \sum_{i=1}^n \frac{1}{w_i} \sup_{z_i^* \in \mathbb{R}^d} \left\{ \left\langle \frac{w_i}{\lambda}(x - p_i), z_i^* \right\rangle - \|z_i^*\| \right\} \right\}$$

$$= \inf_{\substack{\lambda > 0,\ x \in \mathbb{R}^d, \\ w_i \|x - p_i\| \leq \lambda,\ i=1,\ldots,n}} \lambda = \inf_{x \in \mathbb{R}^d} \max_{1 \leq i \leq n} \{w_i \|x - p_i\|\}.$$

We conclude, on the one hand, that the Lagrange dual of the dual problem $(\widetilde{D}_N^S)$ *(i.e. the bidual of the primal location problem* (P^S)*) is the problem* (P^S)*. On the other hand, we see that the Lagrange multipliers of the dual* $(D\widetilde{D}_N^S)$ *characterize the optimal solution and the optimal objective value of the primal problem* (P^S)*. Therefore, we have a complete symmetry between the primal problem* (P_N^S)*, the dual problem* $(\widetilde{D}_N^S)$ *and its Lagrange dual problem* $(D\widetilde{D}_N^S)$*.*

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