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Non-smooth atomic decomposition of 2-microlocal spaces and application to pointwise multipliers

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Non-smooth atomic decomposition of 2-microlocal spaces and application to pointwise multipliers

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Abstract

We provide non-smooth atomic characterizations for 2-microlocal Besov and Triebel-Lizorkin spaces with variable integrability $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. These spaces cover the classical Besov and Triebel-Lizorkin spaces as well as spaces of variable smoothness and integrability. As an application, we state a pointwise multipliers result for these spaces.

Keywords: 2-microlocal spaces, Besov and Triebel-Lizorkin spaces, variable integrability, atomic decomposition, pointwise multipliers.

1 Introduction

In this article we generalize the atomic decomposition theorem for 2-microlocal Besov and Triebel-Lizorkin spaces $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and present an application to pointwise multipliers.

The 2-microlocal function spaces initially appeared in the book of Peetre [20] and have also been studied by Bony [2] in connection with pseudodifferential operators. Later on, they were investigated by Jaffard [9] as well as Jaffard and Meyer [10]. In [16] and [17], Levy Véhel and Seuret showed that they are an useful tool to measure local regularity and to describe the oscillatory behavior of functions near singularities.

Spaces of variable integrability, also known as variable exponent functions spaces $L_{p(\cdot)}(\mathbb{R}^n)$, can be traced back to Orlicz [19] 1931, but the modern development started with the papers [11] of Kováčik and Rákosník as well as [3] of Diening. The spaces $L_{p(\cdot)}(\mathbb{R}^n)$ have interesting applications in fluid dynamics, image processing, PDE and variational calculus, see the introduction of [6]. For an overview we refer to [5].

The concept of function spaces with variable smoothness and the concept of variable integrability were firstly mixed up by Diening, Hästö and Roudenko in [6]. They defined Triebel-Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and from the trace theorem on $\mathbb{R}^{n-1}$ it became clear why it is natural to have all parameters variable. Due to

$$\mathrm{Tr}\ F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = F^{s(\cdot)-\frac{1}{p(\cdot)}}_{p(\cdot),p(\cdot)}(\mathbb{R}^{n-1}),\ \text{with}\ s(\cdot)-\frac{1}{p(\cdot)} > (n-1)\max\left(\frac{1}{p(\cdot)}-1,0\right),$$

(Theorem 3.13 in [6]) we see the necessity of taking s and q variable if p is not constant. Moreover, Almeida and Hästö also introduced Besov spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with all three indices variable in [1].

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The scale we consider here - mixing 2-microlocal weights with variable integrability - was introduced in [13, 14] and provides a unified approach that covers many spaces related with variable smoothness and generalized smoothness. Many results have been studied regarding these spaces, in particular the possibility of decomposing functions $f \in B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ or $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ as linear combinations of smooth atoms, which are the building blocks for atomic decompositions.

The study of smooth atomic decompositions for Besov spaces with p and q variable was firstly done by Drihem in [7], where he proved the result for Besov spaces with variable smoothness and integrability $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Following the same ideas, we present in Section 2 the smooth atomic decomposition result for 2-microlocal Besov spaces $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

In Section 3 we develop non-smooth atomic decompositions for 2-microlocal Besov and Triebel-Lizorkin spaces $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. We show that we can replace the usual atoms used in smooth atomic decompositions by more general ones, whose assumptions on the smoothness are relaxed and, nevertheless, we keep all the crucial information compared to smooth atomic decompositions. This modification appears in [25], where Triebel and Winkelvoß suggested the use of these weaker conditions to define function spaces intrinsically on domains.

In this direction, recently Schneider and Vybíral in [23] and Scharf in [22] proved some results about non-smooth atoms for the classical Besov space $B^s_{p,q}(\mathbb{R}^n)$, but using different approaches. For the case of function spaces with variable exponents this is the first work concerning non-smooth atomic decompositions. Our approach follows Scharf in [22] and generalizes and extends the smooth atomic decomposition results previously referred.

In Section 4, we use the non-smooth atomic representation theorem to deal with pointwise multipliers in the function spaces $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Our result covers the corresponding results obtained in [23] and [22] for classical Besov spaces $B^s_{p,q}(\mathbb{R}^n)$.

2 Notation and definitions

We shall adopt the following general notation: $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}^n$ for $n \in \mathbb{N}$ denotes the n-dimensional real Euclidean space with $|x|$, for $x \in \mathbb{R}^n$, denoting the Euclidean norm of x. For a real number a, let $a_+ := \max(a, 0)$.

If $s \in \mathbb{R}$, then there are uniquely determined $\lfloor s \rfloor^- \in \mathbb{Z}$ and $\{s\}^+ \in (0,1]$ with $s = \lfloor s \rfloor^- + \{s\}^+$.

Definition 2.1. *Let $s > 0$. Then the Hölder space with index s is defined as*

$$\mathcal{C}^s = \Big\{ f \in C^{\lfloor s \rfloor^-} : \|f \mid \mathcal{C}^s\| := \sum_{|\alpha| \le \lfloor s \rfloor^-} \sup_{x \in \mathbb{R}^n} |D^\alpha f(x)| + \sum_{|\alpha| = \lfloor s \rfloor^-} \sup_{x,y \in \mathbb{R}^n, x \ne y} \frac{|f(x) - f(y)|}{|x-y|^{\{s\}^+}} < \infty \Big\}.$$

If $s = 0$, then we set $\mathcal{C}^0 = L_\infty$.

For $q \in (0, \infty]$, ℓ_q stands for the linear space of all complex sequences $f = (f_j)_{j \in \mathbb{N}_0}$ endowed with the quasi-norm

$$\|f \mid \ell_q\| = \Big(\sum_{j=0}^{\infty} |f_j|^q \Big)^{1/q},$$

with the usual modification if $q = \infty$. By c, c_1, c_2, etc. we denote positive constants independent of appropriate quantities. For two non-negative expressions (*i.e.*, functions or functionals) $\mathcal{A}$, $\mathcal{B}$, the symbol $\mathcal{A} \lesssim \mathcal{B}$ (or $\mathcal{A} \gtrsim \mathcal{B}$) means that $\mathcal{A} \le c\,\mathcal{B}$ (or $c\,\mathcal{A} \ge \mathcal{B}$), for some $c > 0$. If $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{A} \gtrsim \mathcal{B}$, we write $\mathcal{A} \sim \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are equivalent.

In order to define 2-microlocal Besov and Triebel-Lizorkin spaces with variable integrability, we start by recalling the definition of admissible weight sequences. We follow [14].

Definition 2.2. *Let $\alpha \geq 0$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 \leq \alpha_2$. A sequence of non-negative measurable functions in $\mathbb{R}^n$ $\boldsymbol{w} = (w_j)_{j \in \mathbb{N}_0}$ belongs to the class $\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbb{R}^n)$ if the following conditions are satisfied:*

(i) There exists a constant $c > 0$ such that

$$0 < w_j(x) \leq c\, w_j(y)\, (1 + 2^j |x - y|)^{\alpha} \quad \text{for all } j \in \mathbb{N}_0 \text{ and all } x, y \in \mathbb{R}^n.$$

(ii) For all $j \in \mathbb{N}_0$ it holds

$$2^{\alpha_1} w_j(x) \leq w_{j+1}(x) \leq 2^{\alpha_2} w_j(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Such a system $(w_j)_{j \in \mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbb{R}^n)$ is called admissible weight sequence.

Properties of admissible weights may be found in [12, Remark 2.4].

Before introducing the function spaces under consideration we still need to recall some notation. By $\mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we denote by $\widehat{f}$ the Fourier transform of f and by $f^{\vee}$ the inverse Fourier transform of f.

Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\varphi_0(x) = 1 \quad \text{if} \quad |x| \leq 1 \quad \text{and} \quad \operatorname{supp} \varphi_0 \subset \{x \in \mathbb{R}^n : |x| \leq 2\}. \tag{2.1}$$

Now define $\varphi(x) := \varphi_0(x) - \varphi_0(2x)$ and set $\varphi_j(x) := \varphi(2^{-j}x)$ for all $j \in \mathbb{N}$. Then the sequence $(\varphi_j)_{j \in \mathbb{N}_0}$ forms a smooth dyadic partition of unity.

By $\mathcal{P}(\mathbb{R}^n)$ we denote the class of exponents, which are measurable functions $p : \mathbb{R}^n \to (c, \infty]$ for some $c > 0$. Let $p \in \mathcal{P}(\mathbb{R}^n)$. Then, $p^+ := \text{ess-sup}_{x \in \mathbb{R}^n} p(x)$, $p^- := \text{ess-inf}_{x \in \mathbb{R}^n} p(x)$ and $L_{p(\cdot)}(\mathbb{R}^n)$ is the variable exponent Lebesgue space, which consists of all measurable functions f such that for some $\lambda > 0$ the modular $\varrho_{L_{p(\cdot)}(\mathbb{R}^n)}(f/\lambda)$ is finite, where

$$\varrho_{L_{p(\cdot)}(\mathbb{R}^n)}(f) := \int_{\mathbb{R}^n_0} |f(x)|^{p(x)}\, dx + \text{ess-sup}_{x \in \mathbb{R}^n_\infty} |f(x)|.$$

Here $\mathbb{R}^n_\infty$ denotes the subset of $\mathbb{R}^n$ where $p(x) = \infty$ and $\mathbb{R}^n_0 = \mathbb{R}^n \setminus \mathbb{R}^n_\infty$. The Luxemburg norm of a function $f \in L_{p(\cdot)}(\mathbb{R}^n)$ is given by

$$\|f \mid L_{p(\cdot)}(\mathbb{R}^n)\| := \inf \left\{ \lambda > 0 : \varrho_{L_{p(\cdot)}(\mathbb{R}^n)} \left(\frac{f}{\lambda} \right) \leq 1 \right\}.$$

In order to define the mixed spaces $\ell_{q(\cdot)}(L_{p(\cdot)})$, we need to define another modular. For $p, q \in \mathcal{P}(\mathbb{R}^n)$ and a sequence $(f_\nu)_{\nu \in \mathbb{N}_0}$ of complex-valued Lebesgue measurable functions on $\mathbb{R}^n$, we define

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_{\nu=0}^{\infty} \inf \left\{ \lambda_\nu > 0 : \varrho_{p(\cdot)} \left(\frac{f_\nu}{\lambda_\nu^{1/q(\cdot)}} \right) \leq 1 \right\}. \tag{2.2}$$

If $q^+ < \infty$, then we can replace (2.2) by the simpler expression

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_{\nu=0}^{\infty} \left\| |f_\nu|^{q(\cdot)} \mid L_{\frac{p(\cdot)}{q(\cdot)}} \right\|. \tag{2.3}$$

The (quasi-)norm in the $\ell_{q(\cdot)}(L_{p(\cdot)})$ spaces is defined as usual by

$$\|f_\nu \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\| = \inf\left\{\mu > 0 : \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}\left(\frac{f_\nu}{\mu}\right) \leq 1\right\}. \tag{2.4}$$

For the sake of completeness, we state also the definition of the space $L_{p(\cdot)}(\ell_{q(\cdot)})$. At first, one just takes the norm $\ell_{q(\cdot)}$ of $(f_\nu(x))_{\nu\in\mathbb{N}_0}$ for every $x \in \mathbb{R}^n$ and then the $L_{p(\cdot)}$-norm with respect to $x \in \mathbb{R}^n$, i.e.

$$\|f_\nu \mid L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbb{R}^n))\| = \left\| \left(\sum_{\nu=0}^{\infty} |f_\nu(x)|^{q(x)}\right)^{1/q(x)} \mid L_{p(\cdot)}(\mathbb{R}^n)\right\|.$$

We introduce now the Hardy-Littlewood maximal operator $\mathcal{M}_t$, which is defined for a locally integrable function $f \in L_1^{loc}$ and for $0 < t \leq 1$ by

$$\mathcal{M}_t(f)(x) = \left(\sup_{x\in Q} \int_Q |f(y)|^t \, dy\right)^{1/t},$$

and $\mathcal{M}(f)(x) = \mathcal{M}_1(f)(x)$. In order to guarantee the boundedness of $\mathcal{M}_t$ in $L_{p(\cdot)}(\mathbb{R}^n)$ for non-constant exponent $p(\cdot)$ we require certain regularity conditions on $p \in \mathcal{P}(\mathbb{R}^n)$. Let us define the important classes.

Definition 2.3. *Let $g \in C(\mathbb{R}^n)$. We say that g is locally log-Hölder continuous, abbreviated $g \in C^{\log}_{loc}(\mathbb{R}^n)$, if there exists $c_{\log}(g) > 0$ such that*

$$|g(x) - g(y)| \leq \frac{c_{\log}(g)}{\log(e + 1/|x-y|)} \quad \text{for all } x, y \in \mathbb{R}^n. \tag{2.5}$$

We say that g is globally log-Hölder continuous, abbreviated $g \in C^{\log}(\mathbb{R}^n)$, if g is locally log-Hölder continuous and there exists $g_\infty \in \mathbb{R}$ such that

$$|g(x) - g_\infty| \leq \frac{c_{\log}}{\log(e + |x|)} \quad \text{for all } x \in \mathbb{R}^n. \tag{2.6}$$

We use the notation $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ if $p \in \mathcal{P}(\mathbb{R}^n)$ and $1/p \in C^{\log}(\mathbb{R}^n)$. It was proved in [4] that the maximal operator $\mathcal{M}$ is bounded in $L_{p(\cdot)}(\mathbb{R}^n)$ provided that $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $1 < p^- \leq p^+ \leq \infty$.

The definitions of the spaces below were given in [15]. The restriction $q^+ < \infty$ in the F-case comes from the use of Lemma 3.8, which is essential for studying Triebel-Lizorkin spaces with variable exponents.

Definition 2.4. *Let $(\varphi_j)_{j\in\mathbb{N}_0}$ be a partition of unity as above, $\boldsymbol{w} = (w_j)_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbb{R}^n)$ and $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$.*

(i) *The space $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined as the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|_\varphi := \|(w_j\, (\varphi_j \widehat{f})^\vee)_{j\in\mathbb{N}_0} \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\|$$

is finite.

(ii) *If* $p^+, q^+ < \infty$, *then the space* $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is defined as the collection of all* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that*

$$\|f \mid F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|_\varphi := \|(w_j\,(\varphi_j \widehat{f})^\vee)_{j\in\mathbb{N}_0} \mid L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbb{R}^n))\|$$

is finite.

Remark 2.5. These spaces include very well-known spaces. For p =const and $w_j(x) = 2^{js}$ we get back to the classical Besov and Triebel-Lizorkin spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$.

Also the spaces of generalized smoothness are contained in this approach (see [8], [18]) by taking

$$w_j(x) = 2^{js}\Psi(2^{-j}), \qquad \text{or more general } w_j(x) = \sigma_j.$$

Here, $\{\sigma_j\}_{j\in\mathbb{N}_0}$ is an admissible sequence, which means that there exist $d_0, d_1 > 0$ with $d_0\sigma_j \leq \sigma_{j+1} \leq d_1\sigma_j$ and Ψ is a slowly varying function.

Moreover, these 2-microlocal spaces also cover the spaces of variable smoothness and integrability $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ and $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$, introduced in [6] and [1]. If $s \in C^{\log}_{loc}(\mathbb{R}^n)$ (which is the standard condition on $s(\cdot)$), then $\boldsymbol{w} = (w_j(x))_{j\in\mathbb{N}_0} = (2^{js(x)})_{j\in\mathbb{N}_0}$ belongs to $\mathcal{W}^\alpha_{\alpha_1,\alpha_2}(\mathbb{R}^n)$ with $\alpha_1 = s^-$, $\alpha_2 = s^+$ and $\alpha = c_{log}(s)$, where $c_{log}(s)$ is the constant for $s(\cdot)$ from (2.5).

For $p, q \in \mathcal{P}(\mathbb{R}^n)$, we put

$$\sigma_p := n\left(\frac{1}{p^-} - 1\right)_+ \quad \text{and} \quad \sigma_{p,q} := n\left(\frac{1}{\min(1, p^-, q^-)} - 1\right).$$

2.1 Smooth atomic decompositions of $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$

In [14] it was presented the atomic decomposition theorems for $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and, with $q =$ const, for $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$. The case of having both parameters p and q variable in the Besov scale was studied by Drihem in [7], where the atomic decomposition of Besov spaces with variable smoothness and integrability $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ was stated. Here we present the atomic decomposition for 2-microlocal Besov spaces $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, whose proof follows mainly from [7] using Remark 2 in [15].

At first, we shall introduce some notation. Let $\mathbb{Z}^n$ stand for the lattice of all points in $\mathbb{R}^n$ with integer-valued components, $Q_{\nu,m}$ denotes a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centered at $2^{-\nu}m = (2^{-\nu}m_1 \ldots, 2^{-\nu}m_n)$ and with side length $2^{-\nu}$, where $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $r > 0$ then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side length r times the side length of Q. By $\chi_{\nu,m}$ we denote the characteristic function of the cube $Q_{\nu,m}$.

We define now smooth atoms, which are the building blocks for atomic decompositions.

Definition 2.6. *Let* $K, L \in \mathbb{N}_0$ *and let* $d > 1$. *A* K*-times continuously differentiable complex-valued function* $a \in C^K(\mathbb{R}^n)$ *is called* $[K, L]$*-atom centered at* $Q_{\nu,m}$, *for all* $\nu \in \mathbb{N}_0$ *and* $m \in \mathbb{Z}^n$, *if*

(i) supp $a \subset d\,Q_{\nu,m}$

(ii) $|D^\beta a(x)| \leq 2^{|\beta|\nu}$ *for* $|\beta| \leq K$

(iii) $\displaystyle\int_{\mathbb{R}^n} x^\beta a(x)dx = 0$ *for* $0 \leq |\beta| < L$ *and* $\nu \geq 1$.

Remark 2.7. If an atom a is centered at $Q_{\nu,m}$, *i.e.*, if it fulfills (i), then we denote it by $a_{\nu,m}$. If $L = 0$ or $\nu = 0$, then no moment conditions (iii) are required.

We shall now introduce the sequence spaces $b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ and $f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$, whose use will become clear in the following.

Definition 2.8. *Let $\boldsymbol{w} = (w_\nu)_{\nu\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbb{R}^n)$ and $p, q \in \mathcal{P}(\mathbb{R}^n)$.*

(i) *The sequence space $b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ consists of those complex-valued sequences $\lambda = (\lambda_{\nu,m})_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ such that*

$$\|\lambda \mid b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| := \Big\|\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|\, w_\nu(2^{-\nu}m)\,\chi_{\nu,m}\Big)_{\nu\in\mathbb{N}_0} \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\Big\|$$

is finite.

(ii) *If $p^+, q^+ < \infty$, then the sequence space $f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ consists of those complex-valued sequences $\lambda = (\lambda_{\nu,m})_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ such that*

$$\|\lambda \mid f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| := \Big\|\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|\, w_\nu(2^{-\nu}m)\,\chi_{\nu,m}\Big)_{\nu\in\mathbb{N}_0} \mid L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbb{R}^n))\Big\|$$

is finite.

We present now the atomic decomposition results for the scale of 2-microlocal spaces with variable p and q. While for the Triebel-Lizorkin space $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ the result was stated in [14], the equivalent result for Besov spaces $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is new. However we do not present it here, since the idea of the proof goes back to [7] where the atomic decomposition was stated for $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, see also Remark 2 in [15].

Theorem 2.9 (Corollary 5.6 in [14])**.** *Let $\boldsymbol{w} = (w_\nu)_{\nu\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbb{R}^n)$ and $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $0 < p^- \le p^+ < \infty$ and $0 < q^- \le q^+ < \infty$. Furthermore, let $d > 1$, $K, L \in \mathbb{N}_0$ with $K > \alpha_2$ and $L > \sigma_{p,q} - \alpha_1$ be fixed. Then every $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if an only if it can be represented as*

$$f = \sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m}\, a_{\nu,m}, \quad \textit{convergence being in } \mathcal{S}'(\mathbb{R}^n), \tag{2.7}$$

for $(a_{\nu,m})_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ $[K, L]$-atoms according to Definition 2.6 and $\lambda \in f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Moreover,

$$\|f \mid F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \sim \inf \|\lambda \mid f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|,$$

where the infimum is taken over all possible representations of f.

Theorem 2.10. *Let $\boldsymbol{w} = (w_\nu)_{\nu\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbb{R}^n)$ and $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $0 < q^- \le q^+ < \infty$. Furthermore, let $d > 1$, $K, L \in \mathbb{N}_0$ with $K > \alpha_2$ and $L > \sigma_p - \alpha_1$ be fixed. Then every $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if an only if it can be represented as*

$$f = \sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m}\, a_{\nu,m}, \quad \textit{convergence being in } \mathcal{S}'(\mathbb{R}^n), \tag{2.8}$$

for $(a_{\nu,m})_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ $[K, L]$-atoms according to Definition 2.6 and $\lambda \in b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Moreover,

$$\|f \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \sim \inf \|\lambda \mid b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|,$$

where the infimum is taken over all possible representations of f.

3 Non-smooth atomic decompositions

3.1 Non-smooth atoms

At first, we present the concept of *non-smooth atoms*, slightly adapted from [22] to our scale. Note that the usual parameters K and L are now nonnegative real numbers instead of natural numbers.

Definition 3.1. *Let $K, L \geq 0$, $d > 1$ and $c > 0$. A function $a : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called a non-smooth $[K, L]$-atom centered at $Q_{\nu,m}$, for all $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, if*

(i) $\operatorname{supp} a \subset d\, Q_{\nu,m}$,

(ii) $\|a(2^{-\nu}\cdot) \mid \mathcal{C}^K\| \leq c$,

(iii) *and for every $\psi \in \mathcal{C}^L$ it holds*

$$\Big|\int_{d\, Q_{\nu,m}} \psi(x)a(x)dx\Big| \leq c\, 2^{-\nu(L+n)}\|\psi \mid \mathcal{C}^L\|.$$

Remark 3.2. (a) As in the smooth case, if $L = 0$, then condition (iii) can be ignored since it follows from conditions (i) and (ii) with $K = 0$. If $K = 0$, then by Definition 2.1 we only require a to be suitable bounded.

(b) The modification of condition (ii) here was suggested in [25] (with some minor adjustments) and one can see that the usual formulation as in Definition 2.6-(ii) follows from this one if K is a natural number, since $C^K(\mathbb{R}^n) \hookrightarrow \mathcal{C}^K(\mathbb{R}^n)$.

(c) Regarding condition (iii), the modification here was suggested by Skrzypczak in [24] for natural numbers $L+1$ (replacing $\mathcal{C}^L(\mathbb{R}^n)$ by $C^L(\mathbb{R}^n)$). Here, as in [22], we extended this definition to general positive numbers L.

3.2 Local means

Let us recall the characterization of $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ by local means, proved in Theorem 6 of [15] and Theorem 2.2 of [13], respectively. For each system $(\Psi_k)_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$, for each distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ and to each number $a > 0$, we define the Peetre maximal operator by

$$(\Psi_k^* f)_a(x) = \sup_{y\in\mathbb{R}^n} \frac{|(\Psi_k * f)(y)|}{1 + |2^k(y-x)|^a}, \qquad \text{where } k \in \mathbb{N}_0 \text{ and } x \in \mathbb{R}^n.$$

We start with two given functions $\psi_0, \psi_1 \in \mathcal{S}(\mathbb{R}^n)$. We define

$$\psi_j(x) = \psi_1(2^{-j}x), \quad \text{for } x \in \mathbb{R}^n \text{ and } j \in \mathbb{N}. \tag{3.1}$$

Furthermore, for all $k \in \mathbb{N}_0$ we write $\Psi_k = \hat{\psi}_k$.

Remark 3.3. Before presenting the results, we would like to highlight that a careful look into the proof of the Theorem 6 in [15] shows that the condition $a > \frac{n+c_{\log}(1/q)}{p^-} + \alpha$ should be replaced by

$$a > \frac{n}{p^-} + c_{\log}(1/q) + \alpha.$$

Proposition 3.4. *Let $\boldsymbol{w} = (w_j)_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbb{R}^n)$, $p, q \in \mathcal{P}^{log}(\mathbb{R}^n)$ and let $a > 0$, $R \in \mathbb{N}_0$ with $R > \alpha_2$. Further, let ψ_0, ψ_1 belong to $\mathcal{S}(\mathbb{R}^n)$ with*

$$D^\beta \psi_1(0) = 0, \quad \textit{for } 0 \leq |\beta| < R, \tag{3.2}$$

and

$$|\psi_0(x)| > 0 \quad \text{on} \quad \{x \in \mathbb{R}^n : |x| < \varepsilon\} \tag{3.3}$$

$$|\psi_1(x)| > 0 \quad \text{on} \quad \{x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon\} \tag{3.4}$$

for some $\varepsilon > 0$.

(i) *For* $a > \frac{n}{p^-} + c_{\log}(1/q) + \alpha$ *and for all* $f \in \mathcal{S}'(\mathbb{R}^n)$ *we have*

$$\|f \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \sim \|(\Psi_k * f)w_k \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\| \sim \|(\Psi_k^* f)_a w_k \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\|$$

(i) *If* $p^+, q^+ < \infty$, *then for* $a > \frac{n}{\min(p^-,q^-)} + \alpha$ *and for all* $f \in \mathcal{S}'(\mathbb{R}^n)$ *we have*

$$\|f \mid F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \sim \|(\Psi_k * f)w_k \mid L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbb{R}^n))\| \sim \|(\Psi_k^* f)_a w_k \mid L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbb{R}^n))\|$$

Remark 3.5. (a) This result shows that the definition of the 2-microlocal spaces of variable integrability given in Definition 2.4 is independent of the resolution of unity used.

(b) The conditions (3.2) are usually called *moment conditions*, while (3.3) and (3.4) are the so-called *Tauberian conditions*.

(c) If $R = 0$, then no moment conditions (3.2) on ψ_1 are required.

(d) The notation $c_{\log}(1/q)$ stands for the constant from (2.5) with $1/q(\cdot)$.

3.2.1 Helpful lemmas

Before presenting the main theorem of this section, we provide some technical lemmas which will be useful later on. The first result is an adaptation of Lemma 3.8 in [22] and it shows that local means are related to non-smooth $[K, L]$-atoms.

Lemma 3.6. *Let* $j \in \mathbb{N}_0$. *If* $\psi_j = \psi_1(2^{-j}\cdot)$ *is a local mean as in Proposition* 3.4, *then* $2^{-jn}\,\psi_j$ *is a non-smooth* $[K, L]$*-atom centered at* $Q_{j,0}$ *for arbitrary large* $K > 0$ *and for* $L \le R + 1$ *with* R *from* (3.2).

The next result can be obtained following the steps of the proof of Theorem 3.12 in [22] with our normalization. The idea is to use the fact that not only $a_{\nu,m}$ but also Ψ_j can be understood as atoms in the sense of Definition 3.1. In this way, we can use estimates of the type (ii) and (iii) of Definition 3.1 in both functions and get this result.

Lemma 3.7. *Let* $j \in \mathbb{N}_0$, $\psi_j = \psi_1(2^{-j}\cdot)$ *be a local mean as in Proposition* 3.4 *and let* $(a_{\nu,m})_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ *be non-smooth* $[K, L]$*-atoms. Then*

$$\left|\int_{\mathbb{R}^n} \Psi_j(y) a_{\nu,m}(x-y)\,dy\right| \le c\,2^{-(j-\nu)K}\chi(c\,Q_{\nu,m})(x), \qquad \text{for } j \ge \nu$$

and

$$\left|\int_{\mathbb{R}^n} \Psi_j(x-y) a_{\nu,m}(y)\,dy\right| \le c\,2^{-(\nu-j)(L+n)}\chi(c\,2^{\nu-j}Q_{\nu,m})(x), \qquad \text{for } j < \nu.$$

Since the maximal operator is of no use in the case when q is a variable function (see [1] and [6]), we have to work with another tool. We use convolution inequalities from [6] and [15] for $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, respectively. Therefore, we introduce the functions

$$\eta_{\nu,R}(x) = \frac{2^{n\nu}}{(1 + 2^\nu|x|)^R},$$

for $\nu \in \mathbb{N}_0$ and $R > 0$.

Lemma 3.8 (Theorem 3.2 in [6]). *Let $p, q \in C^{log}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$ and $1 < q^- \le q^+ < \infty$. Then the inequality*

$$\|\eta_{\nu,R} * f_\nu \mid L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbb{R}^n))\| \le c\,\|f_\nu \mid L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbb{R}^n))\|$$

holds for every sequence $(f_\nu)_{\nu\in\mathbb{N}_0}$ of $L_1^{loc}(\mathbb{R}^n)$ functions and constant $R > n$.

Lemma 3.9 (Lemma 10 in [15]). *Let $p, q \in \mathcal{P}(\mathbb{R}^n)$ with $p(\cdot) \ge 1$. For all $R > n + c_{log}(1/q)$, there exists a constant $c > 0$ such that for all sequences $(f_j)_{j\in\mathbb{N}_0} \in \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))$ it holds*

$$\|\eta_{\nu,R} * f_\nu \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\| \le c\,\|f_\nu \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\|.$$

Lemma 3.10 (Lemma 9 in [15]). *Let $p, q \in \mathcal{P}(\mathbb{R}^n)$ and $\delta > 0$. Let $(g_k)_{k\in\mathbb{Z}}$ be a sequence of non-negative measurable functions on $\mathbb{R}^n$ and denote*

$$G_\nu(x) = \sum_{k\in\mathbb{Z}} 2^{-|\nu-k|\delta} g_k(x), \qquad \text{for } x \in \mathbb{R}^n \text{ and } \nu \in \mathbb{Z}.$$

Then there exist constants $C_1, C_2 > 0$, depending on $p(\cdot), q(\cdot)$ and δ, such that

$$\|G_\nu \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\| \le C_1\,\|g_k \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\|$$

and

$$\|G_\nu \mid L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbb{R}^n))\| \le C_2\,\|g_k \mid L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbb{R}^n))\|.$$

Remark 3.11. Naturally, Lemma 3.10 holds true also if the indices k and ν run only over natural numbers.

Lemma 3.12 (Lemma 5.5 in [14]). *Let $0 < t \le 1$, $j, \nu \in \mathbb{N}_0$ and $(h_{\nu m})_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ be positive real numbers. Then we have for $R > 0$ and $x \in \mathbb{R}^n$ that*

$$\sum_{m\in\mathbb{Z}^n} h_{\nu m}(1 + 2^j|x - 2^{-\nu}m|)^{-R} \le c\,\max(1, 2^{(\nu-j)R}) \left(\left[\eta_{\nu,Rt} * \left(\sum_{m\in\mathbb{Z}^n} h_{\nu m}^t \chi_{\nu m}(\cdot) \right) \right](x) \right)^{1/t}.$$

3.3 A more general atomic representation theorem

In this section we shall prove the more general atomic decomposition theorem. We will follow the approach of the proof of Theorem 3.12 in [22]. But first we will prove the convergence of the atomic series in $\mathcal{S}'(\mathbb{R}^n)$.

Theorem 3.13. *Let $\boldsymbol{w} = (w_j)_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbb{R}^n)$ and $p, q \in \mathcal{P}^{log}(\mathbb{R}^n)$. Let $K, L \ge 0$ with $K > \alpha_2$ and $L > \sigma_p - \alpha_1$. Then*

$$\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m}\, a_{\nu,m} \tag{3.5}$$

converges unconditionally in $\mathcal{S}'(\mathbb{R}^n)$, where $a_{\nu,m}$ are non-smooth $[K, L]$-atoms located at $Q_{\nu,m}$ and $\lambda \in b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ or $\lambda \in f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$.

Proof. According to [1] and [14] (see also the explanation before Proposition 3.9 in [14]), for $t < \min(1, p^-)$ we have the follow sequences of embeddings

$$b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)} \hookrightarrow b^{\boldsymbol{\varrho}}_{\frac{p(\cdot)}{t},q(\cdot)} \hookrightarrow b^{\boldsymbol{\varrho}}_{\frac{p(\cdot)}{t},\infty}$$

and

$$f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)} \hookrightarrow f^{\boldsymbol{\varrho}}_{\frac{p(\cdot)}{t},q(\cdot)} \hookrightarrow b^{\boldsymbol{\varrho}}_{\frac{p(\cdot)}{t},\infty},$$

where $\varrho_j(x) = w_j(x)2^{-j\frac{t}{p(x)}\sigma_t}$. Since $2^{-j\frac{t}{p(x)}\sigma_t} =: 2^{-js(x)\sigma_t}$ is again a 2-microlocal weight sequence with $s(\cdot) \in C^{log}(\mathbb{R}^n)$ and $0 \le s(x) < 1$, then $\boldsymbol{\varrho}$ is also an admissible weight sequence and it can be shown that $\boldsymbol{\varrho} \in \mathcal{W}^{\beta}_{\alpha_1-\sigma_t,\alpha_2}(\mathbb{R}^n)$ with $\beta \ge 0$ large enough. In fact, it can be shown that $\beta = \alpha + t\,\sigma_t\, c_{log}(1/p)$, where the constant $c_{log}(1/p)$ comes from (2.5).

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We can assume that $\lambda \in b^{\boldsymbol{\varrho}}_{\frac{p(\cdot)}{t},\infty}$, having in mind the above embeddings. From the conditions (i) and (iii) of the Definition 3.1 we obtain

$$\Big|\int_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} a_{\nu_m}(x)\varphi(x)\Big| \le c\, 2^{-\nu(L+n)} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|\, \|\varphi(\cdot)\,\psi(2^\nu\cdot - m) \mid \mathcal{C}^L(\mathbb{R}^n)\|, \tag{3.6}$$

where $\psi \in C^\infty(\mathbb{R}^n)$, $\psi(x) = 1$ for $x \in d\,Q_{0,0}$ and supp $\psi \subset (d+1)\,Q_{0,0}$. Since $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we can estimate the norm in (3.6) from above in the following way:

$$\|\varphi(\cdot)\,\psi(2^\nu\cdot - m) \mid \mathcal{C}^L(\mathbb{R}^n)\| \le C_M\,(1+|2^{-\nu}m|)^{-M} \sim C_M\,(1+|y|)^{-M}, \quad y \in Q_{\nu,m},$$

where $M \in \mathbb{N}_0$ is at our disposal and C_M does not depend on ν and m. According to the properties of the admissible weight sequence (3.6) becomes

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} a_{\nu_m}(x)\varphi(x)dx\Big| &\le c\, 2^{-\nu(L+n)} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|\, \varrho_\nu(y)\,(1+|y|)^{-M}\,\chi_{\nu,m}(y)\,\varrho_\nu(y)^{-1} \\
&\le c\, 2^{-\nu(L+n)} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|\, \varrho_\nu(2^{-\nu}m)\,(1+2^\nu|y-2^{-\nu}m|)^{-\beta}\, 2^{\nu(\alpha_1-\sigma_t)}\,(1+|y|)^{-M+\beta}\,\chi_{\nu,m}(y) \\
&\sim c\, 2^{-\nu(L+\alpha_1-\sigma_t)} \int_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|\, \varrho_\nu(2^{-\nu}m)\,(1+2^\nu|x-2^{-\nu}m|)^{-\beta}\,(1+|x|)^{-M+\beta}\,\chi_{\nu,m}(x)\,dx.
\end{aligned}$$

We estimate now the integral and from Hölder's inequality for $L_{p(\cdot)}(\mathbb{R}^n)$ with $p(\cdot)/t > 1$ and choosing M large enough we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|\, \varrho_\nu(2^{-\nu}m)\,(1+2^\nu|x-2^{-\nu}m|)^{-\beta}\,(1+|x|)^{-M+\beta}\,\chi_{\nu,m}(x)\,dx \\
&\qquad\qquad \le c\, \Big\| \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|\, \varrho_\nu(2^{-\nu}m)\,(1+2^\nu|x-2^{-\nu}m|)^{-\beta} \mid L_{\frac{p(\cdot)}{t}}(\mathbb{R}^n)\Big\|.
\end{aligned}$$

Finally, we use Lemma 3.7 of [14] with $\beta > \frac{n}{t}$ and the boundedness of the maximal operator and obtain

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} a_{\nu_m}(x)\varphi(x)\Big| &\le c\, 2^{-\nu(L+\alpha_1-\sigma_t)} \Big\| \mathcal{M}\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|\, \varrho_\nu(2^{-\nu}m)\,\chi_{\nu,m}(\cdot)\Big) \mid L_{\frac{p(\cdot)}{t}}(\mathbb{R}^n)\Big\| \\
&\le c\, 2^{-\nu(L+\alpha_1-\sigma_t)} \Big\| \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|\, \varrho_\nu(2^{-\nu}m)\,\chi_{\nu,m}(\cdot) \mid L_{\frac{p(\cdot)}{t}}(\mathbb{R}^n)\Big\|.
\end{aligned}$$

Summing up over $\nu \in \mathbb{N}_0$ and since $L > \sigma_t - \alpha_1 > \sigma_p - \alpha_1$, we arrive at

$$\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \Big|\int_{\mathbb{R}^n} \lambda_{\nu,m} a_{\nu_m}(x)\varphi(x)dx\Big| \le C\,\|\lambda \mid b^{\boldsymbol{\varrho}}_{\frac{p(\cdot)}{t},\infty}\|,$$

from which the absolute convergence of (3.5) and the unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ follow. □

Now we state the atomic decomposition result.

Theorem 3.14. *Let $\boldsymbol{w} = (w_j)_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbb{R}^n)$ and $p, q \in \mathcal{P}^{log}(\mathbb{R}^n)$.*

(i) *Let $K, L \geq 0$ with $K > \alpha_2$ and $L > \sigma_p - \alpha_1 + c_{log}(1/q)$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if and only if it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m}\, a_{\nu,m}, \quad \text{convergence being in } \mathcal{S}'(\mathbb{R}^n), \tag{3.7}$$

for $(a_{\nu,m})_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ non-smooth $[K, L]$-atoms according to Definition 3.1 and $\lambda \in b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Moreover,

$$\|f \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \sim \inf \|\lambda \mid b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|,$$

where the infimum is taken over all possible representations of f.

(ii) *Let $K, L \geq 0$ with $K > \alpha_2$ and $L > \sigma_{p,q} - \alpha_1$. If $p^+, q^+ < \infty$, then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if and only if it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m}\, a_{\nu,m}, \quad \text{convergence being in } \mathcal{S}'(\mathbb{R}^n),$$

for $(a_{\nu,m})_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ non-smooth $[K, L]$-atoms according to Definition 3.1 and $\lambda \in f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Moreover,

$$\|f \mid F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \sim \inf \|\lambda \mid f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|,$$

where the infimum is taken over all possible representations of f.

Proof. We rely on the proof of Theorem 3.12 in [22].
There are two directions we shall prove. Observing that every smooth $[K, L]$-atom is a non-smooth $[K, L]$-atom one direction follows directly from Theorems 2.10 and 2.9 (see the proof of Theorem 3.12 in [22]). We will focus on the other direction, which is the main part of the proof. We need to show that, even with weaker conditions on the atoms, we still get an element of $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ or $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ when considering a linear combination of these non-smooth atoms. For this, we will use the equivalent characterization by local means given in Proposition 3.4. As usual, we divide the following summation in dependence on $j \in \mathbb{N}_0$ into two parts

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m}\, a_{\nu,m} = \sum_{\nu=0}^{j} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m}\, a_{\nu,m} + \sum_{\nu=j+1}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m}\, a_{\nu,m} = f_j + f^j.$$

We have

$$\begin{aligned} \|f \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| &\leq c \left(\|f_j \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| + \|f^j \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \right) \\ &= c\, \|(\Psi_j * f_j) w_j \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\| \\ &\quad + c\, \|(\Psi_j * f^j) w_j \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\| \\ &= \mathrm{I} + \mathrm{II}. \end{aligned} \tag{3.8}$$

In both parts of the summation, the crucial part is the estimate of

$$\left| \int_{\mathbb{R}^n} \Psi_j(y) a_{\nu,m}(x-y)\, dy \right|,$$

which we already have from Lemma 3.7. As stated before, we use that not only $a_{\nu,m}$ but also Ψ_j can be interpreted as non-smooth atoms and admit estimates as (ii) and (iii) in Definition 3.1.

Let us first consider the case $j \geq \nu$. We have, from Lemma 3.7, the estimate

$$\left| \int_{\mathbb{R}^n} \Psi_j(y) a_{\nu,m}(x-y)\, dy \right| \leq c\, 2^{-(j-\nu)K} \chi(c\, Q_{\nu,m})(x),$$

and from the properties of the weight sequence in Definition 2.2,

$$w_j(x) \leq C\, 2^{(j-\nu)\alpha_2} w_\nu(2^{-\nu}m)(1+2^\nu|x-2^{-\nu}m|)^\alpha.$$

Putting both together, we have

$$|(\Psi_j * f_j)(x)\, w_j(x)| \leq c \sum_{\nu=0}^{j} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|\, 2^{-(j-\nu)(K-\alpha_2)}\, w_\nu(2^{-\nu}m)\, \chi(c\, Q_{\nu,m})(x),$$

using that $|x-2^{-\nu}m| \lesssim 2^{-\nu}$ for $x \in c\, Q_{\nu,m}$. Further, using Lemma 3.10 with $\delta = K-\alpha_2 > 0$ we get

$$\begin{aligned} \mathrm{I} &\leq c \left\| \sum_{\nu=0}^{j} 2^{-(j-\nu)(K-\alpha_2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|\, w_\nu(2^{-\nu}m)\, \chi(c\, Q_{\nu,m}) \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n)) \right\| \\ &\leq c' \left\| \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|\, w_\nu(2^{-\nu}m)\, \chi(c\, Q_{\nu,m}) \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n)) \right\| \\ &= c' \left\| \lambda \mid b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\|. \end{aligned}$$

Now, let $j < \nu$. Lemma 3.7 gives us the estimate

$$\left| \int_{\mathbb{R}^n} \Psi_j(x-y) a_{\nu,m}(y)\, dy \right| \leq c\, 2^{-(\nu-j)(L+n)} \chi(c\, 2^{\nu-j} Q_{\nu,m})(x)$$

and using the properties of the weight sequence again, we get

$$w_j(x) \leq C\, 2^{-(\nu-j)\alpha_1} w_\nu(2^{-\nu}m)(1+2^j|x-2^{-\nu}m|)^\alpha.$$

Note that the following assertion is true: for every $x \in \mathbb{R}^n$ and every $M > 0$, there exist a constant $C > 0$ such that

$$\chi(c\, 2^{\nu-j} Q_{\nu,m})(x) \leq C\, (1+2^j|x-2^{-\nu}m|)^{-M}.$$

Combining the last three estimates, we have

$$\left|(\Psi_j * f^j)(x)\, w_j(x)\right| \leq c \sum_{\nu=j+1}^{\infty} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|\, 2^{-(\nu-j)(L+n+\alpha_1)}\, w_\nu(2^{-\nu}m)\, (1+2^j|x-2^{-\nu}m|)^{\alpha-M}.$$

We can now estimate II. Let $0 < t \leq \min(1, p^-)$. By Lemma 3.12 with $M-\alpha > 0$ we get

$$\begin{aligned} \mathrm{II} &\leq c \left\| \sum_{\nu=j+1}^{\infty} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}| 2^{-(\nu-j)(L+n+\alpha_1)}\, w_\nu(2^{-\nu}m)\, (1+2^j|x-2^{-\nu}m|)^{\alpha-M} \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n)) \right\| \\ &\leq c \left\| \sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)\delta} \left[\eta_{\nu,(M-\alpha)t} * \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|^t\, w^t_\nu(2^{-\nu}m) \chi_{\nu,m}(\cdot) \right) \right]^{1/t} \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n)) \right\| \end{aligned}$$

with $\delta := L + n + \alpha_1 - M + \alpha$ and by Lemma 3.10 with $\delta \geq 0$ we obtain

$$\leq c' \left\| \left[\eta_{\nu,(M-\alpha)t} * \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|^t \, w_\nu^t(2^{-\nu}m)\chi_{\nu,m}(\cdot) \right) \right]^{1/t} \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n)) \right\|$$

$$= c' \left\| \eta_{\nu,(M-\alpha)t} * \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|^t \, w_\nu^t(2^{-\nu}m)\chi_{\nu,m}(\cdot) \right) \mid \ell_{\frac{q(\cdot)}{t}}(L_{\frac{p(\cdot)}{t}}(\mathbb{R}^n)) \right\|^{1/t}.$$

This can be further estimated using Lemma 3.9 with $M - \alpha > \frac{n}{t} + c_{log}(1/q)$ (note that we are applying the result for $\ell_{\frac{q(\cdot)}{t}}(L_{\frac{p(\cdot)}{t}}(\mathbb{R}^n))$)

$$\leq c'' \left\| \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|^t \, w_\nu^t(2^{-\nu}m)\chi_{\nu,m}(\cdot) \mid \ell_{\frac{q(\cdot)}{t}}(L_{\frac{p(\cdot)}{t}}(\mathbb{R}^n)) \right\|^{1/t}$$

$$= c'' \left\| \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}| \, w_\nu(2^{-\nu}m)\chi_{\nu,m}(\cdot) \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n)) \right\|$$

$$= c'' \left\| \lambda \mid b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\|.$$

In the F-case we do the same estimates as above, use the second part of Lemma 3.10 and apply, instead of Lemma 3.9, Lemma 3.8 for $L_{\frac{p(\cdot)}{t}}(\ell_{\frac{q(\cdot)}{t}})$ with $t < \min(1, p^-, q^-)$ and $R = (M - \alpha)t > n$. □

4 Pointwise multipliers

Let φ be a bounded function on $\mathbb{R}^n$. The question is under which conditions the mapping $f \mapsto \varphi \cdot f$ makes sense and generates a bounded operator in a given space $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ or $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

For the classical spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$, Triebel studied this problem in Section 4.2 of [26], where two different approaches were followed: via atoms and via local means. The first idea of taking an atomic decomposition of f required the non-existence of moment conditions as in Definition 2.6-(iii) since the moment conditions are in general destroyed by multiplication with φ. A more general result was then obtained by Triebel using local means. A good overview on pointwise multipliers in constant exponent spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ can be found in Chapter 4 in [21].

Recently, Scharf has shown in [22] that it is possible to get a very general result on pointwise multipliers using atomic decomposition but now with the non-smooth atoms.

Our aim in this section is to extend this result for variable exponent spaces $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Following [22], we take an atomic decomposition of f as in Theorem 3.14, multiply it by φ and prove that the resulting sum is still a sum of non-smooth atoms. We start referring two helpful results.

Lemma 4.1. *(Lemma 4.2 in [22]) Let $s \geq 0$. There exists a constant $c > 0$ such that for all $f, g \in \mathcal{C}^s(\mathbb{R}^n)$, the product $f \cdot g$ belongs to $\mathcal{C}^s(\mathbb{R}^n)$ and it holds*

$$\|f \cdot g \mid \mathcal{C}^s(\mathbb{R}^n)\| \leq c \, \|f \mid \mathcal{C}^s(\mathbb{R}^n)\| \cdot \|g \mid \mathcal{C}^s(\mathbb{R}^n)\|.$$

The next result shows that the product of a non-smooth $[K,L]$-atom with a function $\varphi \in \mathcal{C}^\rho(\mathbb{R}^n)$ is still a non-smooth $[K,L]$-atom. It is a slight normalization of Lemma 4.3 in [22] and so the proof will not be presented here.

Lemma 4.2. *There exists a constant c with the following property: for all $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, all non-smooth $[K,L]$-atoms $a_{\nu,m}$ with support in $d\,Q_{\nu,m}$ and all $\varphi \in \mathcal{C}^\rho(\mathbb{R}^n)$ with $\rho \geq \max(K,L)$, the product*

$$c\,\|\varphi \mid \mathcal{C}^\rho(\mathbb{R}^n)\|^{-1} \cdot \varphi \cdot a_{\nu,m}$$

is a non-smooth $[K,L]$-atom with support in $d\,Q_{\nu,m}$.

We are now in the position of proving the main result.

Theorem 4.3. *Let $\boldsymbol{w} = (w_j)_{j\in\mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}(\mathbb{R}^n)$ and $p, q \in \mathcal{P}^{log}(\mathbb{R}^n)$.*

(i) *Let $\rho > \max(\alpha_2, \sigma_p - \alpha_1 + c_{log}(1/q))$. Then there exists a positive number c such that*

$$\|\varphi \cdot f \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \leq c\,\|\varphi \mid \mathcal{C}^\rho(\mathbb{R}^n)\| \cdot \|f \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|$$

for all $\varphi \in \mathcal{C}^\rho(\mathbb{R}^n)$ and all $f \in B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

(ii) *Let $p^+, q^+ < \infty$ and $\rho > \max(\alpha_2, \sigma_{p,q} - \alpha_1)$. Then there exists a positive number c such that*

$$\|\varphi \cdot f \mid F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \leq c\,\|\varphi \mid \mathcal{C}^\rho(\mathbb{R}^n)\| \cdot \|f \mid F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|$$

for all $\varphi \in \mathcal{C}^\rho(\mathbb{R}^n)$ and all $f \in F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Proof. We will prove (i), since (ii) follows similarly. Let $f \in B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. By Theorem 3.14, there exist $\lambda \in b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and non-smooth $[K,L]$-atoms $(a_{\nu,m})_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ such that f can be represented as

$$f = \sum_{\nu=0}^\infty \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}$$

and

$$\|\lambda \mid b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \leq c\,\|f \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|,$$

where the constant c is independent of $f \in B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. By Lemma 4.2 we know that

$$\begin{aligned}\varphi \cdot f &= \|\varphi \mid \mathcal{C}^\rho(\mathbb{R}^n)\| \sum_{\nu=0}^\infty \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} \left(\|\varphi \mid \mathcal{C}^\rho(\mathbb{R}^n)\|^{-1} \cdot \varphi \cdot a_{\nu,m}\right)\\ &= \sum_{\nu=0}^\infty \sum_{m\in\mathbb{Z}^n} \mu_{\nu,m} \left(\|\varphi \mid \mathcal{C}^\rho(\mathbb{R}^n)\|^{-1} \cdot \varphi \cdot a_{\nu,m}\right)\end{aligned}$$

is still a linear combination of non-smooth $[K,L]$-atoms with coefficients $\mu_{\nu,m} := \|\varphi \mid \mathcal{C}^\rho(\mathbb{R}^n)\|\,\lambda_{\nu,m}$, which means that we obtained an atomic decomposition for the product $\varphi \cdot f$. Using now the other direction of the Theorem 3.14, we know that $\varphi \cdot f$ belongs to $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and it holds

$$\|\varphi \cdot f \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \leq c\,\|\varphi \mid \mathcal{C}^\rho(\mathbb{R}^n)\| \cdot \|\lambda \mid b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \leq c'\,\|\varphi \mid \mathcal{C}^\rho(\mathbb{R}^n)\| \cdot \|f \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|,$$

which completes the proof. □

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