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G. Wanka, O. Wilfer

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A Lagrange Duality Approach for Multi-Composed Optimization Problems*

Gert Wanka† Oleg Wilfer‡

Abstract: In this paper we consider an optimization problem with geometric and cone constraints, whose objective function is a composition of $n+1$ functions. For this problem we calculate its conjugate dual problem, where the functions involved in the objective function of the primal problem will be decomposed. Furthermore, we formulate generalized interior point regularity conditions for strong duality and give necessary and sufficient optimality conditions. As applications of this approach we determine the formulas of the conjugate as well as the biconjugate of the objective function of the primal problem and discuss an optimization problem having as objective function the sum of reciprocals of concave functions.

Key words: Lagrange Duality, Composed Functions, Generalized Interior Point Regularity Conditions, Conjugate Functions, Reciprocals of Concave Functions, Power Functions.

1 Introduction

Conjugate duality is a powerful instrument to analyze optimization problems and has for that reason a wide range of applications. Over the last couple of years, an important field of applications arises in subjects such as facility location theory [19], machine learning [3], image restoration [4] and portfolio optimization [6], to mention only a few of them. In many cases, the objective function of an optimization problem occurring in the mentioned research areas may be written as a composition of two functions. The method presented in this paper can also be seen as a splitting technique, which makes not only the derivation of duality assertions easier, but also the handling of optimization problems from the numerical point of view.

But until now there is no duality approach for the more general situation, namely, where the optimization problem is considered as the minimization of an objective function that is a composition of more than two functions. The advantage of this consideration is that the objective function of a certain optimization problem can be splitted into a certain number of functions to refine and improve some theoretical and numerical aspects. In fact, this study is more general than in [1], [2], [5], [7], [8] and [14] and can furthermore be understood as a union of all kinds of meaningful perturbation methods.

Therefore, the goal of this paper is to consider an optimization problem with geometric and

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†Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, email: gert.wanka@mathematik.tu-chemnitz.de

‡Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, email: oleg.wilfer@mathematik.tu-chemnitz.de

cone constraints, whose objective function is a composition of $n+1$ functions and to deliver a full duality approach for this type of problems. For short, we will call such problems multi-composed optimization problems. As applications we present the formulas of the conjugate and the biconjugate of a multi-composed function, i.e. a function that is a composition of $n+1$ functions. Moreover, we discuss in the last section an optimization problem having as objective function the sum of reciprocals of concave functions.
To this end, we first introduce in Section 2 some definitions and notations from convex analysis. In Section 3 we consider a multi-composed optimization problem with geometric and cone constraints. Then, we give an equivalent formulation of this problem and use the reformulated optimization problem to construct a corresponding conjugate dual problem to the main problem, followed by a weak duality theorem. The convenience of this approach is that the functions involved in the composed objective function of the original problem can be decomposed in the formulation of the conjugate dual problem or, to formulate it more precisely, their conjugates.
Section 4 is devoted to generalized interior point regularity conditions guaranteeing strong duality. Moreover, by using the strong duality theorem we formulate some optimality conditions for the original problem and its corresponding conjugate dual problem. Besides of this approach, we discover in Section 5 the formula of the conjugate of a multi-composed function. We find also a formula of the biconjugate function and close this section with a theorem which characterizes some topological properties of this function.
In Section 6, as a further application of our approach, we consider a convex optimization problem having as objective function the sum of reciprocals of concave functions. For this problem we formulate its corresponding conjugate dual problem and state a strong duality theorem from which we derive necessary and sufficient optimality conditions.

2 Notations and preliminary results

Let X be a Hausdorff locally convex space and X^* its topological dual space endowed with the weak* topology $w(X^*, X)$. For $x \in X$ and $x^* \in X^*$, let $\langle x^*, x\rangle := x^*(x)$ be the value of the linear continuous functional x^* at x. For a subset $A \subseteq X$, its indicator function $\delta_A : X \to \overline{\mathbb{R}} = \mathbb{R}\cup\{\pm\infty\}$ is

$$\delta_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

For a given function $f : X \to \overline{\mathbb{R}}$ we consider its effective domain

$$\operatorname{dom} f := \{x \in X : \ f(x) < +\infty\}.$$

and call the function f proper if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. The conjugate function of f with respect to the non-empty subset $S \subseteq X$ is defined by

$$f_S^* : X^* \to \overline{\mathbb{R}},\ f_S^*(x^*) = (f + \delta_S)^*(x^*) = \sup_{x\in S}\{\langle x^*, x\rangle - f(x)\}.$$

In the case $S = X$, it is clear that f_S^* turns into the classical Fenchel-Moreau conjugate function of f denoted by f^*.
Additionally, we consider a non-empty convex cone $K \subseteq X$, which induces on X a partial ordering relation "$\leqq_K$", defined by

$$\leqq_K := \{(x, y) \in X \times X : y - x \in K\},$$

i.e. for $x, y \in X$ it holds $x \leqq_K y \Leftrightarrow y - x \in K$. Note that we assume that all cones we consider contain the origin. Further, we attach to X a greatest element with respect to "$\leqq_K$", denoted by $+\infty_K$, which does not belong to X and denote $\overline{X} = X \cup \{+\infty_K\}$. Then it holds $x \leqq_K +\infty_K$ for all $x \in \overline{X}$. We also define $x \leq_K y$ if and only if $x \leqq_K y$ and $x \neq y$. Further, we define $\leqq_{\mathbb{R}_+} =: \leq$ and $\leq_{\mathbb{R}_+} =: <$.
On $\overline{X}$ we consider the following operations and conventions: $x + (+\infty_K) = (+\infty_K) + x := +\infty_K\ \forall x \in X \cup \{+\infty_K\}$ and $\lambda \cdot (+\infty_K) := +\infty_K\ \forall \lambda \in [0, +\infty]$. Further, if $K^* := \{x^* \in X^* : \langle x^*, x\rangle \geq 0,\ \forall x \in K\}$ is the dual cone of K, then we define $\langle x^*, +\infty_K\rangle := +\infty$ for all $x^* \in K^*$. On the extended real space $\overline{\mathbb{R}}$ we add the following operations and conventions: $\lambda + (+\infty) = (+\infty) + \lambda := +\infty\ \forall \lambda \in (-\infty, +\infty]$, $\lambda + (-\infty) = (-\infty) + \lambda := -\infty\ \forall \lambda \in [-\infty, +\infty)$, $\lambda \cdot (+\infty) := +\infty\ \forall \lambda \in [0, +\infty]$, $\lambda \cdot (+\infty) := -\infty\ \forall \lambda \in [-\infty, 0)$, $\lambda \cdot (-\infty) := -\infty\ \forall \lambda \in (0, +\infty]$, $\lambda \cdot (-\infty) := +\infty\ \forall \lambda \in [-\infty, 0)$, $(+\infty) + (-\infty) = (-\infty) + (+\infty) := +\infty$, $0(+\infty) := +\infty$ and $0(-\infty) := 0$.
Let Z be another Hausdorff locally convex space ordered by the convex cone $Q \subseteq Z$ and Z^* its topological dual space endowed with the weak* topology $w(Z^*, Z)$, then for a vector function $F : X \to \overline{Z} = Z \cup \{+\infty_Q\}$ the domain is the set $\operatorname{dom} F := \{x \in X : F(x) \neq +\infty_Q\}$. If $\operatorname{dom} F \neq \emptyset$, then the function F is called proper. When $F(\lambda x + (1-\lambda)y) \leqq_Q \lambda F(x) + (1-\lambda)F(y)$ holds for all $x, y \in X$ and all $\lambda \in [0, 1]$ the function F is said to be Q-convex. A function $f : X \to \overline{\mathbb{R}}$ is called convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in X$ and all $\lambda \in [0, 1]$.
The Q-epigraph of a vector function F is $\operatorname{epi}_Q F = \{(x, z) \in X \times Z : F(x) \leqq_Q z\}$ and we say that F is Q-epi closed if $\operatorname{epi}_Q F$ is a closed set.
For a $z^* \in Q^*$ we define the function $(z^*F) : X \to \overline{\mathbb{R}}$ by $(z^*F)(x) := \langle z^*, F(x)\rangle$, where it is not hard to see that $\operatorname{dom}(z^*F) = \operatorname{dom} F$. Moreover, it is easy to see that if F is Q-convex, then (z^*F) is convex for all $z^* \in Q^*$. Let us point out that by the operations we defined on a Hausdorff locally convex space attached with a maximal element and on the extended real space, there holds $0f = \delta_{\operatorname{dom} f}$ and $(0_{Z^*}F) = \delta_{\operatorname{dom} F}$, where 0_{Z^*} denotes the origin of Z^*.
A function $f : X \to \overline{\mathbb{R}}$ is called lower semicontinuous at $\overline{x} \in X$ if $\liminf_{x \to \overline{x}} f(x) \geq f(\overline{x})$ and when this function is lower semicontinuous at all $x \in X$, then we call it lower semicontinuous (l.s.c. for short). The vector function F is called star Q-lower semicontinuous at $x \in X$ if (z^*F) is lower semicontinuous at x for all $z^* \in Q^*$. The function F is called star Q-lower semicontinuous if it is star Q-lower semicontinuous at every $x \in X$. Note that if F is star Q-lower semicontinuous, then it is also Q-epi closed, while the inverse statement is not true in general (see: Proposition 2.2.19 in [1]). Let us mention that in the case $Z = \mathbb{R}$ and $Q = \mathbb{R}_+$, the notion of Q-epi closedness falls into the classical notion of lower semicontinuity.
Let $W \subseteq X$ be a non-empty set, then a function $f : X \to \overline{\mathbb{R}}$ is called K-increasing on W, if from $x \leqq_K y$ follows $f(x) \leq f(y)$ for all $x, y \in W$. When $W = X$, then we call the function f K-increasing.

Definition 2.1. *The vector function $F : X \to \overline{Z}$ is called K-Q-increasing on W, if from $x \leqq_K y$ follows $F(x) \leqq_Q F(y)$ for all $x, y \in W$. For the case $W = X$, we call this function K-Q-increasing.*

Theorem 2.1. *Let V be a Hausdorff locally convex space ordered by the convex cone U, $F : X \to \overline{Z}$ be a proper and Q-convex function and $G : Z \to \overline{V}$ be an U-convex and Q-U-increasing function on $F(\operatorname{dom} F) \subseteq \operatorname{dom} G$ with the convention $G(+\infty_Q) = +\infty_U$. Then the function $(G \circ F) : X \to \overline{V}$ is U-convex.*

The proof of this theorem is straightforward.

Lemma 2.1. *Let $h : X \times Z \to \overline{Z}$ and $F : X \to \overline{Z}$ be proper vector functions, where h is defined by $h(x, v) := F(x) - v$. If we suppose that Q is a closed convex cone such that $\operatorname{int} Q \neq \emptyset$, then h is Q-epi closed if and only if F is Q-epi closed.*

Proof. By Theorem 5.8 of [15] it holds that h is Q-epi closed if and only if the level set of h, defined by $\operatorname{lev}_z h = \{(x, v) \in X \times Z : h(x, v) \leqq_Q z\}$, is closed for all $z \in Z$. Further, as $h(x, v) = F(x) - v \leqq_Q z$, we have

$$\begin{aligned} \operatorname{lev}_z h &= \{(x, v) \in X \times Z : F(x) \leqq_Q z + v\} \\ &= \{(x, v) \in X \times Z : (x, z + v) \in \operatorname{epi}_Q F\} = \operatorname{epi}_Q F - (0_X, z) \end{aligned}$$

for all $z \in Z$. But this is nothing else as a translation of $\operatorname{epi}_Q F$ and means that $\operatorname{epi}_Q F$ is closed if and only if $lev_z h$ is closed for all $z \in Z$. As a consequence we get the desired statement and the proof is complete. □

For a set $S \subseteq X$ the conic hull is defined by $\operatorname{cone}(S) := \{\lambda x : x \in S,\ \lambda \geq 0\}$. Further, the prefix int we use to denote the interior of a set $S \subseteq X$, while the prefixes core and sqri are used to denote the algebraic interior and the strong quasi relative interior, respectively, where in the case of having a convex set $S \subseteq X$ it holds (see [11])

$$\begin{aligned} \operatorname{core}(S) &= \{x \in S : \operatorname{cone}(S - x) = X\}, \\ \operatorname{sqri}(S) &= \{x \in S : \operatorname{cone}(S - x) \text{ is a closed linear subspace}\}. \end{aligned}$$

Note, that if $\operatorname{cone}(S - x)$ is a linear subspace, then $x \in S$. Moreover, it holds the following statement.

Lemma 2.2. *Let $A \subseteq X$ and $B \subseteq Z$ be non-empty convex subsets. Then, it holds*

$$0_{X \times Z} \in \operatorname{sqri}(A \times B) \Leftrightarrow 0_X \in \operatorname{sqri}(A) \text{ and } 0_Z \in \operatorname{sqri}(B).$$

Proof. First, let us recall that if A and B are convex and $0_X \in A$ and $0_Z \in B$, then

$$\operatorname{cone}(A \times B) = \operatorname{cone}(A) \times \operatorname{cone}(B).$$

Now, let us assume that $0_{X \times Z} \in \operatorname{sqri}(A \times B)$, then $\operatorname{cone}(A \times B)$ is a closed linear subspace of $X \times Z$, which implies that $0_{X \times Z} = (0_X, 0_Z) \in A \times B$. But this means that $\operatorname{cone}(A \times B) = \operatorname{cone}(A) \times \operatorname{cone}(B)$ and hence, $\operatorname{cone}(A)$ and $\operatorname{cone}(B)$ are closed linear subspaces, i.e. $0_X \in \operatorname{sqri}(A)$ and $0_Z \in \operatorname{sqri}(B)$.
On the other hand, let $0_X \in \operatorname{sqri}(A)$ and $0_Z \in \operatorname{sqri}(B)$, then $\operatorname{cone}(A)$ and $\operatorname{cone}(B)$ are closed linear subspaces and so, $0_X \in A$ and $0_Z \in B$. From here follows that $\operatorname{cone}(A \times B) = \operatorname{cone}(A) \times \operatorname{cone}(B)$ and thus, $\operatorname{cone}(A \times B)$ is a closed linear subspace, i.e. $0_{X \times Z} \in \operatorname{sqri}(A \times B)$. □

In this paper we do not use the classical differentiability, but we use the notion of subdifferentiability to formulate optimality conditions. If we take an arbitrary $x \in X$ such that $f(x) \in \mathbb{R}$, then we call the set

$$\partial f(x) := \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x \rangle \ \forall y \in X\}$$

the (convex) subdifferential of f at x, where the elements are called the subgradients of f at x. Moreover, if $\partial f(x) \neq \emptyset$, then we say that f is subdifferentiable at x and if $f(x) \notin \mathbb{R}$, then we make the convention that $\partial f(x) := \emptyset$.

3 Lagrange duality for multi-composed optimization problems

As mentioned in the introduction our aim is to formulate a conjugate dual problem to an optimization problem with geometric and cone constraints having as objective function the composition of $n+1$ functions. In other words, we consider the following problem

$$(P^C) \qquad \inf_{x\in\mathcal{A}} (f \circ F^1 \circ ... \circ F^n)(x),$$
$$\mathcal{A} = \{x \in S : g(x) \in -Q\},$$

where X_i is partially ordered by the non-empty convex cone $K_i \subseteq X_i$ for $i = 0, ..., n-1$. Moreover,

- $S \subseteq X_n$ is a non-empty set,
- $f : X_0 \to \overline{\mathbb{R}}$ is proper and K_0-increasing on $F^1(\operatorname{dom} F^1) + K_0 \subseteq \operatorname{dom} f$,
- $F^i : X_i \to \overline{X}_{i-1} = X_{i-1} \cup \{+\infty_{K_{i-1}}\}$ is proper and K_i-K_{i-1}-increasing on $F^{i+1}(\operatorname{dom} F^{i+1}) + K_i \subseteq \operatorname{dom} F^i$ for $i = 1, ..., n-2$,
- $F^{n-1} : X_{n-1} \to \overline{X}_{n-2} = X_{n-2} \cup \{+\infty_{K_{n-2}}\}$ is proper and K_{n-1}-K_{n-2}-increasing on $F^n(\operatorname{dom} F^n \cap \mathcal{A}) + K_{n-1} \subseteq \operatorname{dom} F^{n-1}$,
- $F^n : X_n \to \overline{X}_{n-1} = X_{n-1} \cup \{+\infty_{K_{n-1}}\}$ is a proper function and
- $g : X_n \to \overline{Z}$ is a proper function fulfilling $S \cap g^{-1}(-Q) \cap ((F^n)^{-1} \circ ... \circ (F^1)^{-1})(\operatorname{dom} f) \neq \emptyset$.

Additionally, we make the convention that $f(+\infty_{K_0}) = +\infty$ and $F^i(+\infty_{K_i}) = +\infty_{K_{i-1}}$, i.e. $f : \overline{X}_0 \to \overline{\mathbb{R}}$ and $F^i : \overline{X}_i \to \overline{X}_{i-1}$, $i = 1, ..., n-1$.

Let us now consider the following problem

$$(\widetilde{P}^C) \qquad \inf_{(y^0,...,y^n)\in\widetilde{\mathcal{A}}} \widetilde{f}(y^0, ..., y^n),$$

where

$$\widetilde{\mathcal{A}} = \left\{(y^0, ..., y^{n-1}, y^n) \in X_0 \times ... \times X_{n-1} \times S : g(y^n) \in -Q,\ h^i(y^i, y^{i-1}) \in -K_{i-1},\ i = 1, ..., n\right\}.$$

The functions $\widetilde{f} : X_0 \times ... \times X_n \to \overline{\mathbb{R}}$ and $h^i : X_i \times X_{i-1} \to \overline{X}_{i-1}$ are defined as

$$\widetilde{f}(y^0, ..., y^n) = f(y^0) \text{ and } h^i(y^i, y^{i-1}) = F^i(y^i) - y^{i-1} \text{ for } i = 1, ..., n.$$

Lemma 3.1. *Let $(y^0, ..., y^n)$ be feasible to $(\widetilde{P}^C)$, then it holds $f((F^1 \circ ... \circ F^n)(y^n)) \leq f(y^0)$.*

Proof. Let $(y^0, ..., y^n)$ be feasible to $(\widetilde{P}^C)$, then we have $F^n(y^n) \leqq_{K_{n-1}} y^{n-1}$, ..., $F^1(y^1) \leqq_{K_0} y^0$. Moreover, since F^{n-1} is K_{n-1}-K_{n-2}-increasing on $F^n(\operatorname{dom} F^n \cap \mathcal{A}) + K_{n-1}$ and F^i is K_i-K_{i-1}-increasing on $F^{i+1}(\operatorname{dom} F^{i+1}) + K_i$ for $i = 1, ..., n-2$, it follows $(F^{n-1} \circ F^n)(y^n) \leqq_{K_{n-2}} F^{n-1}(y^{n-1}) \leqq_{K_{n-2}} y^{n-2}$ and so on $(F^1 \circ ... \circ F^n)(y^n) \leqq_{K_0} F^1(y^1) \leqq_{K_0} y^0$. Since f is K_0-increasing on $F^1(\operatorname{dom} F^1) + K_0$ we get the desired inequality $f((F^1 \circ ... \circ F^n)(y^n)) \leq f(y^0)$. □

Remark 3.1. *If F^n is an affine function, then it can be useful to set $K_{n-1} = \{0_{X_{n-1}}\}$, because in this case F^{n-1} does not need to be monotone to ensure the inequality of the previous lemma.*

If we denote by $v(P^C)$ and $v(\widetilde{P}^C)$ the optimal objective values of the problems (P^C) and $(\widetilde{P}^C)$, respectively, then the following relation between the optimal objective values is always true.

Theorem 3.1. *It holds* $v(P^C) = v(\widetilde{P}^C)$.

Proof. Let x be a feasible element to (P^C) and set $y^n = x$, $y^{n-1} = F^n(y^n)$, $y^{n-2} = F^{n-1}(y^{n-1})$, ..., $y^0 = F^1(y^1)$. If there exists an $i \in \{2, ..., n\}$ such that $F^i(y^i) \notin \operatorname{dom} F^{i-1}$ or $F^1(y^1) \notin \operatorname{dom} f$ or there exists an $i \in \{1, ..., n\}$ such that $F^i(y^i) = +\infty_{K_{i-1}}$, then it obviously holds $f((F^1 \circ ... \circ F^n)(y^n)) = +\infty \geq v(\widetilde{P}^C)$. Otherwise it holds $F^i(y^i) - y^{i-1} = 0 \in -K_{i-1}$ for $i = 1, ..., n$. Moreover, by the feasibility of y^n it holds $g(y^n) \in -Q$, which implies the feasibility of $(y^0, ..., y^n)$ to the problem $(\widetilde{P}^C)$ and $f((F^1 \circ ... \circ F^n)(y^n)) = f(y^0) = \widetilde{f}(y^0, ..., y^n) \geq v(\widetilde{P}^C)$. Hence it holds $f((F^1 \circ ... \circ F^n)(y^n)) \geq v(\widetilde{P}^C)$ for all y^n feasible to (P^C), which means that $v(P^C) \geq v(\widetilde{P}^C)$.
Let now $(y^0, ..., y^n)$ be feasible to $(\widetilde{P}^C)$. If $y^0 \notin \operatorname{dom} f$, then obviously we have $v(P^C) \leq f((F^1 \circ ... \circ F^n)(y^n)) \leq f(y^0) = \widetilde{f}(y^0, ..., y^n) = +\infty$. On the other hand, since $(y^0, ..., y^n)$ is feasible to $(\widetilde{P}^C)$ it holds $h^i(y^i, y^{i-1}) \in -K_{i-1}$ for $i = 1, ..., n$ (i.e. $F^i(y^i) - y^{i-1} \in -K_{i-1}$ for $i = 1, ..., n$) and $g(y^n) \in -Q$. By Lemma 3.1 we have $v(P^C) \leq f((F^1 \circ ... \circ F^n)(y^n)) \leq f(y^0) = \widetilde{f}(y^0, ..., y^n)$ and by taking the infimum over $(y^0, ..., y^n)$ on the right-hand side we get $v(P^C) \leq v(\widetilde{P}^C)$.
Summarizing, we get the desired result $v(P^C) = v(\widetilde{P}^C)$. □

Remark 3.2. *The assumption that f is K_0-increasing on $F^1(\operatorname{dom} F^1) + K_0 \subseteq \operatorname{dom} f$ was made to allow functions which are not necessarily monotone on their whole effective domain. But in some situations the inclusion $F^1(\operatorname{dom} F^1) + K_0 \subseteq \operatorname{dom} f$ may not be fulfilled. As an example consider the convex optimization problem (P^G) in Section 6.*
To overcome this circumstances one can alternatively assume that f is K_0-increasing on $\operatorname{dom} f$ and $F^1(\operatorname{dom} F^1) \subseteq \operatorname{dom} f$. For the functions $F^1, ..., F^{n-1}$ one can formulate in the same way alternative assumptions. To be more precise, we can alternatively ask that F^i is K_i-K_{i-1}-increasing on $\operatorname{dom} F^i$ and $F^{i+1}(\operatorname{dom} F^{i+1}) \subseteq \operatorname{dom} F^i$, $i = 1, ..., n-2$, and F^{n-1} is K_{n-1}-K_{n-2}-increasing on $\operatorname{dom} F^{n-1}$ and $F^n(\operatorname{dom} F^n \cap \mathcal{A}) \subseteq \operatorname{dom} F^{n-1}$. One can observe that under this alternative assumptions Lemma 3.1 and especially Theorem 3.1 still hold.

As we have seen by Theorem 3.1, the problem (P^C) can be associated to the problem $(\widetilde{P}^C)$. In the next step we want to determine the corresponding conjugate dual problems to the problems (P^C) and $(\widetilde{P}^C)$.
As we take a careful look at the optimization problem $(\widetilde{P}^C)$, we can see that this problem can be rewritten in the form

$$(\widetilde{P}^C) \quad \inf_{\substack{\widetilde{y} \in \widetilde{S}, \\ \widetilde{h}(\widetilde{y}) \in -\widetilde{K}}} \widetilde{f}(\widetilde{y}), \tag{1}$$

where $\widetilde{y} := (y^0, ..., y^n) \in \widetilde{X} := X_0 \times ... \times X_n$, $\widetilde{Z} := X_0 \times ... \times X_{n-1} \times Z$ ordered by $\widetilde{K} := K_0 \times ... \times K_{n-1} \times Q$, $\widetilde{S} := X_0 \times ... \times X_{n-1} \times S$ and $\widetilde{h} : \widetilde{X} \to \overline{\widetilde{Z}} = \widetilde{Z} \cup \{+\infty_{\widetilde{K}}\}$ is defined as

$$\widetilde{h}(\widetilde{y}) := \begin{cases} (h^1(y^1, y^0), ..., h^n(y^n, y^{n-1}), g(y^n)), & \text{if } (y^i, y^{i-1}) \in \operatorname{dom} h^i, \ i = 1, ..., n, \ y^n \in \operatorname{dom} g, \\ +\infty_{\widetilde{K}}, & \text{otherwise.} \end{cases}$$

Note that by the definition of h^i we have

$$\operatorname{dom} h^i = \operatorname{dom} F^i \times X_{i-1}, \ i = 1, ..., n,$$

which yields

$$\mathrm{dom}\,\widetilde{h} \quad = \quad X_0 \times \mathrm{dom}\,F^1 \times ... \times (\mathrm{dom}\,F^n \cap \mathrm{dom}\,g). \tag{2}$$

At this point, let us additionally remark that the assumption from the beginning, $S \cap g^{-1}(-Q) \cap ((F^n)^{-1} \circ ... \circ (F^1)^{-1})(\mathrm{dom}\,f) \neq \emptyset$, implies also that $\mathrm{dom}\,\widetilde{f} \cap \widetilde{S} \cap \widetilde{h}^{-1}(-\widetilde{K}) \neq \emptyset$, but the inverse is not true. This means

$$\begin{aligned}
& S \cap g^{-1}(-Q) \cap ((F^n)^{-1} \circ ... \circ (F^1)^{-1})(\mathrm{dom}\,f) \neq \emptyset \\
\Leftrightarrow\ & \exists (y^0, y^1, ..., y^{n-1}, y^n) \in \mathrm{dom}\,f \times X_1 \times ... \times X_{n-1} \times S \text{ such that} \\
& F^1(y^1) - y^0 = 0 \in -K_0, ..., F^n(y^n) - y^{n-1} = 0 \in -K_{n-1} \text{ and } g(y^n) \in -Q \\
\Rightarrow\ & \exists \widetilde{y} \in \widetilde{S} \cap \mathrm{dom}\,\widetilde{f} \text{ such that } \widetilde{h}(\widetilde{y}) \in -\widetilde{K} \\
\Leftrightarrow\ & \mathrm{dom}\,\widetilde{f} \cap \widetilde{S} \cap \widetilde{h}^{-1}(-\widetilde{K}) \neq \emptyset.
\end{aligned}$$

The corresponding Lagrange dual problem $(\widetilde{D}^C)$ with $\widetilde{z}^* := (z^{0*}, ..., z^{(n-1)*}, z^{n*}) \in \widetilde{K}^* := K_0^* \times ... \times K_{n-1}^* \times Q^*$ as the dual variable to the problem $(\widetilde{P}^C)$ is

$$(\widetilde{D}^C) \quad \sup_{\widetilde{z}^* \in \widetilde{K}^*} \inf_{\widetilde{y} \in \widetilde{S}} \{\widetilde{f}(\widetilde{y}) + \langle \widetilde{z}^*, \widetilde{h}(\widetilde{y})\rangle\},$$

which can equivalently be written as

$$(\widetilde{D}^C) \quad \sup_{\substack{z^{n*} \in Q^*,\ z^{i*} \in K_i^* \\ i=0,...,n-1}} \inf_{\substack{y^n \in S,\ y^i \in X_i \\ i=0,...,n-1}} \left\{ \widetilde{f}(y^0, ..., y^n) + \sum_{i=1}^{n} \langle z^{(i-1)*}, h^i(y^i, y^{i-1})\rangle + \langle z^{n*}, g(y^n)\rangle \right\}.$$

Through the definitions we made above for $\widetilde{f}$ and h^i and since we set $x = y^n$, we can deduce the conjugate dual problem (D^C) to problem (P^C)

$$\begin{aligned}
(D^C) \quad & \sup_{\substack{z^{n*} \in Q^*,\ z^{i*} \in K_i^* \\ i=0,...,n-1}} \inf_{\substack{x \in S,\ y^i \in X_i \\ i=0,...,n-1}} \Bigg\{ f(y^0) + \langle z^{(n-1)*}, F^n(x) - y^{n-1}\rangle + \langle z^{n*}, g(x)\rangle + \\
& \qquad \sum_{i=1}^{n-1} \langle z^{(i-1)*}, F^i(y^i) - y^{i-1}\rangle \Bigg\} \\
= & \sup_{\substack{z^{n*} \in Q^*,\ z^{i*} \in K_i^*, \\ i=0,...,n-1}} \Bigg\{ \inf_{x \in S} \{\langle z^{(n-1)*}, F^n(x)\rangle + \langle z^{n*}, g(x)\rangle\} - \sup_{y^0 \in X_0} \{\langle z^{0*}, y^0\rangle - f(y^0)\} - \\
& \qquad \sum_{i=1}^{n-1} \sup_{\substack{y^i \in X_i, \\ i=1,...,n-1}} \{\langle z^{i*}, y^i\rangle - \langle z^{(i-1)*}, F^i(y^i)\rangle\} \Bigg\}.
\end{aligned}$$

Hence, the conjugate dual problem (D^C) to problem (P^C) has the following form

$$(D^C) \quad \sup_{\substack{z^{n*} \in Q^*,\ z^{i*} \in K_i^*, \\ i=0,...,n-1}} \left\{ \inf_{x \in S} \{\langle z^{(n-1)*}, F^n(x)\rangle + \langle z^{n*}, g(x)\rangle\} - f^*(z^{0*}) - \sum_{i=1}^{n-1} (z^{(i-1)*} F^i)^*(z^{i*}) \right\}.$$

The optimal objective values of the problems $(\widetilde{D}^C)$ and (D^C) are of course equal, i.e. $v(\widetilde{D}^C) = v(D^C)$.
The next result arises from the definition of the dual problem and is always fulfilled.

Theorem 3.2. *(weak duality) Between the primal problem (P^C) and its conjugate dual problem weak duality always holds, i.e. $v(P^C) \geq v(D^C)$.*

Proof. By Theorem 3.1.1 in [1] it holds $v(\widetilde{P}^C) \geq v(\widetilde{D}^C)$. Moreover, by Theorem 3.1 and since $v(\widetilde{D}^C) = v(D^C)$ we have $v(P^C) = v(\widetilde{P}^C) \geq v(\widetilde{D}^C) = v(D^C)$. □

Remark 3.3. *Let Z_i be a locally convex Hausdorff space partially ordered by the non-empty convex cone Q_i, $i = 0, ..., n-1$. Then the introduced concept in this paper covers also optimization problems of the form*

$$(P^{CC}) \qquad \inf_{x \in \mathcal{L}} \varphi(x),$$

with

$$\mathcal{L} := \{x \in S : (G^1 \circ ... \circ G^n)(x) \in -Q_0\},$$

where $\varphi : X_n \to \overline{\mathbb{R}}$ is proper, $G^i : \overline{Z}_i \to \overline{Z}_{i-1}$ is proper and Q_i-Q_{i-1}-increasing on $G^{i+1}(\operatorname{dom} G^{i+1}) + Q_i \subseteq \operatorname{dom} G^i$, $i = 1, ..., n-1$, and $G^n : X_n \to \overline{Z}_{n-1}$ is proper. The problem (P^{CC}) can equivalently be rewritten as

$$(P^{CC}) \qquad \inf_{x \in X_n} \{\varphi(x) + \delta_S(x) + (\delta_{-Q_0} \circ G^1 \circ ... \circ G^n)(x)\}$$

and by setting $X_0 := \mathbb{R} \times Z_0$, $K_0 := \mathbb{R}_+ \times Q_0$, $X_i := \mathbb{R} \times Z_i$, $K_i := \mathbb{R}_+ \times Q_i$, $i = 1, ..., n-1$ and by defining the following functions

- $f : \overline{X}_0 \times \overline{\mathbb{R}}$, $f(y^0) := y_1^0 + \delta_{-Q_0}(y_2^0)$ *with* $y^0 = (y_1^0, y_2^0) \in X_0$,
- $F^i : \overline{X}_i \to \overline{X}_{i-1}, F^i(y_1^i, y_2^i) := (y_1^i, G^i(y_2^i))$, $i = 1, ..., n-1$ *with* $y^i = (y_1^i, y_2^i) \in X_i$,
- $F^n : X_n \to \overline{X}_{n-1}$, $F^n(x) := (\varphi(x) + \delta_S(x), G^n(x))$,

the problem (P^{CC}) turns into a special case of the problem (P^C)

$$(P^{CC}) \qquad \inf_{x \in X_n} (f \circ F^1 \circ ... \circ F^n)(x)$$

with $\mathcal{A} \equiv X_n$.

4 Regularity conditions and strong duality

In this section we want to characterize strong duality through the so-called generalized interior point regularity conditions. Besides we will provide some optimality conditions for the primal problem and its corresponding conjugate dual problem. For this purpose we additionally assume for the rest of the paper that $S \subseteq X_n$ is a convex set, f is a convex function, F^i is a K_{i-1}-convex function for $i = 1, ..., n$ and g is a Q-convex function. Hence, as can be easily seen, $(f \circ F^1 \circ ... \circ F^n)$ is a convex function and (P^C) is a convex optimization problem. Moreover, the problem $(\widetilde{P}^C)$ is also convex.

Remark 4.1. *Let us point out that for the convexity of* $(f \circ F^1 \circ ... \circ F^n)$ *we ask that the function* f *be convex and* K_0*-increasing on* $F^1(\operatorname{dom} F^1) + K_0$ *and the function* F^i *be* K_{i-1}*-convex and fulfills also the property of monotonicity for* $i = 1, ..., n-1$*, while the function* F^n *need just be* K_{n-1}*-convex (see Theorem 2.1). This means that if* F^n *is an affine function, we do not need the monotonicity of* F^{n-1}*, since the composition of an affine function and a function, which fulfills the property of convexity, fulfills also the property of convexity. In this context let us pay also attention to Remark 3.1, i.e. one can choose* $K_{n-1} = \{0_{X_{n-1}}\}$.

To derive regularity conditions which secure strong duality for the pair (P^C)-(D^C), we first consider regularity conditions for strong duality between the problems $(\widetilde{P}^C)$ and $(\widetilde{D}^C)$, which were presented in [1]. The first one is the well-known Slater constraint qualification

$$(\widetilde{RC}_1^C) \quad \Big| \quad \exists \widetilde{y}' \in \operatorname{dom} \widetilde{f} \cap \widetilde{S} \text{ such that } \widetilde{h}(\widetilde{y}') \in -\operatorname{int} \widetilde{K}.$$

Using the definitions of $\widetilde{f}$ and $\widetilde{h}$ as well as $\widetilde{S}$ and $\widetilde{K}$ we get

$$\begin{aligned} \operatorname{dom} \widetilde{f} \cap \widetilde{S} &= (\operatorname{dom} f \times X_1 \times ... \times X_n) \cap (X_0 \times X_1 \times ... \times X_{n-1} \times S) \\ &= \operatorname{dom} f \times X_1 \times ... \times X_{n-1} \times S \end{aligned} \tag{3}$$

and

$$\operatorname{int} \widetilde{K} = \operatorname{int}(K_0 \times ... \times K_{n-1} \times Q) = \operatorname{int} K_0 \times ... \times \operatorname{int} K_{n-1} \times \operatorname{int} Q.$$

Therefore the condition $(\widetilde{RC}_1^C)$ can in the context of the primal-dual pair (P^C)-(D^C) be rewritten as follows

$$(RC_1^C) \quad \Bigg| \quad \begin{array}{l} \exists (y^{0'}, y^{1'}, ..., y^{(n-1)'}, y^{n'}) \in \operatorname{dom} f \times X_1 \times ... \times X_{n-1} \times S \text{ such that} \\ F^i(y^{i'}) - y^{(i-1)'} \in -\operatorname{int} K_{i-1},\ i = 1, ..., n, \text{ and } g(y^{n'}) \in -\operatorname{int} Q. \end{array}$$

The condition (RC_1^C) can also equivalently be formulated as

$$(RC_{1'}^C) \quad \Bigg| \quad \begin{array}{l} \exists x' \in S \text{ such that } g(x') \in -\operatorname{int}Q \text{ and } F^n(x') \in (F^{n-1})^{-1}((F^{n-2})^{-1}(... \\ (F^1)^{-1}(\operatorname{dom} f - \operatorname{int} K_0) - \operatorname{int} K_1 ...) - \operatorname{int} K_{n-2}) - \operatorname{int} K_{n-1}. \end{array}$$

This can be seen as follows: The assumption that there exists $x' \in S$ such that

$$F^n(x') \in (F^{n-1})^{-1}((F^{n-2})^{-1}(...(F^1)^{-1}(\operatorname{dom} f - \operatorname{int} K_0) - \operatorname{int} K_1 ...) - \operatorname{int} K_{n-2}) - \operatorname{int} K_{n-1}$$

implies that there exists $(y^{0'}, ..., y^{(n-1)'}) \in X_0 \times ... \times X_{n-1}$ such that

$$\begin{array}{l} y^{(n-1)'} \in (F^{n-1})^{-1}((F^{n-2})^{-1}(...(F^1)^{-1}(\operatorname{dom} f - \operatorname{int} K_0) - \operatorname{int} K_1 ...) - \operatorname{int} K_{n-2}) \\ y^{(n-2)'} \in (F^{n-2})^{-1}((F^{n-3})^{-1}(...(F^1)^{-1}(\operatorname{dom} f - \operatorname{int} K_0) - \operatorname{int} K_1 ...) - \operatorname{int} K_{n-3}) \\ \vdots \\ y^{1'} \in (F^1)^{-1}(\operatorname{dom} f - \operatorname{int} K_0) \\ y^{0'} \in \operatorname{dom} f. \end{array}$$

Therefore, by setting $x' = y^{n'}$ the elements $(y^{0'}, ..., y^{n'}) \in \operatorname{dom} f \times X_1 \times ... \times X_{n-1} \times S$ fulfill $F^n(y^{n'}) - y^{(n-1)'} \in -\operatorname{int} K_{n-1}$, ..., $F^1(y^{1'}) - y^{0'} \in -\operatorname{int} K_0$ and from here we can now affirm that the condition (RC_1^C) is fulfilled.

On the other hand, if there exists $(y^{0'},...,y^{n'}) \in \operatorname{dom} f \times X_1 \times ... \times X_{n-1} \times S$ such that $g(y^{n'}) \in -\operatorname{int} Q$ and $F^i(y^{i'}) - y^{(i-1)'} \in -\operatorname{int} K_{i-1}$ for $i = 1,...,n$, then we set $y^{n'} = x'$ and get

$$\begin{aligned} & F^n(x') - y^{(n-1)'} \in -\operatorname{int} K_{n-1} \\ \Rightarrow\ & F^n(x') \in y^{(n-1)'} - \operatorname{int} K_{n-1}. \end{aligned} \tag{4}$$

Further, we have

$$\begin{aligned} & F^{n-1}(y^{(n-1)'}) - y^{(n-2)'} \in -\operatorname{int} K_{n-2} \\ \Rightarrow\ & F^{n-1}(y^{(n-1)'}) \in y^{(n-2)'} - \operatorname{int} K_{n-2} \\ \Rightarrow\ & y^{(n-1)'} \in (F^{n-1})^{-1}(y^{(n-2)'} - \operatorname{int} K_{n-2}). \end{aligned} \tag{5}$$

From (4) and (5) follows

$$F^n(x') \in (F^{n-1})^{-1}(y^{(n-2)'} - \operatorname{int} K_{n-2}) - \operatorname{int} K_{n-1}. \tag{6}$$

Since

$$\begin{aligned} & F^{n-2}(y^{(n-2)'}) - y^{(n-3)'} \in -\operatorname{int} K_{n-3} \\ \Rightarrow\ & F^{n-2}(y^{(n-2)'}) \in y^{(n-3)'} - \operatorname{int} K_{n-3} \\ \Rightarrow\ & y^{(n-2)'} \in (F^{n-2})^{-1}(y^{(n-3)'} - \operatorname{int} K_{n-3}) \end{aligned}$$

we get for (6)

$$F^n(x') \in (F^{n-1})^{-1}((F^{n-2})^{-1}(y^{(n-3)'} - \operatorname{int} K_{n-3}) - \operatorname{int} K_{n-2}) - \operatorname{int} K_{n-1}.$$

If we continue in this manner until $y^{0'} \in \operatorname{dom} f$ we get finally

$$F^n(x') \in (F^{n-1})^{-1}((F^{n-2})^{-1}(...(F^1)^{-1}(\operatorname{dom} f - \operatorname{int} K_0) - \operatorname{int} K_1 ...) - \operatorname{int} K_{n-2}) - \operatorname{int} K_{n-1}.$$

This means that $(RC^C_{1'})$ is fulfilled if it is supposed in its first formulation, i.e. (RC^C_1). Additionally, we consider a class of regularity conditions which assume that the underlying spaces are Fréchet spaces:

$(\widetilde{RC}^C_2)$ | $\widetilde{X}$ and $\widetilde{Z}$ are Fréchet spaces, $\widetilde{S}$ is closed, $\widetilde{f}$ is lower semicontinuous, $\widetilde{h}$ is $\widetilde{K}$-epi closed and $0_{\widetilde{Z}} \in \operatorname{sqri}(\widetilde{h}(\operatorname{dom} \widetilde{f} \cap \widetilde{S} \cap \operatorname{dom} \widetilde{h}) + \widetilde{K})$.

If we exchange sqri for core or int we get stronger versions of this regularity condition:

$(\widetilde{RC}^C_{2'})$ | $\widetilde{X}$ and $\widetilde{Z}$ are Fréchet spaces, $\widetilde{S}$ is closed, $\widetilde{f}$ is lower semicontinuous, $\widetilde{h}$ is $\widetilde{K}$-epi closed and $0_{\widetilde{Z}} \in \operatorname{core}(\widetilde{h}(\operatorname{dom} \widetilde{f} \cap \widetilde{S} \cap \operatorname{dom} \widetilde{h}) + \widetilde{K})$,

$(\widetilde{RC}^C_{2''})$ | $\widetilde{X}$ and $\widetilde{Z}$ are Fréchet spaces, $\widetilde{S}$ is closed, $\widetilde{f}$ is lower semicontinuous, $\widetilde{h}$ is $\widetilde{K}$-epi closed and $0_{\widetilde{Z}} \in \operatorname{int}(\widetilde{h}(\operatorname{dom} \widetilde{f} \cap \widetilde{S} \cap \operatorname{dom} \widetilde{h}) + \widetilde{K})$,

where the last two conditions are equivalent (see [1]). If we work in finite dimensional spaces the regularity condition can be written in the following way (see [1])

$$(\widetilde{RC}_3^C) \quad \left| \begin{array}{l} \dim(\mathrm{lin}(\widetilde{h}(\mathrm{dom}\,\widetilde{f} \cap \widetilde{S} \cap \mathrm{dom}\,\widetilde{h}) + \widetilde{K})) < +\infty \text{ and} \\ 0_{\widetilde{Z}} \in \mathrm{ri}(\widetilde{h}(\mathrm{dom}\,\widetilde{f} \cap \widetilde{S} \cap \mathrm{dom}\,\widetilde{h}) + \widetilde{K}). \end{array} \right.$$

To derive corresponding regularity conditions for the primal-dual pair (P^C)-(D^C) we first consider the formulas (2) and (3), which imply that

$$\begin{aligned} \widetilde{h}(\mathrm{dom}\,\widetilde{f} \cap \widetilde{S} \cap \mathrm{dom}\,\widetilde{h}) &= \widetilde{h}(\mathrm{dom}\,f \times \mathrm{dom}\,F^1 \times ... \times \mathrm{dom}\,F^{n-1} \times (\mathrm{dom}\,F^n \cap \mathrm{dom}\,g \cap S)) \\ &= h^1(\mathrm{dom}\,F^1 \times \mathrm{dom}\,f) \times h^2(\mathrm{dom}\,F^2 \times \mathrm{dom}\,F^1) \times ... \times \\ &\quad\; h^{n-1}(\mathrm{dom}\,F^{n-1} \times \mathrm{dom}\,F^{n-2}) \times \\ &\quad\; h^n((\mathrm{dom}\,F^n \cap \mathrm{dom}\,g \cap S) \times \mathrm{dom}\,F^{n-1}) \times g(\mathrm{dom}\,F^n \cap \mathrm{dom}\,g \cap S) \\ &= (F^1(\mathrm{dom}\,F^1) - \mathrm{dom}\,f) \times (F^2(\mathrm{dom}\,F^2) - \mathrm{dom}\,F^1) \times ... \times \\ &\quad\; (F^{n-1}(\mathrm{dom}\,F^{n-1}) - \mathrm{dom}\,F^{n-2}) \times \\ &\quad\; (F^n(\mathrm{dom}\,F^n \cap \mathrm{dom}\,g \cap S) - \mathrm{dom}\,F^{n-1}) \times g(\mathrm{dom}\,F^n \cap \mathrm{dom}\,g \cap S) \end{aligned}$$

and from here we get by Lemma 2.2 that

$$\begin{aligned} & 0_{\widetilde{Z}} \in \mathrm{sqri}((F^1(\mathrm{dom}\,F^1) - \mathrm{dom}\,f + K_0) \times ... \times (F^{n-1}(\mathrm{dom}\,F^{n-1}) - \mathrm{dom}\,F^{n-2} + K_{n-2}) \\ & \times (F^n(\mathrm{dom}\,F^n \cap \mathrm{dom}\,g \cap S) - \mathrm{dom}\,F^{n-1} + K_{n-1}) \times (g(\mathrm{dom}\,F^n \cap \mathrm{dom}\,g \cap S) + Q)) \\ \Leftrightarrow\; & 0_{X_0} \in \mathrm{sqri}(F^1(\mathrm{dom}\,F^1) - \mathrm{dom}\,f + K_0),\; 0_{X_i} \in \mathrm{sqri}(F^i(\mathrm{dom}\,F^i) - \mathrm{dom}\,F^{i-1} + K_{i-1}), \\ & i = 2, ..., n-1,\; 0_{X_n} \in \mathrm{sqri}(F^n(\mathrm{dom}\,F^n \cap \mathrm{dom}\,g \cap S) - \mathrm{dom}\,F^{n-1} + K_{n-1}) \text{ and} \\ & 0_Z \in \mathrm{sqri}(g(\mathrm{dom}\,F^n \cap \mathrm{dom}\,g \cap S) + Q). \end{aligned}$$

Now, let $\varrho : X_0 \times ... \times X_n \times X_0 \times ... \times X_{n-1} \times Z \to X_0^2 \times ... \times X_{n-1}^2 \times X_n \times Z$ be defined by $\varrho(y^0, ..., y^n, v^0, ..., v^n) := (y^0, v^0, ..., y^n, v^n)$. Further, let us define the functions $\varrho^n_{X_i} : X_i \times X_{i-1} \times X_{i-1} \to X_{i-1} \times X_{i-1} \times X_i$ by $\varrho^n_{X_i}(y^i, y^{i-1}, v^{i-1}) := (y^{i-1}, v^{i-1}, y^i)$, $i = 1, ..., n$. Obviously, the defined functions are homeomorphisms and map open sets to open sets and closed sets to closed sets. More precisely, this means that $\varrho(\mathrm{epi}_{\widetilde{K}}\,\widetilde{h})$ is closed if and only if $\mathrm{epi}_{\widetilde{K}}\,\widetilde{h}$ is a closed set and $\varrho^n_{X_i}(\mathrm{epi}_{K_{i-1}}\,h^i)$ is closed if and only if $\mathrm{epi}_{K_{i-1}}\,h^i$ is a closed set, $i = 1, ..., n$. Furthermore, we have

$$\begin{aligned} \mathrm{epi}_{\widetilde{K}}\widetilde{h} &= \{(y^0, ..., y^n, v^0, ..., v^n) \in X_0 \times ... \times X_n \times X_0 \times ... \times X_{n-1} \times Z : \\ &\quad\; (y^1, y^0, v^0) \in \mathrm{epi}_{K_0} h^1, \\ &\quad\; \vdots \\ &\quad\; (y^n, y^{n-1}, v^{n-1}) \in \mathrm{epi}_{K_{n-1}} h^n, \\ &\quad\; (y^n, v^n) \in \mathrm{epi}_Q g\}, \\ &= \{(y^0, ..., y^n, v^0, ..., v^n) \in X_0 \times ... \times X_n \times X_0 \times ... \times X_{n-1} \times Z : \\ &\quad\; (y^0, v^0, y^1) \in \varrho^n_{X_1}(\mathrm{epi}_{K_0} h^1), \\ &\quad\; \vdots \\ &\quad\; (y^{n-1}, v^{n-1}, y^n) \in \varrho^n_{X_n}(\mathrm{epi}_{K_{n-1}} h^n), \\ &\quad\; (y^n, v^n) \in \mathrm{epi}_Q g\}, \end{aligned}$$

$$
\begin{aligned}
= & \{(y^0,...,y^n,v^0,...,v^n) \in X_0 \times ... \times X_n \times X_0 \times ... \times X_{n-1} \times Z : \\
& (y^0,v^0,y^1,v^1,y^2,v^2,...,y^{n-1},v^{n-1},y^n,v^n) \in \\
& \quad \varrho^n_{X_1}(\mathrm{epi}_{K_0} h^1) \times X_1 \times X_2^2 \times ... \times X_{n-1}^2 \times X_n \times Z, \\
& \vdots \\
& (y^0,v^0,y^1,v^1,...,y^{n-2},v^{n-2},y^{n-1},v^{n-1},y^n,v^n) \in \\
& \quad X_0^2 \times X_1^2 \times ... \times X_{n-2}^2 \times (\varrho^n_{X_n}(\mathrm{epi}_{K_{n-1}} h^n)) \times Z, \\
& (y^0,v^0,...,y^{n-1},v^{n-1},y^n,v^n) \in X_0^2 \times ... \times X_{n-1}^2 \times \mathrm{epi}_Q g\} \\
= & \Big\{(y^0,...,y^n,v^0,...,v^n) \in X_0 \times ... \times X_n \times X_0 \times ... \times X_{n-1} \times Z : \\
& (y^0,v^0,...,y^{i-2},v^{i-2},y^{i-1},v^{i-1},y^i,v^i,y^{i+1},v^{i+1},...,y^n,v^n) \in \\
& \quad X_0^2 \times ... \times X_{i-2}^2 \times \left(\varrho^n_{X_i}(\mathrm{epi}_{K_{i-1}} h^i)\right) \times X_i \times X_{i+1}^2 \times ... \times X_n \times Z,\ i = 1,...n, \\
& (y^0,v^0,...,y^{n-1},v^{n-1},y^n,v^n) \in X_0^2 \times ... \times X_{n-1}^2 \times \mathrm{epi}_Q g\Big\},
\end{aligned}
$$

so we can write

$$
\begin{aligned}
\varrho(\mathrm{epi}_{\widetilde{K}} \widetilde{h}) = & \left(\bigcap_{i=1}^n \left(X_0^2 \times ... \times X_{i-2}^2 \times \left(\varrho^n_{X_i}(\mathrm{epi}_{K_{i-1}} h^i)\right) \times X_i \times X_{i+1}^2 \times ... \times X_n \times Z\right)\right) \\
& \bigcap \left(X_0^2 \times ... \times X_{n-1}^2 \times \mathrm{epi}_Q g\right)
\end{aligned}
$$

and get as a consequence that $\mathrm{epi}_{\widetilde{K}} \widetilde{h}$ is closed if $\mathrm{epi}_{K_{i-1}} h^i$, $i = 1,...,n$, and $\mathrm{epi}_Q\, g$ are closed sets. Besides, we know by Lemma 2.1 that for a non-empty closed convex cone K_{i-1} with $\mathrm{int}\, K_{i-1} \neq \emptyset$ it holds that $\mathrm{epi}_{K_{i-1}} h^i$ is closed if and only if $\mathrm{epi}_{K_{i-1}} F^i$ is closed, $i = 1,...,n$. Bringing now the last facts together implies that for a non-empty closed convex cone K_{i-1} with $\mathrm{int}\, K_{i-1} \neq \emptyset$ it holds that $\mathrm{epi}_{\widetilde{K}} \widetilde{h}$ is closed if $\mathrm{epi}_Q\, g$ and $\mathrm{epi}_{K_{i-1}} F^i$ are closed sets, $i = 1,...,n$.
Moreover, since $\widetilde{S}$ is closed if and only if S is closed and $\widetilde{f}$ is lower semicontinuous if and only if f is lower semicontinuous (follows from the fact that $\mathrm{epi}\, f$ is closed $\Leftrightarrow$ $\mathrm{epi}\, \widetilde{f}$ is closed), we get the following regularity condition for the primal-dual pair (P^C)-(D^C) (call to mind that if X_i is a Fréchet space, $i = 0,...,n$, then $\widetilde{X} = X_0 \times ... \times X_n$ is a Fréchet space, too)

(RC_2^C) | $X_0,...,X_n$ and Z are Fréchet spaces, f is l.s.c., S is closed, g is Q-epi closed, K_{i-1} is closed, $\mathrm{int}\, K_{i-1} \neq \emptyset$, F^i is K_{i-1}-epi closed, $i = 1,...,n$,
$0_{X_0} \in \mathrm{sqri}(F^1(\mathrm{dom}\, F^1) - \mathrm{dom}\, f + K_0)$,
$0_{X_{i-1}} \in \mathrm{sqri}(F^i(\mathrm{dom}\, F^i) - \mathrm{dom}\, F^{i-1} + K_{i-1}), i = 2,...,n-1$,
$0_{X_{n-1}} \in \mathrm{sqri}(F^n(\mathrm{dom}\, F^n \cap \mathrm{dom}\, g \cap S) - \mathrm{dom}\, F^{n-1} + K_{n-1})$ and
$0_Z \in \mathrm{sqri}(g(\mathrm{dom}\, F^n \cap \mathrm{dom}\, g \cap S) + Q)$.

In the same way we get equivalent formulations of the regularity conditions $(RC_{2'}^C)$ and $(RC_{2''}^C)$ using core and int, respectively, instead sqri. The same holds also for the condition (RC_3^C).

As we have seen the condition (RC_i^C) implies the condition $(\widetilde{RC}_i^C)$, $i \in \{2, 2', 2'', 3\}$, while the condition (RC_1^C) implies the condition $(\widetilde{RC}_1^C)$ and vice versa. Moreover, since on the one hand Theorem 3.1 is always fulfilled and on the other hand the optimal objective values between $(\widetilde{D}^C)$ and (D^C) are equal, it holds the following theorem (see Theorem 3.2.9 and 3.2.10 in [1]).

Theorem 4.1. *(strong duality) If one of the conditions* (RC_i^C), $i \in \{1, 1', 2, 2', 2'', 3\}$, *is fulfilled, then between* (P^C) *and* (D^C) *strong duality holds, i.e.* $v(P^C) = v(D^C)$ *and the conjugate dual problem has an optimal solution.*

Remark 4.2. *If for some* $i \in \{1, ..., n\}$ *the function* F^i *is star* K_{i-1}*-lower semicontinuous, then we can omit asking that* K_{i-1} *is closed,* $\operatorname{int} K_{i-1} \neq \emptyset$ *and* F^i *is* K_{i-1}*-epi closed in the regularity conditions* (RC_i^C), $i \in \{2, 2', 2''\}$, *because the star* K_{i-1}*-lower semicontinuity of* F^i *implies the star* K_{i-1}*-lower semicontinuity of* h^i*, which then again implies the* K_{i-1}*-epi closedness of* h^i.

We come now to the point where we can give necessary and sufficient optimality conditions for the primal-dual pair $v(P^C)$-$v(D^{C_L})$.

Theorem 4.2. *(optimality conditions) (a) Suppose that one of the regularity conditions* (RC_i^C), $i \in \{1, 1', 2, 2', 2'', 3\}$, *is fulfilled and let* $\overline{x} \in \mathcal{A}$ *be an optimal solution of the problem* (P^C)*. Then there exists* $(\overline{z}^{0*}, ..., \overline{z}^{(n-1)*}, \overline{z}^{n*}) \in K_0^* \times ... \times K_{n-1}^* \times Q^*$, *an optimal solution to* (D^C), *such that*

(i) $f((F^1 \circ ... \circ F^n)(\overline{x})) + f^*(\overline{z}^{0*}) = \langle \overline{z}^{0*}, (F^1 \circ ... \circ F^n)(\overline{x}) \rangle$,

(ii) $(\overline{z}^{(i-1)*}F^i)((F^{i+1} \circ ... \circ F^n)(\overline{x})) + (\overline{z}^{(i-1)*}F^i)^*(\overline{z}^{i*}) = \langle \overline{z}^{i*}, (F^{i+1} \circ ... \circ F^n)(\overline{x}) \rangle$, $i = 1, ..., n-1$,

(iii) $(\overline{z}^{(n-1)*}F^n)(\overline{x}) + (\overline{z}^{n*}g)(\overline{x}) + ((\overline{z}^{(n-1)*}F^n) + (\overline{z}^{n*}g))_S^*(0_{X_n^*}) = 0$,

(iv) $\langle \overline{z}^{n*}, g(\overline{x}) \rangle = 0$,

(b) If there exists $\overline{x} \in \mathcal{A}$ *such that for some* $(\overline{z}^{0*}, ..., \overline{z}^{(n-1)*}, \overline{z}^{n*}) \in K_0^* \times ... \times K_{n-1}^* \times Q^*$ *the conditions (i)-(v) are fulfilled, then* $\overline{x}$ *is an optimal solution of* (P^C), $(\overline{z}^{0*}, ..., \overline{z}^{n*})$ *is an optimal solution for* (D^C) *and* $v(P^C) = v(D^C)$.

Proof. By Theorem 4.1 strong duality holds between the primal-dual pair (P^C)-(D^C), which means that there exists $\overline{x} \in \mathcal{A}$ and $(\overline{z}^{0*}, ..., \overline{z}^{(n-1)*}, \overline{z}^{n*}) \in K_0^* \times ... \times K_{n-1}^* \times Q^*$, an optimal solution to (D^C), such that the following equality holds

$$(f \circ F^1 \circ ... \circ F^n)(\overline{x}) = \inf_{x \in S} \{ \langle \overline{z}^{(n-1)*}, F^n(x) \rangle + \langle \overline{z}^{n*}, g(x) \rangle \} - f^*(\overline{z}^{0*}) - \sum_{i=1}^{n-1} (\overline{z}^{(i-1)*}F^i)^*(\overline{z}^{i*}).$$

Furthermore, since by definition it holds

$$\begin{aligned} & \sum_{i=1}^{n} (\overline{z}^{(i-1)*}F^i)((F^{i+1} \circ ... \circ F^n)(\overline{x})) \\ = \quad & \langle z^{0*}, (F^1 \circ ... \circ F^n)(\overline{x}) \rangle + \sum_{i=1}^{n-1} \langle \overline{z}^{i*}, (F^{i+1} \circ ... \circ F^n)(\overline{x}), \end{aligned}$$

the assertions (i)-(iv) can be deduced immediately by the following consideration

$$\begin{aligned} & (f \circ F^1 \circ ... \circ F^n)(\overline{x}) + f^*(\overline{z}^{0*}) + \sum_{i=1}^{n-1} (\overline{z}^{(i-1)*}F^i)^*(\overline{z}^{i*}) + \\ & ((z^{(n-1)*}F^n) + (\overline{z}^{n*}g))_S^*(0_{X_n^*}) = 0 \end{aligned}$$

$$\Leftrightarrow \quad (f \circ F^1 \circ ... \circ F^n)(\overline{x}) + f^*(\overline{z}^{0*}) + \sum_{i=1}^{n-1} (\overline{z}^{(i-1)*} F^i)^* (\overline{z}^{i*}) + ((z^{(n-1)*} F^n) + (\overline{z}^{n*} g))^*_S (0_{X_n^*})$$
$$+ (\overline{z}^{n*} g)(\overline{x}) - \langle \overline{z}^{n*}, g(\overline{x}) \rangle$$
$$+ \sum_{i=1}^{n} (\overline{z}^{(i-1)*} F^i)((F^{i+1} \circ ... \circ F^n)(\overline{x}))$$
$$- \langle z^{0*}, (F^1 \circ ... \circ F^n)(\overline{x}) \rangle - \sum_{i=1}^{n-1} \langle \overline{z}^{i*}, (F^{i+1} \circ ... \circ F^n)(\overline{x}) \rangle = 0$$
$$\Leftrightarrow \quad [(f \circ F^1 \circ ... \circ F^n)(\overline{x}) + f^*(\overline{z}^{0*}) - \langle \overline{z}^{0*}, (F^1 \circ ... \circ F^n)(\overline{x}) \rangle] +$$
$$\sum_{i=1}^{n-1} [(\overline{z}^{(i-1)*} F^i)((F^{i+1} \circ ... \circ F^n)(\overline{x})) + (\overline{z}^{(i-1)*} F^i)^* (\overline{z}^{i*}) - \langle \overline{z}^{i*}, (F^{i+1} \circ ... \circ F^n)(\overline{x}) \rangle] +$$
$$[(\overline{z}^{(n-1)*} F^n)(\overline{x}) + (\overline{z}^{n*} g)(\overline{x}) + ((\overline{z}^{(n-1)*} F^n) + (\overline{z}^{n*} g))^*_S (0_{X_n^*})] + [-\langle \overline{z}^{n*}, g(\overline{x}) \rangle] = 0.$$

By the Young-Fenchel inequality and the constraints of the primal and dual problem all the terms within the brackets are non-negative and must be equal to zero. □

Remark 4.3. *The conditions (i)-(iv) can equivalently be expressed as*

(i) $\overline{z}^{0*} \in \partial f((F^1 \circ ... \circ F^n)(\overline{x}))$,

(ii) $\overline{z}^{i*} \in \partial (\overline{z}^{(i-1)*} F^i)((F^{i+1} \circ ... \circ F^n)(\overline{x})),\ i = 1, ..., n-1$,

(iii) $0_{X_n^*} \in \partial ((\overline{z}^{(n-1)*} F^n) + (\overline{z}^{n*} g) + \delta_S)(\overline{x})$,

(iv) $\langle \overline{z}^{n*}, g(\overline{x}) \rangle = 0$.

5 The conjugate function of a multi-composed function

Before we continue with our further approach we want to calculate the conjugate of the function $(f \circ F^1 \circ ... \circ F^n)$, or, to be more precise, we will determine of the function

$$\gamma(x) = (f \circ F^1 \circ ... \circ F^n)(x),\ x \in X_n,$$

its conjugate function

$$\gamma^*(x^*) = \sup_{x \in X_n} \{\langle x^*, x \rangle - (f \circ F^1 \circ ... \circ F^n)(x)\},\ x^* \in X_n^*.$$

With this in mind, we consider for fixed $x^* \in X_n^*$ the problem

$$(P^K) \quad \inf_{x \in X_n} \{(f \circ F^1 \circ ... \circ F^n)(x) - \langle x^*, x \rangle\}$$

and the primal problem

$$(\widetilde{P}^K) \quad \inf_{\substack{F^i(y^i) - y^{i-1} \in -K_{i-1}, \\ y^i \in X_i,\ i=0,...,n}} \{\widetilde{f}(y^0, y^1, ..., y^n) - \langle x^*, y^n \rangle\}.$$

In the same way like in the proof of Theorem 3.1 one can show that it holds $v(P^K) = v(\widetilde{P}^K)$ (where $v(P^K)$ and $v(\widetilde{P}^K)$ denote the optimal objective values of the problems (P^K) and $(\widetilde{P}^K)$, respectively). The corresponding Lagrange dual problem to problem $(\widetilde{P}^K)$ looks like

$$
\begin{aligned}
(\widetilde{D}^K) \qquad & \sup_{\substack{z^{i*}\in K_i^*,\\ i=0,...,n-1}} \inf_{\substack{y^i\in X_i,\\ i=0,...,n}} \left\{ \widetilde{f}(y^0,y^1,...,y^n) + \sum_{i=1}^n \langle z^{(i-1)*}, F^i(y^i) - y^{i-1}\rangle - \langle x^*, y^n\rangle \right\} \\
= \; & \sup_{\substack{z^{i*}\in K_i^*,\\ i=0,...,n-1}} \Bigg\{ - \sup_{y^0\in X_0} \{\langle z^{0*}, y^0\rangle - f(y^0)\} - \\
& \sup_{\substack{y^n\in S,\ y^i\in X_i,\\ i=1,...,n-1}} \left\{ \sum_{i=1}^{n-1} \langle z^{i*}, y^i\rangle + \langle x^*, y^n\rangle - \sum_{i=1}^n \langle z^{(i-1)*}, F^i(y^i)\rangle \right\}\Bigg\} \\
= \; & \sup_{\substack{z^{i*}\in K_i^*,\\ i=0,...,n-1}} \left\{ - f^*(z^{0*}) - (z^{(n-1)*}F^n)^*(x^*) - \sum_{i=1}^{n-1} (z^{(i-1)*}F^i)^*(z^{i*}) \right\}.
\end{aligned}
$$

Hence, we define the conjugate dual problem corresponding to the primal problem (P^K) as

$$
(D^K) \qquad \sup_{\substack{z^{i*}\in K_i^*,\\ i=0,...,n-1}} \left\{ - f^*(z^{0*}) - (z^{(n-1)*}F^n)^*(x^*) - \sum_{i=1}^{n-1} (z^{(i-1)*}F^i)^*(z^{i*}) \right\}.
$$

Let us notice that for all $x^* \in X_n^*$ one has $\operatorname{dom} \widetilde{f} = \operatorname{dom}(\widetilde{f} + \langle x^*, \cdot\rangle)$. To guarantee strong duality between the problem (P^K) and its conjugate dual problem (D^K), we use the regularity conditions we introduced above. Therefore, we set $Z = X$ ordered by the trivial cone $Q = X$ and define the function $g : X \to X$ by $g(x) := x$ such that g is Q-epi closed and

$$
0_X \in \operatorname{sqri}(g(X) + Q) = \operatorname{sqri}(X + Q) = X.
$$

Hence, we get for the pair (P^K)-(D^K) the following regularity conditions. The first one looks like

$$
(RC_1^K) \quad \left| \begin{array}{l} \exists (y^{0'}, y^{1'}, ..., y^{n'}) \in \operatorname{dom} f \times X_1 \times ... \times X_n \text{ such that} \\ F^i(y^{i'}) - y^{(i-1)'} \in -\operatorname{int} K_{i-1},\ i = 1, ..., n \end{array} \right.
$$

and can also be written as

$$
(RC_{1'}^K) \quad \left| \begin{array}{l} \exists x' \in X_n \text{ such that } F^n(x') \in (F^{n-1})^{-1}((F^{n-2})^{-1}(... \\ (F^1)^{-1}(\operatorname{dom} f - \operatorname{int} K_0) - \operatorname{int} K_{n-1} ...) - \operatorname{int} K_{n-2}) - \operatorname{int} K_{n-1}. \end{array} \right.
$$

For the interior point regularity condition we get

$$
(RC_2^K) \quad \left| \begin{array}{l} X_0, ..., X_n \text{ are Fréchet spaces, } f \text{ is l.s.c., } K_{i-1} \text{ is closed,} \\ \operatorname{int} K_{i-1} \neq \emptyset,\ F^i \text{ is } K_{i-1}\text{-epi closed, } i = 1, ..., n, \\ 0_{X_0} \in \operatorname{sqri}(F^1(\operatorname{dom} F^1) - \operatorname{dom} f + K_0) \text{ and} \\ 0_{X_{i-1}} \in \operatorname{sqri}(F^i(\operatorname{dom} F^i) - \operatorname{dom} F^{i-1} + K_{i-1}), i = 2, ..., n. \end{array} \right.
$$

In the same way we get representations for (RC_i^K), $i = 2', 2'', 3$.

Let us denote by $v(D^K)$ the optimal objective value of the problem (D^K), then by Theorem 4.1 we can state the following one:

Theorem 5.1. *(strong duality) If one of the conditions* (RC_i^K), $i \in \{1, 1', 2, 2', 2'', 3\}$, *is fulfilled, then between* (P^K) *and* (D^K) *strong duality holds, i.e.* $v(P^K) = v(D^K)$ *and the conjugate dual problem has an optimal solution.*

Furthermore, it holds the following theorem.

Theorem 5.2. *Let* $f : \overline{X}_0 \to \overline{\mathbb{R}}$ *be proper, convex and* K_0*-increasing on* $F^1(\operatorname{dom} F^1) + K_0$, $F^i : \overline{X}_i \to \overline{X}_{i-1}$, *be proper,* K_{i-1}*-convex and* K_i*-*K_{i-1}*-increasing on* $F^{i+1}(\operatorname{dom} F^{i+1}) + K_i$, $i = 1, ..., n-1$ *and* $F^n : X_n \to \overline{X}_{n-1}$ *be proper and* K_{n-1}*-convex. If one of the regularity conditions* (RC_i^K), $i \in \{1, 1', 2, 2', 2'', 3\}$, *is fulfilled, then the conjugate function of* γ *is given by*

$$\gamma^*(x^*) \quad = \quad \min_{\substack{z^{i*} \in K_i^*, \\ i=0,...,n-1}} \left\{ f^*(z^{0*}) + (z^{(n-1)*}F^n)^*(x^*) + \sum_{i=1}^{n-1} (z^{(i-1)*}F^i)^*(z^{i*}) \right\} \; \forall x^* \in X_n^*. \quad (7)$$

Proof. By using Theorem 5.1 it follows that

$$\begin{aligned} \gamma^*(x^*) &= \sup_{x \in X} \{ \langle x^*, x \rangle - (f \circ F^1 \circ ... \circ F^n)(x) \} \\ &= \min_{\substack{y^{i*} \in K_i^*, \\ i=0,...,n-1}} \left\{ f^*(y^{0*}) + (y^{(n-1)*}F^n)^*(x^*) + \sum_{i=1}^{n-1} (y^{(i-1)*}F^i)^*(y^{i*}) \right\} \; \forall x^* \in X_n^*. \end{aligned}$$

□

Remark 5.1. *The advantage of the introduced concept is that a "complicated" function* γ *can be splitted into* $n+1$ *"simple" functions such that the calculation of the conjugate can be simplified by calculating just the conjugates of the* $n+1$ *"simple" functions.*

Example 5.1. Let us consider the following generalized signomial function $\gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\gamma(x, y) = \begin{cases} \sup \left\{ \frac{1}{x_1^{p_1} y_1^{q_1}}, ..., \frac{1}{x_n^{p_n} y_n^{q_n}} \right\}, & \text{if } (x, y) \in \operatorname{int} \mathbb{R}_+^n \times \operatorname{int} \mathbb{R}_+^n \\ +\infty, & \text{otherwise,} \end{cases}$$

with $p_i, q_i \geq 0$ for all $i = 1, ..., n$, and $x = (x_1, ..., x_n)^T \in \mathbb{R}^n$, $y = (y_1, ..., y_n)^T \in \mathbb{R}^n$. Then, we split the function γ into the functions

- $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}}$ defined by

$$f(y^0) := \begin{cases} \sup\{y_1^0, ..., y_n^0\}, & \text{if } y^0 = (y_1^0, ..., y_n^0)^T \in \mathbb{R}_+^n, \\ +\infty, & \text{otherwise,} \end{cases}$$

- $F^1 : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$, defined by

$$F^1(y^1) := \begin{cases} (e^{y_1}, ..., e^{y_n})^T, & \text{if } y^1 = (y_1^1, ..., y_n^1)^T \in \mathbb{R}^n \\ +\infty_{\mathbb{R}_+^n}, & \text{otherwise,} \end{cases}$$

and

- $F^2 : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}^n}$, defined by

$$F^2(x,y) := \begin{cases} (-p_1 \ln x_1 - q_1 \ln y_1, ..., -p_n \ln x_n - q_n \ln y_n)^T, \text{ if } x, y \in \operatorname{int} \mathbb{R}^n_+, \\ +\infty_{\mathbb{R}^n_+}, \text{ otherwise,} \end{cases}$$

such that γ is writeable as

$$\gamma(x,y) = (f \circ F^1 \circ F^2)(x,y) \tag{8}$$

and set $K_0 = K_1 = \mathbb{R}^n_+$. Without much effort one can observe that f is proper, convex and $\mathbb{R}^n_+$-increasing on $F^1(\operatorname{dom} F^1) + \mathbb{R}^n_+ = \operatorname{int} \mathbb{R}^n_+ + \mathbb{R}^n_+ = \operatorname{int} \mathbb{R}^n_+ \subseteq \mathbb{R}^n_+$, F^1 is proper, $\mathbb{R}^n_+$-convex and $\mathbb{R}^n_+$-$\mathbb{R}^n_+$-increasing on $F^2(\operatorname{dom} F^2) + \mathbb{R}^n_+ = \mathbb{R}^n$ and F^2 is proper and $\mathbb{R}^n_+$-convex. Moreover, it is easy to verify that the regularity condition $(RC^K_{1'})$ looks in this special case like

$$(RC^{K_e}_{1'}) \quad \Big| \quad \exists (x', y') \in \mathbb{R}^n \times \mathbb{R}^n \text{ such that } -p_i \ln x'_i - q_i \ln y'_i \in \mathbb{R},\ i = 1, ..., n,$$

which, of course, is always fulfilled. Thus, we can apply the formula (7) of Theorem 5.2 for the determination of the conjugate function of γ:

$$\gamma^*(x^*, y^*) = \min_{z^{0*},\ z^{1*} \in \mathbb{R}^n_+} \left\{ f^*(z^{0*}) + (z^{0*}F^1)^*(z^{1*}) + (z^{1*}F^2)^*(x^*, y^*) \right\},\ \forall (x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}^n \tag{9}$$

Now, we have to calculate the conjugate functions involved in the formula (9). We have for $z^{0*} = (z^{0*}_1, ..., z^{0*}_n)^T \in \mathbb{R}^n_+$:

$$\begin{aligned} f^*(z^{0*}) &= \sup_{(y^0_1,...,y^0_n)^T \in \mathbb{R}^n} \left\{ \sum_{i=1}^n z^{0*}_i y^0_i - f(y^0) \right\} = \sup_{(y^0_1,...,y^0_n)^T \in \mathbb{R}^n_+} \left\{ \sum_{i=1}^n z^{0*}_i y^0_i - \sup\{y^0_1, ..., y^0_n\} \right\} \\ &= \sup_{(y^0_1,...,y^0_n)^T \in \mathbb{R}^n_+} \left\{ \sum_{i=1}^n z^{0*}_i y^0_i - \inf_{\substack{t \in \mathbb{R}_+,\ y^0_i \le t, \\ i=1,...,n}} t \right\} = \sup_{\substack{y^0_i \in \mathbb{R}_+,\ t \in \mathbb{R}_+,\ y^0_i \le t, \\ i=1,...,n}} \left\{ \sum_{i=1}^n z^{0*}_i y^0_i - t \right\}. \end{aligned}$$

As one may see, f^* is for fixed $z^{0*} \in \mathbb{R}^n_+$ a linear problem and thus, by elementary calculation, we have that

$$f^*(z^{0*}) = \begin{cases} 0, \text{ if } \sum\limits_{i=1}^n z^{0*}_i \le 1, \\ +\infty, \text{ otherwise.} \end{cases} \tag{10}$$

From (9) and (10) follows for the conjugate function of γ

$$\gamma^*(x^*, y^*) = \min_{\substack{z^{0*}_i,\ z^{1*}_i \in \mathbb{R}_+,\ i=1,...,n, \\ \sum_{i=1}^n z^{0*}_i \le 1}} \left\{ (z^{0*}F^1)^*(z^{1*}) + (z^{1*}F^2)^*(x^*, y^*) \right\}. \tag{11}$$

Furthermore, we have for $z^{0*}_i \ge 0$, $i = 1, ..., n$,

$$\begin{aligned} (z^{0*}F^1)^*(z^{1*}) &= \sup_{y_i \in \mathbb{R},\ i=1,...,n} \left\{ \sum_{i=1}^n z^{1*}_i y_i - \sum_{i=1}^n z^{0*}_i e^{y_i} \right\} \\ &= \sum_{i=1}^n \sup_{y_i \in \mathbb{R}} \{ z^{1*}_i y_i - z^{0*}_i e^{y_i} \} \end{aligned}$$

with (see [1] or also [9])

$$\sup_{y_i\in\mathbb{R}}\{z_i^{1*}y_i - z_i^{0*}e^{y_i}\} = \begin{cases} z_i^{1*}\left(\ln\frac{z_i^{1*}}{z_i^{0*}} - 1\right), \text{ if } z_i^{0*},\ z_i^{1*} > 0, \\ 0, \text{ if } z_i^{1*} = 0,\ z_i^{0*} \geq 0, \\ +\infty, \text{ otherwise,} \end{cases} \tag{12}$$

for $i = 1, ..., n$ and for $z_i^{1*} \geq 0$, $i = 1, ..., n$, it holds

$$\begin{aligned}(z^{1*}F^2)^*(x^*, y^*) &= \sup_{x_i, y_i > 0,\ i=1,...,n}\left\{\sum_{i=1}^n x_i^* x_i + \sum_{i=1}^n y_i^* y_i + \sum_{i=1}^n z_i^{1*}p_i \ln x_i + \sum_{i=1}^n z_i^{1*}q_i \ln y_i\right\} \\ &= \sum_{i=1}^n\left(\sup_{x_i>0}\{x_i^*x_i + z_i^{1*}p_i\ln x_i\} + \sup_{y_i>0}\{y_i^*y_i + z_i^{1*}q_i\ln y_i\}\right)\end{aligned}$$

for all $x^* = (x_1^*, ..., x_n^*)^T,\ y^* = (y_1^*, ..., y_n^*)^T \in \mathbb{R}^n$, where (see [9])

$$\sup_{x_i>0}\{x_i^*x_i + z_i^{1*}p_i\ln x_i\} = \begin{cases} -z_i^{1*}p_i\left(1 + \ln\left(-\frac{x_i^*}{z_i^{1*}p_i}\right)\right), \text{ if } x_i^* < 0,\ z_i^{1*},\ p_i > 0, \\ 0, \text{ if } x_i^* \leq 0 \text{ and } z_i^{1*} = 0 \text{ or } x_i^* \leq 0 \text{ and } p_i = 0, \\ +\infty, \text{ otherwise,} \end{cases} \tag{13}$$

and likewise

$$\sup_{y_i>0}\{y_i^*y_i + z_i^{1*}q_i\ln y_i\} = \begin{cases} -z_i^{1*}q_i\left(1 + \ln\left(-\frac{y_i^*}{z_i^{1*}q_i}\right)\right), \text{ if } y_i^* < 0,\ z_i^{1*},\ q_i > 0, \\ 0, \text{ if } y_i^* \leq 0 \text{ and } z_i^{1*} = 0 \text{ or } y_i^* \leq 0 \text{ and } q_i = 0, \\ +\infty, \text{ otherwise,} \end{cases} \tag{14}$$

for $i = 1, ..., n$. Finally, we define the function $\xi : \mathbb{R} \to \{0, 1\}$ by

$$\xi(x) = \begin{cases} 1, \text{ if } x > 0, \\ 0, \text{ otherwise,} \end{cases} \tag{15}$$

which leads, by using (11), (12), (13), (14) and (15), to the following formula of the conjugate function of γ

$$\begin{aligned}\gamma^*(x^*, y^*) = \min_{\substack{\sum_{n=1}^n z_i^{0*} \leq 1,\ z_i^{0*} \geq 0, \\ z_i^{1*} \geq 0,\ i=1,...,n}} &\left\{\sum_{i=1}^n z_i^{1*}[\left(\ln z_i^{1*} - \ln z_i^{0*} - 1\right)\xi(z_i^{0*})\right. \\ &\left. -p_i\left(1 + \ln x_i^* - \ln z_i^{1*}p_i\right) - q_i\left(1 + \ln y_i^* - \ln z_i^{1*}q_i\right)]\right\}\end{aligned}$$

for all $x_i^*,\ y_i^* \geq 0$, $i = 1, ..., n$, with the convention $0\ln 0 = 0$.

In the next, we will give an alternative representation for γ. But, first pay attention to the following function

$$\beta(x^*) \quad := \quad \inf_{\substack{z^{i*}\in X_i^*, \\ i=0,...,n-1}}\left\{f^*(z^{0*}) + (z^{(n-1)*}F^n)^*(x^*) + \sum_{i=1}^{n-1}(z^{(i-1)*}F^i)^*(z^{i*})\right\},\ \forall x^* \in X_n^*.$$

If $f : X_0 \to \overline{\mathbb{R}}$ is a K_0-increasing function on $\{F^1(\operatorname{dom} F^1) + K_0\} - K_0$, it follows by Proposition 2.3.11. in [1] that

$$f^*(z^{0*}) = +\infty,\ \forall z^{0*} \notin K_0^*,\ \text{i.e.}\ \operatorname{dom} f^* \subseteq K_0^*$$

and thus, it holds

$$\beta(x^*) = \inf_{\substack{z^{0*}\in K_0^*,\ z^{i*}\in X_i^*,\\ i=1,\dots,n-1}} \left\{ f^*(z^{0*}) + (z^{(n-1)*}F^n)^*(x^*) + \sum_{i=1}^{n-1} (z^{(i-1)*}F^i)^*(z^{i*}) \right\},\ \forall x^* \in X_n^*.$$

Moreover, if $F^1 : X_1 \to \overline{X}_0$ is K_1-K_0-increasing on $\{F^2(\operatorname{dom} F^2) + K_1\} - K_1$, then $(z^{0*}F^1) : X_1 \to \overline{\mathbb{R}}$ is K_1-increasing on $\{F^2(\operatorname{dom} F^2) + K_1\} - K_1$ for $z^{0*} \in K_0^*$. By using again Proposition 2.3.11. in [1] one gets for $z^{0*} \in K_0^*$

$$(z^{0*}F^1)^*(z^{1*}) = +\infty,\ \forall z^{1*} \notin K_1^*,\ \text{i.e.}\ \operatorname{dom}(z^{0*}F^1) \subseteq K_1^*$$

and we can write

$$\beta(x^*) = \inf_{\substack{z^{0*}\in K_0^*,\ z^{1*}\in K_1^*,\\ z^{i*}\in X_i^*,\ i=2,\dots,n-1}} \left\{ f^*(z^{0*}) + (z^{(n-1)*}F^n)^*(x^*) + \sum_{i=1}^{n-1} (z^{(i-1)*}F^i)^*(z^{i*}) \right\},\ \forall x^* \in X_n^*.$$

If we proceed in this way, it reveals that

$$(z^{(i-1)*}F^i)^*(z^{i*}) = +\infty,\ \forall z^{i*} \notin K_i^*,\ \text{i.e.}\ \operatorname{dom}(z^{(i-1)*}F^i)^* \subseteq K_i^*,\ i = 2, ..., n-1,$$

and therefore, it holds

$$\begin{aligned}
\beta(x^*) &= \inf_{\substack{z^{i*}\in X_i^*,\\ i=0,\dots,n-1}} \left\{ f^*(z^{0*}) + (z^{(n-1)*}F^n)^*(x^*) + \sum_{i=1}^{n-1} (z^{(i-1)*}F^i)^*(z^{i*}) \right\} \\
&= \inf_{\substack{z^{i*}\in K_i^*,\\ i=0,\dots,n-1}} \left\{ f^*(z^{0*}) + (z^{(n-1)*}F^n)^*(x^*) + \sum_{i=1}^{n-1} (z^{(i-1)*}F^i)^*(z^{i*}) \right\},\ \forall x^* \in X_n^*.
\end{aligned}$$

For the conjugate function of β one has

$$\begin{aligned}
\beta^*(x) &= \sup_{x^*\in X_n^*} \{\langle x^*, x\rangle - \beta(x^*)\} \\
&= \sup_{x^*\in X_n^*} \Bigg\{ \langle x^*, x\rangle - \\
&\qquad \inf_{\substack{z^{i*}\in X_i^*,\\ i=0,\dots,n-1}} \left\{ f^*(z^{0*}) + (z^{(n-1)*}F^n)^*(x^*) + \sum_{i=1}^{n-1} (z^{(i-1)*}F^i)^*(z^{i*}) \right\} \Bigg\} \\
&= \sup_{\substack{x^*\in X_n^*,\ z^{i*}\in X_i^*,\\ i=0,\dots,n-1}} \left\{ \langle x^*, x\rangle - f^*(z^{0*}) - (z^{(n-1)*}F^n)^*(x^*) - \sum_{i=1}^{n-1} (z^{(i-1)*}F^i)^*(z^{i*}) \right\} \\
&= \sup_{\substack{z^{i*}\in X_i^*,\\ i=0,\dots,n-1}} \left\{ \sup_{x^*\in X_n^*} \{\langle x^*, x\rangle - (z^{(n-1)*}F^n)^*(x^*)\} - f^*(z^{0*}) - \sum_{i=1}^{n-1} (z^{(i-1)*}F^i)^*(z^{i*}) \right\} \\
&= \sup_{\substack{z^{i*}\in X_i^*,\\ i=0,\dots,n-1}} \left\{ (z^{(n-1)*}F^n)^{**}(x)\} - f^*(z^{0*}) - \sum_{i=1}^{n-1} (z^{(i-1)*}F^i)^*(z^{i*}) \right\},\ \forall x \in X_n. \qquad (16)
\end{aligned}$$

Since F^n is proper and K_{n-1}-convex and if we ask that F^n is also star K_{n-1}-lower semicontinuous, (16) can by using the Fenchel-Moreau Theorem be written as

$$\beta^*(x) = \sup_{\substack{z^{i*}\in X_i^*,\\ i=0,\dots,n-1}} \left\{ (z^{(n-1)*}F^n)(x) - f^*(z^{0*}) - \sum_{i=1}^{n-1} (z^{(i-1)*}F^i)^*(z^{i*}) \right\}, \ \forall x \in X_n. \quad (17)$$

If we additionally ask that the function F^i is star K_{i-1}-lower semicontinuous, $i = 1, \dots, n$, and if we assume that f is lower semicontinuous, then one gets for (17) by using again the Fenchel-Moreau Theorem

$$\begin{aligned}
\beta^*(x) &= \sup_{\substack{z^{i*}\in X_i^*,\\ i=0,\dots,n-2}} \Bigg\{ \sup_{z^{(n-1)*}\in X_{n-1}^*} \{\langle z^{(n-1)*}, F^n(x)\rangle - (z^{(n-2)*}F^{(n-1)})^*(z^{(n-1)*})\} - \\
&\quad f^*(z^{0*}) - \sum_{i=1}^{n-2} (z^{(i-1)*}F^i)^*(z^{i*}) \Bigg\} \\
&= \sup_{\substack{z^{i*}\in X_i^*,\\ i=0,\dots,n-2}} \left\{ (z^{(n-2)*}F^{n-1})^{**}(F^n(x)) - f^*(z^{0*}) - \sum_{i=1}^{n-2} (z^{(i-1)*}F^i)^*(z^{i*}) \right\} \\
&= \sup_{\substack{z^{i*}\in X_i^*,\\ i=0,\dots,n-2}} \left\{ (z^{(n-2)*}F^{n-1})(F^n(x)) - f^*(z^{0*}) - \sum_{i=1}^{n-2} (z^{(i-1)*}F^i)^*(z^{i*}) \right\} \\
&= \sup_{\substack{z^{i*}\in X_i^*,\\ i=0,\dots,n-3}} \Bigg\{ \sup_{z^{(n-2)*}\in X_{n-2}^*} \{\langle z^{(n-2)*}, F^{n-1}(F^n(x))\rangle - (z^{(n-3)*}F^{(n-2)})^*(z^{(n-2)*})\} - \\
&\quad f^*(z^{0*}) - \sum_{i=1}^{n-3} (z^{(i-1)*}F^i)^*(z^{i*}) \Bigg\} \\
&= \sup_{\substack{z^{i*}\in X_i^*,\\ i=0,\dots,n-3}} \left\{ (z^{(n-3)*}F^{n-2})(F^{n-1}(F^n(x))) - f^*(z^{0*}) - \sum_{i=1}^{n-3} (z^{(i-1)*}F^i)^*(z^{i*}) \right\} \\
&\ \vdots \\
&= \sup_{z^{0*}\in X_0^*} \{\langle z^{0*}, (F^1 \circ \dots \circ F^n)(x)\rangle - f^*(z^{0*})\} = f^{**}((F^1 \circ \dots \circ F^n)(x)) \\
&= (f \circ F^1 \circ \dots \circ F^n)(x) = \gamma(x), \ \forall x \in X_n.
\end{aligned}$$

Since the weak duality always holds, i.e. $v(P^K) \geq v(D^K)$, we have $\gamma^*(x^*) \leq \beta(x^*)$ for all $x^* \in X_n^*$. Moreover, it holds $\gamma(x) \geq \gamma^{**}(x)$ for all $x \in X_n$ and from here it follows that $\gamma(x) \geq \gamma^{**}(x) \geq \beta^*(x) = \gamma(x)$, $x \in X_n$, i.e. $\gamma(x) = \gamma^{**}(x)$ for all $x \in X_n$. The latter means that γ is proper, convex and lower semicontinuous. Summarizing, we get the following theorem:

Theorem 5.3. *Let $f : \overline{X}_0 \to \overline{\mathbb{R}}$ be a proper, convex, K_0-increasing on $\{F^1(\operatorname{dom} F^1) + K_0\} - K_0$ and lower semicontinuous function, $F^i : \overline{X}_i \to \overline{X}_{i-1}$ be a proper, K_{i-1}-convex, K_i-K_{i-1}-increasing on $\{F^{i+1}(\operatorname{dom} F^{i+1}) + K_i\} - K_i$ and star K_{i-1}-lower semicontinuous function, $i = 1, \dots, n-1$, and $F^n : X_n \to \overline{X}_{n-1}$ be a proper, K_{n-1}-convex and star K_{n-1}-lower semicontinuous function. Then the function $\gamma = f \circ F^1 \circ \dots \circ F^n : X_n \to \overline{\mathbb{R}}$ is proper, convex and lower* *semicontinuous and can alternatively be written as*

$$\gamma(x) = \beta^*(x) = \sup_{\substack{z^{i*}\in X_i^*,\\ i=0,\ldots,n-1}} \left\{ (z^{(n-1)*}F^n)(x) - f^*(z^{0*}) - \sum_{i=1}^{n-1} (z^{(i-1)*}F^i)^*(z^{i*}) \right\},\ \forall x \in X_n.$$

6 An Optimization problem having as objective function the sum of reciprocals of concave functions

Let E_i be a non-empty convex subset of X, $i = 1, ..., n$, where X is defined like in the beginning, a locally convex Hausdorff space partially ordered by the non-empty convex cone K. Then, we consider a convex optimization problem having as objective function the sum of reciprocals of concave functions $h_i : E_i \to \mathbb{R}$ with strict positive values, $i = 1, .., n$, and geometric and cone constraints, i.e. the optimization problem that we discuss in this section (cf. the definitions from Section 3) is given by

$$(P^G) \quad \inf_{\substack{x\in S,\\ g(x)\in -Q}} \left\{ \sum_{i=1}^{n} \frac{1}{h_i(x)} \right\}.$$

Optimization problems of this type arise, for instance, in the study of power functions by setting $h_i : \mathbb{R}_+ \to \mathbb{R},\ h_i(x) = c_i x^{p_i}$ with $c_i p_i (p_i - 1) \leq 0,\ i = 1, ..., n$, (see [17]) and have a wide range of applications in economics, engineering and finance.
To apply the results from the previous section to (P^G), i.e. to characterize strong duality and to derive optimality conditions, we assume that the function $-h_i$ is K-increasing on E_i, $i = 1, ..., n$, and set $X_0 = \mathbb{R}^n$, $K_0 = \mathbb{R}^n_+$, $X_1 = X^n$, $K_1 = K^n$ and $X_2 = X$. Additionally, we define the following functions

- $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}},\ f(y^0) = \begin{cases} -\sum\limits_{i=1}^{n} \frac{1}{y_i^0},\ \text{if } y_i^0 < 0,\ i = 1, ..., n, \\ +\infty,\ \text{otherwise}, \end{cases}$
- $F^1 : X^n \to \overline{\mathbb{R}^n},\ F^1(y^1) = \begin{cases} (-h_1(y_1^1), ..., -h_n(y_n^1))^T,\ \text{if } y_i^1 \in E_i,\ i = 1, ..., n, \\ +\infty_{\mathbb{R}^n_+},\ \text{otherwise} \end{cases}$

  and
- $F^2 : X \to X^n,\ F^2(x) := (x, ..., x) \in X^n$

and we assume that $F^2(S \cap \operatorname{dom} g) \subseteq E_1 \times ... \times E_n$ (cf. Remark 3.2). From here, it reveals that the problem (P^G) can equivalently be written as

$$(P^G) \quad \inf_{\substack{x\in S,\\ g(x)\in -Q}} \left\{ (f \circ F^1 \circ F^2)(x) \right\}$$

and by using the formula from Section 3 its corresponding conjugate dual problem (D^G) turns into

$$(D^G) \quad \sup_{\substack{z^{0*}\in\mathbb{R}^n_+,\ z^{1*}\in(K^*)^n,\\ z^{2*}\in Q^*}} \left\{ \inf_{x\in S} \left\{ \left\langle \sum_{i=1}^{n} z_i^{1*}, x \right\rangle + \langle z^{2*}, g(x)\rangle \right\} - f^*(z^{0*}) - (z^{0*}F^1)^*(z^{1*}) \right\}.$$

Furthermore, one has (see [2], [12] or [13])

$$f^*(z^{0*}) = \sum_{i=1}^{n} \sup_{y_i<0} \left\{ z_i^{0*} y_i^0 + \frac{1}{y_i^0} \right\} = -2 \sum_{i=1}^{n} \sqrt{z_i^{0*}}$$

for all $z_i^{0*} \geq 0$, $i = 1, ..., n$, and since, it holds

$$(z^{0*}F^1)^*(z^{1*}) = \sum_{i=1}^{n} \sup_{y_i^1 \in E_i} \left\{ \langle z_i^{1*}, y_i^1 \rangle + z_i^{0*} h_i(y_i^1) \right\} = \sum_{i=1}^{n} (-z_i^{0*} h_i)_{E_i}^*(z_i^{1*}),$$

one gets for the conjugate dual problem

$$(D^G) \quad \sup_{\substack{z^{0*}\in\mathbb{R}^n_+,\ z^{1*}\in(K^*)^n,\\ z^{2*}\in -Q^*}} \left\{ -(z^{2*}g)_S^* \left(-\sum_{i=1}^{n} z_i^{1*} \right) + \sum_{i=1}^{n} \left(2\sqrt{z_i^{0*}} - (-z_i^{0*} h_i)_{E_i}^*(z_i^{1*}) \right) \right\}.$$

Remark 6.1. *We want to note that for* $i = 1, ..., n$ *the representation of the conjugate function of* $(-z_i^{0*} h_i)$ *can be improved by the following one (see [2], [12] or [13])*

$$(-z_i^{0*} h_i)_{E_i}^*(z_i^{1*}) = \begin{cases} z_i^{0*}(-h_i)_{E_i}^* \left(\frac{z_i^{1*}}{z_i^{0*}} \right), & \text{if } z_i^{0*} > 0 \\ \sigma_{E_i}(z_i^{1*}), & \text{if } z_i^{0*} = 0. \end{cases}$$

Therefore, by using (15) one can write for the problem (D^G) *also*

$$(D^G) \quad \sup_{\substack{z^{0*}\in\mathbb{R}^n_+,\ z^{1*}\in(K^*)^n,\\ z^{2*}\in -Q^*}} \Bigg\{ -(z^{2*}g)_S^* \left(-\sum_{i=1}^{n} z_i^{1*} \right) + \sum_{i=1}^{n} \Bigg(2\sqrt{z_i^{0*}} - z_i^{0*}(-h_i)_{E_i}^* \left(\frac{z_i^{1*}}{z_i^{0*}} \right) \xi(z_i^{0*}) - \sigma_{E_i}(z_i^{1*})(1 - \xi(z_i^{0*})) \Bigg) \Bigg\}.$$

Remark 6.2. *One may see that the function* F^2 *has been introduced in order to decompose the functions* h_i*,* $i = 1, ..., n$*, and* g *or, more precisely, to decompose its conjugate functions in the formulation of the dual problem* (D^G)*.*

It is easy to observe that f is proper, $\mathbb{R}^n_+$-increasing on $\operatorname{dom} f = \operatorname{int}(-\mathbb{R}^n_+)$, convex and lower semicontinuous, F^1 is proper, K^n-$\mathbb{R}^n_+$-increasing on $\operatorname{dom} F^1 = E_1 \times ... \times E_n$ and $\mathbb{R}^n_+$-convex and that $F^1(\operatorname{dom} F^1) \subseteq \operatorname{int}(-\mathbb{R}^n_+) = \operatorname{dom} f$ (in this context pay attention on Remark 3.2). For that reason we can now attach the regularity condition (RC_1^C), specialized for the optimization problem (P^G),

$$(RC_1^G) \quad \Big| \begin{array}{l} \exists (y^{0'}, y^{1'}, y^{2'}) \in (-\infty, 0)^n \times X^n \times S \text{ such that } h_i(y_i^{1'}) + y_i^{0'} > 0, \\ y^{2'} - y_i^{1'} \in -\operatorname{int} K,\ i = 1, ..., n, \text{ and } g(y^{2'}) \in -\operatorname{int} Q. \end{array}$$

As h_i is a concave function with strict positive values on E_i, there exist $y_i^{0*} < 0$ and $y_i^{1*} \in E_i$ such that $h_i(y_i^{1'}) + y_i^{0'} > 0$, $i = 1, ..., n$, and hence, (RC_1^G) reduces to

$$(RC_1^G) \quad \Big| \quad \exists (y^{1'}, y^{2'}) \in X^n \times S \text{ such that } y^{2'} - y_i^{1'} \in -\operatorname{int} K,\ i = 1, ..., n, \text{ and } g(y^{2'}) \in -\operatorname{int} Q.$$

or, equivalently, in the light of $(RC^C_{1'})$,

$$(RC^G_{1'}) \quad \Big| \quad \exists x' \in S \text{ such that } x' \in E_i - \operatorname{int} K,\ i = 1, ..., n, \text{ and } g(x') \in -\operatorname{int} Q.$$

The generalized interior point regularity conditions (RC^C_2), specialized for (P^G), looks like

$$(RC^G_2) \quad \left| \begin{array}{l} X \text{ and } Z \text{ are Fréchet spaces, } S \text{ is closed, } g \text{ is } Q\text{-epi closed,} \\ -h_i \text{ is lower semicontinuous, } 0_X \in \operatorname{sqri}(\operatorname{dom} g \cap S - E_i + K), \\ i = 1, ..., n, \text{ and } 0_Z \in \operatorname{sqri}(g(\operatorname{dom} g \cap S) + Q). \end{array} \right.$$

In the same way one can formulate a specialized regularity condition (RC^G_i) in respect to the condition (RC^C_i) for $i \in \{2', 2'', 3\}$.

Remark 6.3. *Recall, that in respect to Remark 3.1 and 4.1 the function F^1 does not need to be monotone, because F^2 is a linear function. In this case we set, like mentioned in Remark 3.1, $K_1 = \{0_{X^n}\} = \{0_X\}^n$. But take attention to the circumstance that the regularity conditions (RC^G_1) and $(RC^G_{1'})$ are no more applicable in this framework, as $\operatorname{int}\{0_X\} = \emptyset$.*

By Theorem 4.1 and 4.2 the strong duality statement and the optimality conditions follows immediately.

Theorem 6.1. *(strong duality) If one of the conditions (RC^G_i), $i \in \{1, 1', 2, 2', 2'', 3\}$, is fulfilled, then between (P^G) and (D^G) strong duality holds, i.e. $v(P^G) = v(D^G)$ and the conjugate dual problem has an optimal solution.*

Theorem 6.2. *(optimality conditions) (a) Suppose that one of the regularity conditions (RC^G_i), $i \in \{1, 1', 2, 2', 2'', 3\}$, is fulfilled and let $\overline{x} \in S$ be an optimal solution of the problem (P^G). Then there exists $(\overline{z}^{0*}, \overline{z}^{1*}, \overline{z}^{2*}) \in \mathbb{R}^n_+ \times (K^*)^n \times Q^*$, an optimal solution to (D^C), such that*

(i) $\sum\limits_{i=1}^n \frac{1}{h_i(\overline{x})} - 2 \sum\limits_{i=1}^n \sqrt{\overline{z}^{0*}_i} = - \sum\limits_{i=1}^n \overline{z}^{0*}_i h_i(\overline{x})$,

(ii) $\sum\limits_{i=1}^n (-\overline{z}^{0*}_i h_i)^*_{E_i}(\overline{z}^{1*}_i) - \sum\limits_{i=1}^n \overline{z}^{0*}_i h_i(\overline{x}) = \left\langle \sum\limits_{i=1}^n \overline{z}^{1*}_i, \overline{x} \right\rangle$,

(iii) $\langle \overline{z}^{2*}, g(\overline{x}) \rangle + (\overline{z}^{2*} g)^*_S \left(- \sum\limits_{i=1}^n \overline{z}^{1*}_i \right) = \left\langle - \sum\limits_{i=1}^n \overline{z}^{1*}_i, \overline{x} \right\rangle$,

(iv) $\langle \overline{z}^{2*}, g(\overline{x}) \rangle = 0$.

(b) If there exists $\overline{x} \in S$ such that for some $(\overline{z}^{0*}, \overline{z}^{1*}, \overline{z}^{2*}) \in \mathbb{R}^n_+ \times (K^*)^n \times Q^*$ the conditions (i)-(iv) are fulfilled, then $\overline{x}$ is an optimal solution of (P^G), $(\overline{z}^{0*}, \overline{z}^{1*}, \overline{z}^{2*})$ is an optimal solution for (D^G) and $v(P^G) = v(D^G)$.

Remark 6.4. *In view of the Young-Fenchel inequality, we can refine the conditions (i) and (ii) of Theorem 6.2 like follows*

(i) $\overline{z}^{0*}_i h_i(\overline{x}) = 2\sqrt{\overline{z}^{0*}_i} - \frac{1}{h_i(\overline{x})}$, $i = 1, ..., n$,

(ii) $(-\overline{z}^{0*}_i h_i)^*_{E_i}(\overline{z}^{1*}_i) - \overline{z}^{0*}_i h_i(\overline{x}) = \langle \overline{z}^{1*}_i, \overline{x} \rangle$, $i = 1, ..., n$.

In the end of this paper we give, for completeness, alternative representations of the optimality conditions presented in Theorem 6.2 and refined in the previous remark.

Remark 6.5. *In accordance with Remark 4.3 and 6.4 the optimality conditions* (*i*)*-*(*iv*) *of Theorem 6.2 can equivalently be rewritten as*

(i) $\overline{z}_i^{0*} \in \partial\left(-\frac{1}{\cdot}\right)(-h_i(\overline{x}))$, $i = 1, ..., n$,

(ii) $\overline{z}_i^{1*} \in \partial(-\overline{z}_i^{0*} h_i)(\overline{x})$, $i = 1, ..., n$,

(iii) $-\sum\limits_{i=1}^{n} \overline{z}_i^{1*} \in \partial((\overline{z}^{2*} g) + \delta_S)(\overline{x})$,

(iv) $\langle \overline{z}^{2*}, g(\overline{x}) \rangle = 0$.

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