Multiple Rank-1 Lattices as Sampling Schemes for Multivariate Trigonometric Polynomials

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We present a new sampling method that allows the unique reconstruction of (sparse) multivariate trigonometric polynomials. The crucial idea is to use several rank-1 lattices as spatial discretization in order to overcome limitations of a single rank-1 lattice sampling method. The structure of the corresponding sampling scheme allows for the fast computation of the evaluation and the reconstruction of multivariate trigonometric polynomials, i.e., a fast Fourier transform. Moreover, we present a first algorithm that constructs a reconstructing sampling scheme consisting of several rank-1 lattices for a given frequency index set. Various numerical tests indicate the advantages of the constructed sampling schemes.

Keywords and phrases : sparse multivariate trigonometric polynomials, lattice rule, multiple rank-1 lattice, fast Fourier transform

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1 Introduction

During the last years, sparse mathematical models has become very popular. Particularly, the numerical treatment of high dimensional problems ask for suitable sparsity. However, even sparse problems may deal with huge amounts of data such that efficient numerical methods are on the main focus in various fields of research.

In this paper, we consider multivariate trigonometric polynomials and introduce a new suitable spatial discretization scheme that allows for fast evaluation and reconstruction algorithms. Previous papers on multivariate trigonometric polynomials that deal with single rank-1 lattices as sampling schemes, cf. [16, 9], or randomly chosen sampling sets [5] led to different limitations in the number of used sampling nodes and computational expense, respectively. On the other hand, widely used concepts in higher dimensions are so-called function spaces of dominating mixed smoothness and efficient approximation methods that use hyperbolic cross trigonometric polynomials that are computed from sampling values along sparse grids, cf. [6, 17, 14, 3, 4],. However, research results on hyperbolic cross trigonometric

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polynomials [10] point out that even sparse grid discretizations suffer from limitations, particularly, the condition number of the corresponding Fourier matrix increases with growing problem sizes.

Anyhow, our new sampling method is motivated by the structure of sparse grids. A sparse grid is a composition of specific rank-1 up to rank-d lattices, cf. [1, 18, 15], that are well-suited in order to sample trigonometric polynomials with frequencies supported on hyperbolic crosses. In the present paper, we introduce sampling schemes that are compositions of multiple rank-1 lattices. We investigate necessary and sufficient conditions on the sampling scheme in order to guarantee a unique reconstruction of multivariate trigonometric polynomials. We emphasize that we are interested in arbitrary multivariate trigonometric polynomials, i.e., we do not require any structure of the frequency support.

Furthermore, we present fast algorithms, i.e., fast Fourier transforms (FFTs), that computes

- the evaluation of multivariate trigonometric polynomials at all sampling nodes, cf. Algorithm 3, and
- the reconstruction of multivariate trigonometric polynomials from the function values at the sampling nodes, cf. Section 4.

As a matter of course, we are interested in sampling sets that allow for the unique reconstruction of multivariate trigonometric polynomials supported on specific frequency index set. For a given frequency index set, we present a method, cf. Algorithm 5, that determines a set of rank-1 lattices that guarantees the unique reconstruction of multivariate trigonometric polynomials supported on this frequency index set. The constructed sampling sets allow for a specific fast algorithm that computes the reconstruction in a direct way, cf. Algorithm 6.

The present paper deals only with multivariate trigonometric polynomials. We stress on the fact that one may use the new sampling method in an adaptive way similar to the ideas from [13]. The main ingredient therein is to use reconstructing sampling schemes for low-dimensional frequency index sets in order to approximate functions using multivariate trigonometric polynomials in a dimension incremental way.

2 Prerequisites

In this paper, we deal with multivariate trigonometric polynomials

$$p\colon \mathbb{T}^d \to \mathbb{C}, \qquad \boldsymbol{x} \mapsto \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}},$$

where $\mathbb{T}^d \cong [0,1)^d$ is the d-dimensional torus, the complex numbers $(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I} \in \mathbb{C}^{|I|}$ are called Fourier coefficients of p, and the frequency index set $I \subset \mathbb{Z}^d$ is of finite cardinality. The term $\boldsymbol{k}\cdot\boldsymbol{x} = \sum_{j=1}^d k_j x_j$ is the usual scalar product of two d-dimensional vectors. Furthermore, the space of all trigonometric polynomials supported on the frequency index set I is denoted by $\Pi_I := \operatorname{span}\{\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\colon \boldsymbol{k}\in I\}$. The evaluation of the multivariate trigonometric polynomial p at a finite set $\mathcal{X} \subset \mathbb{T}^d$ of sampling nodes is specified by the matrix-vector product

$$\boldsymbol{A}(\mathcal{X}, I)\hat{\boldsymbol{p}} = \boldsymbol{p},$$

where $\hat{\boldsymbol{p}} = (\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I} \in \mathbb{C}^{|I|}$ is the vector of the Fourier coefficients of the trigonometric polynomial p, the right hand side $\boldsymbol{p} = (p(\boldsymbol{x}))_{\boldsymbol{x}\in\mathcal{X}}$ contains the function values of p at all nodes that belong to the sampling set $\mathcal{X}$, and the Fourier matrix $\boldsymbol{A}(\mathcal{X}, I)$ is given by

$$\boldsymbol{A}(\mathcal{X}, I) = \left(\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right)_{\boldsymbol{x}\in\mathcal{X},\boldsymbol{k}\in I}.$$

As usual, we assume a fixed order of the elements of the sampling set $\mathcal{X}$ and the frequency index set I in order to use the matrix-vector notation.

A very specific kind of sampling schemes are so-called rank-1 lattices

$$\mathcal{X} = \Lambda(\boldsymbol{z}, M) := \left\{\frac{j}{M}\boldsymbol{z} \bmod \boldsymbol{1} \colon j = 0, \ldots, M-1\right\} \subset \mathbb{T}^d,$$

which are well-investigated in the field of numerical integration, cf. [15, 2]. The d-dimensional integer vector $\boldsymbol{z} \in \mathbb{Z}^d$ is called generating vector and the positive integer $M \in \mathbb{N}$ the lattice size of the rank-1 lattice $\Lambda(\boldsymbol{z}, M)$. The author studied such sampling schemes for the reconstruction of trigonometric polynomials in [8]. Various specific examples illustrate the advantages of rank-1 lattices for the unique reconstruction of trigonometric polynomials. The main disadvantage of a single rank-1 lattice sampling is the necessarily growing oversampling, i.e., $M/|I|$ necessarily increases for growing $|I|$ in order to ensure a unique reconstruction of trigonometric polynomials supported on specifically structured frequency index sets I, e.g., hyperbolic cross frequency index sets I, cf. also [11, 7] for more details on this specific topic.

In order to overcome this problem, we would like to extend the method of rank-1 lattice sampling in a more or less usual manner by sampling along multiple rank-1 lattices. The corresponding sampling sets are joined rank-1 lattices

$$\mathcal{X} = \Lambda(\boldsymbol{z}_1, M_1, \boldsymbol{z}_2, M_2, \ldots, \boldsymbol{z}_s, M_s) := \bigcup_{r=1}^{s} \Lambda(\boldsymbol{z}_r, M_r)$$

and called multiple rank-1 lattice.

The $\boldsymbol{0}$ is contained in each of the rank-1 lattices $\Lambda(\boldsymbol{z}_r, M_r), r = 1, \ldots, s$. Consequently, the number of distinct elements in $\Lambda(\boldsymbol{z}_1, M_1, \boldsymbol{z}_2, M_2, \ldots, \boldsymbol{z}_s, M_s)$ is bounded from above by $|\Lambda(\boldsymbol{z}_1, M_1, \boldsymbol{z}_2, M_2, \ldots, \boldsymbol{z}_s, M_s)| \le 1-s+\sum_{r=1}^{s} M_r$. Due to the fact that there may be $1 \le r_1 < r_2 \le s$ such that $|\Lambda(\boldsymbol{z}_{r_1}, M_{r_1}) \cap \Lambda(\boldsymbol{z}_{r_2}, M_{r_2})| > 1$ or $1 \le r_3 \le s$ such that $|\Lambda(\boldsymbol{z}_{r_3}, M_{r_3})| < M_{r_3}$, we have to expect a lower number of elements within $\Lambda(\boldsymbol{z}_1, M_1, \boldsymbol{z}_2, M_2, \ldots, \boldsymbol{z}_s, M_s)$ than the upper bound promises. However, we do not care about duplicate rows within the Fourier matrix with the exception of the duplicates that arises from $\boldsymbol{x} = \boldsymbol{0}$. We define the Fourier matrix

$$\boldsymbol{A} := \boldsymbol{A}(\Lambda(\boldsymbol{z}_1, M_1, \boldsymbol{z}_2, M_2, \ldots, \boldsymbol{z}_s, M_s), I) := \begin{pmatrix} \left(\mathrm{e}^{2\pi\mathrm{i}\frac{j}{M_1}\boldsymbol{k}\cdot\boldsymbol{z}_1}\right)_{j=0,\ldots,M_1-1,\,\boldsymbol{k}\in I} \\ \left(\mathrm{e}^{2\pi\mathrm{i}\frac{j}{M_2}\boldsymbol{k}\cdot\boldsymbol{z}_2}\right)_{j=1,\ldots,M_2-1,\,\boldsymbol{k}\in I} \\ \vdots \\ \left(\mathrm{e}^{2\pi\mathrm{i}\frac{j}{M_s}\boldsymbol{k}\cdot\boldsymbol{z}_s}\right)_{j=1,\ldots,M_s-1,\,\boldsymbol{k}\in I} \end{pmatrix},$$

where we assume that the frequency indices $\boldsymbol{k} \in I$ are in a fixed order.

In the following, we prove some basics about multiple rank-1 lattices.

Lemma 2.1. *Let $\Lambda(\boldsymbol{z}_1, M_1)$ and $\Lambda(\boldsymbol{z}_2, M_2)$ be two rank-1 lattices with relative prim lattice sizes M_1 and M_2. Then, the rank-1 lattice $\Lambda(M_2\boldsymbol{z}_1 + M_1\boldsymbol{z}_2, M_1M_2)$ is a supset of $\Lambda(\boldsymbol{z}_1, M_1)$ and $\Lambda(\boldsymbol{z}_2, M_2)$, respectively. Thus, the multiple rank-1 lattice $\Lambda(\boldsymbol{z}_1, M_1, \boldsymbol{z}_2, M_2) \subset \Lambda(M_2\boldsymbol{z}_1 + M_1\boldsymbol{z}_2, M_1M_2)$ is a subset of the rank-1 lattice $\Lambda(M_2\boldsymbol{z}_1 + M_1\boldsymbol{z}_2, M_1M_2)$. Furthermore, the cardinality of $\Lambda(\boldsymbol{z}_1, M_1, \boldsymbol{z}_2, M_2)$ is given by $|\Lambda(\boldsymbol{z}_1, M_1)| + |\Lambda(\boldsymbol{z}_2, M_2)| - 1$.*

Proof. Due to the coprimality of the numbers M_1 and M_2, the chinese reminder theorem implies that there exists one $\ell \in \{0, M_1M_2 - 1\}$ such that

$$\ell \equiv j_1 \pmod{M_1} \qquad \text{and} \qquad \ell \equiv j_2 \pmod{M_2}.$$

Consequently, we obtain

$$\frac{\ell(M_2\boldsymbol{z}_1 + M_1\boldsymbol{z}_2)}{M_1M_2} \bmod \boldsymbol{1} = \left(\frac{\ell\boldsymbol{z}_1}{M_1} + \frac{\ell\boldsymbol{z}_2}{M_2}\right) \bmod \boldsymbol{1} = \left(\frac{j_1\boldsymbol{z}_1}{M_1} \bmod \boldsymbol{1} + \frac{j_2\boldsymbol{z}_2}{M_2} \bmod \boldsymbol{1}\right) \bmod \boldsymbol{1}.$$

This yields

$$\frac{\ell(M_2\boldsymbol{z}_1 + M_1\boldsymbol{z}_2)}{M_1M_2} \bmod \boldsymbol{1} = \frac{j_1\boldsymbol{z}_1}{M_1} \bmod \boldsymbol{1}$$

and

$$\frac{\ell(M_2\boldsymbol{z}_1 + M_1\boldsymbol{z}_2)}{M_1M_2} \bmod \boldsymbol{1} = \frac{j_2\boldsymbol{z}_2}{M_2} \bmod \boldsymbol{1}$$

for $j_2 = 0$ and $j_1 = 0$, respectively.

The coprimality of M_1 and M_2 directly implies that $\Lambda(\boldsymbol{z}_1, M_1) \cap \Lambda(\boldsymbol{z}_2, M_2) = \{\boldsymbol{0}\}$ and the assertion follows. ∎

Corollary 2.2. *Let the multiple rank-1 lattice $\Lambda(\boldsymbol{z}_1, M_1, \dots, \boldsymbol{z}_s, M_s)$ with pairwise coprime lattice sizes $M_1, \dots, M_s$ be given. Then, the cardinality of $\Lambda(\boldsymbol{z}_1, M_1, \dots, \boldsymbol{z}_s, M_s)$ is given by*

$$|\Lambda(\boldsymbol{z}_1, M_1, \dots, \boldsymbol{z}_s, M_s)| = 1 - s + \sum_{r=1}^{s} |\Lambda(\boldsymbol{z}_r, M_r)|$$

and the embedding

$$\Lambda(\boldsymbol{z}_1, M_1, \dots, \boldsymbol{z}_s, M_s) \subset \Lambda(\boldsymbol{z}, M)$$

holds, where the generating vector $\boldsymbol{z} = \sum_{r=1}^{s}(\prod_{\substack{l=1\\l\neq r}}^{s} M_l)\boldsymbol{z}_r$ and the lattice size $M = \prod_{r=1}^{s} M_r$ are explicitely given.

Proof. An iterative application of Lemma 2.1 yields the assertion. ∎

As a consequence of Corollary 2.2, a multiple rank-1 lattice $\Lambda(\boldsymbol{z}_1, M_1, \dots, \boldsymbol{z}_s, M_s)$ with pairwise coprime $M_1, \dots, M_s$ and a full rank matrix $\boldsymbol{A}(\Lambda(\boldsymbol{z}_1, M_1, \boldsymbol{z}_2, M_2, \dots, \boldsymbol{z}_s, M_s), I)$ is a subset of the rank-1 lattice $\Lambda(\boldsymbol{z}, M)$, $\boldsymbol{z} = \sum_{r=1}^{s}(\prod_{\substack{l=1\\l\neq r}}^{s} M_l)\boldsymbol{z}_r$ and $M = \prod_{r=1}^{s} M_r$. The matrix $\boldsymbol{A}(\Lambda(\boldsymbol{z}, M), I)$ has full column rank and, in particular, pairwise orthogonal columns, cf. [8, Lemma 3.1]. Further restrictions on the rank-1 lattices $\Lambda(\boldsymbol{z}_1, M_1), \dots, \Lambda(\boldsymbol{z}_s, M_s)$ allow for an easy determination of the number of distinct sampling values that are contained in $\Lambda(\boldsymbol{z}_1, M_1, \dots, z_s, M_s)$.

Corollary 2.3. *Under the assumptions of Corollary 2.2 and the additional requirements*

- M_r *is prim for all* $r = 1, \ldots, s$ *and*
- $\mathbf{0} \neq \boldsymbol{z}_r \in [0, M_r - 1]^d \cap \mathbb{Z}^d$,

we conclude

$$|\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)| = 1 - s + \sum_{r=1}^{s} M_r.$$

Proof. The additional requirements of the corollary guarantee $|\Lambda(\boldsymbol{z}_r, M_r)| = M_r$, $r = 1, \ldots, s$. Hence, the statement follows directly from Corollary 2.2. ■

3 Evaluation of Trigonometric Polynomials

The evaluation of a trigonometric polynomial at all nodes of a multiple rank-1 lattice $\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)$ is simply the evaluation at the s different rank-1 lattices $\Lambda(\boldsymbol{z}_1, M_1)$, $\ldots$, $\Lambda(\boldsymbol{z}_s, M_s)$. A corresponding fast Fourier transform is given by using s-times the evaluation along a single rank-1 lattice, cf. Algorithm 1, which uses only a single one-dimensional fast Fourier transform. This yields to a complexity of the fast evaluation at all nodes of the whole multiple rank-1 lattice $\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)$, cf. Algorithm 3, that is bounded by terms contained in $\mathcal{O}\left(\sum_{r=1}^{s} M_r \log M_r + sd|I|\right)$. For details on the single rank-1 lattice Fourier transform confer [12, 8].

Algorithm 1 Single Lattice Based FFT (`LFFT`)

Input:	$M \in \mathbb{N}$	lattice size of rank-1 lattice $\Lambda(\boldsymbol{z}, M)$
	$\boldsymbol{z} \in \mathbb{Z}^d$	generating vector of $\Lambda(\boldsymbol{z}, M)$
	$I \subset \mathbb{Z}^d$	frequency index set
	$\hat{\boldsymbol{p}} = (\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k} \in I}$	Fourier coefficients of $p \in \Pi_I$

$\hat{\boldsymbol{g}} = (0)_{l=0}^{M_s-1}$
for each $\boldsymbol{k} \in I$ ***do***
  $\hat{g}_{\boldsymbol{k} \cdot \boldsymbol{z}_l \bmod M_l} = \hat{g}_{\boldsymbol{k} \cdot \boldsymbol{z}_l \bmod M_l} + \hat{p}_{\boldsymbol{k}}$
end for
$\boldsymbol{p} = \text{iFFT\_1D}(\hat{\boldsymbol{g}})$
$\boldsymbol{p} = M\boldsymbol{p}$

Output:	$\boldsymbol{p} = \boldsymbol{A}\hat{\boldsymbol{p}}$	function values of $p \in \Pi_I$

Complexity: $\mathcal{O}(M \log M + d|I|)$

The evaluation of multivariate trigonometric polynomials along multiple rank-1 lattices is guaranteed by the indicated algorithm. Hence, we shift our attention to the reconstruction problem, i.e., to necessary and sufficient conditions on the sampling set $\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)$ such that each trigonometric polynomial $p \in \Pi_I$, that belongs to a given space of trigonometric polynomials, is uniquely specified by its sampling values $p(\boldsymbol{x})$, $\boldsymbol{x} \in \Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)$ along the multiple rank-1 lattice $\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)$. Moreover, we are interested in a fast and unique reconstruction of each trigonometric polynomial $p \in \Pi_I$ from its values along the sampling set $\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)$.

Algorithm 2 Adjoint Single Lattice Based FFT (`aLFFT`)

Input:	$M \in \mathbb{N}$	lattice sizes of rank-1 lattice $\Lambda(\boldsymbol{z}, M)$
	$\boldsymbol{z} \in \mathbb{Z}^d$	generating vector of $\Lambda(\boldsymbol{z}, M)$
	$I \subset \mathbb{Z}^d$	frequency index set
	$\boldsymbol{p} = \left(p\left(\frac{j}{M}\boldsymbol{z}\right)\right)_{j=0,\ldots,M-1}$	function values of $p \in \Pi_I$

$\hat{\boldsymbol{g}} = \mathrm{FFT\_1D}(\boldsymbol{p})$
$\hat{\boldsymbol{a}} = (0)_{\boldsymbol{k}\in I}$
for each $\boldsymbol{k} \in I$ ***do***
  $\hat{a}_{\boldsymbol{k}} = \hat{a}_{\boldsymbol{k}} + \hat{g}_{\boldsymbol{k}\cdot\boldsymbol{z}_l \bmod M_l}$
end for

Output:	$\hat{\boldsymbol{a}} = \boldsymbol{A}^*\boldsymbol{p}$	adjoint evaluation of the Fourier matrix $\boldsymbol{A}^*$

Complexity: $\mathcal{O}\left(M \log M + d|I|\right)$

Algorithm 3 Evaluation at multiple rank-1 lattices

Input:	$M_1, \ldots, M_s \in \mathbb{N}$	lattice sizes of rank-1 lattices $\Lambda(\boldsymbol{z}_l, M_l)$
	$\boldsymbol{z}_1, \ldots \boldsymbol{z}_s \in \mathbb{Z}^d$	generating vectors of $\Lambda(\boldsymbol{z}_l, M_l)$
	$I \subset \mathbb{Z}^d$	frequency index set
	$\hat{\boldsymbol{p}} = (\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$	Fourier coefficients of $p \in \Pi_I$

1: ***for*** $l = 1, \ldots, s$ ***do***
2:   $\boldsymbol{p}_l$=`LFFT`(M_l, $\boldsymbol{z}_l$, I, $\hat{\boldsymbol{p}}$)
3: ***end for***
4: $\boldsymbol{p} = (\boldsymbol{p}_1[1], \ldots, \boldsymbol{p}_1[M_1], \boldsymbol{p}_2[2], \ldots, \boldsymbol{p}_2[M_2], \ldots, \boldsymbol{p}_s[2], \ldots \boldsymbol{p}_s[M_s])^\top$

Output:	$\boldsymbol{p} = \boldsymbol{A}\hat{\boldsymbol{p}}$	function values of $p \in \Pi_I$

Complexity: $\mathcal{O}\left(\sum_{l=1}^{s} M_l \log M_l + sd|I|\right)$

4 Reconstruction Properties of Multiple Rank-1 Lattices

In order to investigate the reconstruction properties of a sampling set $\mathcal{X} \subset \mathbb{T}^d$, $|\mathcal{X}| < \infty$, with respect to a given frequency index set I, we have to consider the corresponding Fourier matrix $\boldsymbol{A}(\mathcal{X}, I)$. A unique reconstruction of all trigonometric polynomials $p \in \Pi_I$ from the sampling values $(p(\boldsymbol{x}))_{\boldsymbol{x}\in\mathcal{X}}$ implies necessarily a full column rank of the matrix $\boldsymbol{A}(\mathcal{X}, I)$. We may use single rank-1 lattices $\Lambda(\boldsymbol{z}, M)$ as sampling set $\mathcal{X}$, i.e., $\mathcal{X} = \Lambda(\boldsymbol{z}, M)$. In [8], we investigated necessary and sufficient conditions on the reconstruction property of single rank-1 lattices $\Lambda(\boldsymbol{z}, M)$ as spatial discretizations. Roughly speaking, we may have to expect oversampling that increase with growing cardinality of the frequency index set I, due to the group structure of the lattice nodes.

Motivated by the construction idea of sparse grids, which are in general a union of different lattices of several ranks, we will join a few rank-1 lattices as spatial discretization in order to construct sampling schemes $\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)$ that guarantee full column ranks of the Fourier matrices $\boldsymbol{A}(\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s), I)$ for given frequency index sets I. In general, we are interested in practically suitable construction strategies of such multiple rank-1 lattices

Algorithm 4 Adjoint evaluation at multiple rank-1 lattices

Input:	$M_1, \ldots, M_s \in \mathbb{N}$	lattice sizes of rank-1 lattices $\Lambda(\boldsymbol{z}_l, M_l)$
	$\boldsymbol{z}_1, \ldots \boldsymbol{z}_s \in \mathbb{Z}^d$	generating vectors of $\Lambda(\boldsymbol{z}_l, M_l)$
	$I \subset \mathbb{Z}^d$	frequency index set
	$\boldsymbol{p} = \begin{pmatrix} \boldsymbol{p}_1 \\ \vdots \\ \boldsymbol{p}_s \end{pmatrix}$	sampling values of $p \in \Pi_I$, $\boldsymbol{p}_l = \left(p(\frac{j}{M_l}\boldsymbol{z}_l)\right)_{j=1-\delta_{1,l},\ldots,M_l}$

$\hat{\boldsymbol{a}} = (0)_{\boldsymbol{k}\in I}$
for $l = 1, \ldots, s$ ***do***
  $\boldsymbol{g} = (\delta_{1,l}\boldsymbol{p}[1], \boldsymbol{p}[M_1 + \cdots + M_{l-1} - l + 3], \ldots, \boldsymbol{p}[M_1 + \cdots + M_l - l + 1])^\top$
  $\hat{\boldsymbol{a}} = \hat{\boldsymbol{a}} + \texttt{aLFFT}(M_l, \boldsymbol{z}_l, I, \boldsymbol{g})$
end for

Output:	$\hat{\boldsymbol{a}} = \boldsymbol{A}^*\boldsymbol{f}$	result of adjoint matrix times vector product

Complexity: $\mathcal{O}\left(\sum_{l=1}^{s} M_l \log M_l + sd|I|\right)$

$\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)$. We call a sampling set $\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)$ with

$$\det\left(\boldsymbol{A}^*(\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s), I)\boldsymbol{A}(\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s), I)\right) > 0$$

a reconstructing multiple rank-1 lattice for the frequency index set I.

For a given frequency index set I and given sampling set $\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)$, $|\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)| \geq |I|$, we can check the reconstruction property in different ways. For instance, one can compute the echelon form or (lower bounds on) the smallest singular value of the matrices $\boldsymbol{A}$ or $\boldsymbol{A}^*\boldsymbol{A}$ in order to check whether the rank of the matrix is full or not. We emphasize, that the complexity of the test methods is at least $\Omega\left(|I|^2\right)$ and in the case of the computation of lower bounds on the smallest singular values, cf. [19], the test may fail.

However, if the matrix $\boldsymbol{A}$ is of full column rank, one can reconstruct the Fourier coefficients of a multivariate trigonometric polynomial $p \in \Pi_I$ by solving the normal equation

$$\boldsymbol{A}^*\boldsymbol{A}\hat{\boldsymbol{p}} = \boldsymbol{A}^*\boldsymbol{p}.$$

Usually, one approximates the inverse of the matrix $\boldsymbol{A}^*\boldsymbol{A}$ using a conjugate gradient method and fast algorithms that compute the matrix-vector products associated with $\boldsymbol{A}$ and $\boldsymbol{A}^*$, cf. Algorithm 3 and Algorithm 4 associated with Algorithm 2. Thus, the fast reconstruction of a trigonometric polynomial $p \in \Pi_I$ using the samples along a reconstructing multiple rank-1 lattice $\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)$ for I is guaranteed. Anyway, we are still interested in a practically suitable construction strategy in order to determine reconstructing multiple rank-1 lattices $\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)$ for given frequency index sets I. In the following subsection, we present one possibility of such a construction.

4.1 Construction of reconstructing multiple rank-1 lattices

In this subsection we characterize an algorithm that determines reconstructing multiple rank-1 lattice for given frequency index sets $I \subset \mathbb{Z}^d$, $|I| < \infty$. Accordingly, we assume the frequency index set I of finite cardinality being given and fixed. We suggest to reconstruct a

Algorithm 5 Determining reconstructing multiple rank-1 lattices

Input:	$I \subset \mathbb{Z}^d$	frequency index set
	$\sigma \in \mathbb{R}$	oversampling factor $\sigma \geq 1$
	$n \in \mathbb{N}$	number of random test vectors

$l = 0$
while $|I| > 1$ ***do***
  $l = l + 1$
  $M_l = \text{nextprime}(\sigma|I|)$
  while $M_l \in \{M_1, \ldots, M_{l-1}\}$ ***do***
    $M_l = \text{nextprime}(M_l)$
  end while
  for $j = 1, \ldots, n$ ***do***
    choose random integer vector $\boldsymbol{v}_j^{(l)} \in [1, M_l - 1]^d$
    compute $K_j^{(l)} = |I \setminus \{\boldsymbol{k} \in I \colon \exists \boldsymbol{h} \in I \setminus \{\boldsymbol{k}\} \text{ with } \boldsymbol{k} \cdot \boldsymbol{v}_j^{(l)} \equiv \boldsymbol{h} \cdot \boldsymbol{v}_j^{(l)} \pmod{M_l}\}|$
  end for
  determine $\boldsymbol{z}_l = \boldsymbol{v}_{j_0}^{(l)}$ such that $K_{j_0}^{(l)} = \max_{j \in \{1,\ldots,n\}} K_j^{(l)}$
  $I = \{\boldsymbol{k} \in I \colon \exists \boldsymbol{h} \in I \setminus \{\boldsymbol{k}\} \text{ with } \boldsymbol{k} \cdot \boldsymbol{z}_l \equiv \boldsymbol{h} \cdot \boldsymbol{z}_l \pmod{M_l}\}$
end while

Output:	$M_1, \ldots, M_l$	lattice sizes of rank-1 lattices and
	$\boldsymbol{z}_1, \ldots, \boldsymbol{z}_l$	generating vectors of rank-1 lattices such that
	$\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_l, M_l)$	is a reconstructing multiple rank-1 lattice

trigonometric polynomial $p \in \Pi_I$ step by step. Independent of the structure of the frequency index set I, we fix a lattice size $M_1 \sim |I|$ and a generating vector $\boldsymbol{z}_1 \in [1, M_1 - 1]^d$ and reconstruct only these frequencies $\hat{p}_{\boldsymbol{k}}$ that can be uniquely reconstructed by the means of the used rank-1 lattice. The indices $\boldsymbol{k}$ of these frequencies $\hat{p}_{\boldsymbol{k}}$ are simply given by

$$I_1 = \{\boldsymbol{k} \in I \colon \boldsymbol{k} \cdot \boldsymbol{z}_1 \not\equiv \boldsymbol{h} \cdot \boldsymbol{z}_1 \bmod M_1 \text{ for all } \boldsymbol{h} \in I \setminus \{\boldsymbol{k}\}\}.$$

Assuming that the frequencies $(\hat{p}_{\boldsymbol{k}})_{\boldsymbol{k} \in I_1}$ are already determined, we have to reconstruct a trigonometric polynomial $p_1 \in \Pi_{I \setminus I_1}$ supported by the frequency index set $I \setminus I_1$ now. We determine the sampling values of this trigonometric polynomial by

$$p_1(\boldsymbol{x}) = p(\boldsymbol{x}) - \sum_{\boldsymbol{k} \in I_1} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}.$$

We apply this strategy successively as long as there are frequencies that have to be reconstructed.

Algorithm 5 indicates one possibility of an algorithm that determines a set of rank-1 lattices that allows for the application of the mentioned reconstruction strategy. Obviously, in each step we are interested in a rank-1 lattice that allows for the reconstruction of as many as possible frequencies. For that reason, we check a few generating vectors for its number of reconstructable frequency indices in Algorithm 5.

We show that Algorithm 5 determines a sampling set such that $\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)$ implies a full column rank matrix $\boldsymbol{A}(\Lambda(\boldsymbol{z}_1, M_1, \boldsymbol{z}_2, M_2, \ldots, \boldsymbol{z}_s, M_s), I)$.

Lemma 4.1. *Let the matrix $\boldsymbol{B} \in \mathbb{C}^{n\times m}$ of the following form*

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_1 & \boldsymbol{B}_2 \\ \boldsymbol{B}_3 & \boldsymbol{B}_4 \end{pmatrix}$$

be given. The matrices $\boldsymbol{B}_1 \in \mathbb{C}^{n_1\times m_1}$, ..., $\boldsymbol{B}_4 \in \mathbb{C}^{n_2\times m_2}$ are submatrices of $\boldsymbol{B}$, i.e., $n = n_1 + n_2$ and $m = m_1 + m_2$. In addition, we assume that

- *$\boldsymbol{B}_1$ has full column rank, i.e., the columns of $\boldsymbol{B}_1$ are linear independent,*
- *$\boldsymbol{B}_4$ has full column rank, i.e., the columns of $\boldsymbol{B}_4$ are linear independent, and*
- *the columns of $\boldsymbol{B}_2$ are not in the span of the columns of $\boldsymbol{B}_1$.*

Then the matrix $\boldsymbol{B}$ has full column rank.

Proof. The matrix $\boldsymbol{B} \in \mathbb{C}^{n\times m}$, $n \geq m$, has full column rank iff the columns of the matrix $\boldsymbol{B}$ are all linear independent, i.e. the formula

$$\sum_{j=1}^{m} \lambda_j \boldsymbol{b}_j = \boldsymbol{0} \tag{4.1}$$

has the unique solution $\boldsymbol{\lambda} = \boldsymbol{0}$. We will exploit the full column rank of the matrix $\boldsymbol{B}_1 \in \mathbb{C}^{n_1\times m_1}$, $n_1 \geq m_1$, and, thus, we consider the sum

$$\sum_{j=1}^{m} \lambda_j \boldsymbol{b}'_j = \boldsymbol{0},$$

where $\boldsymbol{b}'_j = (b_{j,l})_{l=1}^{n_1}$ are vectors consisting of the first n_1 elements of the vectors $\boldsymbol{b}_j$. Due to the fact that the columns of $\boldsymbol{B}_2 \in \mathbb{C}^{n_1\times m_2}$ are not in the span of the columns of $\boldsymbol{B}_1$ and the columns of $\boldsymbol{B}_1$ are linear independent, we achieve $\lambda_j = 0$ for all $j = 1, \ldots, m_1$.

Accordingly, (4.1) simplifies to

$$\sum_{j=m_1+1}^{m} \lambda_j \boldsymbol{b}_j = \boldsymbol{0}.$$

For the remaining vectors $\boldsymbol{b}_j$, $j = m_1 + 1, \ldots, m$, we know that the vectors of the last m_2 components of $\boldsymbol{b}_j$ are linear independent and, consequently, we obtain $\lambda_j = 0$ for all $j = 1, \ldots, m$. ■

Theorem 4.2. *Algorithm 5 determines sampling sets such that the corresponding Fourier matrix $\boldsymbol{A}(\Lambda(\boldsymbol{z}_1, M_1, \boldsymbol{z}_2, M_2, \ldots, \boldsymbol{z}_s, M_s), I)$ is a full column rank matrix.*

Proof. In order to exploit Lemma 4.1 we will need to rearrange the order of the columns of $\boldsymbol{A}(\Lambda(\boldsymbol{z}_1, M_1, \boldsymbol{z}_2, M_2, \ldots, \boldsymbol{z}_s, M_s), I)$ in a suitable way. We assume that $M_1, \ldots, M_s$ and $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_s$ is the output of Algorithm 5. Consequently, we can determine the following frequency index sets

$$\begin{aligned}
I_1^{\complement} &= \{\boldsymbol{k} \in I \colon \exists \boldsymbol{h} \in I \setminus \{\boldsymbol{k}\} \text{ with } \boldsymbol{k}\cdot\boldsymbol{z}_1 \equiv \boldsymbol{h}\cdot\boldsymbol{z}_1 \pmod{M_1}\} && \text{and} \quad I_1 = I \setminus I_1^{\complement}, \\
I_2^{\complement} &= \{\boldsymbol{k} \in I_1^{\complement} \colon \exists \boldsymbol{h} \in I_1^{\complement} \setminus \{\boldsymbol{k}\} \text{ with } \boldsymbol{k}\cdot\boldsymbol{z}_2 \equiv \boldsymbol{h}\cdot\boldsymbol{z}_2 \pmod{M_2}\} && \text{and} \quad I_2 = I_1^{\complement} \setminus I_2^{\complement}, \\
&\vdots && \qquad\qquad \vdots \\
I_s^{\complement} &= \{\boldsymbol{k} \in I_{s-1}^{\complement} \colon \exists \boldsymbol{h} \in I_{s-1}^{\complement} \setminus \{\boldsymbol{k}\} \text{ with } \boldsymbol{k}\cdot\boldsymbol{z}_s \equiv \boldsymbol{h}\cdot\boldsymbol{z}_s \pmod{M_s}\} = \emptyset && \text{and} \quad I_s = I_{s-1}^{\complement} \setminus I_s^{\complement}.
\end{aligned}$$

The resulting frequency index sets I_j, $j = 1, \ldots, s$, yield a disjoint partition of I. We rearrange the columns of the matrix $\boldsymbol{A}(\Lambda(\boldsymbol{z}_1, M_1, \boldsymbol{z}_2, M_2, \ldots, \boldsymbol{z}_s, M_s), I)$ and we achieve a matrix

$$\tilde{\boldsymbol{A}} := \begin{pmatrix} \boldsymbol{K}^{1,1} & \ldots & \boldsymbol{K}^{1,s} \\ \vdots & \ddots & \vdots \\ \boldsymbol{K}^{s,1} & \ldots & \boldsymbol{K}^{s,s} \end{pmatrix},$$

where the submatrices $\boldsymbol{K}^{r,l}$ are given by $\boldsymbol{K}^{r,l} := \left(\mathrm{e}^{2\pi\mathrm{i}\frac{j}{M_r}\boldsymbol{k}\cdot\boldsymbol{z}_r}\right)_{j=1-\delta_{1,r},\ldots,M_r-1,\,\boldsymbol{k}\in I_l}$, $(r,l) \in [1,s]^2 \cap \mathbb{N}^2$.

We define the matrices

$$\tilde{\boldsymbol{A}}_l := \left(\boldsymbol{K}^{r,t}\right)_{r=1,\ldots,l,\,t=1,\ldots,l} = \begin{pmatrix} & \tilde{\boldsymbol{A}}_{l-1} & & \begin{matrix}\boldsymbol{K}^{1,l} \\ \vdots \\ \boldsymbol{K}^{l-1,l}\end{matrix} \\ \boldsymbol{K}^{l,1} & \ldots & \boldsymbol{K}^{l,l-1} & \boldsymbol{K}^{l,l} \end{pmatrix},$$

which are in fact submatrices of $\tilde{\boldsymbol{A}}$. In particular, we obtain $\tilde{\boldsymbol{A}}_1 = \boldsymbol{K}^{1,1}$ and $\tilde{\boldsymbol{A}}_s = \tilde{\boldsymbol{A}}$.

In the following, we conclude the full column rank of $\tilde{\boldsymbol{A}}_l$ from the full column rank of $\tilde{\boldsymbol{A}}_{l-1}$, the full column rank of $\boldsymbol{K}^{l,l}$, and the linear independence of each column of the matrix

$$\begin{pmatrix} \boldsymbol{K}^{1,l} \\ \vdots \\ \boldsymbol{K}^{l-1,l} \end{pmatrix}$$

and all columns of the matrix $\tilde{\boldsymbol{A}}_{l-1}$, cf. Lemma 4.1.

We start with $l = 1$, i.e., $\tilde{\boldsymbol{A}}_1 = \boldsymbol{K}^{1,1}$. Since $\Lambda(\boldsymbol{z}_1, M_1)$ is a reconstructing rank-1 lattice for I_1, the matrix $\tilde{\boldsymbol{A}}_1$ has linear independent columns. Due to the construction of I_1 and $I_1^{\complement}$, each column of one of the matrices $\boldsymbol{K}^{1,r}$, $r = 2, \ldots, s$, is not in the span of the columns of $\boldsymbol{K}^{1,1}$.

We prove the full rank of $\tilde{\boldsymbol{A}}_l$, $l = 2, \ldots, s$, inductively. For that, we assume $\tilde{\boldsymbol{A}}_{l-1}$ to be of full column rank. Additionally, we know that $\boldsymbol{K}^{l,l}$ is of full column rank, since $\Lambda(\boldsymbol{z}_l, M_l)$ is a reconstructing rank-1 lattice for I_l. In order to apply Lemma 4.1, we have to show, that no column of the matrix

$$\begin{pmatrix} \boldsymbol{K}^{1,l} \\ \vdots \\ \boldsymbol{K}^{l-1,l} \end{pmatrix} \tag{4.2}$$

is a linear combination of the columns of $\tilde{\boldsymbol{A}}_{l-1}$. We assume the contrary, i.e., let $\boldsymbol{k} \in I_l$ be a frequency index, $\boldsymbol{a_k}$ the corresponding column of the matrix in (4.2), such that

$$\boldsymbol{a_k} = \tilde{\boldsymbol{A}}_{l-1}\boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{K}^{1,1} & \ldots & \boldsymbol{K}^{1,l-1} \\ \vdots & \ldots & \vdots \\ \boldsymbol{K}^{l-1,1} & \ldots & \boldsymbol{K}^{l-1,l-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda}_1 \\ \vdots \\ \boldsymbol{\lambda}_{l-1} \end{pmatrix}, \tag{4.3}$$

where we look for the vectors $\boldsymbol{\lambda}_r \in \mathbb{C}^{|I_r|}$, $r = 1, \ldots, l-1$. We solve this linear equation recursively. The first M_1 rows of $\boldsymbol{a_k}$ are not in the span of the pairwise orthogonal columns of $\boldsymbol{K}^{1,1}$ and the columns of the first M_1 rows of $\boldsymbol{K}^{1,r}$, $r = 2, \ldots, l-1$ are also not in the span of the columns of $\boldsymbol{K}^{1,1}$. Consequently, we observe that $\boldsymbol{\lambda}_1 = \boldsymbol{0} \in \mathbb{C}^{|I_1|}$ holds.

Accordingly, the solution of (4.3) is given by $\boldsymbol{\lambda} := (\boldsymbol{\lambda}_1^\top, \ldots, \boldsymbol{\lambda}_{l-1}^\top)^\top = (\underbrace{0, \ldots, 0}_{|I_1| \text{ times}}, \boldsymbol{\lambda}_2^\top, \ldots, \boldsymbol{\lambda}_{l-1}^\top)^\top$ and we search for the solution of

$$\boldsymbol{a}_{\boldsymbol{k}} = \begin{pmatrix} \boldsymbol{K}^{1,2} & \ldots & \boldsymbol{K}^{1,l-1} \\ \vdots & \ldots & \vdots \\ \boldsymbol{K}^{l-1,2} & \ldots & \boldsymbol{K}^{l-1,l-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda}_2 \\ \vdots \\ \boldsymbol{\lambda}_{l-1} \end{pmatrix}.$$

Next, we consider the rows numbered by $M_1 + 1, \ldots, M_1 + M_2 - 1$ and obtain

$$(a_{\boldsymbol{k},j})_{j=1+M_1}^{-1+M_1+M2} = \left(\boldsymbol{K}^{2,2}, \ldots, \boldsymbol{K}^{2,l-1}\right) \begin{pmatrix} \boldsymbol{\lambda}_2 \\ \vdots \\ \boldsymbol{\lambda}_{l-1} \end{pmatrix}.$$

Since $\Lambda(\boldsymbol{z}_2, M_2)$ is a reconstructing rank-1 lattice for I_2 and there is no $\boldsymbol{k}' \in \bigcup_{j=3}^{l} I_j$ that aliases to a frequency index $\boldsymbol{k}'' \in I_2$ with respect to $\Lambda(\boldsymbol{z}_2, M_2)$, we obtain the result $\boldsymbol{\lambda}_2 = (0, \ldots, 0)^\top \in \mathbb{C}^{|I_2|}$. These considerations lead inductively to the formulas

$$(a_{\boldsymbol{k},j})_{j=3-t+\sum_{r=1}^{t-1} M_r}^{1-t+\sum_{r=1}^{t} M_r} = \left(\boldsymbol{K}^{t,t}, \ldots, \boldsymbol{K}^{t,l-1}\right) \begin{pmatrix} \boldsymbol{\lambda}_t \\ \vdots \\ \boldsymbol{\lambda}_{l-1} \end{pmatrix}, \qquad t = 1, \ldots, l-1$$

and the result $\boldsymbol{\lambda}_t = \boldsymbol{0} \in \mathbb{C}^{|I_t|}$, $t = 1, \ldots, l-1$, in (4.3), which implies $\boldsymbol{a}_{\boldsymbol{k}} = \boldsymbol{0}$ and is in contradiction to $\|\boldsymbol{a}_{\boldsymbol{k}}\|_1 = 1 - (l-1) + \sum_{r=1}^{l-1} M_r > 0$.

Consequently, we apply Lemma 4.1 on the matrix $\tilde{\boldsymbol{A}}_l$ and observe the full column rank of $\tilde{\boldsymbol{A}}_l$, in particular for $l = s$. ■

Algorithm 5 determines a reconstructing sampling scheme for all multivariate trigonometric polynomials with frequencies supported on the index set I. Moreover, the idea behind Algorithm 5 is the stepwise reconstruction of trigonometric polynomials as mentioned above. Accordingly, we indicate the corresponding reconstruction strategy in Algorithm 6, where we use Algorithms 1 and 2 in order to compute the required single lattice based discrete Fourier transforms. As a consequence, we achieve a computational complexity of this algorithm which is bounded by $C\left(\sum_{l=1}^{s} M_l \log M_l + s(d + \log|I|)|I|\right)$, where the term C does not depend on the multiple rank-1 lattice $\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)$, the frequency index set I, or the spatial dimension d.

5 Numerics

Basically, we use Algorithm 5 with oversampling parameter $\sigma = 1$ and we choose $n = 10d$ for our numerical examples. There is only a slight modification of the implemented algorithm: If there is an l such that $\max_{j \in \{1,\ldots,n\}} K_j^{(l)} = 0$, then we do not use the corresponding sampling values, since the sampling values do not ensure additional information for the reconstruction.

Algorithm 6 Direct reconstruction of trigonometric polynomials $p \in \Pi_I$ from samples along reconstructing multiple rank-1 lattices that are determined by Algorithm 5

Input:	$M_1, \dots, M_s \in \mathbb{N}$	lattice sizes of rank-1 lattices $\Lambda(\boldsymbol{z}_l, M_l)$
	$\boldsymbol{z}_1, \dots z_s \in \mathbb{Z}^d$	generating vectors of $\Lambda(\boldsymbol{z}_l, M_l)$
	$I \subset \mathbb{Z}^d$	frequency index set
	$\boldsymbol{p} = \begin{pmatrix} \boldsymbol{p}_1 \\ \vdots \\ \boldsymbol{p}_s \end{pmatrix}$	sampling values of $p \in \Pi_I$, $\boldsymbol{p}_l = \left(p(\frac{j}{M_l}\boldsymbol{z}_l)\right)_{j=1-\delta_{1,l},\dots,M_l}$

$I_R = \{\}$
$\hat{\boldsymbol{p}} = (0)_{\boldsymbol{k}\in I}$
for $l = 1, \dots, s$ ***do***
  $I_l = \{\boldsymbol{k} \in I \setminus I_R \colon \boldsymbol{k} \cdot \boldsymbol{z}_l \not\equiv \boldsymbol{h} \cdot \boldsymbol{z}_l \pmod{M_l} \text{ for all } \boldsymbol{h} \in I \setminus (I_R \cup \{\boldsymbol{k}\})\}$
  $\boldsymbol{g}_l = (\boldsymbol{p}[1], \boldsymbol{p}[3 - l + \sum_{r=1}^{l-1} M_r], \dots, \boldsymbol{p}[1 - l + \sum_{r=1}^{l} M_r])^\top - \mathtt{LFFT}(M_l, \boldsymbol{z}_l, I_R, (\hat{\boldsymbol{p}}_{\boldsymbol{k}})_{\boldsymbol{k}\in I_R})$
  $(\hat{\boldsymbol{p}}_{\boldsymbol{k}})_{\boldsymbol{k}\in I_l} = \mathtt{aLFFT}(M_l, \boldsymbol{z}_l, I_l, \boldsymbol{g}_l)$
  $I_R = I_R \cup I_l$
end for

Output:	$\hat{\boldsymbol{p}} = (\boldsymbol{A}^*\boldsymbol{A})^{-1}\boldsymbol{A}^*\boldsymbol{p}$	Fourier coefficients of $p \in \Pi_I$

Complexity: $\mathcal{O}\left(\sum_{l=1}^{s} M_l \log M_l + s(d + \log |I|)|I|\right)$

5.1 Axis Crosses

We consider so-called axis crosses of a specific width $N = 2^n \in \mathbb{N}$ defined by

$$I = I_{\mathrm{ac},N}^d := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \|\boldsymbol{k}\|_\infty = \|\boldsymbol{k}\|_1 \leq 2^n\}$$

as frequency index sets. One can use sampling values along single rank-1 lattices $\Lambda(\boldsymbol{z}, M)$ for the reconstruction of multivariate trigonometric polynomials supported on axis crosses, cf. [8, Ex. 3.27]. The main disadvantage of this approach is the necessary oversampling, i.e., one needs at least $|\Lambda(\boldsymbol{z}, M)| \geq (N+1)^2$ sampling values in order to ensure a unique reconstruction of trigonometric polynomials $p \in \Pi_{I_{\mathrm{ac},N}^d}$. We compare this lower bound on the number of sampling values to the number of frequency indices within $I_{\mathrm{ac},N}^d$. This yields oversampling factors of

$$\frac{|\Lambda(\boldsymbol{z}, M)|}{|I_{\mathrm{ac},N}^d|} \geq \frac{(N+1)^2}{2dN+1} \geq \frac{N}{2d}$$

and thus the oversampling increases linearly in N.

On the other hand, we want to use multiple rank-1 lattices as spatial discretizations. Our numerical tests, illustrated in Figure 5.1, allows for the conjecture that the oversampling factors stagnate for growing width N of the considered axis crosses of fixed dimension d. Moreover, we observe small oversampling factors which are always below three. For comparison, sampling along a reconstructing single rank-1 lattice necessarily implies oversampling factors greater than 400 for $N \geq 2^{14}$ and all dimensions $2 \leq d \leq 20$.

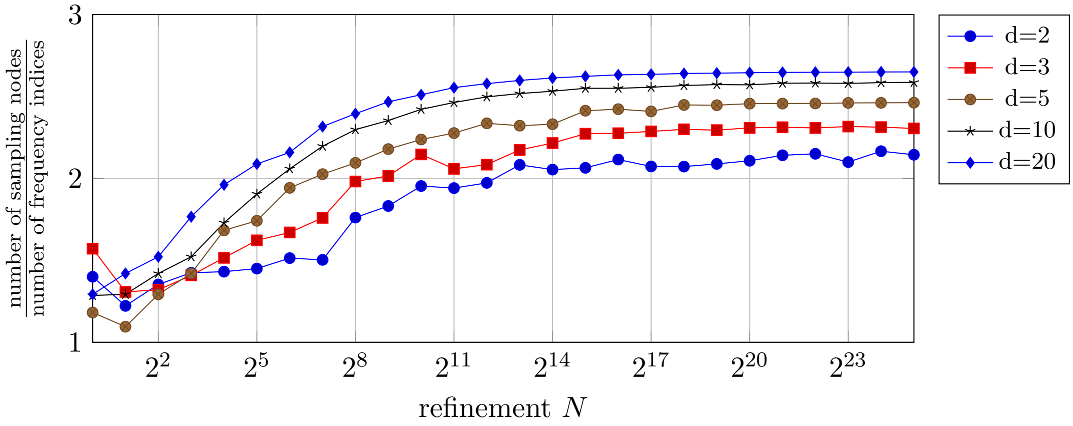

Figure 5.1: Oversampling factors of reconstructing multiple rank-1 lattices for axis crosses $I^d_{\mathrm{ac},N}$.

5.2 Dyadic Hyperbolic Crosses

In this secion, we consider so-called dyadic hyperbolic crosses defined by

$$I = I^d_{\mathrm{dhc},N} := \bigcup_{\substack{\boldsymbol{l}\in\mathbb{N}_0^d \\ \|\boldsymbol{l}\|_1=n}} \hat{G}_{\boldsymbol{l}}, \qquad \hat{G}_{\boldsymbol{l}} = \mathbb{Z}^d \cap \bigtimes_{s=1}^{d} (-2^{l_s-1}, 2^{l_s-1}],$$

where the number $N = 2^n$, $n \in \mathbb{N}_0$, is a power of two.

We applied Algorithm 5 to dyadic hyperbolic crosses of different refinements and dimensions in order to determine reconstructing multiple rank-1 lattices. The resulting oversampling factors slightly grows with respect to the refinement and dimension, cf. Figure 5.2. We observe that the oversampling factors seem to stagnate. In particular, Euler's number e bounds them from above in all our numerical tests.

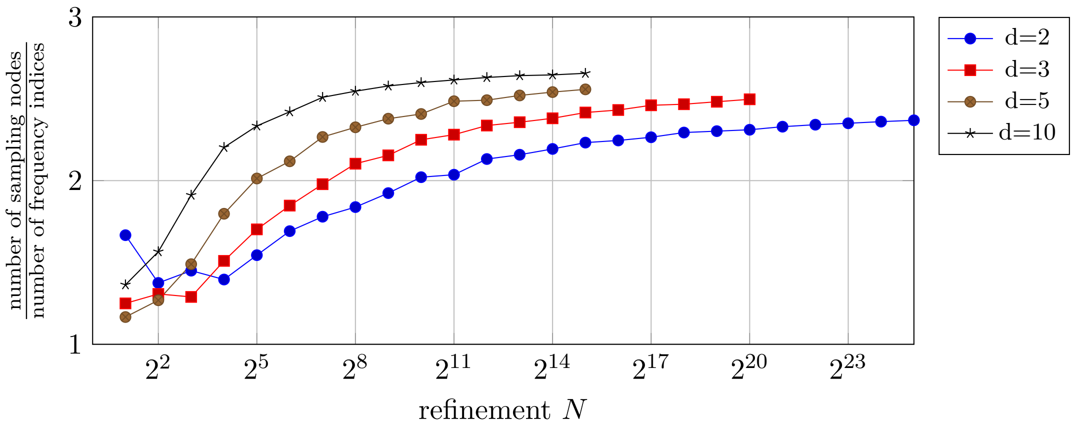

Figure 5.2: Oversampling factors of reconstructing multiple rank-1 lattices for dyadic hyperbolic crosses $I^d_{\mathrm{dhc},N}$.

In the following, we compare three different sampling methods in order to reconstruct

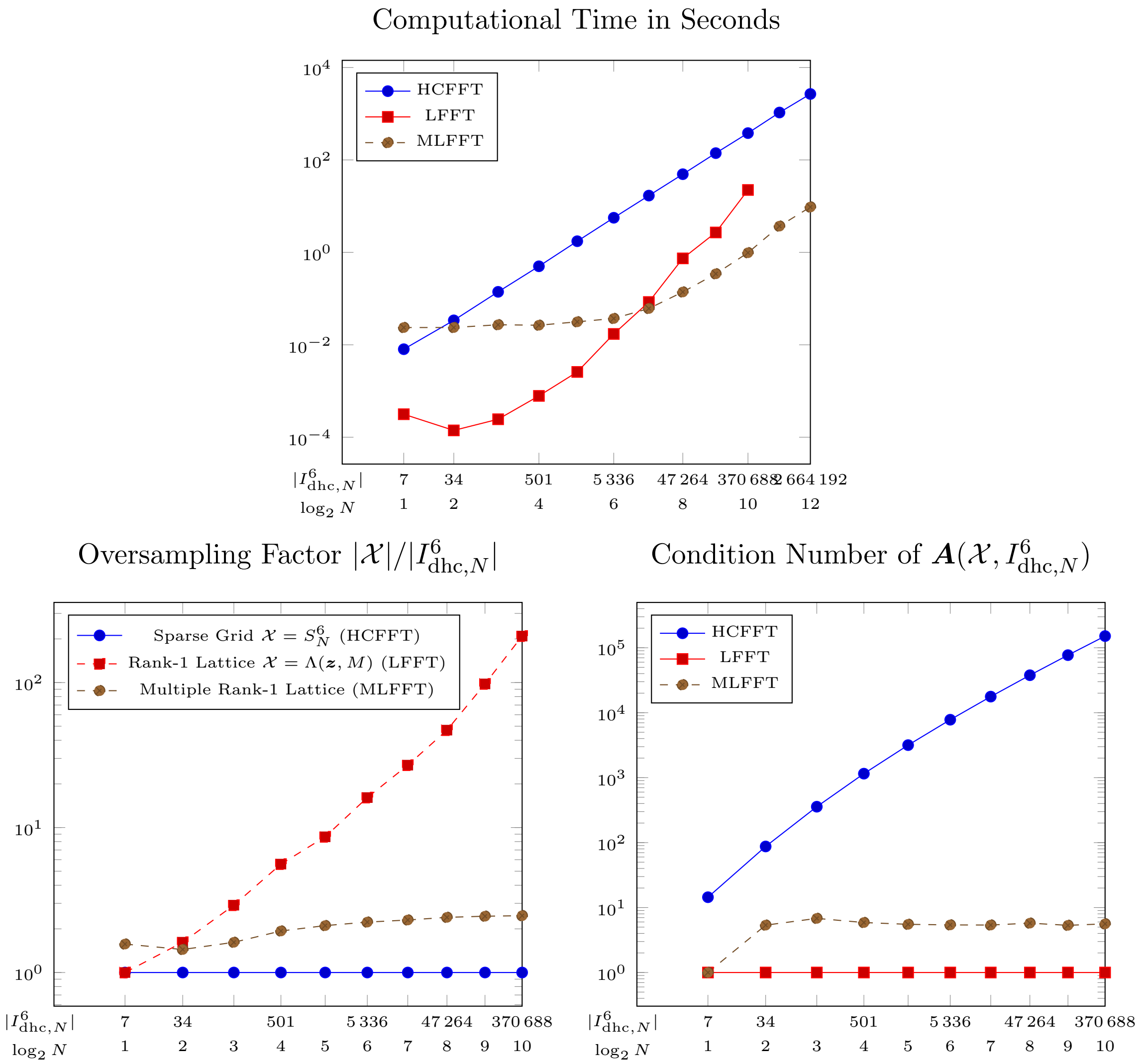

Figure 5.3: Six-dimensional hyperbolic cross fast Fourier transforms for comparison.

multivariate trigonometric polynomials with frequencies supported on dyadic hyperbolic cross index sets:

- Sampling along sparse grids (HCFFT, cf. [6]),
- sampling along a reconstructing single rank-1 lattices (LFFT, cf. [7]),
- sampling along multiple rank-1 lattices (MLFFT).

Our numerical tests illustrate the characteristics of the discrete Fourier transforms (condition number of $\boldsymbol{A}(\mathcal{X}, I)$, spent oversampling $|\mathcal{X}|/|I|$) and the corresponding fast algorithms (computational times).

Fixed Dimension $d = 6$

Since we are interested in numerical tests that visualize in some sense the asymptotic behavior of the three different sampling methods, we fix the moderate dimension $d = 6$. Thus, we can compute condition numbers even for moderate refinements $N = 2^n$ up to $N = 1024$, cf. Figure 5.3. We observe, that the discrete Fourier transform based on multiple rank-1 lattice sampling need only low overampling factors and that the corresponding Fourier matrices are well-conditioned. Moreover, the computational complexity of the fast Algorithm of the Fourier transform (MLFFT) is illustrated and at least almost as good as the computational complexity of the HCFFT with respect to N. Accordingly, the MLFFT seems to avoid both the disadvantage of the HCFFT (growing condition numbers, cf. [10]) and the disadvantage of the LFFT (growing oversampling factors and the associated fast growing computational times, cf. [11]).

Fixed Refinement $N = 2^2, \ldots, 2^5$

In [11, Fig. 4.2] we compared the computational times of sampling along sparse grids to sampling along reconstructing single rank-1 lattices. The lesson of this figure is clear: Unique sampling along reconstructing single rank-1 lattices may be not optimal due to the fact that the number of necessarily used sampling values, cf. [7], may be not optimal, in general.

Similar to this, we illustrate the computational times of different fast algorithms in Figure 5.4. In particular, we mapped the computational times of the

- hyperbolic cross fast Fourier transform (HCFFT), i.e., the fast algorithm for the evaluation of hyperbolic cross trigonometric polynomials at all nodes of a sparse grid and the fast algorithm for the reconstruction of hyperbolic cross trigonometric polynomials from the sampling values at sparse grid nodes,
- multiple lattice fast Fourier transform (MLFFT) applied to hyperbolic cross trigonometric polynomials, i.e., the fast algorithm for the evaluation of multivariate trigonometric polynomials at all nodes of a reconstructing multiple rank-1 lattice and different fast algorithms for the reconstruction of multivariate trigonometric polynomials from the sampling values at a reconstructing multiple rank-1 lattice.

In addition to the direct reconstruction, cf. Algorithm 6, we applied a CG method with starting vector $\hat{\boldsymbol{p}} = \mathbf{0}$ and a CG method with starting vector $\hat{\boldsymbol{p}}$ that is the result of the direct reconstruction. We are interested in the application of such a CG method since the direct reconstruction suffers from growing relative errors

$$\mathrm{err}_2 := \frac{\|\tilde{\hat{\boldsymbol{p}}} - \hat{\boldsymbol{p}}\|_2}{\|\hat{\boldsymbol{p}}\|_2}, \tag{5.1}$$

where $\tilde{\hat{\boldsymbol{p}}}$ is the result of the reconstruction from sampling values of the hyperbolic cross trigonometric polynomial with frequencies $\hat{\boldsymbol{p}}$, cf. Figure 5.5. This figure also contains the relative errors of the fast reconstruction method from sampling values at sparse grids (HCFFT). We point out that this fast algorithm also suffers from growing relative errors. We did not apply a conjugate gradient method on the HCFFT since we expect huge computational costs due to the expected number of iterations of the CG method which is indicated by the growing condition numbers of the corresponding Fourier matrices.

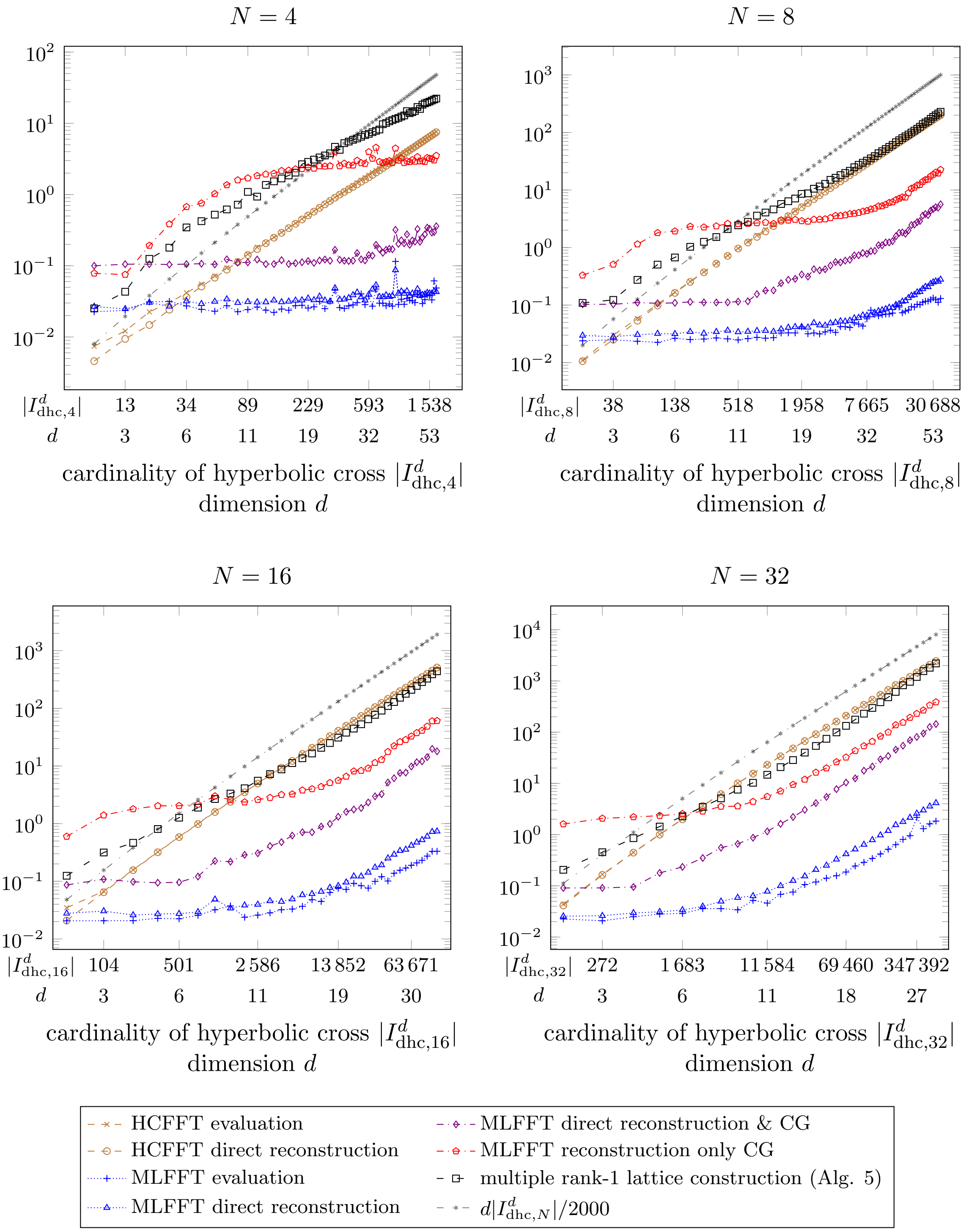

Figure 5.4: Computational times in seconds of the fast algorithms computing the dyadic hyperbolic cross discrete Fourier transforms with respect to the problem size $|I^d_{\mathrm{dhc},N}|$.

However, the runtime of the new fast algorithms (MLFFT) behave similar to the runtime of the HCFFT with respect to the dimension d. Even the CG methods have a similar runtime

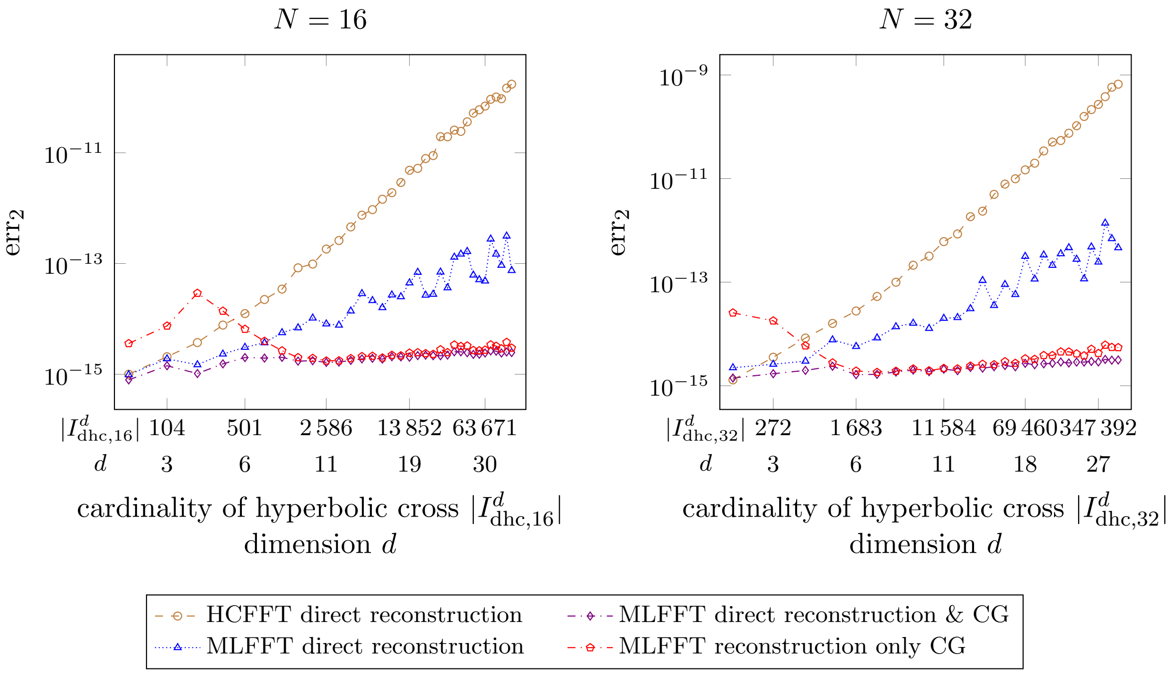

Figure 5.5: Relative errors, cf. (5.1), of the fast algorithms computing the dyadic hyperbolic cross discrete Fourier transforms with respect to the problem size $|I^d_{\mathrm{dhc},N}|$.

behavior. This indicates that the number of iterations that are used during the CG method only slightly increases with growing dimension which may be caused by bounded condition numbers of the Fourier matrices, cf. Figure 5.6.

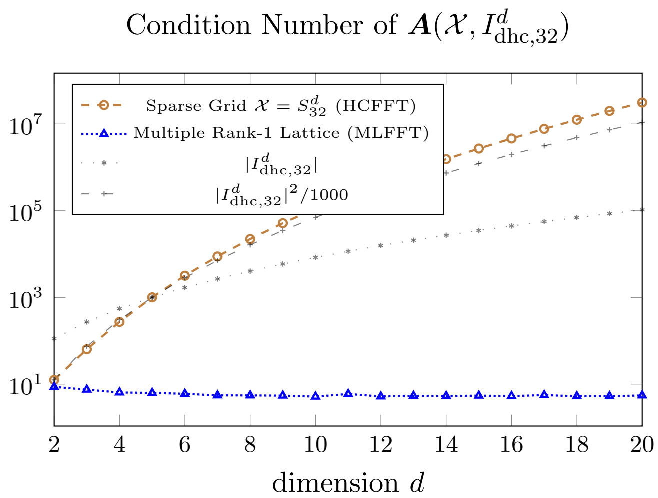

Figure 5.6: Condition numbers of the Fourier matrices using corresponding sparse grids and reconstructing multiple rank-1 lattices as sampling schemes.

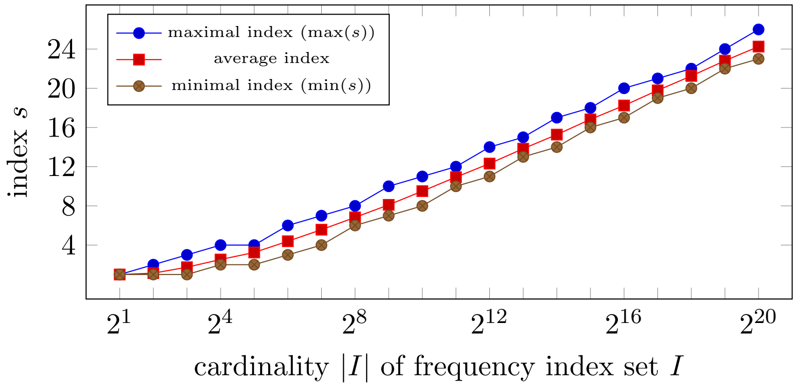

Figure 5.7: Indices s of reconstructing multiple rank-1 lattices for randomly chosen ten-dimensional frequency index sets I.

5.3 Random frequency index sets

The complexity of the new fast Fourier transform algorithms, cf. Algorithms 3, 4, and 6, mainly depends on the number s of rank-1 lattices that are joined in order to build the multiple rank-1 lattice $\Lambda(\boldsymbol{z}_1, M_1, \dots, \boldsymbol{z}_s, M_s)$. We demonstrate the behavior of s for ten-dimensional randomly chosen frequency index sets. Each component of the indices $\boldsymbol{k} \in I \subset \mathbb{Z}^{10}$ are rounded values that are chosen from a normal distribution with mean zero and variance 10 000.

We considered frequency index sets I of different cardinalities which are powers of two, i.e. $|I| \in \{2, 4, 8, \dots, 2^{20}\}$. We produced 1 000 different frequency index sets for each of the cardinalities and constructed one reconstructing multiple rank-1 lattice for each of the frequency index sets. Figure 5.7 plots the cardinality of the frequency index set I against the maximal number s, the minimal number s and the average of the numbers s that occurred in our numerical tests. We observe that the number s behave logarithmically with respect to the cardinality of I. We would like to point out that we observed a similar behavior in all numerical tests that we treated before.

Furthermore, the oversampling factors $\sum_{r=1}^{s} M_r/|I|$ are less than Euler's number e. In addition, we computed the condition numbers of the Fourier matrices $\boldsymbol{A}(\Lambda(\boldsymbol{z}_1, M_1, \dots, \boldsymbol{z}_s, M_s), I)$ for the cases $|I| \leq 2^{12}$ and obtained low condition numbers less than 16. The average of the condition numbers were less than seven in each of the cases $|I| = 2^n$, $n = 1, \dots, 12$.

6 Conclusion

The paper presents a new sampling method that allows for the unique sampling of sparse multivariate trigonometric polynomials. The main idea is to sample along more than one lattice grid similar to the sparse grids. In contrast to sparse grids we restrict the used lattices to rank-1 lattices. The constructive idea that determines reconstructing multiple rank-1 lattices, cf. Algorithm 5, allows for another point of view. The resulting multiple rank-1 lattices $\Lambda(\boldsymbol{z}_1, M_1, \dots, \boldsymbol{z}_s, M_s)$ are subsampling schemes of huge single reconstructing rank-1 lattices $\Lambda(\boldsymbol{z}, M)$ for the frequency index set I, where $\boldsymbol{z}$ and M are specified in Corollary 2.2. The Fourier matrix $\boldsymbol{A}(\Lambda(\boldsymbol{z}, M), I)$ that belongs to this huge rank-1 lattice $\Lambda(\boldsymbol{z}, M)$ has orthogonal columns. All numerical tests indicate that the reconstructing multiple rank-1 lattice $\Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_s, M_s)$ is a stable subsampling scheme of the corresponding reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M)$.

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