

**Spectral threshold dominance,  
Brouwer's conjecture  
and maximality of Laplacian energy**

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# Spectral threshold dominance, Brouwer's conjecture and maximality of Laplacian energy

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## Abstract

The Laplacian energy of a graph is the sum of the distances of the eigenvalues of the Laplacian matrix of the graph to the graph's average degree. The maximum Laplacian energy over all graphs on  $n$  nodes and  $m$  edges is conjectured to be attained for threshold graphs. We prove the conjecture to hold for graphs with the property that for each  $k$  there is a threshold graph on the same number of nodes and edges whose sum of the  $k$  largest Laplacian eigenvalues exceeds that of the  $k$  largest Laplacian eigenvalues of the graph. We call such graphs spectrally threshold dominated. These graphs include split graphs and cographs and spectral threshold dominance is preserved by disjoint unions and taking complements. We conjecture that all graphs are spectrally threshold dominated. This conjecture turns out to be equivalent to Brouwer's conjecture concerning a bound on the sum of the  $k$  largest Laplacian eigenvalues.

**Keywords:** Laplacian Energy, threshold graph, Brouwer conjecture, Grone-Merris-Bai Theorem

**MSC 2010:** 05C50, 05C35

## 1 Introduction

Let  $G = (N, E)$  be a simple graph with node set  $N = \{1, \dots, n\}$  and edge set  $E \subseteq \{\{i, j\} : i, j \in N, i \neq j\}$ . For brevity, we will usually write  $ij$  instead of  $\{i, j\}$  for edges and put  $m = |E|$ . It will be convenient to assume that the nodes are numbered so that their degrees  $d_i = |\{j : ij \in E\}|$  are sorted nonincreasingly. Let  $e_i$  denote the  $i$ -th column of the  $n \times n$  identity matrix  $I_n$  and define the positive semidefinite matrices  $E_{ij} := (e_i - e_j)(e_i - e_j)^T$ , then the Laplacian matrix of  $G$  is defined to be  $L(G) = \sum_{ij \in E} E_{ij}$ . If  $G$  is clear from the context, we drop the argument and simply write  $L$ . The Laplacian is a positive semidefinite matrix with a trivial eigenvalue 0 and the vector of all ones  $\mathbf{1}$  as associated eigenvector. In this paper we denote the eigenvalues of  $L$  in nonincreasing order by  $\lambda_1(L) \geq \dots \geq \lambda_{n-1}(L) \geq \lambda_n(L) = 0$ . As the trace of  $L$  is  $2m$  there holds  $\sum_{i=1}^n \lambda_i(L) = 2m$  and for  $m > 0$  at least one eigenvalue has value greater than the average degree  $2m/n$ .

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The Laplacian energy is defined to be

$$LE(G) := \sum_{i=1}^n \left| \lambda_i(L) - \frac{2m}{n} \right|.$$

For  $i = 1, \dots, n$  the conjugate degree  $d_i^*(G) = |\{i: d_i \geq i\}|$  gives the number of nodes of  $G$  of degree at least  $i$ . Each degree sequence satisfying  $d_i^* = d_i + 1$  for  $i = 1, \dots, f$  with  $trace\ f = \max\{i: d_i \geq i\}$  uniquely defines a graph and these graphs form the so called threshold graphs [8]. In our context a central property of threshold graphs is that the conjugate degrees are exactly the eigenvalues of their Laplacian matrix,  $\lambda_i = d_i^*$  for  $i = 1, \dots, n$  [7]. It has been conjectured that among all connected graphs on  $n$  nodes the threshold graph called pineapple with trace  $\lfloor \frac{2n}{3} \rfloor$  maximizes the Laplacian energy (see [10]). Among connected threshold graphs the pineapple is indeed the maximizer; for general threshold graphs on  $n$  nodes the clique of size  $\lfloor \frac{2n+1}{3} \rfloor + 1$  together with  $\lfloor \frac{n-3}{3} \rfloor$  isolated vertices is a threshold graph maximizing Laplacian energy [5] and we conjecture that this graph has maximum Laplacian energy among all graphs on  $n$  nodes.

In this paper we prove that the general conjecture holds for graphs that are spectrally dominated by threshold graphs in the following sense.

**Definition 1** *A graph  $G$  on  $n$  nodes with  $m$  edges is spectrally threshold dominated if for each  $k \in \{1, \dots, n\}$  there is a threshold graph  $T_k$  having the same number of nodes and edges satisfying  $\sum_{i=1}^k d_i^*(T_k) = \sum_{i=1}^k \lambda_i(L(T_k)) \geq \sum_{i=1}^k \lambda_i(L(G))$ .*

This definition was in part motivated by the Grone-Merris conjecture, proved by Bai [1] – from here on called the Grone-Merris-Bai Theorem – which states that for any graph  $G$  on  $n$  vertices with degree sequence  $d_1 \geq \dots \geq d_n$  and for any  $k \in \{1, \dots, n\}$ ,

$$\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k d_i^*. \tag{1}$$

Note that equality holds in (1) for threshold graphs.

Our main result (proved in Section 2) is the following.

**Theorem 2** *For each spectrally threshold dominated graph  $G$  there exists a threshold graph with the same number of nodes and edges whose Laplacian energy is at least as large as that of  $G$ .*

We conjecture that all graphs are spectrally threshold dominated, in which case the maximum Laplacian energy would be attained by threshold graphs for any given number of nodes and edges. We prove that this class goes well beyond threshold graphs (definitions of the graph classes will be given along with the proofs in Section 3 and Section 4).

**Theorem 3** *Split graphs are spectrally threshold dominated.*

**Theorem 4** *Disjoint unions and complements of spectrally threshold dominated graphs are spectrally threshold dominated.*

This has the following immediate consequence.

**Corollary 5** *Cographs are spectrally threshold dominated.*

The search for further examples of graph classes whose sum of the  $k$  largest Laplacian eigenvalues can be bounded by threshold graphs leads to Brouwer's conjecture. It is related to (and motivated by) the Grone-Merris conjecture [4] and states that for any graph  $G$  on  $n$  vertices and  $m$  edges,

$$\sum_{i=1}^k \lambda_i \leq m + \binom{k+1}{2}. \quad (2)$$

One may ask whether the bound given by the Grone-Merris-Bai theorem is sharper than Brouwer's conjecture, because it uses more detailed information from the graph. Indeed, it has been shown in [6] that for split graphs this is the case. However, more generally, it is shown that there is a  $k$  such that the  $k$ -th inequality of Brouwer's conjecture is sharper than the  $k$ -th Grone-Merris inequality if and only if the graph is non-split. Brouwer's conjecture remains unproven to this date.

It turns out (see Section 5) that Brouwer's conjecture is, in fact, equivalent to spectral threshold dominance.

**Theorem 6** *A graph  $G$  satisfies Brouwer's conjecture if and only if it is spectrally threshold dominated.*

Quite likely this relation to threshold graphs has been part of the motivation for Brouwer's conjecture. Recognizing this equivalence also opens the door to previous, rather different proofs of theorem 3, theorem 4, and corollary 5 by [6], who proved that Brouwer's conjecture holds in these cases.

Establishing Brouwer's conjecture would prove the Laplacian energy conjecture in the non connected case. The requirement of spectral threshold dominance is, however, stronger than needed for the Laplacian energy conjecture. A counterexample for Brouwer's conjecture might not be sufficient to disprove the Laplacian energy conjecture. On the other hand, a counterexample for the non connected Laplacian energy conjecture would immediately disprove the spectral threshold dominance conjecture and thus also Brouwer's conjecture.

## 2 Spectral threshold dominance and Laplacian energy

Before embarking on the proof of theorem 2 we illustrate spectral threshold dominance by an example involving cycles (these are neither cographs nor split graphs). Note that in the definition of spectral threshold dominance, the threshold graph is allowed to depend on  $G$  and  $k$ . In the example, we are able to supply a single threshold graph that spectrally dominates  $G$  for all values of  $k$ . Our constructions here and later are inspired by the characterization of threshold graphs by their Ferrers (or Young) diagram (see for example [8]) of their nonincreasing degree sequence, i. e. row  $i$  displays  $d_i$  boxes aligned on the left. For threshold graphs, the shape described by the boxes on and above the diagonal is exactly the transpose of the shape of the boxes below the diagonal. The  $f$  boxes on the diagonal will be displayed in black.

**Example 7** *Figure 1 depicts the cycle  $C_8$  on 8 vertices, while Figure 2 shows a threshold graph that spectrally dominates  $C_8$  for all  $k \in \{1, \dots, 8\}$ . Indeed the spectrum of  $C_8$  is the multiset  $\{4, 2+\sqrt{2}, 2+\sqrt{2}, 2, 2, 2-\sqrt{2}, 2-\sqrt{2}, 0\}$  whose partial sums are  $4, 6+\sqrt{2}, 8+2\sqrt{2}, 10+2\sqrt{2}, 12+$*

$2\sqrt{2}, 14 + \sqrt{2}, 16, 16$ , whereas the partial sums for the eigenvalues of the threshold graph of Figure 2 are  $5, 10, 14, 16, 16, 16, 16, 16$ . Note, the resulting threshold graph is disconnected.

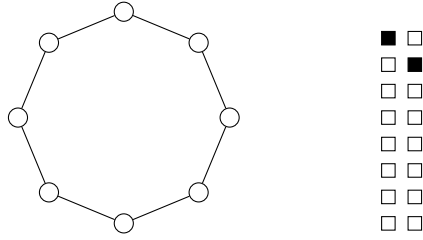


Figure 1:  $C_8$  and its Ferrers diagram

This procedure can be generalized for a general cycle  $C_n$  on  $n$  vertices. Indeed consider  $C_n$  with  $n \geq 8$  vertices (hence with  $m = n \geq 8$  edges). Let  $h = \lfloor \sqrt{2n} \rfloor$ . We observe that  $2n - (h^2 - h) \geq h$  and  $2n - (h^2 + h) \leq h + 1$ . Define  $T$  to be the threshold graph whose Ferrers diagram has Durfee square (trace)  $f = h - 1$ , if  $2n - (h^2 + h) < 0$ , otherwise  $T$  will have trace  $f = h$ . The remaining boxes will be placed as a last  $(f + 1)$ th-column (half of them) and the corresponding last  $(f + 2)$ th-row in the Ferrers diagram. As the Laplacian spectrum of  $C_n$  is the set  $\{2 - 2 \cos \frac{2\pi}{n} i : i = 1, \dots, n\}$  we observe that for  $i = 1, \dots, f$ ,  $\lambda_i(C_n) \leq 4 = \sqrt{2 \cdot 8} = \lfloor \sqrt{2 \cdot 8} \rfloor \leq \lfloor \sqrt{2 \cdot n} \rfloor \leq \lambda_i(T)$ . Hence it holds that  $\sum_{i=1}^k \lambda_i(C_n) \leq 4 \cdot k \leq h \cdot k \leq \sum_{i=1}^k \lambda_i(T)$ , for  $k = 1, \dots, f$ . If  $k \geq f$ , we observe that  $\sum_{i=1}^k \lambda_i(T) = 2n = 2m \geq \sum_{i=1}^k \lambda_i(C_n)$ .

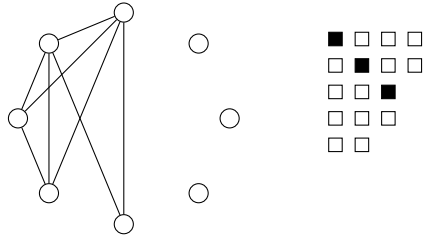


Figure 2: A spectrally threshold dominant graph of  $C_8$

The Laplacian energy of a graph  $G$  is actually fully determined by the sum of the  $k$  eigenvalues whose values exceed the average degree  $\frac{2m}{n}$ . For providing a threshold graph  $T$  on the same number of nodes and edges with the same or higher Laplacian energy it suffices to find one with  $\sum_{i=1}^k \lambda_i(T) \geq \sum_{i=1}^k \lambda_i(G)$  for this specific  $k$ , as proved in the following lemma.

**Lemma 8** *Let  $G$  be a graph on  $n$  nodes with  $m$  edges and conjugate degree sequence  $d_i^*$ ,  $i = 1, \dots, n$  and let  $k \in \{1, \dots, n\}$  be the index satisfying  $\lambda_k(L(G)) > \frac{2m}{n} \geq \lambda_{k+1}(L(G))$ . Any threshold graph  $T$  on  $n$  nodes with  $m$  edges satisfying  $\sum_{i=1}^k d_i^*(T) \geq \sum_{i=1}^k \lambda_i(L(G))$  also satisfies  $LE(T) \geq LE(G)$ .*

**Proof** By  $\sum_{i=0}^n \lambda_i(G) = 2m$  there holds  $\sum_{i=k+1}^n \lambda_i(L(G)) = 2m - \sum_{i=1}^k \lambda_i(L(G))$  and therefore

$$\begin{aligned}
LE(G) &= \sum_{i=1}^n |\lambda_i(L(G)) - \frac{2m}{n}| \\
&= \sum_{i=1}^k (\lambda_i(L(G)) - \frac{2m}{n}) + \sum_{i=k+1}^n (\frac{2m}{n} - \lambda_i(L(G))) \\
&= 2 \sum_{i=1}^k \lambda_i(L(G)) - 2m - k \frac{2m}{n} + (n-k) \frac{2m}{n} \\
&= 2 \sum_{i=1}^k (\lambda_i(L(G)) - \frac{2m}{n}) \\
&\leq 2 \sum_{i=1}^k (d_i^*(T) - \frac{2m}{n}) \\
&= 2 \sum_{i=1}^k (\lambda_i(L(T)) - \frac{2m}{n}) \\
&\leq 2 \sum_{i \in \{j: \lambda_j(L(T)) > \frac{2m}{n}\}} (\lambda_i(L(T)) - \frac{2m}{n}) = LE(T).
\end{aligned}$$

The last equation follows by repeating the initial arguments on  $G$  for  $T$ . □

If a graph is spectrally threshold dominated, an appropriate threshold graph  $T_k$  exists for all  $k$ , in particular for the  $k$  required in lemma 8. This proves theorem 2.

### 3 Split graphs are spectrally threshold dominated

Recall that  $G$  is a split graph if its set of vertices can be partitioned in two sets  $A$  and  $B$  such that  $A$  induces a clique in  $G$  and  $B$  does not contain any edge. The key for proving that split graphs are spectrally threshold dominated is the characterization of split graphs and threshold graphs by their Ferrers diagram. Split graphs have the same number of boxes above and on the diagonal as below the diagonal. Threshold graphs are special split graphs in that the shape below is the transposed of the shape above and on the diagonal. This forms the basis of the proof of the following lemma, which directly establishes theorem 3.

**Lemma 9** *Given a split graph  $G$  on  $n$  nodes with  $m$  edges and  $k \in \{1, \dots, n\}$ , there is a threshold graph  $T$  on  $n$  nodes with  $m$  edges so that  $\sum_{i=1}^k \lambda_i(L(T)) \geq \sum_{i=1}^k \lambda_i(L(G))$ .*

**Proof** Let  $f(G) = \max\{i: d_i(G) \geq i\}$  be the trace of the Ferrers diagram for  $G$ . We discern the cases  $k < f(G)$  and  $k \geq f(G)$ .

$k < f(G)$ : leaving the boxes below the diagonal unchanged and by copying its shape in transposed form to the part above the diagonal we obtain a diagram uniquely defining a threshold graph  $T$  with the property  $\sum_{i=1}^k \lambda_i(L(T)) = \sum_{i=1}^k d_i^*(T) = \sum_{i=1}^k d_i^*(G) \geq \sum_{i=1}^k \lambda_i(L(G))$ , where the last inequality follows from the Grone-Merris-Bai theorem. We refer to Figure 3 for an illustration of the graph transform. In the example  $k = 2$  and the hatched box is moved.



Figure 3: A transform for  $k < f(G)$ .

$k \geq f(G)$ : construct the desired threshold graph  $T$  by filling up the diagram above and on the diagonal in columnwise order by the  $m$  boxes, but only up to and including row  $f(G)$  (and in rowwise order below the diagonal up to column  $f(G)$  for the transposed shape). Thus  $f(T) = f(G)$ . Figure 4 shows the threshold graph  $T$  for a particular graph  $G$ . In the example the hatched boxes are moved. The construction never moves boxes across the diagonal.

First consider the case  $k = f(G)$ . Because the number of boxes below the diagonal is the same for  $G$  and  $T$ , we obtain

$$\sum_{i=1}^{f(G)} \lambda_i(T) = \sum_{i=1}^{f(G)} d_i^*(T) = m + f(G)(f(G) + 1)/2 = \sum_{i=1}^{f(G)} d_i^*(G). \quad (3)$$

The claim follows from  $\sum_{i=1}^k d_i^*(G) \geq \sum_{i=1}^k \lambda_i(L(G))$  by the Grone-Merris-Bai theorem.

Finally, for  $k > f(G)$  observe that for  $j \in \{f(G) + 1, \dots, n\}$  there holds  $\sum_{i=f(G)+1}^j d_i^*(T) \geq \sum_{i=f(G)+1}^j d_i^*(G)$ , because in  $T$  the boxes have been rearranged to maximally fill up the first columns after column  $f(G)$ . This, (3) and the Grone-Merris-Bai theorem yield  $\sum_{i=1}^k d_i^*(T) \geq \sum_{i=1}^k d_i^*(G) \geq \sum_{i=1}^k \lambda_i^*(G)$ .  $\square$

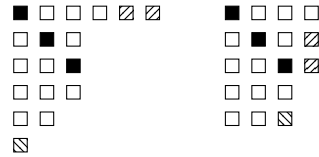


Figure 4: A transform for  $k \geq f(G)$ .

Note that the construction of the proof for  $k \geq f(G)$  may generate a threshold graph that is not connected even if  $G$  is. Indeed, at this point we do not know how to construct for a general connected split graph  $G$  and given  $k$  a spectrally dominating connected threshold graph  $T_k$ .

## 4 Disjoint unions and complements preserve spectral threshold dominance

In order to increase the class of spectrally threshold dominated graphs a bit further, we consider taking the union and complements of spectrally threshold dominated graphs.

**Lemma 10** *Let  $G$  be a (disjoint) union of spectrally threshold dominated graphs with  $n$  nodes and  $m$  edges and let  $k \in \{1, \dots, n\}$ . There is a threshold graph  $T$  on  $n$  nodes and  $m$  edges so that  $\sum_{i=1}^k \lambda_i(L(T)) \geq \sum_{i=1}^k \lambda_i(L(G))$ .*

**Proof** Suppose  $G = \bigcup_{j=1}^h G_j$  with each  $G_j$  spectrally threshold dominated. Let the first  $k$  eigenvalues of  $G$  consist of the first  $k_j$  eigenvalues of  $G_j$ ,  $j = 1, \dots, h$  with  $\sum_{j=1}^h k_j = k$ . For each  $G_j$  there is threshold graph  $T_j$  so that  $\sum_{i=1}^{k_j} \lambda_i(L(T_j)) \geq \sum_{i=1}^{k_j} \lambda_i(L(G_j))$ . Thus it suffices to prove the result under the assumption that each  $G_j$  is a threshold graph  $T_j = (N_j, E_j)$  with  $n_j = |N_j|$ ,  $m_j = |E_j|$  so that  $\sum_{j=1}^h n_j = n$  and  $\sum_{j=1}^h m_j = m$ . Put  $H = \{(j, i) : j \in \{1, \dots, h\}, i \in \{1, \dots, n_j\}\}$  with index  $(j, i)$  representing  $d_{(j,i)}^* = d_i^*(T_j) = \lambda_i(T_j)$ . Represent the ordering of the eigenvalues of  $G$  by a bijection

$$\sigma: \{1, \dots, n\} \rightarrow H \text{ so that } d_{\sigma(p)}^* \geq d_{\sigma(q)}^* \text{ for } p \leq q \text{ and so that } i \leq q \text{ for } (j, i) = \sigma(q) \quad (4)$$

(this is always possible, because the  $d_i^*(T_j)$  are sorted nonincreasingly). Consider a diagram  $D_G$  having  $d_{\sigma(i)}^*$  boxes in column  $i$ , then  $\sum_{i=1}^k \lambda_i(L(G))$  counts the boxes in columns one to  $k$ . The Ferrers diagram  $D_T$  of the intended threshold graph  $T$  will be obtained from  $D_G$  by only moving boxes to columns with smaller or equal index, then  $\sum_{i=1}^k \lambda_i(L(T)) = \sum_{i=1}^k d_i^*(L(T)) \geq \sum_{i=1}^k d_{\sigma(i)}^* = \sum_{i=1}^k \lambda_i(L(G))$ .

For column  $q = 1, \dots, n$  in  $D_G$  and  $(j, i) = \sigma(q)$  place the  $d_i^*(T_j)$  boxes of this column by the following algorithm in the new diagram  $D_T$ . The box of row  $r \in \{1, \dots, \min\{i, d_i^*(T_j)\}\}$  is placed on or above the diagonal of  $D_T$ , concretely in row  $r$  in the next free column  $c = q - |\{p \in \{1, \dots, q-1\} : (\bar{j}, \bar{i}) = \sigma(p) \wedge \bar{i} < r\}|$ , thus (4) implies  $r \leq c \leq q$ ; the box of row  $r \in \{i+1, \dots, d_i^*(T_j)\}$  is placed in column  $i$  (recall that  $i \leq q$  by (4)) in the next free row  $i + \sum_{(j,i) \in \{(\bar{j}, \bar{i}) = \sigma(p) : \bar{i} = i \wedge p \in \{1, \dots, q-1\}\}} \max\{0, d_i^*(T_j) - i\} + (r - i)$ , thus below the diagonal of  $D_T$ .

We complete the proof by showing that  $D_T$  is the Ferrers diagram of a threshold graph with trace  $f = \max\{i : \exists (j, i) \in H \text{ with } d_i^*(T_j) > i\}$ . Note that no boxes are placed on or above the diagonal of  $D_T$  for rows  $r > f$  (indeed,  $d_i^*(T_j) \neq i$  for all threshold graphs  $T_j$  and  $i$  due to the transposed structure of their diagrams) and no boxes are placed below the diagonal for columns  $c > f$ . Column  $c = 1, \dots, f$  contains  $\sum_{j=1}^h \max\{0, d_c^*(T_j) - c\}$  boxes below the diagonal. The number of boxes in row  $r = 1, \dots, f$  of  $D_T$  on or above the diagonal computes to  $\sum_{j=1}^h |\{i \in \{r, \dots, n_j\} : d_i^*(T_j) \geq r\}| = \sum_{j=1}^h \max\{0, d_r(T_j) - r + 1\} = \sum_{j=1}^h \max\{0, d_r^*(T_j) - r\}$ , where the last equation uses the defining property  $d_i^*(T_j) = d_i(T_j) + 1$  for  $i = 1, \dots, f(T_j)$  (while  $\max\{d_i^*(T_j), d_i(T_j)\} \leq f(T_j)$  for  $i > f(T_j)$ ). Thus, for  $c = r \in \{1, \dots, f\}$  the counts coincide and  $D_T$  is the Ferrers diagram of a threshold graph  $T$  with  $\sum_{i=1}^k \lambda_i(L(T)) \geq \sum_{i=1}^k \lambda_i(L(G))$ .  $\square$

Forming the complement does not pose a problem as we show next.

**Lemma 11** *Let  $G$  be a graph on  $n$  nodes with  $m$  edges and suppose that for each  $k \in \{1, \dots, n-1\}$  there is a threshold graph  $T$  on  $n$  nodes and  $m$  edges so that  $\sum_{i=1}^k \lambda_i(L(T)) \geq \sum_{i=1}^k \lambda_i(L(G))$ . Then for the complement graphs  $\bar{G}$  and  $\bar{T}$  there holds  $\sum_{i=1}^{n-k-1} \lambda_i(L(\bar{T})) \geq \sum_{i=1}^{n-k-1} \lambda_i(L(\bar{G}))$ . Because of  $\lambda_n = 0$  any threshold graph  $T$  having the same number of nodes and edges satisfies  $\sum_{i=1}^{n-1} \lambda_i(L(\bar{T})) = \sum_{i=1}^{n-1} \lambda_i(L(\bar{G}))$ .*

**Proof** The equation  $L(\bar{G}) = nI_n - \mathbf{1}\mathbf{1}^\top - L(G)$  gives rise to the well known relation  $\lambda_i(L(\bar{G})) = n - \lambda_{n-i}(L(G))$  for  $i = 1, \dots, n-1$  (and  $\lambda_n(L(\bar{G})) = 0$  as usual). Because the sum of the



eigenvalues of the Laplacian equals twice the number of edges of the graph, we get first  $\sum_{i=k+1}^{n-1} \lambda_i(L(T)) = \sum_{i=1}^{n-k-1} \lambda_{n-i}(L(T)) \leq \sum_{i=1}^{n-k-1} \lambda_{n-i}(L(G))$  and then

$$\sum_{i=1}^{n-k-1} \lambda_i(L(\bar{T})) = (n-k-1)n - \sum_{i=1}^{n-k-1} \lambda_{n-i}(L(T)) \geq (n-k-1)n - \sum_{i=1}^{n-k-1} \lambda_{n-i}(L(G)) = \sum_{i=1}^{n-k-1} \lambda_i(L(\bar{G})). \quad \square$$

The two previous lemmas establish theorem 4.

Cographs are defined recursively as (i)  $K_1$  is a cograph, (ii) the disjoint union of cographs is a cograph and (iii) the complement of a cograph is a cograph. Since  $K_1$  is a threshold graph, in view of Lemmas 10 and 11 we have proved corollary 5,

## 5 Equivalence with Brouwer's conjecture

In this section we prove that, together with the Grone-Merris-Bai theorem, Brouwer's conjecture is equivalent to conjecturing that every graph is spectrally threshold dominated. Since threshold graphs are known to satisfy Brouwer's conjecture, it is clear that any spectrally threshold dominated graph  $G$  satisfies Brouwer's conjecture. The core of the proof is therefore to construct for arbitrary  $n$  and  $m \leq \binom{n}{2}$  a threshold graph that attains Brouwer's eigenvalue bound.

**Proof (of theorem 6)** Note that by the Grone-Merris-Bai theorem Brouwer's conjecture is equivalent to  $\sum_{i=1}^k \lambda_i(L(G)) \leq \min\{kn, m + k(k+1)/2, 2m\}$  holding for  $k \in \{1, \dots, n\}$ , because no conjugate degree exceeds  $n$  and the sum of all eigenvalues is  $2m$ .

Thus the equivalence is proven if for arbitrary  $k \in \{1, \dots, n\}$  we show  $\min\{kn, m + k(k+1)/2, 2m\} = \max\{\sum_{i=1}^k d_i^*(T) : T \text{ threshold graph on } n \text{ nodes and } m \text{ edges}\}$ . Depending on the relation between  $k$ ,  $n$  and  $m$ , we discern the following cases:

**Case 1.**  $\min\{kn, m + k(k+1)/2, 2m\} = kn$ : The threshold graph  $T$  constructed by filling up the Ferrers diagram below the diagonal in columnwise order (on and above the diagonal in corresponding rowwise order) satisfies  $d_i^*(T) = n$  for  $i = \{1, \dots, k\}$ , so  $\sum_{i=1}^k \lambda_i(T) = \sum_{i=1}^k d_i^*(T) = kn$  and this is the maximum attainable over all threshold graphs on  $n$  nodes.

**Case 2.**  $\min\{kn, m + k(k+1)/2, 2m\} = m + k(k+1)/2$ : In this case put  $h := \lfloor \frac{m}{k} + \frac{k+1}{2} \rfloor < n$  and  $r := m + k(k+1)/2 - kh < k$ . Note that this implies  $h \geq k+1$ . Define a threshold graph  $T$  on  $n$  nodes with  $m$  edges of trace  $k$  by the conjugate degrees

$$d_i^*(T) = \begin{cases} h+1 & i \leq r, \\ h & r < i \leq k, \end{cases}$$

then  $\sum_{i=1}^k \lambda_i(T) = \sum_{i=1}^k d_i^*(T) = m + k(k+1)/2$ . This value cannot be exceeded by any threshold graph on  $n$  nodes with  $m$  edges by the Grone-Merris-Bai Majorization theorem, because in the Ferrers diagram of the conjugate degrees up to column  $k$  all boxes are used on and above the diagonal, while all possible  $m$  boxes are included below the diagonal.

**Case 3.**  $\min\{kn, m + k(k+1)/2, 2m\} = 2m$ : Put  $h := \max\{h \in \{1, \dots, n\} : h(h+1) \leq 2m\} < k$  and  $r := (2m - h(h+1))/2 < h+1$ , then the threshold graph  $T$  of trace  $h$  with

conjugate degrees

$$d_i^*(T) = \begin{cases} h+2 & i \leq r, \\ h+1 & r < i \leq h, \\ r & i = h+1, \\ 0 & h+1 < i, \end{cases}$$

satisfies  $\sum_{i=1}^k \lambda_i(T) = \sum_{i=1}^k d_i^*(T) = 2m$  and this is the maximum attainable over all threshold graphs with  $m$  edges.  $\square$

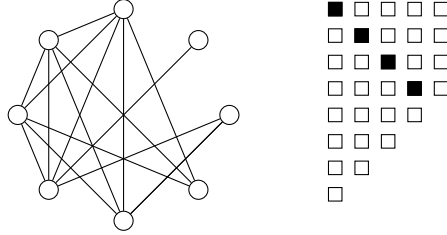


Figure 5: Graph for illustration of the equivalence

**Example 12** Consider the graph and its Ferrers diagram of Figure 5. There are  $n = 8$  vertices and  $m = 15$  edges. For  $k = 1, 2$  we are in Case 1, for  $k = 3, 4, 5$  it is Case 2 and for  $k = 6, 7, 8$  we are in Case 3 of the theorem. We illustrate the construction of the threshold graphs  $T$  for which  $\min\{kn, m + k(k+1)/2, 2m\} = \max\{\sum_{i=1}^k d_i^*(T) : T \text{ threshold graph on } n \text{ nodes and } m \text{ edges}\}$  in Figure 6 for  $k = 2$ , representing Case 1 (left), for  $k = 4$ , representing Case 2 with  $h = 6, r = 1$  (center) and for  $k = 7$ , representing Case 3 with  $h = 5, r = 0$  (right).

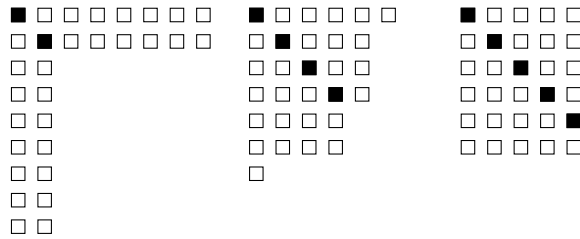


Figure 6: Transforms for  $\min\{kn, m + k(k+1)/2, 2m\} = kn, m + k(k+1)/2, 2m$

We remark that in view of this result, whenever a class of graphs is shown to satisfy Brouwer's conjecture, the class also is spectrally threshold dominated and, in particular, the Laplacian energy is bounded by the Laplacian energy of threshold graphs.

**Corollary 13** *Trees, unicyclic and bicyclic graphs are spectrally threshold dominated.*

This follows from the fact that in [4], it is proven that trees satisfy Brouwer's conjecture. Likewise, in [2], it is proven that unicyclic and bicyclic graphs satisfy Brouwer's conjecture. Hence the Laplacian energy of these classes of graphs are also bounded by the Laplacian energy

of threshold graphs. An explicit construction of a threshold graph on  $n$  nodes and  $m$  edges maximizing the Laplacian energy over all such threshold graphs is given in [5]. In the case of trees, it is known that the star on  $n$  vertices (a threshold graph) has largest Laplacian energy among all trees with  $n$  vertices [3]

In the same direction in [9] and in [11], it is proven that the conjecture of Brouwer holds for further classes of graphs.

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