SAMPLING INEQUALITY FOR L^2-NORMS OF EIGENFUNCTIONS OF SCHRÖDINGER OPERATORS

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Theorem 1. *There exists a constant $K_0 := K_0(d) \in (0,\infty)$ depending merely on the dimension d, such that for any $M > 0$, $\delta \in (0, M/2]$, any subset $\{z_k\}_{k\in(M\mathbb{Z})^d}$ of $\mathbb{R}^d$ with*

$$B(z_k,\delta) := \{x \in \mathbb{R}^d \colon |x - z_k|_2 < \delta\} \subset \Lambda_M(k) := k + (-M/2, M/2)^d \quad \text{for all } k \in (M\mathbb{Z})^d,$$

any measurable and bounded $V\colon \mathbb{R}^d \to \mathbb{R}$, and any real-valued $\varphi \in W^{2,2}(\mathbb{R}^d)$ satisfying $|\Delta\varphi| \leq |(V - E)\varphi|$ a. e. on $\mathbb{R}^d$ the inequality

$$\left(\frac{\delta}{K_0 M}\right)^{K_0(1+M^{4/3}\|V-E\|_\infty^{2/3})} \|\varphi\| \leq \|\chi_S \varphi\| \leq \|\varphi\|$$

holds, where $S := \bigcup_{k\in(M\mathbb{Z})^d} B(z_k,\delta)$.

This is a corollary of the proof of the main result of [RMV13], where Schrödinger operators on a sequence of boxes $\Lambda_L(0)$, $L \in \mathbb{N}$, are considered. Let us sketch which modifications are necessary in the proof of [RMV13, Theorem 2.1] to obtain the above result.

Proof. (1) By scaling it suffices to consider $M = 1$. (2) One does not need to extend φ further, since it is already defined on the whole of $\mathbb{R}^d$. (3) A site $k \in \mathbb{Z}^d$ is called dominant if

$$\int_{\Lambda_1(k)} |\varphi|^2 \geq \frac{1}{2T^d} \int_{\Lambda_T(k)} |\varphi|^2 \quad \text{with} \quad T = 62\lceil\sqrt{d}\rceil,$$

and otherwise weak. This corresponds to the notion chosen in [RMV13] in the case of periodic boundary conditions. (4) Estimating the contribution of boxes centered at weak sites (Now the sum is over the infinite set $\mathbb{Z}^d$, but all sums are finite since $\varphi \in L^2(\mathbb{R}^d)$.) gives again $\|\varphi\|^2 < 2\|\chi_D\varphi\|^2$, where D denotes the union of those boxes $\Lambda_1(k)$ such that $k \in \mathbb{Z}^d$ is dominant. (5) For a dominating site $k \in \mathbb{Z}^d$ we define the right near-neighbor by $k^+ := k + (\lceil\sqrt{d}\rceil + 1)\mathbf{e_1}$ and set $R := R_k := \lceil\sqrt{d}\rceil + y_k$ with $y_k := \langle e_1, z_{k^+}\rangle - \langle e_1, k^+\rangle + 1/2 \in [0,1]$. Then $\Theta := \Lambda_1(k)$ is disjoint from the open ball $B(z_{k^+}, R)$. On the other hand there exists an $a \in \Lambda_1(k)$ with $|a - z_{k^+}|_2 = R$. Thus for any $b \in \Lambda_1(k)$ we have $|b - z_{k^+}|_2 \leq |b - a|_2 + |a - z_{k^+}|_2 \leq \sqrt{d} + R \leq 2R$. Thus $\Theta \subset \overline{B(z_{k^+}, 2R)} \setminus B(z_{k^+}, R)$. (6) Once this geometric condition is satisfied, the proof of Corollary 3.2 in [RMV13] applies. (7) Hence for every dominating site $k \in \mathbb{Z}^d$ we have

$$\|\chi_{B(z_{k^+},\delta)}\varphi\|^2 \geq C_{\mathrm{qUC}} \|\chi_{\Lambda_1(k)}\varphi\|^2$$

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with the constant C_{qUC} arising from the quantitative unique continuation estimate, i.e. Corollary 3.2 in [RMV13]. (8) Taking the sum over all dominating sites $k \in \mathbb{Z}^d$ we obtain

$$\sum_{k\in\mathbb{Z}^d \text{ dominating}} \|\chi_{B(z_{k^+},\delta)}\varphi\|^2 \geq \frac{C_{\mathrm{qUC}}}{2}\|\varphi\|^2.$$

The result follows by using the quantitative estimate of C_{qUC} provided in [RMV13]. □

Remark 2. We use the opportunity to correct a minor mistake in [RMV13]. There, in the statement of Theorem 3.1 and Corollary 3.2 the geometric condition $\operatorname{diam}\Theta \leq R = \operatorname{dist}(x,\Theta)$ should be replaced by $\Theta \subset \overline{B(x,2R)} \setminus B(x,R)$. The latter is the property actually used in the proof. This property is satisfied in the application, i.e. the proof of [RMV13, Theorem 2.1], as explained in the Step (5) above.

Remark 3. While for Schrödinger operators on a box eigenfunctions capture the whole spectrum, for analogs on $\mathbb{R}^d$ continuous spectrum exists as well. Thus studying unique continuation of spectral projectors (rather than of eigenfunctions) is here even more important than for operators on cubes, as done in [Kle13] and [NTTV14]. Nevertheless there are important classes of potentials (short range or random, for instance) where a substantial part of the spectrum consists of eigenvalues.

In the analysis of random Schrödinger operators $H_\omega = H_0 + V_\omega$ with $H_0 = -\Delta + V_{\mathrm{per}}$ one often studies finite volume/finite box subsystems. This can be achieved by restricting H_ω itself to Λ_L with Dirichlet or periodic boundary conditions. Alternatively, if one is interested only in energies in the resolvent set of H_0, the finite scale approximation $H_L := H_0 + \chi_{\Lambda_L} V_\omega$ is possible as well, to which the Theorem above applies (under standard model assumptions).

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