Fast, exact and stable reconstruction of multivariate algebraic polynomials in Chebyshev form

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***Abstract*—We describe a fast method for the evaluation of an arbitrary high-dimensional multivariate algebraic polynomial in Chebyshev form at the nodes of an arbitrary rank-1 Chebyshev lattice. Our main focus is on conditions on rank-1 Chebyshev lattices allowing for the exact reconstruction of such polynomials from samples along such lattices and we present an algorithm for constructing suitable rank-1 Chebyshev lattices based on a component-by-component approach. Moreover, we give a method for the fast, exact and stable reconstruction.**

I. Introduction

We denote the Chebyshev polynomials of the first kind by $T_k : [-1,1] \to [-1,1]$, $T_k(x) := \cos(k \arccos x)$, $k \in \mathbb{N}_0$. Note that for each $k \in \mathbb{N}_0$, T_k is an algebraic polynomial of degree $\deg(T_k) = k$ restricted to the domain $[-1,1]$. Moreover, we define the multivariate Chebyshev polynomials $T_{\boldsymbol{k}} : [-1,1]^d \to [-1,1]$, $T_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{t=1}^d T_{k_t}(x_t)$ for $d \in \mathbb{N}$, $\boldsymbol{x} := (x_1,\ldots,x_d)^\top \in [-1,1]^d$ and $\boldsymbol{k} := (k_1,\ldots,k_d)^\top \in \mathbb{N}_0^d$.

Let $\Pi_I := \operatorname{span}\{T_{\boldsymbol{k}}(\circ) \colon \boldsymbol{k} \in I\}$, where $I \subset \mathbb{N}_0^d$, $d \in \mathbb{N}$, is a non-negative index set of finite cardinality, $|I| < \infty$. Then, each multivariate polynomial $p \in \Pi_I$ can be written as

$$p(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} a_{\boldsymbol{k}}\, T_{\boldsymbol{k}}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} a_{\boldsymbol{k}} \prod_{t=1}^d T_{k_t}(x_t), \quad a_{\boldsymbol{k}} \in \mathbb{R}, \quad (1)$$

where $\boldsymbol{x} \in [-1,1]^d$. We remark that if the index set $I = I_n^d := \{\boldsymbol{k} \in \mathbb{N}_0^d \colon \|\boldsymbol{k}\|_1 \le n\}$, $n \in \mathbb{N}_0$, is the ℓ_1-ball, then Π_I is the space of all algebraic polynomials of (total) degree $\le n$ in d variables restricted to the domain $[-1,1]^d$. Moreover, polynomials with hyperbolic cross index sets $I = H_n^d := \left\{\boldsymbol{k} \in \mathbb{N}_0^d \colon \prod_{t=1}^d \max(1,|k_t|) \le n\right\}$, where $n, d \in \mathbb{N}$, have already been used for approximations in sparse high-dimensional spectral Galerkin methods, cf. [1, Section 8.5].

In this paper, for a given arbitrary index set $I \subset \mathbb{N}_0^d$ of finite cardinality, we present a method for the fast evaluation of a polynomial p from (1) at the nodes $\boldsymbol{x}_j := \cos(\frac{j}{M}\pi \boldsymbol{z})$, $j = 0,\ldots,M$, of a d-dimensional rank-1 Chebyshev lattice

$$\tilde{\Lambda}(\boldsymbol{z}, M) := \left\{\boldsymbol{x}_j := \cos\left(\tfrac{j}{M}\pi \boldsymbol{z}\right) : j = 0,\ldots,M\right\},$$

where the generating vector $\boldsymbol{z} \in \mathbb{N}_0^d$ and the size parameter $M \in \mathbb{N}_0$, cf. [2] for a more general definition of d-dimensional rank-k Chebyshev lattices. Moreover, we discuss conditions on a rank-1 Chebyshev lattice $\tilde{\Lambda}(\boldsymbol{z}, M)$ such that the fast, exact and stable reconstruction of all coefficients $a_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, from sampling values $p(\boldsymbol{x}_j)$ taken at the corresponding nodes $\boldsymbol{x}_j$, $j = 0,\ldots,M$, is possible. Both, for the fast evaluation and reconstruction, we only apply a single one-dimensional discrete cosine transform of type I (DCT-I) and additionally compute simple index transforms, see also [3]. Note that for the special case $I = I_n^d$, constructions of rank-1 Chebyshev lattices suitable for the exact reconstruction were already discussed in [2], [4] and the references therein. Here, we present an algorithm based on component-by-component (CBC) construction for arbitrary index sets $I \subset \mathbb{N}_0^d$ using ideas from [5]–[7].

We remark that our considerations for the reconstruction of the coefficients $a_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, of a polynomial p from (1) with known index set $I \subset \mathbb{N}_0^d$ in this paper establish a basis for the reconstruction of a polynomial p with unknown index set I using a method similar to the one presented in [8].

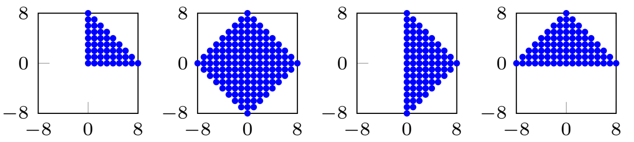
The remaining parts of this paper are organized as follows: In Secion II, we give prerequisites for the subsequent sections. We discuss the fast evaluation and reconstruction in Section III. In Section IV, we point out relations of our results to existing work. Afterwards, in Section V, we present computed rank-1 Chebyshev lattices suitable for reconstruction. Finally, in Section VI, we summarize the results of this paper.

II. Prerequisites

A. One-dimensional DCT-I

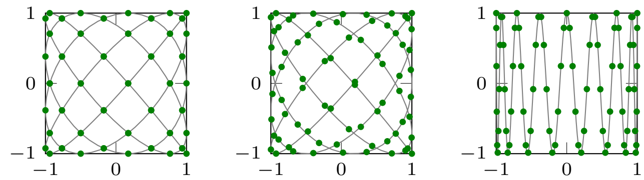
First, we recall results for the fast reconstruction of a (one-dimensional algebraic) polynomial p. We are able to reconstruct the coefficients $a_0,\ldots,a_n \in \mathbb{R}$ of a polynomial p from (1) with $I := I_n^1$ from sampling values $p(x_j)$ at the Chebyshev nodes $x_j := \cos(j\pi/n)$, $j = 0,\ldots,n$. For this, we apply a one-dimensional DCT-I to the sampling values $p(x_j)$ and we obtain $\sum_{j=0}^n (\varepsilon_j^n)^2\, p(x_j)\, \cos(jk\pi/n) = \sum_{k'\in I_n^1} a_{k'} \sum_{j=0}^n (\varepsilon_j^n)^2 \cos(jk'\pi/n) \cos(jk\pi/n)$ for $k \in I_n^1$, $\varepsilon_l^n := 1/\sqrt{2}$ for $l \in \{0,n\}$ and $\varepsilon_l^n := 1$ for $l \in \{1,\ldots,n-1\}$, since $T_k(x_j) = T_k(\cos(j\pi/n)) = \cos(jk\pi/n)$. Due to

$$\frac{2}{n} \varepsilon_k^n \varepsilon_{k'}^n \sum_{j=0}^n (\varepsilon_j^n)^2 \cos\left(\frac{jk\pi}{n}\right) \cos\left(\frac{jk'\pi}{n}\right) = \delta_{k,k'},\ k, k' \in I_n^1, \quad (2)$$



$|I_8^2| = 45$ $|\mathcal{M}(I_8^2)| = 145$ $|\mathcal{M}_1(I_8^2)| = 81$ $|\mathcal{M}_2(I_8^2)| = 81$

Fig. 1. Index sets I_8^2, $\mathcal{M}(I_8^2)$, $\mathcal{M}_1(I_8^2)$, $\mathcal{M}_2(I_8^2)$ (from left to right).



(a) $\boldsymbol{z} := (8,9)^\top$, $M := 72$, $|\tilde{\Lambda}(\boldsymbol{z}, M)| = 45$. (b) $\boldsymbol{z} := (8,9)^\top$, $M := 73$, $|\tilde{\Lambda}(\boldsymbol{z}, M)| = 74$. (c) $\boldsymbol{z} := (1,16)^\top$, $M := 76$, $|\tilde{\Lambda}(\boldsymbol{z}, M)| = 77$.

Fig. 2. Rank-1 Chebyshev lattices $\tilde{\Lambda}(\boldsymbol{z}, M)$.

where $\delta_{k,k'}$ is Kronecker's delta, see e.g. [9, Section 2.4], this yields $a_k = \frac{2(\varepsilon_k^n)^2}{n} \sum_{j=0}^n (\varepsilon_j^n)^2 \; p(x_j) \;\; \cos(jk\pi/n)$ for $k \in I_n^1$. Note that the DCT-I can be computed by means of a fast algorithm in $\mathcal{O}(n \log n)$ arithmetic operations.

B. Index sets and tensor-products of cosines

Let $I \subset N_0^d$ be an arbitrary index set of finite cardinality. For the description of the approach for the fast evaluation and reconstruction, we define the extended symmetric index set

$$\mathcal{M}(I) := \{\boldsymbol{h} \in \mathbb{Z}^d \colon (|h_1|, \ldots, |h_d|)^\top \in I\},$$

which contains all frequencies $\boldsymbol{k} \in I$ and versions of these frequencies $\boldsymbol{k}$ mirrored at all coordinate axes. Moreover, we define the index sets

$$\mathcal{M}_s(I) := \{\boldsymbol{h} \in \mathcal{M}(I) \colon h_s \geq 0\}, \; s \in \{1, \ldots, d\},$$

which contain all frequencies $\boldsymbol{k} \in I$ and versions of these frequencies mirrored at all coordinate axes except the s-th. For instance, in the case $d = 2$ and $n = 8$, the index set I_8^2 as well as the corresponding extended symmetric index set $\mathcal{M}(I_8^2)$ and mirrored index sets $\mathcal{M}_1(I_8^2)$, $\mathcal{M}_2(I_8^2)$ are depicted in Fig. 1.

Next, we remark that for $y_1, y_2 \in \mathbb{R}$, we have $\cos(y_1)\cos(y_2) = \frac{1}{2}(\cos(y_1 + y_2) + \cos(y_1 - y_2))$. Using induction on the dimension $d \in \mathbb{N}$ and due to $\cos(x) = \cos(-x)$ for all $x \in \mathbb{R}$, we obtain for $\boldsymbol{y} := (y_1, \ldots, y_d)^\top \in \mathbb{R}$

$$\prod_{t=1}^d \cos(y_t) = \sum_{\boldsymbol{m} \in \mathcal{M}_s(\{\boldsymbol{1}\})} \frac{1}{2^{d-1}} \cos(\boldsymbol{m} \cdot \boldsymbol{y}) \tag{3}$$

$$= \sum_{\boldsymbol{m} \in \mathcal{M}(\{\boldsymbol{1}\})} \frac{1}{2^d} \cos(\boldsymbol{m} \cdot \boldsymbol{y}), \tag{4}$$

where $\boldsymbol{1} := (1, \ldots, 1)^\top \in \mathbb{N}^d$ and $\boldsymbol{m} \cdot \boldsymbol{y} := \sum_{t=1}^d m_t y_t$.

III. Fast evaluation and reconstruction of multivariate polynomials from Π_I along rank-1 Chebyshev lattices using DCT-I

A. Fast evaluation at the nodes of rank-1 Chebyshev lattices

Briefly, we describe a simple method for the fast evaluation of a polynomial p from (1) with arbitrary index set $I \subset \mathbb{N}_0^d$ at the nodes $\boldsymbol{x}_j := \cos(\frac{j}{M}\pi\boldsymbol{z})$, $j = 0, \ldots, M$, of an arbitrary d-dimensional rank-1 Chebyshev lattice $\tilde{\Lambda}(\boldsymbol{z}, M)$. Examples for two-dimensional rank-1 Chebyshev lattices are shown in Fig. 2. We remark that not all $(M+1)$ nodes $\boldsymbol{x}_j$, $j = 0, \ldots, M$, have to be distinct, i.e., $|\tilde{\Lambda}(\boldsymbol{z}, M)| \in \{1, \ldots, M+1\}$, see Fig. 2a. Due to (3), we have

$$p(\boldsymbol{x}_j) = \sum_{\boldsymbol{k} \in I} \frac{a_{\boldsymbol{k}}}{2^{d-1}} \sum_{\boldsymbol{m} \in \mathcal{M}_s(\{\boldsymbol{1}\})} \cos\left(\frac{j}{M}\pi\,(\boldsymbol{m} \odot \boldsymbol{k}) \cdot \boldsymbol{z}\right),$$

$j = 0, \ldots, M$, for any $s \in \{1, \ldots, d\}$ and for each polynomial p from (1), where $\boldsymbol{m} \odot \boldsymbol{k} := (m_1 k_1, \ldots, m_d k_d)^\top$. For $M \in \mathbb{N}$ and $l \in \mathbb{Z}$, we define the even-mod relation

$$l \operatorname{emod} M := \begin{cases} l \bmod (2M), & l \bmod (2M) \leq M, \\ 2M - (l \bmod (2M)) & \text{else}, \end{cases}$$

as well as in the special case $M = 0$, $l \operatorname{emod} 0 := 0$ for $l \in \mathbb{Z}$. For each $l \in I_M^1$, we consider the frequencies $\boldsymbol{k} \in I$ and $\boldsymbol{m} \in \mathcal{M}_s(\{\boldsymbol{1}\})$, such that $l = (\boldsymbol{m} \odot \boldsymbol{k}) \cdot \boldsymbol{z} \operatorname{emod} M$. Since we have $\cos(jl\pi/M) = \cos\left(\frac{j}{M}\pi\,(\boldsymbol{m} \odot \boldsymbol{k}) \cdot \boldsymbol{z}\right)$ for $j = 0, \ldots, M$ in the case $l = (\boldsymbol{m} \odot \boldsymbol{k}) \cdot \boldsymbol{z} \operatorname{emod} M$, we obtain $p(\boldsymbol{x}_j) = \sum_{l=0}^M (\varepsilon_l^M)^2\, \hat{b}_l \cos(jl\pi/M)$, where the coefficients

$$\hat{b}_l := \sum_{\boldsymbol{k} \in I} \sum_{\substack{\boldsymbol{m} \in \mathcal{M}_s(\{\boldsymbol{1}\}) \\ (\boldsymbol{m} \odot \boldsymbol{k}) \cdot \boldsymbol{z} \operatorname{emod} M = l}} \frac{a_{\boldsymbol{k}}}{2^{d-1}\,(\varepsilon_l^M)^2} \quad \text{for } l \in I_M^1. \tag{5}$$

Therefore, for any $s \in \{1, \ldots, d\}$, we build the index set $\mathcal{M}_s(I)$ and we compute the coefficients $\hat{b}_l$ by (5) for $l \in I_M^1$. Then, we apply a one-dimensional DCT-I to these coefficients $\hat{b}_l$ and this yields the function values $p(\boldsymbol{x}_j)$ for $j = 0, \ldots, M$. In total, we require $\mathcal{O}(M \log M + d\, 2^d |I|)$ arithmetic operations.

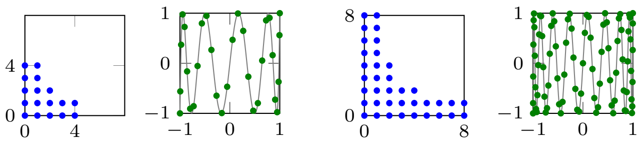
B. Fast, exact and stable reconstruction

In this section, we consider the fast reconstruction of a polynomial p from (1) with arbitrary index set $I \subset \mathbb{N}_0^d$, $|I| < \infty$. Our approach is based on applying a one-dimensional DCT-I to the sampling values $p(\boldsymbol{x}_j)$ at the nodes $\boldsymbol{x}_j := \cos(j\pi\boldsymbol{z}/M)$, $j = 0, \ldots, M$, of a rank-1 Chebyshev lattice $\tilde{\Lambda}(\boldsymbol{z}, M)$ fulfilling a certain property. Concretely, we compute the coefficients

$$\hat{a}_l := \sum_{j=0}^M (\varepsilon_j^M)^2 \; p(\boldsymbol{x}_j) \; \cos\left(\frac{jl}{M}\pi\right) \tag{6}$$

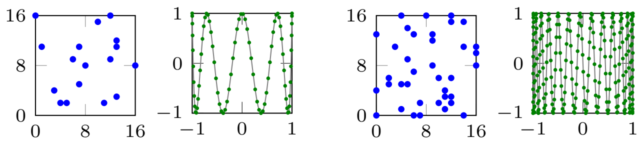
$$= \sum_{j=0}^M (\varepsilon_j^M)^2 \sum_{\boldsymbol{k} \in I} a_{\boldsymbol{k}} \left(\prod_{t=1}^d \cos\left(\frac{j}{M}\pi\, k_t\, z_t\right)\right) \cos\left(\frac{jl}{M}\pi\right)$$

for $l \in I_M^1$. Due to (4), this means $\hat{a}_l =$
$\sum_{\boldsymbol{k} \in I} \frac{a_{\boldsymbol{k}}}{2^d} \sum_{\boldsymbol{m} \in \mathcal{M}(\{\boldsymbol{1}\})} \sum_{j=0}^M (\varepsilon_j^M)^2 \cos\left(\frac{j}{M}\pi\,(\boldsymbol{m} \odot \boldsymbol{k}) \cdot \boldsymbol{z}\right) \cos\left(\frac{jl}{M}\pi\right)$



(a) H_4^2 and $\tilde{\Lambda}\left((1,9)^\top,29\right)$. (b) H_8^2 and $\tilde{\Lambda}\left((1,17)^\top,82\right)$.

Fig. 3. Examples for hyperbolic cross index sets $I = H_n^2$ and corresponding rank-1 Chebyshev lattices $\tilde{\Lambda}(\boldsymbol{z}, M)$ fulfilling condition (7).



(a) I ($|I| = 17$), $\tilde{\Lambda}\left((1,8)^\top,77\right)$. (b) I ($|I| = 37$), $\tilde{\Lambda}\left((1,27)^\top,249\right)$.

Fig. 4. Examples for arbitrarily chosen index sets $I \subset \mathbb{N}_0^2$ and corresponding rank-1 Chebyshev lattices $\tilde{\Lambda}(\boldsymbol{z}, M)$ fulfilling condition (7).

for $l \in I_M^1$ and we consider the indices $l := \boldsymbol{k} \cdot \boldsymbol{z} \,\mathrm{emod}\, M$ for $\boldsymbol{k} \in I$. Since we have $\{\boldsymbol{m} \odot \boldsymbol{k} \colon \boldsymbol{m} \in \mathcal{M}(\{\boldsymbol{1}\})\} = \mathcal{M}(\{\boldsymbol{k}\})$ for $\boldsymbol{k} \in I$ and due to the orthogonality condition (2), we are able to exactly reconstruct all the coefficients $a_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, of the polynomial p from (1) using the computed coefficients $\hat{a}_l$, $l := \boldsymbol{k} \cdot \boldsymbol{z} \,\mathrm{emod}\, M$ for $\boldsymbol{k} \in I$, from (6) if and only if

$$\begin{aligned} &\boldsymbol{k} \cdot \boldsymbol{z} \,\mathrm{emod}\, M \neq \boldsymbol{h} \cdot \boldsymbol{z} \,\mathrm{emod}\, M \\ &\text{for all } \boldsymbol{k} \in I \text{ and } \boldsymbol{h} \in \mathcal{M}(I),\ \boldsymbol{k} \neq (|h_1'|, \ldots, |h_d'|)^\top. \end{aligned} \tag{7}$$

Examples for two-dimensional hyperbolic cross index sets $I = H_4^2$ and $I = H_8^2$ with corresponding rank-1 Chebyshev lattices $\tilde{\Lambda}(\boldsymbol{z}, M)$ fulfilling condition (7) are depicted in Fig. 3 as well as two-dimensional examples for index sets I with less structure with corresponding $\tilde{\Lambda}(\boldsymbol{z}, M)$ in Fig. 4. Moreover, the rank-1 Chebyshev lattices $\tilde{\Lambda}(\boldsymbol{z}, M)$ in Fig. 2 fulfill condition (7) for the ℓ_1-ball index set $I = I_8^2$ in Fig. 1.

Due to the symmetry of the emod operator, we can reduce the number of tests in condition (7) by a factor of (about) two.

Lemma III.1. *For $M \in \mathbb{N}_0$ and $l \in \mathbb{Z}$, we have $l \,\mathrm{emod}\, M = (-l) \,\mathrm{emod}\, M$.*

Proof. Considering the two different cases in the definition of the emod operator, the assertion follows straight forward. □

Lemma III.2. *For a given arbitrary index set $I \subset \mathbb{N}_0^d$ of finite cardinality, $|I| < \infty$, let $\tilde{I} \subset \mathbb{Z}^d$ be an arbitrary index set with the property $\mathcal{M}(I) = \tilde{I} \cup \{-\boldsymbol{h} \colon \boldsymbol{h} \in \tilde{I}\}$. Then, condition (7) is equivalent to*

$$\begin{aligned} &\boldsymbol{k} \cdot \boldsymbol{z} \,\mathrm{emod}\, M \neq \boldsymbol{h} \cdot \boldsymbol{z} \,\mathrm{emod}\, M \\ &\textit{for all } \boldsymbol{k} \in I \textit{ and } \boldsymbol{h} \in \tilde{I},\ \boldsymbol{k} \neq (|h_1'|, \ldots, |h_d'|)^\top. \end{aligned}$$

Proof. Due to $(-\boldsymbol{h}) \cdot \boldsymbol{z} = -(\boldsymbol{h} \cdot \boldsymbol{z})$ for $\boldsymbol{h} \in \mathbb{Z}^d$, we obtain

$$(-\boldsymbol{h}) \cdot \boldsymbol{z} \,\mathrm{emod}\, M = \boldsymbol{h} \cdot \boldsymbol{z} \,\mathrm{emod}\, M \text{ for } \boldsymbol{h} \in \mathbb{Z}^d \tag{8}$$

from Lemma III.1 and the assertion follows. □

Input: index set $I_{\text{input}} \subset \mathbb{N}_0^d$, parameter $s \in \{1, \ldots, d\}$.

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Determine suitable initial size parameter M_start, see e.g. Remark IV.4.
for t := 1, ..., d do
  for z_t := 0, ..., M_start do
    if Condition (9) is valid for
       I := {(k_1, ..., k_t)^T : k ∈ I_input},
       z := (z_1, ..., z_t)^T, M := M_start then
      break
    end if
  end for
end for
for M := |I_input| − 1, ..., M_start do
  if Condition (9) is valid for I := I_input,
     z := (z_1, ..., z_d)^T, M then
    break
  end if
end for
```

Output: generating vector $\boldsymbol{z} \in \mathbb{N}_0^d$ and size parameter $M \in \mathbb{N}_0$ fulfilling condition (7) for index set $I := I_{\text{input}}$.

Fig. 5. Algorithm for construction of rank-1 Chebyshev lattice $\tilde{\Lambda}(\boldsymbol{z}, M)$ suitable for reconstruction of multivariate polynomials (1) supported on the index set $I := I_{\text{input}}$.

Corollary III.3. *For any $s \in \{1, \ldots, d\}$, condition (7) is equivalent to*

$$\begin{aligned} &\boldsymbol{k} \cdot \boldsymbol{z} \,\mathrm{emod}\, M \neq \boldsymbol{h} \cdot \boldsymbol{z} \,\mathrm{emod}\, M \\ &\textit{for all } \boldsymbol{k} \in I \textit{ and } \boldsymbol{h} \in \mathcal{M}_s(I),\ \boldsymbol{k} \neq (|h_1'|, \ldots, |h_d'|)^\top. \end{aligned} \tag{9}$$

If condition (7) or (9) is fulfilled, we can reconstruct the coefficients $a_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, in the following way. We apply a DCT-I to the sampling values $p(\boldsymbol{x}_j) = p(\cos(j\pi\boldsymbol{z}/M))$, $j = 0, \ldots, M$, which yields the coefficients $\hat{a}_l$, $l \in I_M^1$, in (6). Then, we obtain the coefficients of the polynomial p by $a_{\boldsymbol{k}} =$

$$\frac{2(\varepsilon^M_{\boldsymbol{k}\cdot\boldsymbol{z} \,\mathrm{emod}\, M})^2}{M}\,\hat{a}_{\boldsymbol{k}\cdot\boldsymbol{z} \,\mathrm{emod}\, M} \cdot \frac{2^{d-1}}{|\{\boldsymbol{m} \in \mathcal{M}_s(\{\boldsymbol{1}\}) \colon (\boldsymbol{m} \odot \boldsymbol{k}) \cdot \boldsymbol{z} \,\mathrm{emod}\, M = \boldsymbol{k} \cdot \boldsymbol{z} \,\mathrm{emod}\, M\}|}$$

for all $\boldsymbol{k} \in I$ and any $s \in \{1, \ldots, d\}$.

Using a fast algorithm for the DCT-I, this computation can be performed in $\mathcal{O}(M \log M + d\, 2^d |I|)$ arithmetic operations.

Again, we stress the fact that the index set $I \subset \mathbb{N}_0^d$, $|I| < \infty$, may be arbitrarily chosen. Upper bounds on the size parameter M for the existence of a rank-1 Chebyshev lattice $\tilde{\Lambda}(\boldsymbol{z}, M)$ fulfilling condition (7) are discussed in Section IV-B. A method for the construction of a suitable generating vector $\boldsymbol{z} \in \mathbb{N}_0^d$ is described in the following subsection.

C. Construction of suitable rank-1 Chebyshev lattices

In Fig. 5, we present an algorithm for the construction of a rank-1 Chebyshev lattice $\tilde{\Lambda}(\boldsymbol{z}, M)$ which allows for the exact reconstruction of the coefficients $a_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, of a polynomial p from (1) based on samples taken at the nodes of $\tilde{\Lambda}(\boldsymbol{z}, M)$, where $I \subset \mathbb{N}_0^d$, $|I| < \infty$, is an arbitrary index set. Our algorithm is based on [7, Algorithm 1 and 2] and uses a CBC search for the generating vector $\boldsymbol{z} \in \mathbb{N}_0^d$.

IV. Relations to existing work

A. Padua points and higher-dimensional rank-s Chebyshev lattices

In [10], special sampling points were discussed in the two-dimensional case, so-called Padua points. For a parameter $n \in \mathbb{N}$, these are the nodes $\boldsymbol{x}_j := (\cos(j\pi/(n+1)), \cos(j\pi/n))^\top = \cos(j\pi\boldsymbol{z}/M)$, $j = 0, \ldots, M$, of the rank-1 Chebyshev lattice $\mathcal{A}_n := \tilde{\Lambda}(\boldsymbol{z}, M)$, where the generating vector $\boldsymbol{z} := (n, n+1)^\top$ and the size parameter $M := n(n+1)$. As discussed in [10, Section 2], the Padua point set $\mathcal{A}_n$ only consists of $\binom{n+2}{2} = \frac{n^2}{2} + \frac{3}{2}n + 1$ distinct points, whereas $M = n^2 + n$.

Lemma IV.1. *Let the index set $I = I_n^2 := \{\boldsymbol{k} \in \mathbb{N}_0^2 \colon k_1 + k_2 \leq n\}$, $n \in \mathbb{N}_0$, be the ℓ_1-ball. Then, condition* (7) *is fulfilled and we can exactly reconstruct the coefficients $a_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, of a polynomial p from* (1) *from sampling values at the nodes of the Padua point set $\mathcal{A}_n$ using* (6)*.*

Proof. The case $n = 0$ is trivial. For $n \in \mathbb{N}$, we show condition (9) for $s = 1$, which is equivalent to condition (7) due to Corollary III.3. Let $\boldsymbol{z} := (n, n+1)^\top$ and $M := n(n+1)$ as well as let arbitrary frequencies $\boldsymbol{k} \in I$ and $\boldsymbol{h} \in \mathcal{M}_1(I) = \{\boldsymbol{h} \in \mathbb{N}_0 \times \mathbb{Z} \colon h_1 + |h_2| \leq n\}$ with $\boldsymbol{k} \neq (|h_1|, \ldots, |h_d|)^\top$ be given. We show that $\boldsymbol{k} \cdot \boldsymbol{z} \operatorname{emod} M \neq \boldsymbol{h} \cdot \boldsymbol{z} \operatorname{emod}$ follows. For this, we assume the contrary, i.e., $\boldsymbol{k} \cdot \boldsymbol{z} \operatorname{emod} M = \boldsymbol{h} \cdot \boldsymbol{z} \operatorname{emod} M$. We obtain that the only solution for this condition is $\boldsymbol{k} = \boldsymbol{h}$ for $h_2 \geq 0$ and $h_2 < 0$, which is a contradiction to the requirement $\boldsymbol{k} \neq (|h'_1|, \ldots, |h'_d|)^\top$. $\square$

In [4], an extensive search for higher-rank Chebyshev lattices allowing for the reconstruction of polynomials p from (1) with ℓ_1-ball index sets $I := I_n^d$ was performed and numerical results for the cases $d = 3, 4, 5$ were presented.

B. Reconstructing rank-1 lattices of multivariate trigonometric polynomials

In the following, we briefly show the relation to reconstructing rank-1 lattices of multivariate trigonometric polynomials from [7].

Theorem IV.2. *Let $I \subset \mathbb{N}_0^d$ be an arbitrary index set of finite cardinality, $|I| < \infty$. Moreover, let $\Lambda(\boldsymbol{z}, \hat{M}) := \{\boldsymbol{y}_j := \frac{j}{\hat{M}}\boldsymbol{z} \bmod \boldsymbol{1} \colon j = 0, \ldots, \hat{M} - 1\}$ be a reconstructing rank-1 lattice with generating vector $\boldsymbol{z} \in \mathbb{N}_0^d$ and even rank-1 lattice size $\hat{M} \in 2\mathbb{N}$ for the extended symmetric index set $\mathcal{M}(I)$, i.e.,*

$$\boldsymbol{h} \cdot \boldsymbol{z} \not\equiv \boldsymbol{h'} \cdot \boldsymbol{z} \pmod{\hat{M}} \text{ for all } \boldsymbol{h}, \boldsymbol{h'} \in \mathcal{M}(I),\ \boldsymbol{h} \neq \boldsymbol{h'}. \quad (10)$$

Then, the rank-1 Chebyshev lattice $\tilde{\Lambda}(\boldsymbol{z}, \frac{\hat{M}}{2})$ fulfills condition (7)*, i.e., we are able to exactly reconstruct the coefficients of a polynomial from* (1) *using samples at the nodes of $\tilde{\Lambda}(\boldsymbol{z}, \frac{\hat{M}}{2})$.*

Proof. We consider the values

$$\boldsymbol{h} \cdot \boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2} = \begin{cases} \boldsymbol{h} \cdot \boldsymbol{z} \bmod \hat{M}, & \boldsymbol{h} \cdot \boldsymbol{z} \bmod \hat{M} \leq \frac{\hat{M}}{2}, \\ \hat{M} - (\boldsymbol{h} \cdot \boldsymbol{z} \bmod \hat{M}) & \text{else}, \end{cases}$$

for $\boldsymbol{h} \in \mathcal{M}(I)$. Due to property (10), all values $\boldsymbol{h} \cdot \boldsymbol{z} \bmod \hat{M}$ are distinct for $\boldsymbol{h} \in \mathcal{M}(I)$ and we obtain for each $l \in I^1_{\hat{M}/2}$ that one of the following three cases may occur: Either

1. exactly two distinct frequencies $\boldsymbol{h}, \boldsymbol{h'} \in \mathcal{M}(I)$ exist such that $\boldsymbol{h} \cdot \boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2} = \boldsymbol{h'} \cdot \boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2} = l$, or
2. exactly one frequency $\boldsymbol{h} \in \mathcal{M}(I)$ exists such that $\boldsymbol{h} \cdot \boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2} = l$, or
3. such a frequency does not exist for l.

In the first case, $\boldsymbol{h'} = -\boldsymbol{h}$ follows, since for each $\boldsymbol{h} \in \mathcal{M}(I) \setminus \{\boldsymbol{0}\}$, also the frequency $-\boldsymbol{h} \in \mathcal{M}(I) \setminus \{\boldsymbol{0}\}$ and we have (8) with $M := \frac{\hat{M}}{2}$, i.e., $(-\boldsymbol{h}) \cdot \boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2} = \boldsymbol{h} \cdot \boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2} = l$. The second case can only occur for $\boldsymbol{h} = \boldsymbol{0}$, since otherwise the (non-zero) frequency $-\boldsymbol{h} \in \mathcal{M}(I) \setminus \{\boldsymbol{0}\}$, $-\boldsymbol{h} \neq \boldsymbol{h}$, and this would yield $(-\boldsymbol{h}) \cdot \boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2} = \boldsymbol{h} \cdot \boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2}$ which corresponds to the first case. In total, we obtain $\boldsymbol{h} \cdot \boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2} \neq \boldsymbol{h'} \cdot \boldsymbol{z} \operatorname{emod} \frac{\hat{M}}{2}$ for all $\boldsymbol{h}, \boldsymbol{h'} \in \mathcal{M}(I)$, $(|h_1|, \ldots, |h_d|)^\top \neq (|h'_1|, \ldots, |h'_d|)^\top$, implying condition (7). $\square$

Remark IV.3. *Condition* (7) *and* (10) *with $\hat{M} = 2M$ are not equivalent in general. For instance, the generating vector $\boldsymbol{z} := (8, 9)^\top$ and size parameter $M := 72$ from Fig. 2a fulfill condition* (7) *for $I = I_8^2$ but not condition* (10) *with $\hat{M} = 2M$. However, there exist special cases where both conditions are fulfilled, see e.g. the examples in Fig. 2b and 2c which fulfill condition* (7) *as well as condition* (10)*.*

Remark IV.4. *There always exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, \hat{M})$ for $\mathcal{M}(I)$ with even rank-1 lattice size*

$$\hat{M} \leq 2 \max\left\{\frac{2}{3}(|\mathcal{M}(I)|^2 - |\mathcal{M}(I)| + 8), \max_{\boldsymbol{k} \in I} 3\|\boldsymbol{k}\|_\infty\right\}$$

and consequently a rank-1 Chebyshev lattice $\tilde{\Lambda}(\boldsymbol{z}, M)$ with size parameter $M := \hat{M}/2$. This result is due to [8, Theorem 2.1] which is a direct consequence of the results from [7].

C. Tent-transformed rank-1 lattices for cosine polynomials

In [11], [12], tent-transformed rank-1 lattices $P_\phi(\boldsymbol{z}, \hat{M}) := \{\phi(j\boldsymbol{z}/\hat{M} \bmod \boldsymbol{1}) \colon j = 0, \ldots, \hat{M} - 1\}$, fulfilling a condition equivalent to (10) are used, where $\boldsymbol{z} \in \mathbb{N}_0^d$, $\hat{M} \in \mathbb{N}$ and the tent transform $\phi \colon [0,1] \to [0,1]$, $\phi(x) := 1 - |2x - 1|$, is component-wise applied. Then, the exact reconstruction of cosine polynomials $\tilde{p} \colon [0,1] \to \mathbb{R}$, $\tilde{p}(\boldsymbol{x}) := \sum_{\boldsymbol{k} \in I} \tilde{a}_{\boldsymbol{k}} \prod_{t=1}^d \cos(\pi k_t x_t)$, $I \subset \mathbb{N}_0^d$, can be performed by applying a fast Fourier transform to samples at these nodes, cf. [12]. Note that these polynomials $\tilde{p}$ are not algebraic polynomials in general.

TABLE I

CARDINALITIES OF ℓ_1-BALL INDEX SETS I_n^d AS WELL AS SIZE PARAMETERS M OF CORRESPONDING RANK-1 CHEBYSHEV LATTICES $\tilde{\Lambda}(\boldsymbol{z}, M)$, WHERE M FULFILLS CONDITION (7) AND $\hat{M} = 2M$ CONDITION (10) FOR $I := I_n^d$.

Parameters		Cardinalities		Condition (7) / (9) / (10)	
d	n	$\lvert I_n^d\rvert$	$\lvert\mathcal{M}_1(I_n^d)\rvert$	$M = \frac{\hat{M}}{2}$	$\frac{M+1}{\lvert I_n^d\rvert}$
2	64	2 145	4 225	4 192	1.95
2	128	8 385	16 641	16 576	1.98
2	256	33 153	66 049	65 920	1.99
3	16	969	3 281	4 265	4.40
3	32	6 545	23 969	33 361	5.10
3	64	47 905	183 105	264 353	5.52
4	8	495	2 241	2 693	5.44
4	16	4 845	28 033	37 865	7.82
4	32	58 905	396 033	565 073	9.59
5	4	126	501	630	5.01
5	8	1 287	8 361	14 276	11.09
5	16	20 349	192 593	393 361	19.33
6	4	210	985	1 461	6.96
6	8	3 003	26 577	63 369	21.10
6	16	74 613	1 110 049	3 242 322	43.46
7	4	330	1 765	2 777	8.42
7	8	6 435	74 313	223 332	34.71
7	16	245 157	5 529 233	21 254 517	86.70
8	2	45	129	116	2.60
8	4	495	2 945	5 645	11.41
8	8	12 870	187 137	733 748	57.01
9	2	55	163	152	2.78
9	4	715	4 645	10 760	15.05
9	8	24 310	432 073	2 252 367	92.65
10	2	66	201	202	3.08
10	4	1 001	7 001	19 423	19.40
10	8	43 758	927 441	5 912 807	135.13

V. NUMERICAL RESULTS

Using the algorithm in Fig. 5, we construct rank-1 Chebyshev lattices $\tilde{\Lambda}(\boldsymbol{z}, M)$ fulfilling condition (7) for the ℓ_1-ball index sets $I := I_n^d$ for various refinements $n \in \mathbb{N}$ and dimensions d. The corresponding size parameters M and oversampling factors $(M+1)/\lvert I_n^d\rvert$ are shown in Table I. Additionally, we apply [7, Algorithm 1 and 2] to the extended symmetric index sets $\mathcal{M}(I_n^d)$ with the modification that an even rank-1 lattice size $\hat{M} \in 2\mathbb{N}$ is returned. We obtain reconstructing rank-1 lattices $\Lambda(\boldsymbol{z}, \hat{M})$ for $\mathcal{M}(I_n^d)$ and consequently rank-1 Chebyshev lattices $\tilde{\Lambda}(\boldsymbol{z}, \hat{M}/2)$ fulfilling condition (7) for I_n^d due to Theorem IV.2. For the dimensions d and refinements n in Table I except the case $d = 7$ and $n = 16$, these rank-1 Chebyshev lattices are identical to the ones constructed by the algorithm in Fig. 5. In the mentioned case, the algorithm in Fig. 5 yielded a slightly larger size parameter $M = 21\,344\,934$. The reason for this is the greedy search for the generating vector $\boldsymbol{z}$ with fixed initial size parameter $M = M_{\text{start}}$ and both approaches returned a distinct generating vector $\boldsymbol{z}$. If we run the algorithm in Fig. 5 setting $M_{\text{start}} := 21\,254\,517$, then both approaches yield an identical rank-1 Chebyshev lattice.

TABLE II

CARDINALITIES OF HYPERBOLIC CROSS INDEX SETS H_n^d AS WELL AS SIZE PARAMETERS $M := \widetilde{M}$ AND $M := \hat{M}/2$ OF CORRESPONDING RANK-1 CHEBYSHEV LATTICES $\tilde{\Lambda}(\boldsymbol{z}, M)$ FULFILLING CONDITION (7) AND (10) FOR $I := H_n^d$, RESPECTIVELY.

Parameters		Card.	Condition (7) / (9)		Condition (10)
d	n	$\lvert H_n^d\rvert$	$\widetilde{M}$	$\frac{\widetilde{M}+1}{\lvert H_n^d\rvert}$	$\hat{M}/2$
2	256	1 979	66 050	33.38	66 050
2	512	4 305	263 170	61.13	263 170
2	1 024	9 311	1 050 626	112.84	1 050 626
3	256	10 303	302 883	29.40	359 075
3	512	23 976	1 424 613	59.42	1 424 662
3	1 024	55 202	4 600 672	83.34	5 560 838
4	128	17 700	860 284	48.60	1 083 747
4	256	44 403	3 136 383	70.63	4 355 469
4	512	109 395	14 659 035	134.00	19 550 612
5	64	23 853	1 382 832	57.97	1 703 741
5	128	64 373	6 843 471	106.31	9 138 634
5	256	170 299	31 997 990	187.89	41 255 293
6	16	8 684	303 396	34.94	557 773
6	32	26 088	1 751 513	67.14	2 867 903
6	64	76 433	8 979 932	117.49	13 603 339
7	8	7 184	291 267	40.54	529 877
7	16	23 816	1 659 143	69.67	3 575 914
7	32	75 532	10 375 340	137.36	21 375 543
8	4	5 120	196 522	38.38	629 597
8	8	18 176	1 334 559	73.42	2 975 159
8	16	63 328	8 615 461	136.05	22 270 727
9	2	2 816	132 708	47.13	473 013
9	4	12 032	781 974	64.99	3 449 019
9	8	45 056	6 329 397	140.48	16 125 059

Moreover, we consider hyperbolic cross index sets $I := H_n^d$. Again, we apply both algorithms for the construction of rank-1 Chebyshev lattices $\tilde{\Lambda}(\boldsymbol{z}, M)$ suitable for reconstruction. The results of these construction processes are shown in Table II. We remark that the size parameters M of the rank-1 Chebyshev lattices $\tilde{\Lambda}(\boldsymbol{z}, M)$ are distinctly larger for $d \geq 3$ when using [7, Algorithm 1 and 2], which itself uses condition (10).

VI. CONCLUSION

In this paper, we considered the fast evaluation as well as the fast, exact and stable reconstruction of high-dimensional multivariate algebraic polynomials in Chebyshev form at the nodes of rank-1 Chebyshev lattices. Moreover, we presented an algorithm for the construction of such lattices based on ideas for the CBC construction in the periodic case.

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