THE ANDERSON MODEL ON THE BETHE LATTICE: LIFSHITZ TAILS

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ABSTRACT. This paper is devoted to the study of the (discrete) Anderson Hamiltonian on the Bethe lattice, which is an infinite tree with constant vertex degree. The Hamiltonian we study corresponds to the sum of the graph Laplacian and a diagonal operator with non-negative bounded, i.i.d. random coefficients on its diagonal. We study in particular the asymptotic behavior of the integrated density of states near the bottom of the spectrum. More precisely, under a natural condition on the random variables, we prove the conjectured double-exponential Lifschitz tail with exponent 1/2. We study the Laplace transform of the density of states. It is related to the solution of the parabolic Anderson problem on the tree. These estimates are linked to the asymptotic behavior of the ground state energy of the Anderson Hamiltonian restricted to trees of finite length. The proofs make use of Tauberian theorems, a discrete Feynman–Kac formula, a discrete IMS localization formula, the spectral theory of the free Laplacian on finite rooted trees, an uncertainty principle for low-energy states, an epsilon-net argument and the standard concentration inequalities.

CONTENTS

1. Introduction

In this paper we are interested in a tight-binding, one-body Hamiltonian of a disordered alloy. This Hamiltonian is known as the Anderson model, and it was introduced in its most simple form by the American physicist Phillip W. Anderson in 1958 [And58]. Given the extensive mathematical and physical literature on the subject, see e. g. [Abr10, and references therein]. We defer the discussion and review of the literature until after the rigorous statement of our results.

The underlying physical space of our model is assumed to be a *Bethe lattice*, this is, an infinite regular graph with no loops and constant coordination number k (see fig. 1). The Anderson model in this setting was introduced very early by Abou-Chacra, Thouless and Anderson in [ATA73]. A number of physical and numerical (e. g. [KH85; MF91; MD94; BAF04; AF05; MG09]) as well as rigorous mathematical works (e. g. [KS83; Aiz94; AK92; Kle96; Kle98; ASW06; AW11b; AW11a; War13]) in this setting have been since published.

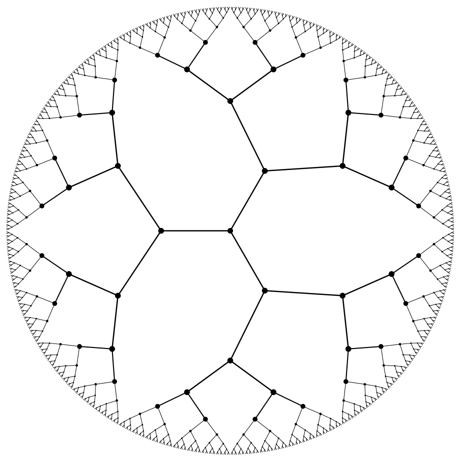

FIGURE 1. The Bethe lattice with coordination number $k = 3$.

The study of transport properties of disordered models leads to the spectral theory of random Schrödinger operators. The prototypical example of these operators is the Anderson Hamiltonian. An important quantity of study is the *integrated density of states*, which is a function we denote by $\mathcal{N}$. The numerical value $\mathcal{N}(E)$ counts the available energy levels below the energy E per unit volume. Under very general assumptions, the support of this function coincides with the spectrum of the Hamiltonian in consideration. The study of its asymptotic behavior when we approach the bottom of the spectrum E_0 has attracted a lot of attention since Lifshitz' remark [Lif65]. The physicist noted that in presence of disorder this asymptotic behavior is drastically different from the one of the free operator. Indeed, as soon as the disorder is non-trivial, this function exhibits a very fast decay at the bottom of the spectrum E_0. This behavior has also drawn the attention of many mathematicians, as it can be used as one of the main ingredients of the rigorous proofs of the ocurrence of *Anderson localization*. In the setting of our paper, it was conjectured, see [KH85; BST10; BS11], that the integrated density of states exhibits a double exponential decay with exponent 1/2, i. e. that for some suitable $\epsilon > 0$

$$\exp\left(-\mathrm{e}^{\epsilon^{-1}(E-E_0)^{-1/2}}\right) \leqslant \mathcal{N}(E) \leqslant \exp\left(-\mathrm{e}^{\epsilon(E-E_0)^{-1/2}}\right) \quad \text{as } E \searrow E_0. \tag{1.1}$$

In the literature (e.g. [War13, eq. 5]) one also finds this written in the somewhat weaker form

$$\lim_{E\searrow E_0} \frac{\log\log|\log \mathcal{N}(E)|}{\log(E-E_0)} = -\frac{1}{2}.$$

The purpose of this paper is to prove this conjecture. To do so, we study the Laplace transform $t \mapsto \tilde{\mathcal{N}}(t)$ of the measure $\mathrm{d}\mathcal{N}(\,\cdot\,)$ and we establish asymptotic bounds for large t. We will see that for a suitable $\epsilon > 0$

$$\mathrm{e}^{-t\left(E_0+\frac{\epsilon^{-1}}{(\log t)^2}\right)} \leqslant \tilde{\mathcal{N}}(t) \leqslant \mathrm{e}^{-t\left(E_0+\frac{\epsilon}{(\log t)^2}\right)} \quad \text{as } t \to +\infty. \tag{1.2}$$

These bounds are of independent interest, as they are related to the long-time behavior of the so called *parabolic Anderson problem* in the annealed regime. This long-time behavior is in turn related to the location of the ground state energy of suitable finite-dimensional approximations of the Anderson Hamiltonian. We discuss this circle of ideas, which is well known in the literature, after stating rigorously our results.

Most of the novelty lies in the proof of the bounds on the ground state energy E^L_{GS} of the Hamiltonian restricted to finite symmetric rooted trees $\mathcal{T}^L$ of length L (see fig. 2, where $L = \infty$). In absence of disorder, it behaves as

$$E_0 + CL^{-2}. \tag{1.3}$$

up to smaller terms. In presence of disorder, one expects heuristically that the ground state of the disordered Hamiltonian restricted to $\mathcal{T}^L$ lives in some smaller subtree of length $r = C' \log L$ on which the random potential is essentially zero. Hence, with good probability we should have

$$E^L_{GS} = E_0 + C''(\log L)^{-2}, \tag{1.4}$$

which is the order of the ground state energy of the free operator restricted to this subtree. The length scale $\log L$ appears naturally as one balances out the probability that the random potential is small in a subtree, which is exponential in the number of random variables (we find about k^r of them in a subtree of length r), and the number of trees of length r (there are about $k^{L-r} \leqslant k^L$). Note also that because of symmetry reasons, one might expect the ground state to localize to a ball.

It turns out that symmetric rooted trees provide a tractable, good approximation for the balls of the Bethe lattice (which are also finite trees).

Let us finish this short summary by emphasizing that the usual rigorous argument does not work in our setting, the culprit being (i) the exponential growth of the trees and (ii) the spectral gap of the free Laplacian restricted to trees, which is of order L^{-3} and thus too small, compared with (1.3) and (1.4). As a consequence of (i) we are not able to use Dirichlet–Neumann bracketing and (ii) renders impossible the approximation of the ground state of the perturbed operator by the ground state of the free one. We discuss later the new ideas required to overcome these two problems.

Let us now introduce some notation and the rigorous statements of our results.

1.1. **Main results.** To fix the ideas, let Γ be an infinite graph and denote by $\ell^2(\Gamma)$ the space of square summable functions defined on the vertices of Γ. Let Δ_Γ be the associated (negative definite) Laplacian operator, i. e.

$$\begin{aligned}\Delta_\Gamma \colon \ell^2(\Gamma) &\to \ell^2(\Gamma)\\ (\Delta_\Gamma f)(v) &:= \sum_{w\sim v}\big(f(w)-f(v)\big).\end{aligned}$$

Here the index $w \sim v$ runs over all neighboring nodes $w \in \Gamma$ of the node $v \in \Gamma$. Let us define a *random potential* on this graph, i. e. a diagonal operator

$$\begin{aligned}V^\Gamma_\omega \colon \ell^2(\Gamma) &\to \ell^2(\Gamma)\\ (V^\Gamma_\omega \varphi)(v) :=& \omega_v \varphi(v), \quad v \in \Gamma,\ \varphi \in \ell^2(\Gamma),\end{aligned}$$

where $\omega := \{\omega_v\}_{v\in\Gamma}$ is a sequence of non-trivial, bounded, non-negative, independent and identically distributed random variables. We will also assume that

$$\operatorname{ess\,inf} \omega_0 = 0.$$

This is no additional restriction given that we can always shift the energy through a translation. We are interested in the random operator

$$H^\Gamma_\omega := -\Delta_\Gamma + \lambda V^\Gamma_\omega, \tag{1.5}$$

where λ denotes a (strictly) positive coupling constant. We will call this Hamiltonian the *Anderson model on* Γ. Choose some $0 \in \Gamma$ and let us define its associated *integrated density of states* as

$$\mathcal{N}^\Gamma(E) := \mathbb{E}[\langle \delta_0, \mathbf{1}_{(-\infty,E]}(H^\Gamma_\omega)\delta_0\rangle], \tag{1.6}$$

which is a function of the energy $E \in \mathbb{R}$ defined on $\mathbb{R}$. Here, and in the rest of the paper, $\mathbf{1}_S$ denotes the indicator function of the set S, the operator $\mathbf{1}_{(-\infty,E]}(H^\Gamma_\omega)$ is the spectral projector on $(-\infty, E]$, defined by functional calculus and $\delta_v \in \ell^2(\Gamma)$ denotes Kronecker's delta. The function $\mathcal{N}^\Gamma(E)$ is positive, increasing and take values in $[0, 1]$. It is the cumulative distribution of the *density of states measure*, which we denote by $\mathrm{d}\mathcal{N}$.

If one assumes that H^Γ_ω is *ergodic* [PF92; CL90], then (1.6) is independent of the choice of $0 \in \Gamma$ and we know that there exists some set $\Sigma \subset \mathbb{R}$ such that

$$\Sigma := \sigma(H^\Gamma_\omega) = \sigma(-\Delta_\Gamma) + \lambda \operatorname{supp} \omega_0 = \operatorname{supp} \mathrm{d}\mathcal{N}^\Gamma, \tag{1.7}$$

for almost every ω. This is the case if Γ is the graph $\mathbb{Z}^d$ or the Bethe lattice $\mathcal{B}$ defined below. We will denote by E_0 the bottom of the almost sure spectrum, i. e.

$$E_0 := \inf \Sigma = \inf \sigma(-\Delta_\Gamma).$$

It is well known that the asymptotic behavior of the integrated density of states close to the bottom of the spectrum E_0 is very different in the presence of disorder (see remark 1.2 below or [KM06] for a survey). In this work, we study this behavior on a graph known as the *Bethe lattice*, which we define as an infinite connected graph, with no closed loops and degree constant and equal to $k+1$. If $k=1$, we obtain with this definition the graph $\mathbb{Z}$. From now on we fix $k \geqslant 2$ for the rest of this paper and we denote this graph by $\mathcal{B}$. Whenever we omit the index Γ it will be assumed that $\Gamma = \mathcal{B}$.

This paper is devoted to the proof of the following theorem.

Theorem 1.1. *Let $k \geqslant 2$ and H_ω be the Anderson model on the Bethe lattice of degree $k+1$. Then, if*

$$\nu := \limsup_{\kappa \searrow 0} \sqrt{\kappa} \log\left|\log \mathbb{P}(V_\omega(v) \leqslant \kappa)\right| < 1. \tag{1.8}$$

then inequalities (1.1) hold and thus

$$\lim_{E \to E_0} \frac{\log\log|\log N(E-E_0)|}{\log(E-E_0)} = -\frac{1}{2}. \tag{1.9}$$

Remark 1.2.
- The fact that the integrated density of states decays faster in presence of disorder has been known to hold rigorously since the works of Nakao [Nak77] and Pastur [Pas77] on $\mathbb{R}^d$. Analogous results have also been obtained in the discrete setting $\Gamma = \mathbb{Z}^d$. In this case, if $\lambda = 0$ in (1.5), then

$$\lim_{E \searrow E_0} \frac{\log \mathcal{N}^{\mathbb{Z}^d}(E)}{\log(E-E_0)} = \frac{d}{2}, \qquad \text{(Van Hove singularity)}$$

  while as soon as $\lambda \neq 0$, (and with a restriction analogous to (1.8))

$$\lim_{E \searrow E_0} \frac{\log|\log \mathcal{N}^{\mathbb{Z}^d}(E)|}{\log(E-E_0)} = -\frac{d}{2}. \qquad \text{(Lifshitz tails)}$$

- In absence of disorder, the density of states of the free Laplacian on the Bethe lattice can be calculated explicitly (see [Kes59; McK81]),

$$\begin{aligned} \mathrm{d}\mathcal{N}_0(E) :=& \mathrm{d}\langle \delta_0, \mathbf{1}_{(-\infty,E]}(-\Delta_{\mathcal{B}})\delta_0\rangle \\ =& \mathbf{1}_I(E) \frac{k+1}{2\pi} \frac{\sqrt{4k-(E-k-1)^2}}{(k+1)^2-(E-k-1)^2} dE, \end{aligned} \tag{1.10}$$

with $I := \sigma(-\Delta_{\mathcal{B}}) = \operatorname{supp} \mathrm{d}\mathcal{N}_0 = [(\sqrt{k}-1)^2, (\sqrt{k}+1)^2]$. In particular, we see that for any $k \geqslant 2$

$$\lim_{E \searrow E_0} \frac{\log \mathcal{N}_0(E)}{\log(E - E_0)} = \frac{3}{2}.$$

- The double exponential decay of the integrated density of states in (1.9) stems from concentration inequalities, which are exponential in the volume of shells of the Bethe lattice, and the fact that the volume of these shells grows exponentially with their radius.
- Condition (1.8) tells us that the distribution of the random variables should not decay too fast when we approach 0. It is satisfied, for example, by uniform or Bernoulli random variables. We provide in the text a slightly weaker version for which we can prove (1.9) but not (1.1). If this last condition is not satisfied, it is indeed possible to show that the lower bound fails (see lemma 2.3). Similar results are known to hold true in the Euclidean settings (see [KM06]).

To establish our main result, we will study the Laplace transform of $\mathrm{d}\mathcal{N}$. The study of the integrated density of states through the Laplace transform of its derivative goes back at least to Pastur [Pas71]. This last work together with the celebrated result of Donsker and Varadhan [DV75] on the asymptotics of the Wiener sausage were used to give a rigorous proof of the Lifshitz tails for the continuous Anderson model with Poisson impurities, see [Pas77; Nak77]. The same ideas work in the discrete setting [BK01]. The spectral theorem shows that the Laplace transform of the *density of states measure* $\mathrm{d}\mathcal{N}$ is the continuous solution $u\colon [0, +\infty) \times \mathcal{B} \to [0, +\infty)$ of a heat equation associated to H_ω evaluated at one point. Thus, the proof of our main theorem will be a consequence of our next result, which is related to the following Cauchy problem:

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,v) = \Delta_{\mathcal{B}} u(t,v) - V_\omega u(t,v), & \text{for } (t,v) \in (0,\infty) \times \mathcal{B} \\ u(0,v) = \delta_0(v) & \text{for } v \in \mathcal{B}. \end{cases} \tag{1.11}$$

The solution $t \mapsto u(t,\,\cdot\,)$ is the solution to the *heat equation with random coefficients* and localised initial datum δ_0. Again, $0 \in \Gamma$ is here any point of the lattice and the results are independent of this choice.

Theorem 1.3 (Annealed regime)**.** *Assume* (1.8)*. There exists some $\epsilon > 0$ and $t^* > 0$ such that for all $t > t^*$,*

$$\exp\Big(-t\Big(E_0 + \frac{\epsilon^{-1}}{(\log t)^2}\Big)\Big) \leqslant \mathbb{E}\big[u(t,0)\big] \leqslant \exp\Big(-t\Big(E_0 + \frac{\epsilon}{(\log t)^2}\Big)\Big) \tag{1.12}$$

Remark 1.4. Obviously $\exp(-t(E_0 + |O((\log t)^{-2})|)) = \exp(-|O(t(\log t)^{-2})|)$ but the quantity $E_0 + |O((\log t)^{-2})|$ can be regarded as an energy in the spectrum Σ close to the bottom E_0.

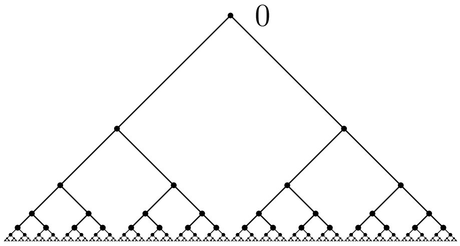

FIGURE 2. The infinite rooted tree.

The long term behavior (1.12) at the node $0 \in \Gamma$ of the solution to the heat equation (1.11) is well approximated by finite volume versions of the same problem (using e. g. Feynman–Kac formula). More precisely, we will look at the solution to the Cauchy problem on a ball of radius $L \asymp t$ with *Dirichlet* boundary conditions, i. e. we require that the solution is zero outside this ball. The solution of the finite dimensional problem is then bounded above by a term of the form $\mathrm{e}^{-tE_{GS}(H_\omega|\mathcal{B}^L)}$, where $E_{GS}(H_\omega|\mathcal{B}^L)$ denotes the smallest eigenvalue of H_ω restricted to the ball $\mathcal{B}^L$. A crucial ingredient of our proof consists in replacing the balls $\mathcal{B}^L$ by *finite symmetric roooted trees* $\mathcal{T}^L$.

Let us introduce some more notation. We let $\mathcal{T}$ be a rooted tree with branching number k, this is, an infinite connected graph which has no closed loops and such that the degree is constant and equal to $k+1$ on every site, except at one particular site 0, which is called the root of the tree (see fig. 2). Note that we can embed this infinite graph into the Bethe lattice $\mathcal{B}$. In this note we consider finite versions of this tree, namely, for any natural number $L > 0$ we denote by $\mathcal{T}^L$ the subtree of $\mathcal{T}$ of finite depth L, consisting on all those sites at a distance $L-1$ or smaller from the root 0:

$$\mathcal{T}^L := \{v \in \mathcal{T} : d^{\mathcal{T}}(0, v) \leqslant L-1\}.$$

Here $d^\Gamma(\,\cdot\,,\,\cdot\,)$ denotes the graph distance associated to the graph Γ. By introducing the notation (which we use later) $|v| = d^{\mathcal{T}}(0, v) + 1$ for the "level" of the node v, we can also write $\mathcal{T}^L = \{v \in \mathcal{T} : |v| \leqslant L\}$.

These finite symmetric rooted trees look like the balls $\mathcal{B}^L \subset \mathcal{B}$, centered at 0, after removing entirely one of the branches attached to the center of the ball . Note that $\mathcal{T}^L$ is a finite connected graph which has no closed loops and such that the degree is constant at each site except at the root and at the leaves $\{|v| = L\}$. We also picture these finite subtrees as subsets of the infinite Bethe lattice $\mathcal{B}$. Let now

$$H^L_\omega := H_\omega|\mathcal{T}^L$$

be the Hamiltonian H_ω restricted to the subtree $\mathcal{T}^L$ of length L with *Dirichlet* (also called *simple*) boundary conditions. We denote by E^L_{GS} the ground state energy of H^L_ω, i. e.

$$E^L_{GS} := \inf_{\|\varphi\|_2=1} \left\langle H^L_\omega \varphi, \varphi \right\rangle .$$

We can now state our last result.

Theorem 1.5. *Asssume* (1.8). *There exists some $\epsilon > 0$ and $L^* > 1$ such that for all $L > L^*$ we have*

$$E_0 + \epsilon(\log L)^{-2} \leqslant E^L_{\mathit{GS}} \leqslant E_0 + \epsilon^{-1}(\log L)^{-2}$$

with probability at least

$$1 - \exp(-\mathrm{e}^{\epsilon L}).$$

To finish the presentation of our results, let us note that an immediate corollary of theorem 1.5 is that

$$\lim_{L\to\infty} \frac{\log(E^L_{\mathit{GS}} - E_0)}{\log\log L} = -2 \quad \text{a.s.} \tag{1.13}$$

In fact, the Borel–Cantelli lemma implies this, since

$$\sum_{L=L^*}^{\infty} \mathbb{P}\Big(\frac{\log(E^L_{\mathit{GS}} - E_0)}{\log\log L} > -2 - \frac{\log\epsilon}{\log\log L}\Big) \leqslant \sum_{L=L^*}^{\infty} \exp(-\mathrm{e}^{\epsilon L}) < \infty,$$

so that $\limsup_{L\to\infty} \frac{\log(E^L_{\mathit{GS}} - E_0)}{\log\log L} \leqslant 2$ a.s., and analogously for the other direction. For comparison, in absence of disorder we obtain (see section 3):

$$E^L_0 := E_{\mathit{GS}}(-\Delta_{\mathcal{B}}|\mathcal{T}^L) = E_0 + \frac{\sqrt{k}\pi^2}{(L+1)^2} + O(L^{-4}) \tag{1.14}$$

which implies

$$\lim_{L\to\infty} \frac{\log(E^L_0 - E_0)}{\log L} = -2 \quad \text{a.s.}$$

1.2. **Discussion.** The results we presented concern an operator which appears naturally in the study of the macroscopic properties of crystals, alloys, glasses, and other materials. If one looks at the Schrödinger equation

$$\begin{cases} i\frac{d\varphi}{dt} = H^\Gamma_\omega \varphi, & i^2 = -1 \\ \varphi(0) = \varphi_0 \in \ell^2(\Gamma), & \|\varphi_0\|^2_2 = 1 \end{cases}, \tag{1.15}$$

then the Anderson model H^Γ_ω defined by (1.5) describes the Hamiltonian governing the behavior of a quantum particle having an initial state φ_0 in a disordered medium. Its integrated density of states measures the "number of energy levels per unit volume" and is a concept of fundamental importance in condensed matter physics, as it encodes various thermodynamical quantities of the material, spectral features of the operator and properties of the underlying geometry.

The Anderson model has been the subject of hundreds of physical and mathematical papers. One of the most studied mathematical features of this model is the phenomenon known as *Anderson localization*, i. e. that the spectrum of the random operator exhibits pure point spectrum with probability 1, for *any* strenght of disorder, whereas the free operator has only absolutely continuous spectrum.

We invite the interested reader to consult the monographs [CL90], [PF92], [Sto01], [His08], [Kir07].

One of the hallmarks of Anderson localization is the so-called *Lifshitz tails behavior*, i. e. the exponential decay of the integrated density of states at the bottom of the spectrum. It is well known that such a decay, together with additional assumptions on the regularity of the random variables, provides one of the main ingredients to start the *multi-scale analysis*, see e. g. [GK13], or to satisfy the *fractional moment criterion* ([AM93]). These strategies have been succesfully applied to prove the existence of localization in a neighborhood of the spectral edges, for example when the graph is $\mathbb{Z}^d$ (the model introduced originally by P. W. Anderson in [And58]) or its continuous version on $\mathbb{R}^d$.

The Bethe lattice is of interest in statistical mechanics because of its symmetry properties and the absence of loops. It allows to obtain closed solutions for some models, e. g. in percolation theory and the non-rigorous scaling theory of Anderson localization. In our setting, the resolvent of the operator H_ω on the Bethe lattice admits a recursive representation (see e. g. [Ros12]), but in this work we make no use of these formulae. It was for these reasons that the model was studied in [ATA73] and it enjoys some renewed interest in the physical community (see e. g. [BST10; BS11]). Because of its exponential growth, it is also of interest in connection with the configuration space of many-body problems [Alt+97].

Perhaps one of the most striking features of the operator defined on the infinite tree $\mathcal{B}$ is the *absence of pure point spectrum* at weak disorder [Kle96; AW11b; AW11a]. For a survey on recent progress on the spectral properties of the Anderson model on the Bethe lattice see [War13]. At weak disorder, thus, this model exhibits no Anderson localization, even near the spectral edges where Lifshitz tails take place. For the Anderson model on the Bethe lattice, the existence of a Lifshitz tail *does not imply localization*. We remind the reader that in the Euclidean case, the absence of localization at higher energies and therefore the existence of a spectral transition is still an open problem.

The parabolic problem (1.11) is the heat equation associated with the Anderson Hamiltonian and is well studied under the name of *Parabolic Anderson model*. It describes a random particle flow in the tree $\mathcal{B}$ through a random field of soft sinks, which can also be seen as traps or obstacles via the Feynman–Kac formula. There is an additional interpretation in terms of a branching process in a field of random branching rates. We refer the reader to [GK05; KW] for a survey. In this context, one is usually interested in the behavior of the total mass $\|u(t,\,\cdot\,)\|_1$ of the solution for large $t > 0$, which is also the behavior of the solution to the heat equation with initial datum $u(\,\cdot\,, 0) \equiv 1$ at one point. Because of the exponential growth of the graph, we were not able to study this quantity. However, theorem 1.3 is a first step in this direction.

A related question of interest is whether there is intermittency in this setting. Following the heuristics described in [KW], intermittency can be understood as a consequence of Anderson localization. On the other hand, Lifshitz tails have been proven as a by-product of the proof of intermittency in the parabolic Anderson model in [BK01]. Given that we may have absolutely continuous spectrum in spite of the existence of Lifshitz tails, the answer to this question is an interesting subject for further studies.

This discussion would not be complete without citing some previous results. The Lifschitz tails behavior for a percolation model on the Bethe lattice was studied in [Rei09], see also [MS11]. In [Sni89] similar bounds to ours are obtained for $\mathcal{N}$ and for $\tilde{\mathcal{N}}$ in the hyperbolic space, which is the continuous analog of the Bethe lattice. In our setting, Lifshitz tails were studied in [BS11; Ros12], where in particular a rigorous lower bound

$$\liminf_{E\to E_0} \frac{\log\log|\log N(E-E_0)|}{\log(E-E_0)} \geqslant -\frac{1}{2}.$$

is established. A proof of any type of decay other than the trivial one has resisted several attempts to be rigorously proven. We will try to explain now why.

The first problem concerns the finite-dimensional approximation of the infinite dimensional operator. In the standard setting of the Anderson model $\Gamma = \mathbb{Z}^d$, if we let

$$\Lambda_L := \{v \in \mathbb{Z}^d : \|v\|_\infty \leqslant L\},$$

then the thermodynamic limit of the *normalized eigenvalue counting function*

$$\lim_{L\to\infty} \frac{\#(\sigma(H_\omega|\Lambda_L) \cap [0,E])}{\#\Lambda_L} = \lim_{L\to\infty} \frac{\operatorname{tr} \mathbf{1}_{[0,E]}(H_\omega|\Lambda_L)}{\#\Lambda_L} \tag{1.16}$$

converges almost surely for every E where $\mathcal{N}$ is continuous. In this setting as well as in the continuous version on $\mathbb{R}^d$, the limit defined in (1.16) is *independent of the boundary conditions*. That this limit coincides with the averaged spectral function $\mathcal{N}$ at these points is known as the *Pastur–Shubin formula*. There is a wealth of results in this direction in different settings. It holds in particular on any amenable graph Γ (like $\mathbb{Z}^d$) if we choose the sequence Λ_L as a Følner sequence.

The limit (1.16) may have different limits depending on the choice of boundary conditions when the graph is not amenable [AW06]. In particular, the usual Dirichlet–Neumann bracketing is of no hope. A similar phenomenon occurs on the hyperbolic space [Sni89; Sni90]. This leaves us with the problem of finding the right finite volume approximations. In [SS14] it is proven that the Pastur–Shubin formula holds in great generality. This setting includes the Cayley graph of a free group. The finite volume approximations are the analogous of the periodic boundary conditions in the Euclidean case. Unfortunately, the rate of convergence of the approximations to the averaged spectral function (1.6) is unknown. An approach of a different vein was explored in [Gei14]. Here one looks at this problem on random regular graphs. It is indeed known, since the pioneering works of McKay [McK81] and Kesten [Kes59], that the density of eigenvalues of the free Laplacian of random regular graphs converges to the measure given by (1.10). This approach introduces another source of randomness and seems difficult to study. We avoid these difficulties alltogether by approximating the integrated density of states only at low energies.

In this work we consider Dirichlet restrictions of the operator to finite trees $\mathcal{T}^L$. We show that these are good approximations as long as we look at the bottom of the spectrum. The problem then reduces, as usual, to that of finding good upper bounds on the ground state energy $E_{GS}(H_\omega|\mathcal{T}^L)$ with good probability. In the standard Anderson model, one usually uses Temple's, [Sim85], or Thirring's inequality, [KM83], or perturbation theory, [Sto99]. Unfortunately, they are all based on the premise that the ground state of the perturbed operator, modulo some small error, should look like the one of the free operator. This reduces the problem to study only the effect of the random potential on the ground state of the free operator. The error term in these methods is related to the spectral gap of the free operator, i. e. the distance between the first and the second eigenvalue. In the Euclidean setting both the first and the second eigenvalue of the free operator with Dirichlet boundary conditions are of the same order, whereas in our setting we have (1.14) but the scond eigenvalue of $-\Delta_{\mathcal{B}}|\mathcal{T}^L$ behaves as

$$E_{GS}(-\Delta_{\mathcal{B}}|\mathcal{T}^L) + O(L^{-3}).$$

See section 3 for these calculations.

1.3. **Strategy of proof.** In order to make the reading of this paper easier, we provide here a road map for the proofs and a table of notations in page 16. This will also allow us to comment on some results and acknowledge some sources. To simplify the (not necessarily rigorous) exposition, we assume $k = 2$ and $\epsilon > 0$ a small constant which may change from line to line. We also write $A \lesssim B$,$B \gtrsim A$ if there exists a constant c such that $A < cB$; $A \asymp B$ if $A \lesssim B$ and $A \gtrsim B$. As usual, to prove theorem 1.1 we prove lower and upper bounds separately.

1.3.1. *Lower bound.* We first prove lower bounds on the integrated density of states, which is usually easier. This is done in section 2. Note that a rigorous proof of the lower bound was already obtained in [BS11] and [Ros12]. Our method is not very different from the one in [Ros12], but we do identify the sharper condition (1.8). Another novelty is that we also obtain a precise lower bound for the ground state energy on finite rooted trees. Indeed, as will be clear, to obtain the lower bound of the Lifshitz tails it is enough to prove that for large L

$$E^L_{GS} \leqslant E_0 + \epsilon L^{-2} \tag{1.17}$$

holds at least with probability

$$\mathrm{e}^{-\mathrm{e}^{\epsilon^{-1} L}}. \tag{1.18}$$

This is usually proven by finding a suitable test function. In [Ros12] the test function corresponds to the ground state of the free Laplacian and the probability

(1.18) corresponds simply to the probability that all the random variables on the tree are small.

Our proposition 2.1 corresponds to the upper bound in theorem 1.5. It is proven by localizing the test function to a subtree of length $\log L$ (see fig. 3 in page 18). This is crucial to prove that the almost sure behavior of E^L_{GS} is of order $(\log L)^{-2}$ for large L. We show then that this upper bound implies the lower bound in (1.2) (corollary 2.5) and a Tauberian theorem (lemma 2.6 and proposition 2.7) gives the lower bound in (1.1).

1.3.2. *Upper bound.* In section 3, we introduce some elements and tools we will need in the course of the proof. Because these calculations make no use of randomness, we decided to isolate them in their own section.

We first study the spectral theory of the free Laplacian on finite trees, calculating explicitly all the eigenvalues and an orthonormal basis of eigenfunctions (lemmas 3.1 to 3.3 and 3.5). The eigenfunctions have their support confined to disjoint subtrees. This property is crucial when we study the action of the random potential in section 4.

We then prove an analog of the Ismagilov–Morgan–Sigal (IMS) localization formula on trees (proposition 3.6). The proof is of interest on its own as it can be adapted to very general discrete settings. In this work, we have decided to prove it only for functions in $\ell^2(\mathbb{Z})$ (lemma 3.7) and then carry it over to the tree by means of the spectral theory of the free Laplacian.

We finally prove in this section an uncertainty relation for truncated eigenfunctions on the tree. Explaining the details here would make this road map too long, but see below to see how this truncation is used. We also prove first this property for functions in $\ell^2(\mathbb{Z})$ and then use the spectral theory to prove it on $\ell^2(\mathcal{T}^L)$.

In section 4 we prove the upper bound in (1.1). It proceeds roughly as follows. The first step is a Tauberian theorem. It is an elementary consequence of an upper bound on the integrated density of states for energies close to E_0 using the large time decay of the Laplace transform $\tilde{\mathcal{N}}(t)$ of its derivative (proposition 4.2). We include it for the sake of completeness. It says the following: if for some $c > 0$

$$\tilde{\mathcal{N}}(t) \leqslant \exp(-t(E_0 + c(\log t)^{-2})) \quad \text{for } t \gg 1 \tag{1.19}$$

then

$$\mathcal{N}(E) \leqslant \exp(-\mathrm{e}^{\epsilon(E-E_0)^{-1/2}}) \quad \text{for } E - E_0 \ll 1.$$

We then do a reduction to a finite scale (proposition 4.3). We show that we can restrict the operator to a ball, as long as the ball grows linearly with the time t:

$$\tilde{\mathcal{N}}(t) = \mathbb{E}\langle \delta_0, \mathrm{e}^{-tH_\omega}\delta_0\rangle \leqslant \mathbb{E}\langle \delta_0, \mathrm{e}^{-tH_\omega|\mathcal{B}^L}\delta_0\rangle + \mathrm{e}^{-\zeta t} \quad \text{with } t = \zeta L \text{ and } \zeta \gg 1.$$

The approximations $H_\omega|\mathcal{B}^L$ considered in this step correspond to the Hamiltonian H_ω restricted to balls of radius L with *simple* (also called *Dirichlet*) boundary conditions. A proof using the Feynman–Kac formula is contained in [Ros12] (following [BK01]). We provide a somewhat elementary proof of this fact, which appears to be new in this context. The idea is to compare the series expansion of both e^{-tH_ω} and $\mathrm{e}^{-tH_\omega|\mathcal{B}^L}$ and expand the matrix products of the terms $\langle\delta_0, (-tH_\omega)^n\delta_0\rangle$ as products of paths from δ_0 to δ_0 of length at most $n/2$. This is a discrete version of the Feynman–Kac formula. Because the coefficients of both H_ω and its approximation coincide in a large ball, the error we make is easily estimated by the tail of the exponential series.

The next step consists in simply replacing the operator by its ground state energy (lemma 4.4). Using a spectral decomposition of δ_0 in terms of the eigenfunctions of H_ω leads to an upper bound of the form

$$\langle\delta_0, \mathrm{e}^{-tH_\omega|\mathcal{B}^L}\delta_0\rangle \leqslant \mathrm{e}^{-tE_{GS}(H_\omega|\mathcal{B}^L)}. \tag{1.20}$$

It is easy to see (lemmas 4.5 and 4.6) that every ball is embedded in a finite symmetric rooted tree and that we can replace the ball by a tree $\mathcal{T}^L$ because

$$\mathrm{e}^{-tE_{GS}(H_\omega|\mathcal{B}^L)} \leqslant \mathrm{e}^{-tE_{GS}(H_\omega^L)} \quad \text{where } H_\omega^L = H_\omega|\mathcal{T}^L.$$

This is crucial in our argument, as we are able to use the spectral theory on the tree as a makeshift "Fourier transform" in the probability estimates we describe below. Using the two last inequalities and taking the expectation in (1.20) we see that (lemma 4.7) for any $E \geqslant 0$

$$\mathbb{E}\langle\delta_0, \mathrm{e}^{-tH_\omega|\mathcal{B}^L}\delta_0\rangle \leqslant \mathrm{e}^{-tE} + \mathrm{e}^{-tE_0}\mathbb{P}[E_{GS}^L \leqslant E].$$

As the (1.19) and the proven lower bound (1.17) suggest, one should take $E = E_0 + C(\log L)^{-2}$ in the last inequality. Note that $E_{GS}(-\Delta|\mathcal{T}^L) = E_0 + CL^{-2}$ is not even of the same order. We perform then a reduction to an even smaller scale. By using the IMS localization formula we go from the scale L to $r = \epsilon^{-1}\log L$ (proposition 4.9). By doing this, we trade energy for probability: the number of subtrees of length $\log L$ is about $k^{L-\log L} \asymp k^L = k^{\mathrm{e}^{\epsilon r}}$ and thus

$$\mathbb{P}[E_{GS}^L \leqslant E_0 + C(\log L)^{-2}] \leqslant k^{\mathrm{e}^{\epsilon r}}\mathbb{P}[E_{GS}^r \leqslant E_0 + Cr^{-2}].$$

It is clear now that we need to prove a bound of the form

$$\mathbb{P}[E_{GS}^L \leqslant E_0 + CL^{-2}] \leqslant \mathrm{e}^{\mathrm{e}^{-\epsilon' L}}, \quad L \gg 1 \tag{1.21}$$

to obtain (1.19). Proving this bound occupies the last section of this paper.

To prove (1.21) we proceed as follows. The idea is that functions with low kinetic energy average the random potential and this pushes their energy away from the bottom of the spectrum. First we show that (up to an error) we can assume that the (random) ground state φ^L of H_ω^L is a linear combination of low energy eigenfunctions. Using our "unique continuation principle", we can furthermore replace these eigenfunctions by truncated versions. The rest of the proof is a careful analysis of the action of the random potential on functions of this type.

Let us be more precise. Every eigenfunction of $\Delta|\mathcal{T}^L$ is supported in some subtree of $\mathcal{T}_v^L$ of length $L-|v|$ rooted at $v \in \mathcal{T}^L$ (see section 3). We index the eigenfunctions $\Psi_{v,m}$ by their *anchor* $v \in \mathcal{T}^L$ and their *mode* (or *frequency*) $m = 1, \ldots L - |v|$. It is not hard to see (lemma 3.3) that the eigenfunctions $\Psi_{v,m}$ close to the bottom E_0 satisfy

$$\langle \Delta_{\mathcal{T}} \Psi_{v,m}, \Psi_{v,m} \rangle \approx E_0 + \Big(\frac{Cm}{L - |v| + 1} \Big)^2$$

We deduce then that the eigenfunctions $\Psi_{v,m}$ having small energy, i. e. those for which

$$\Big(\frac{Cm}{L - |v| + 1} \Big)^2 \leqslant \beta L^{-2}$$

satisfy both that their modes are bounded by β (uniformly in L), i. e.

$$m \leqslant \beta \tag{1.22}$$

and that the distance of their anchors to the root of $\mathcal{T}^L$ is bounded linearly in L, i. e.

$$|v| \leqslant C_A L, \qquad 0 < C_A < 1. \tag{1.23}$$

The susbscript A here stands for "anchor". The reader may imagine that the ground state is (up to an error) "bandlimited in Fourier space".

After this projection in "Fourier space", we introduce a truncation in physical space. This is necessary because low energy functions are not "flat" in the usual sense. The reader will convince herself by looking at the ground state of $\Delta_{\mathcal{T}^L}$, which is radially symmetric and thus exponentially decaying from the root. Nevertheless, these functions distribute evenly their ℓ^2-mass in the transversal direction. We can thus throw away some of the mass close to the anchor and control precisely the error by doing so(with our "unique continuation principle"). Let us call φ^L the (random) ground state of H^L_ω we obtain after applying the spectral projection and the truncation.

Most of the averaging now takes place away from the anchor in the radial direction. Let us look at the potential energy of φ^L. If we center the random variables

$$\langle V_\omega \varphi^L, \varphi^L \rangle = \langle (V_\omega - (\mathbb{E}\omega_0)) \varphi^L, \varphi^L \rangle + (\mathbb{E}\omega_0) \|\varphi^L\|_2$$

then the first term in the sum is close to zero with good probability. The second term of the sum is of order 1, which implies that we are far from the bottom of the spectrum.

We proceed now to explain the probability estimates. To show that the potential energy is concentrated around its mean we may use Hoeffding's inequality, which tells us that for fixed $\varphi \in \ell^2(\mathcal{T}^L)$ we have

$$|\langle (V_\omega - (\mathbb{E}\omega_0)) \varphi, \varphi \rangle| \lesssim \kappa. \tag{1.24}$$

with probability at least

$$1 - \exp(-O(\kappa^2 / \operatorname{Var}[\langle (V_\omega - (\mathbb{E}\omega_0)) \varphi, \varphi \rangle])).$$

Cauchy–Schwarz then tells us that

$$\operatorname{Var}[\langle (V_\omega - (\mathbb{E}\omega_0)) \varphi, \varphi \rangle] \lesssim \|\varphi\|_4^2.$$

We cannot apply directly this inequality with φ^L because it depends on the realization ω. To get rid of this problem, we exploit first the spectral theory of the free Laplacian. Because some spectral projectors have disjoint support, we are able to reduce the metric entropy in the second step, which is a classical ϵ-net argument. The problem is now reduced to estimate the probability that inequality (1.24) holds for every φ chosen from a fixed ϵ-net. The last part of the calculation is thus a uniform estimation of the ℓ^4-norm of functions both restricted in "Fourier" space and truncated in physical space, which decay exponentially fast in $k^{\epsilon L}$.

This finishes our presentation of the proofs and the introduction.

Table of Notation		
Γ	$\triangleq$	A graph. When it is missing from the notation we assume that $\Gamma = \mathcal{B}$
$-\Delta_\Gamma$	$\triangleq$	Graph Laplacian of the graph Γ
$\mathcal{B}$	$\triangleq$	Bethe lattice of degree k (infinite graph)
$\mathcal{B}^L$	$\triangleq$	Ball of radius L of the Bethe lattice $\mathcal{B}$
$\mathcal{T}^L$	$\triangleq$	Rooted tree of length L (every node has k children but the leaves)
E_0	$\triangleq$	Bottom of the spectrum, $E_0 := \inf\sigma(-\Delta_{\mathcal{B}}) = (\sqrt{k}-1)^2 \overset{\text{a.s.}}{=} \inf\sigma(H_\omega)$
$E_{GS}(H)$	$\triangleq$	Ground state energy of H, i. e. $E_{GS}(H) := \inf_{\lVert\varphi\rVert_2=1}\langle\varphi, H\varphi\rangle$
$\mathcal{N}$	$\triangleq$	Integrated density of states of H_ω on $\mathcal{B}$
$\mathcal{N}^L$	$\triangleq$	Expected integrated density of states of H_ω on $\mathcal{B}^L$
$\mathrm{d}\mathcal{N}$	$\triangleq$	Density of states measure
$\tilde{\mathcal{N}}$	$\triangleq$	Laplace transform of the density of states measure $\mathrm{d}\mathcal{N}$
$H\vert\Gamma$	$\triangleq$	Restriction of the operator H with *simple* b.c.
$\mathbb{P}$	$\triangleq$	Probability
$\mathbb{E}$	$\triangleq$	Expectation
δ_i	$\triangleq$	Kronecker's delta, i. e. $\delta_i(j) = 1$ for $i = j$ and zero elsewhere
$d_\Gamma(\,\cdot\,,\,\cdot\,)$	$\triangleq$	Graph distance associated to Γ
$\mathcal{B}_v^L$	$\triangleq$	Ball of the Bethe lattice of radius L centered at v
$\mathcal{T}_v^L$	$\triangleq$	Finite symmetric rooted tree of length L with root v
$\mathcal{T}_*^L$	$\triangleq$	Augmented finite tree, i. e. $\mathcal{T}_*^L := \mathcal{T}^L \uplus \{*\}$ and $d(*,0) = 1$
$\lvert v\rvert$	$\triangleq$	Distance to $* \in \mathcal{T}_*^L$, i. e. $\lvert v\rvert = d(*,v) = d(0,v)+1$.
H_ω^L	$\triangleq$	Anderson Hamiltonian restricted to $\mathcal{T}^L$ with *simple* b.c.
$\mathbf{1}_S$	$\triangleq$	Indicator function of the set S
$u \sim v$	$\triangleq$	u and v are neighbors, i. e. $d(u,v) = 1$
A_Γ	$\triangleq$	Adjacency matrix of the graph Γ
∇, ∇^*	$\triangleq$	Forward gradient on $\mathbb{Z}$ and its adjoint
η_a	$\triangleq$	Partition of unity on $\mathbb{Z}$
$\eta_{a,r}$	$\triangleq$	r-scaled, radially symmetric partition of unity on $\mathcal{T}$
C_{IMS}	$\triangleq$	The constant in the error of the IMS formula
$E_\beta^{(L)}$	$\triangleq$	An energy defined as $E_\beta^{(L)} := 2\sqrt{k}\cos\left(\frac{(\beta+1)\pi}{L+1}\right)$, see lemma 3.3.
$\Pi_E^{(L)}$	$\triangleq$	$\Pi_E^{(L)} := \mathbf{1}_{(-\infty,E]}(k+1+\Delta_{\mathcal{B}}^L\vert\mathcal{T}^L) = \mathbf{1}_{[E,+\infty)}(A_{\mathcal{B}}^L\vert\mathcal{T}^L)$
$\Pi_E^{(L)}$	$\triangleq$	$\Pi_E^{(L)} := 1 - \Pi_E^{(L)}$
$B^{(L)}$	$\triangleq$	unitball of $\ell^2(\mathcal{T}^L)$
$B_v^{(L)}$	$\triangleq$	unitball of $\ell^2(\mathcal{T}_v^L)$
ω_v	$\triangleq$	One of the non-trivial, bounded, i.i.d. random variables having 0 in their support

p.t.o.

Table of Notation (continued)	
$\bar{\omega}$	$\triangleq$ Expectation of the random variable ω_v
$\tilde{\omega}_v$	$\triangleq$ Centered random variable $\tilde{\omega}_v := \omega_v - \bar{\omega}$
ω_+	$\triangleq$ Sup-norm of the random variables $\omega_+ := \|\omega_0\|_\infty$
$\tilde{\omega}_+$	$\triangleq$ Sup-norm of the centered random variables $\tilde{\omega}_+ := \|\omega_0 - \bar{\omega}\|_\infty$

2. Lifschitz tails: The lower bound

In this section we prove the upper bound in theorem 1.5, the lower bound in theorem 1.3 and the lower bound in theorem 1.1.

2.1. Locating the ground state on a finite rooted tree: The upper bound. Denote by E_0 the infimum of the spectrum of the free Laplacian Δ on the infinite rooted tree with k children at each node, i. e. $E_0 := (\sqrt{k} - 1)^2$. As we will see in section 3.1, the ground state energy of the free Laplacian restricted to the finite tree $\mathcal{T}^L$ of length L (with the root on level 1) reads

$$E_{GS}(-\Delta|\mathcal{T}^L) = E_0 + 2\sqrt{k}\big(1 - \cos(\frac{\pi}{L+1})\big) > E_0.$$

The distance between these two values is thus of the order of L^{-2} as $L \to \infty$. By adding a nonnegative random potential V_ω, we increase the ground state energy by at least $\inf V_\omega(\mathcal{T}^L)$. Our first proposition gives a probabilistic upper bound on the random ground state energy of the random operator $H^L_\omega := -\Delta^L + V_\omega$ on $\mathcal{T}^L$.

Proposition 2.1. *Assume that the single-site potentials $V_\omega(v)$, $v \in \mathcal{T}$, satisfy*

$$\nu := \limsup_{\kappa \searrow 0} \sqrt{\kappa} \log\big|\log \mathbb{P}(V_\omega(v) \leqslant \kappa)\big| < 1. \tag{2.1}$$

Fix $C_1 > 1 + \pi^2\sqrt{k}(\log k)^2/(1-\nu)^2$ and $\varepsilon \in (0,1)$. Then there is a scale $L_0 = L_0(k, \nu, C_1, \varepsilon)$ such that, for all $L \geqslant L_0$, we have

$$\mathbb{P}\big(\inf \sigma(H^L_\omega) \leqslant E_0 + C_1(\log L)^{-2}\big) \geqslant 1 - \exp\big(-k^{\varepsilon L}\big).$$

Remark 2.2.

- Condition (2.1) restricts the tail behaviour of the distribution function of the single site potentials at 0. This guarantees enough probability for small single site potentials. The result shows that the ground state is shifted from the scale L^{-2} not further than $(\log L)^{-2}$ with probability exponentially close to 1 as $L \to \infty$. Without condition (2.1) the distribution of the single site potentials could have topological support bounded away from 0, which would shift the spectrum by a positive distance almost surely.
- The upper bound on the ground state provides a lower bound on the integrated density of states, see proposition 2.7. The classical assumption for a lower bound on the IDS on $\mathbb{Z}^d$ is that the cumulative distribution

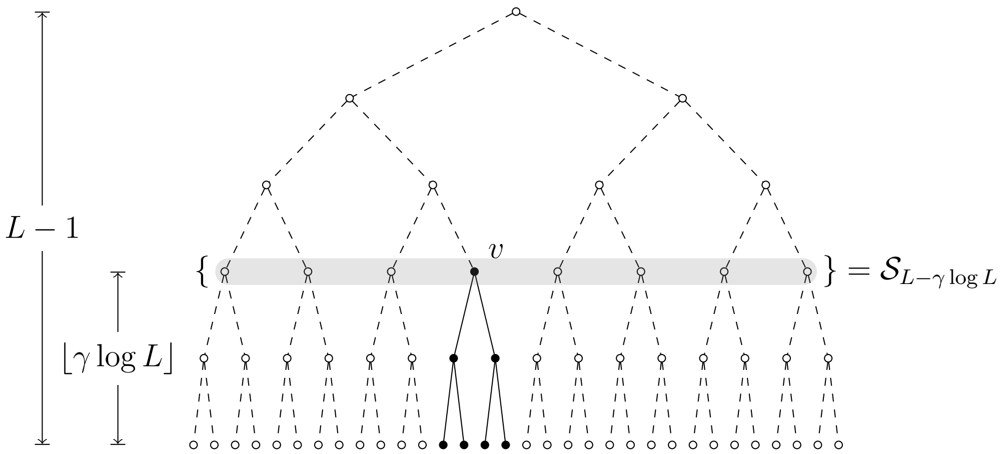

FIGURE 3. Support of test function $\varphi_v := \psi_{v,1}^{\lfloor \gamma \log L \rfloor}$ in the tree based at $v \in \mathcal{S}_{L-\gamma \log L}$.

function of the single site potentials vanishes not faster than a polynomial at 0:

$$\exists C_2, \nu > 0 \colon \forall \kappa > 0 \colon \mathbb{P}(\omega \leqslant \kappa) \geqslant (C_2 \kappa)^{\nu}. \tag{2.2}$$

This is e. g. satisfied for the uniform distribution on an interval $[0, a]$, $a > 0$, and all nondegenerate Bernoulli laws. Condition (2.2) implies $\nu = 0$ and thus (2.1).

Proof of proposition 2.1. Let $\gamma := \pi \sqrt[4]{k} / \sqrt{C_1 - 1}$ and note that

$$0 \leqslant \gamma < (1 - \nu) / \log k.$$

We denote by

$$\mathcal{S}_{L-\gamma \log L} := \{v \in \mathcal{T}^L : |v| = \lceil L - \gamma \log L \rceil\}$$

the sphere of $\mathcal{T}^L$ at level $\lceil L - \gamma \log L \rceil$ and by $\mathcal{T}_v^L$ the subtree of $\mathcal{T}^L$ rooted at v. Define the function $\varphi_v \in \ell^2(\mathcal{T}^L)$ by

$$\varphi_v(w) := \mathbf{1}_{\mathcal{T}_v^L}(w) \sqrt{\frac{2}{(L - |v| + 2) k^{|w| - |v| + 1}}} \sin\Big(\pi \frac{|w| - |v| + 1}{L + |v| + 2}\Big), \quad (w \in \mathcal{T}^L).$$

The support of the function is then $\mathcal{T}_v^L$, see fig. 3. In section 3 we will see that φ_v is the normalized ground state of the free laplacian restricted to $\mathcal{T}_v^L$, trivially embedded in $\mathcal{T}^L$. We also see that the corresponding eigenvalue is

$$k + 1 - 2\sqrt{k} \cos\Big(\frac{\pi}{\lfloor \gamma \log L \rfloor + 1}\Big) = (\sqrt{k} - 1)^2 + 2\sqrt{k}\Big(1 - \cos\Big(\frac{\pi}{\lfloor \gamma \log L \rfloor + 1}\Big)\Big).$$

We will use the states φ_v, $v \in \mathcal{S}_{L-\gamma \log L}$, as test functions to probe for the ground state energy of H_ω^L. In the quadratic form $\langle -\Delta \varphi_v, \varphi_v \rangle$, we sum only over the support of φ_v. Hence, $\langle -\Delta\varphi_v, \varphi_v\rangle$ is the eigenvalue of φ_v on $\mathcal{T}_v^L$. Since $1-\cos(x) \leqslant x^2/2$ for all $x \in \mathbb{R}$, we see that

$$\langle -\Delta\varphi_v, \varphi_v\rangle \leqslant E_0 + \frac{\pi^2\sqrt{k}}{(\lfloor \gamma \log L\rfloor + 1)^2} \leqslant E_0 + \frac{\pi^2\sqrt{k}}{\gamma^2(\log L)^2}.$$

We ask the potential to be small on at least one of the subtrees $\mathcal{T}_v^L$, $v \in \mathcal{S}_{L-\gamma\log L}$. To this end, let $\kappa := (\log L)^{-2}$ and

$$\Omega'_L := \{\omega : \exists v \in \mathcal{S}_{L-\gamma\log L} \colon \max_{w\in\mathcal{T}_v^L} V_\omega(w) \leqslant \kappa\}.$$

For all $\omega \in \Omega'_L$, we have

$$\begin{aligned}\inf\sigma(H_\omega^L) &\leqslant \inf_{v\in\mathcal{S}_{L-\gamma\log L}} \big(\langle -\Delta\varphi_v, \varphi_v\rangle + \langle V_\omega\varphi_v, \varphi_v\rangle\big)\\ &\leqslant E_0 + \frac{\pi^2\sqrt{k}}{\gamma^2(\log L)^2} + \kappa \leqslant E_0 + C_1(\log L)^{-2}.\end{aligned}$$

For the probabilities, this implies

$$\mathbb{P}(\Omega'_L) \leqslant \mathbb{P}\big(\inf\sigma(H_\omega^L) \leqslant E_0 + C_1(\log L)^{-2}\big).$$

We have to estimate $\mathbb{P}(\Omega'_L)$ from below. Choose $\delta \in (0, 1-\nu-\gamma\log k)$. From $\nu < 1$ and (2.1), we get an $L'_0 > 0$ such that, for all $L \geqslant L'_0$ and all $w \in \mathcal{T}^L$,

$$|\log\mathbb{P}(V_\omega(w) \leqslant \kappa)| \leqslant \exp(\kappa^{-1/2}(\nu+\delta)) = L^{\nu+\delta}.$$

We use this to build a lower bound of $\mathbb{P}(\Omega'_L)$ in several steps. Note that for each $v \in \mathcal{S}_{L-\gamma\log L}$, the subtree $\mathcal{T}_v^L$ rooted at v has $\#\mathcal{T}_v^L = \sum_{i=0}^{\lfloor\gamma\log L\rfloor} k^i \leqslant k^{\lfloor\gamma\log L\rfloor+1} \leqslant kL^{\gamma\log k}$ nodes. Therefore, we have

$$\big|\log\mathbb{P}\big(\max_{w\in\mathcal{T}_v^L} V_\omega(w) \leqslant \kappa\big)\big| = \sum_{w\in\mathcal{T}_v^L} \big|\log\mathbb{P}(V_\omega(w) \leqslant \kappa)\big| \leqslant kL^{\gamma\log k+\nu+\delta}.$$

Now we use $\#\mathcal{S}_{L-\gamma\log L} = k^{\lceil L-\gamma\log L\rceil-1} \geqslant k^{L-\gamma\log L-1}$ to find that the probability to find at least one $v \in \mathcal{S}_{L-\gamma\log L}$ where the potential is uniformly bounded by $\kappa > 0$, satisfies

$$\begin{aligned}\mathbb{P}(\Omega'_L) &= 1 - \prod_{v\in\mathcal{S}_{L-\gamma\log L}} \big(1 - \mathbb{P}(\max_{w\in\mathcal{T}_v^L} V_\omega(w) \leqslant \kappa)\big)\\ &= 1 - \prod_{v\in\mathcal{S}_{L-\gamma\log L}} \big(1 - \exp(-|\log\mathbb{P}(\max_{w\in\mathcal{T}_v^L} V_\omega(w) \leqslant \kappa)|)\big)\\ &\geqslant 1 - \Big(1 - \exp\big(-kL^{\gamma\log k+\nu+\delta}\big)\Big)^{\#\mathcal{S}_{L-\gamma\log L}}\\ &\geqslant 1 - \Big(1 - \exp\big(-kL^{\gamma\log k+\nu+\delta}\big)\Big)^{k^{L-\gamma\log L-1}}.\end{aligned}$$

From $\log(1-p) = -\sum_{j=1}^{\infty} p^j/j \leqslant -p$, we see

$$(1-p)^x = \exp(x\log(1-p)) \leqslant \exp(-px)$$

for all $p \in [0,1]$ and $x > 0$. This yields

$$\begin{aligned}\mathbb{P}(\Omega'_L) &\geqslant 1 - \exp\bigl(-\exp\bigl(-kL^{\gamma \log k + \nu + \delta}\bigr)k^{L - \gamma \log L - 1}\bigr) \\ &= 1 - \exp\bigl(-\exp\bigl(L \log k - kL^{\gamma \log k + \nu + \delta + 1} - (\gamma \log L + 1) \log k\bigr)\bigr).\end{aligned}$$

The important fact here is $\gamma \log k + \nu + \delta < 1$. Thus, for all $\varepsilon \in (0,1)$, the exponent satisfies

$$L \log k - kL^{\gamma \log k + \nu + \delta} - (\gamma \log L + 1) \log k \geqslant \varepsilon L \log k$$

as soon as L is large enough, say $L \geqslant L_0 \geqslant L'_0$. The proposition readily follows. ■

We address briefly the question of the optimality of condition (2.1). Let us first note that to prove the lower bound for the Lifshitz tails with exponent $1/2$ it is enough to prove that for every $\eta(0,2)$ we have

$$\inf \sigma(H^L_\omega) \leqslant E_0 + (\log L)^{-\eta}, \quad L \gg 1$$

with good probability (compare this to the consequence of proposition 2.1). This leads us to consider the slightly weaker condition

$$\forall \eta \in (0,2)\colon \limsup_{\kappa \searrow 0} \kappa^{1/\eta} \log\bigl|\log \mathbb{P}(V_\omega(v) \leqslant \kappa)\bigr| = 0,$$

which is implied by condition (2.1). The following lemma shows that we can not expect to do better than this.

Lemma 2.3. *Suppose that for some $\eta > 0$*

$$\limsup_{\kappa \searrow 0} \kappa^{1/\eta} \log\bigl|\log \mathbb{P}(V_\omega(v) \leqslant \kappa)\bigr| > 0. \tag{2.3}$$

Then, if $\eta' > \eta$ and $\zeta > 0$, there is a sequence $L_j \to \infty$ for which

$$\mathbb{P}\bigl(\inf \sigma(H^{L_j}_\omega) \geqslant E_0 + (\log L_j)^{-\eta'}\bigr) \geqslant 1 - \exp\bigl(-\zeta L_j\bigr).$$

Proof. We start with the simple bound

$$\inf \sigma(H^L_\omega) \geqslant E_0 + \min_{v \in \mathcal{T}^L} V_\omega(v).$$

Then, it is enough to prove that for $\eta' > \eta$ and $\zeta > 0$, there is a sequence $L_j \to \infty$ satisfying

$$\mathbb{P}\Bigl(\min_{v \in \mathcal{T}^{L_j}} V_\omega(v) \geqslant (\log L_j)^{-\eta'}\Bigr) \geqslant 1 - \exp\bigl(-\zeta L_j\bigr). \tag{2.4}$$

Condition (2.3) implies for any $\eta'' > 0$ such that $\eta < \eta'' < \eta'$ there exists some sequence $\kappa_j \to 0$ satisfying

$$\bigl|\log \mathbb{P}(V_\omega(v) \leqslant \kappa_j)\bigr| \geqslant \exp(\kappa_j^{-1/\eta''}).$$

We can always assume that the κ_j are small enough by removing some elements of the sequence. Letting $L_j = \lceil \exp(\kappa_j^{-1/\eta'}) \rceil$ this implies that for any $\zeta > 0$ there exists some L^* such that for all $L_j > L^*$

$$\left|\log \mathbb{P}(V_\omega(v) \leqslant (\log L_j)^{-\eta'})\right| \geqslant \exp((\log L_j)^{\eta'/\eta''}) \geqslant \zeta L_j.$$

Using the independence of the random variables and the fact that $|\log(1-p)| \leqslant 2p$ for $0 < p \ll 1$, we see that for any $\zeta > 0$ there is some sequence $L_j \to +\infty$ so that

$$\begin{aligned}
\log \mathbb{P}\big(\min_{v \in \mathcal{T}^{L_j}} V_\omega(v) \geqslant (\log L_j)^{-\eta'}\big) &= \sum_{v \in \mathcal{T}^{L_j}} \log\big(1 - \mathbb{P}\big(V_\omega(v) \leqslant (\log L_j)^{-\eta'}\big)\big) \\
&= |\mathcal{T}^{L_j}| \log\big(1 - \mathbb{P}\big(V_\omega(0) \leqslant (\log L_j)^{-\eta'}\big)\big) \\
&\geqslant -2k^{L_j+1} \mathbb{P}\big(V_\omega(0) \leqslant (\log L_j)^{-\eta'}\big) \\
&\geqslant -2k \mathrm{e}^{(\log k)L_j - \zeta L_j}
\end{aligned} \tag{2.5}$$

In particular, using that (2.5) is small and $\exp(-x) = 1 - x + O(x^2)$, we see that for any $\zeta > 0$ there exists some sequence $L_j \to \infty$ satisfying (2.4). This finishes the proof. ■

If we assume condition (2.3) with $\eta < 2$, this last result and the methods we introduce later in section 4 can be used to prove that there exists some sequence $E'_j \searrow 0$ for which

$$\limsup_{j \to \infty} \frac{\log\log|\log \mathcal{N}(E'_j)|}{\log(E'_j - E_0)} < -\frac{1}{2}.$$

It is thus impossible to obtain the lower bound in proposition 2.7 under this assumption.

2.2. **The lower bound on $\mathcal{N}(E)$.** The upper bound on the ground state in proposition 2.1 implicates a lower bound on the integrated density of states $\mathcal{N}$, formulated in proposition 2.7. The strategy of proof is the same as in [Ros12, section 2.1]. Nonetheless, proposition 2.7 improves the prerequisites under which the lower bound holds, cf. remark 2.2.

The following lemma is taken from [Ros12, section 2.1.1] and adapted for trees $\mathcal{T}^L$ instead of balls of the Bethe lattice.

Lemma 2.4. *For all $L \in \mathbb{N}$, $t > 0$ and $E' \geqslant E_0$, it holds true that*

$$\tilde{\mathcal{N}}(t) \geqslant \mathrm{e}^{-tE'} \mathbb{P}(\inf \sigma(H^L_\omega) \leqslant E') / \#\mathcal{T}^L.$$

Due to the change of notation and for the convenience of the reader, we repeat and detail the proof.

Proof. Let $L \in \mathbb{N}$ and $E \in \mathbb{R}$. $\Pi_E := \mathbf{1}_{(-\infty,E]}(H_\omega)$ is the spectral projection of H_ω. According to (1.6), the integrated density of states is given by

$$\mathcal{N}(E) = \mathbb{E}[\langle \delta_0, \Pi_E \delta_0\rangle] = (\#\mathcal{T}^L)^{-1} \sum_{v\in\mathcal{T}^L} \mathbb{E}[\langle \delta_v, \Pi_E \delta_v\rangle] = (\#\mathcal{T}^L)^{-1}\mathbb{E}[\mathrm{tr}(\mathbf{1}_{\mathcal{T}^L} H_\omega \mathbf{1}_{\mathcal{T}^L})].$$

For the Laplace transform of $\mathcal{N}$, the spectral theorem gives

$$\tilde{\mathcal{N}}(t) = \int \mathrm{e}^{-\lambda t}\mathrm{d}\mathcal{N}(\lambda) = (\#\mathcal{T}^L)^{-1}\mathbb{E}[\mathrm{tr}(\mathbf{1}_{\mathcal{T}^L} \exp(-tH_\omega)\mathbf{1}_{\mathcal{T}^L})]$$

for $t \geqslant 0$. [Sim05, Theorem 8.9] states

$$\mathrm{tr}(\mathbf{1}_{\mathcal{T}^L} \exp(-tH_\omega)\mathbf{1}_{\mathcal{T}^L}) \geqslant \mathrm{tr}(\exp(-tH_\omega^L)), \tag{2.6}$$

where $H_\omega^L := \mathbf{1}_{\mathcal{T}^L} H_\omega \mathbf{1}_{\mathcal{T}^L}$. This is easily seen with help of spectral measures. Due to the convexity of $\lambda \mapsto \mathrm{e}^{-t\lambda}$, for each $v \in \mathcal{T}^L$, the Jensen inequality gives

$$\begin{aligned}\langle \delta_v, \exp(-tH_\omega)\delta_v\rangle &= \int \exp(-t\lambda)\,\mathrm{d}\mu_{\delta_v}(\lambda) \geqslant \exp\Big(-t\int \lambda\,\mathrm{d}\mu_{\delta_v}(\lambda)\Big)\\ &= \exp(-t\langle \delta_v, H_\omega\delta_v\rangle) = \exp(-t\langle \delta_v, \mathbf{1}_{\mathcal{T}^L} H_\omega \mathbf{1}_{\mathcal{T}^L}\delta_v\rangle)\\ &= \langle \delta_v, \exp(-tH_\omega^L)\delta_v\rangle,\end{aligned}$$

where μ_{δ_v} is the spectral measure of H_ω with respect to δ_v. Summing over $v \in \mathcal{T}^L$, we obtain (2.6). The Laplace transform is thus bounded by

$$\tilde{\mathcal{N}}(t) \geqslant (\#\mathcal{T}^L)^{-1}\mathbb{E}[\mathrm{tr}(\exp(-tH_\omega^L))] \geqslant (\#\mathcal{T}^L)^{-1}\mathbb{E}[\exp(-t\inf\sigma(H_\omega^L))].$$

The Markov inequality reduces the last expectation to a probability

$$\mathbb{P}(\inf\sigma(H_\omega^L) \leqslant E') \leqslant \mathrm{e}^{tE'}\mathbb{E}[\exp(-t\inf\sigma(H_\omega^L))]$$

and finishes the proof. ∎

Corollary 2.5. *Let C_1 be the constant in proposition 2.1. For $t > 0$ large enough, then*

$$\tilde{\mathcal{N}}(t) \geqslant \frac{1}{2}\mathrm{e}^{-t(E_0+\frac{C_1}{2(\log L)^2})}$$

Proof. We choose $L = \zeta t$ and $E' = E_0 + C_1(\log L)^{-2}$. Note that

$$\#\mathcal{T}^L = \sum_i = 0^{L-1} k^i \leqslant k^L = \mathrm{e}^{\zeta t\log k}.$$

This, lemma 2.4 and proposition 2.1 gives

$$\begin{aligned}\tilde{\mathcal{N}}(t) &\geqslant \frac{1}{2}\mathrm{e}^{-t(E_0+C_1(\log\zeta t)^{-2})-t\zeta\log k}\\ &\geqslant \frac{1}{2}\mathrm{e}^{-t(E_0+C_1(2\log t)^{-2})}.\end{aligned}$$

∎

As known from Tauberian theorems, the behavior of $\tilde{\mathcal{N}}(t)$ as $t \to \infty$ and the behaviour of $\mathcal{N}(E)$ as $E \searrow E_0$ are related. The following is taken almost verbatim from [Ros12, (2.27)].

Lemma 2.6. *For all $t > 0$ and $E \geqslant E_0$, it holds true that*

$$\mathcal{N}(E) \geqslant \mathrm{e}^{tE_0}\tilde{\mathcal{N}}(t) - \mathrm{e}^{-t(E-E_0)}.$$

For completeness, we give the short proof.

Proof. Integration by parts, with vanishing boundary terms since $\mathcal{N}(E_0) = 0$, gives

$$\begin{aligned}\tilde{\mathcal{N}}(t) &= \int_{E_0}^{\infty} \mathrm{e}^{-t\lambda} \mathrm{d}\mathcal{N}(\lambda) = \int_{E_0}^{\infty} t\mathrm{e}^{-t\lambda}\mathcal{N}(\lambda)\,\mathrm{d}\lambda \\ &\leqslant \mathcal{N}(E)\int_{E_0}^{E} t\mathrm{e}^{-t\lambda}\,\mathrm{d}\lambda + \int_{E}^{\infty} t\mathrm{e}^{-t\lambda}\,\mathrm{d}\lambda \leqslant \mathrm{e}^{-tE_0}\mathcal{N}(E) + \mathrm{e}^{-tE}.\end{aligned}$$

This is equivalent to the claim. ∎

Together with proposition 2.1, lemmas 2.4 and 2.6 are all that is needed to prove the lower bound of the Lifshitz tails. More precisely, we obtain the following.

Proposition 2.7. *Assume (2.1) and fix C_1 as in proposition 2.1. Then there exists $\lambda > E_0$ such that, for all $E \in (E_0, \lambda)$, it holds true that*

$$\mathcal{N}(E) \geqslant k^{-2-2\exp\left(\sqrt{2C_1/(E-E_0)}\right)}/16.$$

In particular,

$$\liminf_{E\searrow E_0} \frac{\log\log|\log\mathcal{N}(E)|}{\log(E-E_0)} \geqslant -\frac{1}{2}.$$

Proof. Lemmas 2.4 and 2.6 concatenate to

$$\begin{aligned}\mathcal{N}(E) &\geqslant (\#\mathcal{T}^L)^{-1}\mathrm{e}^{-t(E'-E_0)}\mathbb{P}(\inf\sigma(H_\omega^L) \leqslant E') - \mathrm{e}^{-t(E-E_0)} \\ &= \left((\#\mathcal{T}^L)^{-1}\mathrm{e}^{t(E-E')}\mathbb{P}(\inf\sigma(H_\omega^L) \leqslant E') - 1\right)\mathrm{e}^{-t(E-E_0)},\end{aligned}$$

which is true for all $t > 0$ and $E, E' > E_0$. We choose $E' := (E_0 + E)/2$. This ensures $E - E' > 0$ and will enable us to choose t large enough to make the lower bound positive.

But first we have to deal with the probability. In order to apply proposition 2.1, we let $L := \left\lceil \exp\left(\sqrt{2C_1/(E-E_0)}\right)\right\rceil$. That way, $E' \geqslant E_0 + C_1/(\log L)^2$ and, provided $E - E_0$ is small enough so that L is large enough,

$$\mathbb{P}\left(\inf\sigma(H_\omega^L) \leqslant E'\right) \geqslant \mathbb{P}\left(\inf\sigma(H_\omega^L) \leqslant E_0 + C_1(\log L)^{-2}\right) \geqslant 1 - \exp(-k^{L/2}) \geqslant 1/2.$$

Up to now we know, for all $t > 0$ and L large enough,

$$\mathcal{N}(E) \geqslant \left((2\#\mathcal{T}^L)^{-1}\mathrm{e}^{t(E-E_0)/2} - 1\right)\mathrm{e}^{-t(E-E_0)}.$$

It is time to choose $t := 2\log(4\#\mathcal{T}^L)/(E-E_0)$, that is, $(2\#\mathcal{T}^L)^{-1}\mathrm{e}^{t(E-E_0)/2} = 2$ and

$$\mathcal{N}(E) \geqslant \exp\bigl(-2\log(4\#\mathcal{T}^L)\bigr) = \frac{(\#\mathcal{T}^L)^{-2}}{16} \geqslant \frac{k^{-2L}}{16} \geqslant \frac{1}{16}k^{-2-2\exp\left(\sqrt{2C_1/(E-E_0)}\right)}.$$

It is now easy to read the exponent of $E-E_0$ from the limit inferior of this lower bound on the Lifshitz tail behaviour , i. e., $-1/2$. This finishes the proof. ■

3. Deterministic preparations

We develop the spectral theory of finite rooted trees. The spectrum was already calculated in [RR07], but we need the eigenfunctions, too. The radially symmetric generalized eigenfunctions for the (infinite) Bethe lattice were calculated in [Bro91].

Recall that we denote by $\mathcal{T}^L$ the (nodes of the) rooted tree of length L with k children at each node except the leaves, by 0 the root of the tree and by $|v| = d(0,v)+1$ the "level" of the node v. For indexing reasons, we introduce the notation $\mathcal{T}_*^L := \mathcal{T}^L \uplus \{*\}$ for the (nodes of the) rooted tree of length L augmented by a vertex $*$ with $|*| = 0$, such that $*$ is a parent of the root. Any function in $\ell^2(\mathcal{T}^L)$ is understood as an element of $\ell^2(\mathcal{T}_*^L)$, too, with the value 0 on $*$.

3.1. The spectrum of the adjacency matrix on a finite rooted tree.

Lemma 3.1. *For each $m \in \{1,\dots,L\}$, the radially symmetric function defined by*

$$\mathcal{T}^L \ni v \mapsto \psi_m^L(v) = \sqrt{\frac{2}{(L+1)k^{|v|-1}}}\sin\Bigl(\frac{m\pi}{L+1}|v|\Bigr)$$

is a normalized eigenfunction of the adjacency matrix $A^{(L)}$ of the rooted tree $\mathcal{T}^L$ with eigenvalue

$$\lambda_m^L := 2\sqrt{k}\cos\Bigl(\frac{m\pi}{L+1}\Bigr).$$

Proof. Let $\theta := \frac{m\pi}{L+1} \in \mathbb{R}$. We check first the eigenvalue equation for $v \in \mathcal{T}^L$:

$$\begin{aligned} A^{(L)}\psi_m^L(v) &= \sum_{w\in\mathcal{T}^L, w\sim v} \psi_m^L(w) \\ &= \sqrt{\tfrac{2}{(L+1)k^{|v|-2}}}\sin((|v|-1)\theta) + k\sqrt{\tfrac{2}{(L+1)k^{|v|}}}\sin(\theta(|v|+1)) \\ &= \sqrt{\tfrac{2}{(L+1)k^{|v|-2}}}\cdot 2\sin(|v|\theta)\cos(\theta) = \lambda_m^L\psi_m^L(v). \end{aligned}$$

The third equation employs $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$, $\alpha, \beta \in \mathbb{R}$. We check now that they are normalized. This is seen via

$$\begin{aligned}\|\psi_m^L\|_2^2 = \sum_{v\in\mathcal{T}^L} |\psi_m^L(v)|^2 = \sum_{\ell=1}^{L} \frac{2\sin^2(\theta\ell)}{L+1} &= \frac{1}{2(L+1)} \sum_{\ell=1}^{L} \big(2 - \mathrm{e}^{2\mathrm{i}\theta\ell} - \mathrm{e}^{-2\mathrm{i}\theta\ell}\big)\\ &= \frac{1}{2(L+1)}\Big(2L - \frac{\mathrm{e}^{2\mathrm{i}\theta} - \mathrm{e}^{2\mathrm{i}(L+1)\theta}}{1-\mathrm{e}^{2\mathrm{i}\theta}} - \frac{\mathrm{e}^{-2\mathrm{i}\theta} - \mathrm{e}^{-2\mathrm{i}(L+1)\theta}}{1-\mathrm{e}^{-2\mathrm{i}\theta}}\Big) = 1,\end{aligned}$$

where we used $\mathrm{e}^{\pm 2\mathrm{i}(L+1)\theta} = \mathrm{e}^{\pm 2\pi i m} = 1$ in the last step. ■

Since the radially symmetric functions on $\mathcal{T}^L$ form a linear subspace of $\ell^2(\mathcal{T}^L)$ of dimension L, lemma 3.1 lists all radially symmetric eigenfunctions of $A^{(L)}$. We now construct the remaining non-radially symmetric eigenfunctions on $\mathcal{T}^L$. Recall that, for each $v \in \mathcal{T}^L$, we denote by $\mathcal{T}_v^L$ the subtree of $\mathcal{T}^L$ rooted at v and of length $L - |v| + 1$.

Let $v \in \mathcal{T}^{L-1} \subseteq \mathcal{T}^L$ and $u \in \mathcal{T}_v^L$, $u \sim v$. The node u is the root of a subtree $\mathcal{T}_u^L$ isomorphic to $\mathcal{T}^{L-|v|}$. According to lemma 3.1, we have $L - |v|$ radially symmetric eigenfunctions $\psi_{u,m}^{L-|v|}$, $m \in \{1, \dots, L-|v|\}$, of the adjacency matrix of $\mathcal{T}_u^L$, given by

$$\psi_{u,m}^{L-|v|}(w) = \sqrt{\frac{2}{(L+1-|v|)k^{|w|-|v|-1}}}\,\sin\Big(\frac{m\pi}{L+1-|v|}(|w|-|v|)\Big) \tag{3.1}$$

for $w \in \mathcal{T}_u^L$. We trivially extend $\psi_{u,m}^{L-|v|}$ to a function on $\mathcal{T}^L$ by assigning 0 to the complement of $\mathcal{T}_u^L$. For a given $v \in \mathcal{T}^L$, we will agglutinate below the functions $\psi_{u,m}^{L-|v|}$, $u \in \mathcal{T}_v^L$, $u \sim v$, at v, see (3.2).

Note that $\mathcal{T}_v^{|v|+1} = \{v\} \cup \{u \in \mathcal{T}_v^L : u \sim v\}$ is isomorphic to $\mathcal{T}^2$ as a graph. The matrix representation of $A^{(2)}$ with respect to a basis $(\delta_v; v \in \mathcal{T}^2)$ with the root as the first entry is

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1\\ 1 & 0 & 0 & \dots & 0\\ 1 & 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

with dimensions $(k+1)\times(k+1)$. The kernel of $A^{(2)}$ on $\mathcal{T}^2$ has dimension $k-1$, so we can find $k-1$ normalized and orthogonal real eigenvectors $\psi_{v,j}^{\perp}$, $j \in \{1, \dots, k-1\}$, of $A^{(2)}$ associated to the eigenvalue 0 on $\mathcal{T}_v^{|v|+1}$. These eigenvectors assign the value 0 to v, since for any $u \in \mathcal{T}_v^{|v|+1}$, $u \neq v$, we have

$$\psi_{v,j}^{\perp}(v) = A^{(2)}\psi_{v,j}^{\perp}(u) = 0 \cdot \psi_{v,j}^{\perp}(u) = 0.$$

We set

$$\Psi^L_{v,j,m} := \sum_{u \in \mathcal{T}^L_v, u\sim v} \psi^\perp_{v,j}(u)\psi^{L-|v|}_{u,m}. \tag{3.2}$$

To unify notation, we define $\Psi^L_{*,1,m} := \psi^L_m$ and $\psi^\perp_{*,1}(v) := 1$ for the root v of $\mathcal{T}^L$, too, as well as

$$J_v := \begin{cases} \{1\} & \text{for } v = * \text{ and} \\ \{1,\dots,k-1\} & \text{if } v \in \mathcal{T}^{L-1}. \end{cases}$$

We call a triple (v,j,m) *L-admissible* if $v \in \mathcal{T}^{L-1}_*$, $j \in J_v$, $m \in \{1,\dots,L-|v|\}$.

Lemma 3.2. *The vectors $\Psi^L_{v,j,m}$ with (v,j,m) L-admissible are normalized eigenvectors of $A^{(L)}$ with eigenvalues $\lambda^L_{v,j,m} := \lambda^{L-|v|}_m = 2\sqrt{k}\cos\left(\frac{m\pi}{L+1-|v|}\right)$, respectively, and form an orthonormal basis of $\ell^2(\mathcal{T}^L)$.*

Proof. Let (v,j,m) be a L-admissible. In the case $v = *$, lemma 3.1 tells us that $\Psi^L_{v,j,m} = \psi^L_m$ is a normalized eigenfunction of $A^{(L)}$. From now on, we let $v \in \mathcal{T}^{L-1}$. Note that

$$\sum_{u \in \mathcal{T}^L_v, u\sim v} \psi^\perp_{v,j}(u) = A^{(2)}\psi^\perp_{v,j}(v) = 0 \cdot \psi^\perp_{v,j}(v) = 0. \tag{3.3}$$

Since $\Psi^L_{v,j,m}$ is pieced together from eigenfunctions on trees with the same eigenvalue, the only node we need to check is v itself. We use (3.3) to see that

$$A^{(L)}\Psi^L_{v,j,m}(v) = \sum_{u \in \mathcal{T}^L_v, w\sim v} \psi^\perp_{v,j}(u)\psi^{L-|v|}_{u,m}(u) = 0 = \lambda^L_{v,j,m}\Psi^L_{v,j,m}(v).$$

Thus, all $\Psi^L_{v,j,m}$ are eigenfunctions of $A^{(L)}$.

Orthonormality is our next goal. For $v \in \mathcal{T}^{L-1}$, $m \in \{1,\dots,L-|v|\}$, $m' \in \{1,\dots,L\}$ and $j \in \{1,\dots,k-1\}$, we have

$$\langle \Psi^L_{*,1,m'}, \Psi^L_{v,j,m}\rangle = \sum_{u \in \mathcal{T}^L_v, u\sim v} \psi^\perp_{v,j}(u)\langle \psi^L_{m'}, \psi^{L-|v|}_{u,m}\rangle = 0,$$

since $\langle \psi^L_{m'}, \psi^{L-|v|}_{u,m}\rangle$ is constant in u and (3.3). For (v,j,m) and (v',j',m') L-admissible with $v, v' \in \mathcal{T}^{L-1}$ we distinguish the following cases.

- If $\lambda^L_{v,j,m} \neq \lambda^L_{v',m',j'}$, then $\langle \Psi^L_{v,j,m}, \Psi^L_{v',m',j'}\rangle = 0$, since $A^{(L)}$ is symmetric.
- Let $v \neq v'$. If $v \in \mathcal{T}^L_{v'}$ or $v' \in \mathcal{T}^L_v$, then the argument from above for $v' = *$ applies. If v and v' have disjoint subtrees, then the supports of $\Psi^L_{v,j,m}$ and $\Psi^L_{v',m',j'}$ are disjoint. Either way we reach $\langle \Psi^L_{v,j,m}, \Psi^L_{v',m',j'}\rangle = 0$.

- Assume $v = v'$, $\lambda^L_{v,j,m} = \lambda^L_{v',m',j'}$. We thus have $\cos(\frac{m\pi}{L+1-|v|}) = \cos(\frac{m'\pi}{L+1-|v|})$. Since $\frac{m\pi}{L-|v|+1} \in (0,\pi)$ and $\cos|_{(0,\pi)}$ is injective, we deduce $m = m'$. Consequently,
$$\langle \Psi^L_{v,j,m}, \Psi^L_{v,j',m}\rangle = \sum_{u,u' \in \mathcal{T}^L_v, u,u' \sim v} \overline{\psi^\perp_{v,j}(u)}\psi^\perp_{v,j'}(u')\langle \psi^{L-|v|}_{u,m}, \psi^{L-|v|}_{u',m}\rangle$$
$$= \sum_{u \in \mathcal{T}^L_v, u \sim v} \overline{\psi^\perp_{v,j}(u)}\psi^\perp_{v,j'}(u) = \delta_{j,j'},$$
since $\psi^\perp_{v,j}$ and $\psi^\perp_{v,j'}$ are orthonormal and $\psi^\perp_{v,j}(v) = 0$.

We now know that the set of all $\Psi^L_{v,j,m}$ with (v,j,m) L-admissible is orthonormal.

To identify this orthonormal set as a basis, we simply count all L-admissible triples:
$$\sum_{v \in \mathcal{T}^{L-1}_*} \sum_{j \in J_v} \sum_{m=1}^{L-|v|} 1 = \sum_{m=1}^{L} 1 + \sum_{v \in \mathcal{T}^{L-1}} \sum_{j=1}^{k-1} \sum_{m=1}^{L-|v|} 1$$
$$= L + \sum_{\ell=1}^{L-1} k^{\ell-1}(k-1)(L-\ell) = L + (k-1)\Big(L\sum_{\ell=1}^{L-1} k^{\ell-1} - \sum_{\ell=1}^{L-1} \ell k^{\ell-1}\Big)$$
$$= L + (k-1)\Big(L\frac{k^{L-1}-1}{k-1} - \frac{Lk^{L-1}(k-1)-(k^L-1)}{(k-1)^2}\Big) = \frac{k^L-1}{k-1}.$$
This is exactly the dimension $\#\mathcal{T}^L = \sum_{\ell=1}^L k^{\ell-1} = \frac{k^L-1}{k-1}$ of $\ell^2(\mathcal{T}^L)$. ■

We study the behaviour of the principal eigenvalue $\lambda^L_{*,1,1}$ of $A^{(L)}$ as a function of L and identify the states in its vicinity. This will be used in section 5, and it is a crucial part of our argument.

Lemma 3.3. *Let $L \in \mathbb{N}$. For $\beta \in \mathbb{R}$ we define $E^{(L)}_\beta := 2\sqrt{k}\cos\big(\frac{(\beta+1)\pi}{L+1}\big)$. For L-admissible (v,j,m) and $\beta \in [0,L]$, we have*
$$\lambda^L_{v,j,m} \in [E^{(L)}_\beta, \lambda^L_{*,1,1}] \iff |v| \leqslant (L+1)\big(1 - \tfrac{m}{\beta+1}\big) \implies m \in \{1,\dots,\lfloor \beta+1 \rfloor\}.$$

Remark 3.4. Note that $E^{(L)}_0 = E^{(L)}_\beta|_{\beta=0} = \lambda^L_{*,1,1}$.

Proof. Remember that $1 - \frac{x^2}{2} \leqslant \cos x \leqslant 1 - \frac{x^2}{2} + \frac{x^4}{24}$ for all $x \in \mathbb{R}$. This reveals
$$-\frac{1}{24}\Big(\frac{(\beta+1)\pi}{L+1}\Big)^4 \leqslant \cos\Big(\frac{\pi}{L+1}\Big) - \cos\Big(\frac{(\beta+1)\pi}{L+1}\Big) - \frac{\pi^2\beta(\beta+2)}{2(L+1)^2} \leqslant \frac{1}{24}\Big(\frac{\pi}{L+1}\Big)^4.$$
For $\beta \leqslant L$, we use that $\cos|_{[0,\pi]}$ is strictly decreasing to obtain
$$\lambda^L_{v,j,m} \geqslant E^{(L)}_\beta \iff \frac{m}{L+1-|v|} \leqslant \frac{\beta+1}{L+1} \iff \frac{m}{\beta+1} + \frac{|v|}{L+1} \leqslant 1$$
$$\iff |v| \leqslant (L+1)\Big(1 - \frac{m}{\beta+1}\Big) \implies m \leqslant \lfloor \beta+1 \rfloor.$$ ■

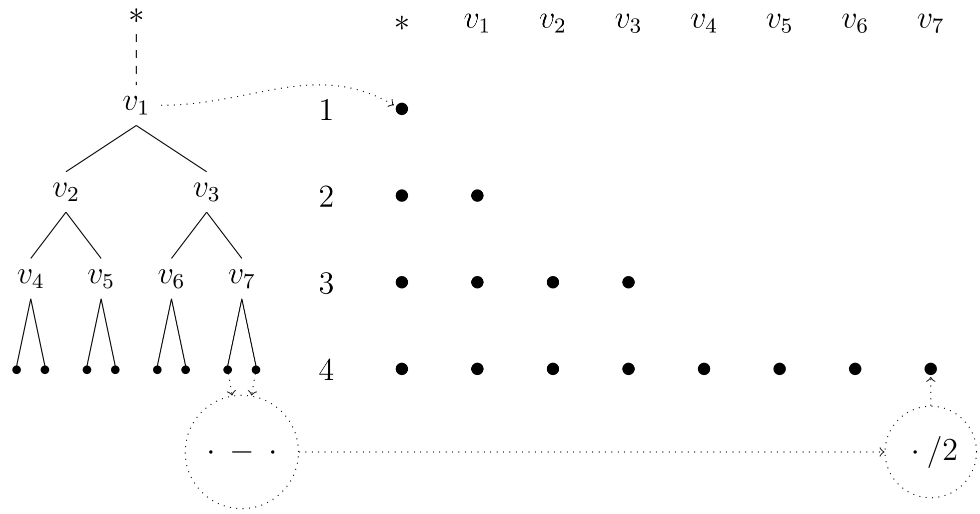

FIGURE 4. The action of the map $\widehat{\cdot}$ is indicated with the dotted arrows.

Next, we study the spectral projections

$$P_{v,j}\colon \ell^2\mathcal{T}^L \to \ell^2\mathcal{T}^L, \quad P_{v,j}\varphi := \sum_{m=1}^{L-|v|} \langle \Psi^L_{v,j,m}, \varphi\rangle \Psi^L_{v,j,m} \tag{3.4}$$

of $A^{(L)}$ for $v \in \mathcal{T}_*^{L-1}$ and $j \in J_v$. We introduce the map

$$\widehat{\cdot}\colon \ell^2(\mathcal{T}^L) \to \bigoplus_{v\in\mathcal{T}_*^{L-1}} \bigoplus_{j\in J_v} \ell^2(\{|v|+1,\dots,L\}),$$

$$\widehat{\psi}_{v,j}(z) := k^{-(z-|v|-1)/2} \sum_{u\in\mathcal{T}_v^L, u\sim v} \overline{\psi^\perp_{v,j}(u)} \sum_{w\in\mathcal{T}_u^L, |w|=z} (P_{v,j}\psi)(w) \tag{3.5}$$

for $z \in \{|v|+1,\dots,L\}$. For a rough illustration see fig. 4. The map $\widehat{\cdot}$ has been sketched in [AW06, Proposition A.2] and it is similar to an infinite dimensional version in [AF00].

The action of $\widehat{\cdot}$ is best illustrated on radially symmetric eigenfunctions ψ^L_m of $A^{(L)}$. As we will see in lemma 3.5, they are mapped to functions supported on $\{1,\dots,L\}$, and in the process, the exponential weights $k^{(|v|-1)/2}$ are removed:

$$(\widehat{\psi^L_m})_{*,1}(z) = \sqrt{\frac{2}{L+1}} \sin\Big(\frac{m\pi}{L+1} z\Big)$$

for $z \in \{1,\dots,L\}$ and $m \in \{1,\dots,L\}$. The result is an eigenfunction of the adjacency matrix of $\mathbb{Z}$ restricted to $\{1,\dots,L\}$. Given a non-radially symmetric eigenfunction $\Psi^L_{v,j,m}$ of $A^{(L)}$, $\widehat{\cdot}$ reconstructs the underlying radially symmetric eigenfunction, removes the exponential weight and presents the result as a function on the copy of $\{|v|+1,\dots,L\}$ which is indexed by (v,j):

$$(\widehat{\Psi^L_{v,j,m}})_{v,j}(z) = \sqrt{\frac{2}{L+1-|v|}}\sin\Big(\frac{m\pi}{L+1-|v|}(z-|v|)\Big) \tag{3.6}$$

for $z\in\{|v|+1,\dots,L\}$ and (v,j,m) L-admissible. This is again an eigenfunction of the adjacency matrix of $\mathbb{Z}$ restricted to $\{|v|+1,\dots,L\}$.

We define the adjacency matrix on the image of $\widehat{\cdot}$, which is the Hilbert sum of the ℓ^2-spaces of segments of $\mathbb{Z}$. The direct sum of the adjacency matrices of the segments of $\mathbb{Z}$ is the natural choice. For $\varphi\in\bigoplus_{v,j}\ell^2(\{|v|+1,\dots,L\})$, it is given by

$$(\widehat{A}\varphi)_{v,j}(z) := (A_{\mathbb{Z}}\varphi_{v,j})(z) = \varphi_{v,j}(z-1)+\varphi_{v,j}(z+1)$$

for $v\in\mathcal{T}^{L-1}$, $j\in J_v$, $z\in\{|v|+1,\dots,L\}$, and with the boundary values $\varphi_{v,j}(|v|):=\varphi_{v,j}(L+1):=0$.

Lemma 3.5. *For all $\psi\in\ell^2(\mathcal{T}^L)$, we have the following.*

(i) *The map $\widehat{\cdot}$ conjugates $A^{(L)}$ and $\sqrt{k}\widehat{A}$: $\widehat{A^{(L)}\psi}=\sqrt{k}\widehat{A}\widehat{\psi}$.*
(ii) *The map $\widehat{\cdot}$ is unitary: $\|\psi\|_2=\|\widehat{\psi}\|_2$. In particular, $\sigma(A^{(L)})=\sqrt{k}\sigma(\widehat{A})$.*
(iii) *Let $v\in\mathcal{T}^{L-1}_*$ and $j\in J_v$. The subspace $P_{v,j}\ell^2(\mathcal{T}^L)$ contains ψ if and only if $\operatorname{supp}(\psi)\subseteq\mathcal{T}^L_v\setminus\{v\}$ and*

$$\psi(w)\psi^{\perp}_{j,v}(u') = \psi(w')\psi^{\perp}_{j,v}(u)$$

for all $u,u'\in\mathcal{T}^L_v$ with $u,u'\sim v$ and all $w\in\mathcal{T}^L_u, w'\in\mathcal{T}^L_{u'}$ such that $|w|=|w'|$.
(iv) *For all radially symmetric functions $\eta\colon\mathcal{T}^L\to\mathbb{C}$, i. e. $\eta(w)=\eta_{\mathbb{Z}}(|w|)$ for all $w\in\mathcal{T}^L$ and an $\eta_{\mathbb{Z}}\colon\{1,\dots,L\}\to\mathbb{C}$, we have $P_{v,j}\eta=\eta P_{v,j}$ and $(\widehat{\eta\psi})_{v,j}=\eta_{\mathbb{Z}}\widehat{\psi}_{v,j}$ for all $v\in\mathcal{T}^{L-1}_*$, $j\in J_v$. Here, $\eta_{\mathbb{Z}}$ denotes the multiplication with the function $\eta_{\mathbb{Z}}|_{\{|v|+1,\dots,L\}}$.*

Proof. Ad (i). We study the linear map $\widehat{\cdot}$ on the orthonormal basis $\Psi^L_{v,j,m}$. To this end, let (v,j,m) be admissible, $v'\in\mathcal{T}^{L-1}_*$, $j'\in J_{v'}$ and $z\in\{|v'|+1,\dots,L\}$. For $v'\neq v$ or $j'\neq j$, we have $P_{v',j'}\Psi^L_{v,j,m}=0$ and ergo $(\widehat{\Psi^L_{v,j,m}})_{v',j'}(z)=0$, too. So from now on, we assume $v'=v$ and $j'=j$. We then have $P_{v,j}\Psi^L_{v,j,m}=\Psi^L_{v,j,m}$. For $u\in\mathcal{T}^L_v$, $u\sim v$ and $w\in\mathcal{T}^L_u$, we find

$$\Psi^L_{v,j,m}(w) = \sum_{u'\in\mathcal{T}^L_v, u'\sim v}\psi^{\perp}_{v,j}(u')\psi^{L-|v|}_{u',m}(w) = \psi^{\perp}_{v,j}(u)x_{|w|,|v|,m,L}$$

with $x_{|w|,|v|,m,L} := \sqrt{\frac{2}{(L+1-|v|)k^{|w|-|v|-1}}}\sin\big(\frac{m\pi}{L+1-|v|}(|w|-|v|)\big)$. We now see

$$\begin{aligned}
(\widehat{\Psi^L_{v,j,m}})_{v,j}(z) &= k^{-(z-|v|-1)/2}\sum_{u\in\mathcal{T}_v^L, u\sim v}\overline{\psi^\perp_{v,j}(u)}\sum_{w\in\mathcal{T}_u^L,|w|=z}\Psi^L_{v,j,m}(w)\\
&= k^{-(z-|v|-1)/2}\sum_{u\in\mathcal{T}_v^L, u\sim v}\overline{\psi^\perp_{v,j}(u)}\psi^\perp_{v,j}(u)\sum_{w\in\mathcal{T}_u^L,|w|=z}x_{z,|v|,m,L}\\
&= k^{-(z-|v|-1)/2}k^{z-|v|-1}x_{z,|v|,m,L}\\
&= \sqrt{\frac{2}{L+1-|v|}}\sin\Big(\frac{m\pi}{L+1-|v|}(z-|v|)\Big).
\end{aligned}$$

We now identify $\widehat{\Psi^L_{v,j,m}}$ as an eigenfunction of $\widehat{A}$. Let $\varphi := \frac{m\pi}{L+1-|v|}$ and note that, for $z\in\{|v|_1,\dots,L\}$, by the angle sum and difference identities,

$$\begin{aligned}
&\sin(\varphi(z-1-|v|))+\sin(\varphi(z+1-|v|))\\
&=\sin(\varphi(z-|v|))\cos(\varphi)+\cos(\varphi(z-|v|))\sin(\varphi)\\
&\quad+\sin(\varphi(z-|v|))\cos(\varphi)-\cos(\varphi(z-|v|))\sin(\varphi)\\
&=2\cos(\varphi)\sin(\varphi(z-|v|)).
\end{aligned}$$

The boundary values $\sin(\varphi(|v|-|v|))=0$ and $\sin(\varphi(L+1-|v|))=0$ are satisfied, too. Thus, $\sqrt{k}\widehat{A}\widehat{\Psi^L_{v,j,m}} = \lambda^L_{v,j,m}\widehat{\Psi^L_{v,j,m}} = \widehat{\lambda^L_{v,j,m}\Psi^L_{v,j,m}} = \widehat{A^{(L)}\Psi^L_{v,j,m}}$ for all L-admissible (v,j,m).

Ad (ii). We have to check that the image of an orthonormal basis is again an orthonormal basis. Let (v,j,m) be admissible. The fact that $\|\widehat{\Psi^L_{v,j,m}}\|_2^2 = 1$ is seen exactly as the normalisation part in lemma 3.1. Let (v',j',m') be another admissible triple. For $(v,j)\neq(v',j')$, $\widehat{\Psi^L_{v,j,m}}$ and $\widehat{\Psi^L_{v,j,m}}$ have disjoint support and are thus orthogonal. In case $(v,j)=(v',j')$ and $m\neq m'$, $\widehat{\Psi^L_{v,j,m}}$ and $\widehat{\Psi^L_{v,j,m}}$ are orthogonal, too, since the corresponding eigenvalues $\lambda^L_{v,j,m}\neq\lambda^L_{v',j',m'}$ with respect to the symmetric operator $\sqrt{k}\widehat{A}$ are not equal. Finally, $\widehat{\cdot}$ is surjective, since the dimensions of its preimage and its image agree.

Ad (iii). Fix $v\in\mathcal{T}_*^{L-1}$ and $j\in J_v$. We denote the linear subspace defined by the condition in (iii) by $\mathcal{D}_{v,j}$. By construction, $\Psi^L_{v,j,m}\in\mathcal{D}_{v,j}$, so $P_{v,j}\ell^2(\mathcal{T}^L)\subseteq\mathcal{D}_{v,j}$. Furthermore, $\dim(\mathcal{D}_{v,j}) = L-|v|$, since the condition allows one degree of freedom per sphere of $\mathcal{T}_v^L\setminus\{v\}$. On the other hand, $\dim(P_{v,j}\ell^2\mathcal{T}_v^L) = L-|v|$, because the vectors $\Psi^L_{v,j,m}$, $m\in\{|v|+1,\dots,L\}$, are a basis of $P_{v,j}\ell^2\mathcal{T}_v^L$. The statement follows.

Ad (iv). Let $\psi\in\ell^2(\mathcal{T}^L)$ and $\eta,\eta_{\mathbb{Z}}$ be given as in the statement. Because of (iii), $\eta P_{v,j}\psi\in P_{v,j}\ell^2(\mathcal{T}^L)$. This and the fact that the spectral projectors are orthogonal implies that $P_{v,j}$ and multiplication with $\eta_{\mathbb{Z}}$ commute:

$$P_{v,j}(\eta\psi) = P_{v,j}\Big(\eta\sum_{v',j'} P_{v',j'}\psi\Big) = \sum_{v',j'} P_{v,j}(\eta P_{v',j'}\psi) = \eta P_{v,j}\psi.$$

We use this in

$$\begin{aligned}\widehat{\eta\psi}_{v,j}(z) &= k^{-(z-|v|-1)/2} \sum_{u\in\mathcal{T}_v^L, u\sim v} \overline{\psi_{v,j}^{\perp}(u)} \sum_{w\in\mathcal{T}_u^L, |w|=z} (P_{v,j}(\eta\psi))(w) \\ &= \eta_{\mathbb{Z}}(z) k^{-(z-|v|-1)/2} \sum_{u\in\mathcal{T}_v^L, u\sim v} \overline{\psi_{v,j}^{\perp}(u)} \sum_{w\in\mathcal{T}_u^L, |w|=z} (P_{v,j}\psi)(w) = \eta_{\mathbb{Z}}(z)\widehat{\psi}_{v,j}(z).\end{aligned}$$

■

3.2. **The IMS localization formula.** In this subsection we provide a proof of the following proposition. It will be needed in section 4 (proposition 4.9).

Proposition 3.6 (IMS localization formula)**.** *There is a constant $C_{IMS} > 0$ such that for each $r > 2$, we have a partition of unity $\{\eta_{a,r}\}_{a\geqslant 0} \subseteq \ell^2(\mathcal{T}^L)$, consisting of radially symmetric functions normalized to $\sum_{a\geqslant 0}\eta_{a,r}^2 = 1$, such that for all $\psi \in \ell^2(\mathcal{T}^L)$ we have*

$$\big\langle A^{(L)}\psi, \psi\big\rangle \geqslant \sum_{a\geqslant 0}\big\langle A^{(L)}(\eta_{a,r}\psi), \eta_{a,r}\psi\big\rangle - \frac{C_{IMS}}{r^2}\|\psi\|_2^2.$$

Furthermore, the support of $\eta_{a,r}$ is a union of disjoint trees of length at most $2r$.

The proof of proposition proposition 3.6 is made in two steps. We first prove this formula for the discrete, one-dimensional Laplacian. Then, we carry this formula onto the tree by means of the spectral theory of the rooted tree.

3.2.1. *The IMS localization formula on $\mathbb{Z}$.* In this subsection we consider the discrete Laplacian

$$\Delta_{\mathbb{Z}} := \tau^{-1} - 2 + \tau$$

on $\ell^2(\mathbb{Z})$, where τ is the translation operator, i.e. given by $(\tau f)(x) = f(x+1)$ for $f\colon \mathbb{Z} \to \mathbb{C}$ and $x \in \mathbb{Z}$. Note that on $\ell^2(\mathbb{Z})$ we have $\tau^{-1} = \tau^*$. We will also employ the discrete gradient

$$\nabla := \tau - 1.$$

Lemma 3.7. *Let $f \in \ell^2(\mathbb{Z})$. For any partition of unity $\{\eta_a\}$, normalized so that $\sum_a \eta_a^2 = 1$, we have*

$$\big\langle -\Delta_{\mathbb{Z}} f, f\big\rangle \leqslant \sum\nolimits_a \big\langle -\Delta_{\mathbb{Z}}(\eta_a f), \eta_a f\big\rangle + \Big\|\sum\nolimits_a (\nabla\eta_a)^2\Big\|_\infty \|f\|_2^2.$$

Remark 3.8. (1) In the proof, we actually show the operator equality

$$-\Delta_{\mathbb{Z}} - \sum\nolimits_a \eta_a(-\Delta_{\mathbb{Z}})\eta_a = -\tfrac{1}{2}\sum\nolimits_a\big((\nabla\eta_a)^2\,\tau + (\nabla^*\eta_a)^2\,\tau^*\big).$$

Thus, the reverse inequality holds, too.

(2) On $\mathbb{Z}^d$, the Laplacian decomposes: $\Delta_{\mathbb{Z}^d} = \sum_{j=1}^d \Delta_{\mathbb{Z}_j}$. Thus, we immediately get the d-dimensional IMS formula

$$-\Delta_{\mathbb{Z}^d} - \sum\nolimits_a \eta_a(-\Delta_{\mathbb{Z}^d})\eta_a = -\tfrac{1}{2}\sum\nolimits_a \sum\nolimits_{j=1}^d \big((\nabla_j\eta_a)^2\,\tau_j + (\nabla_j^*\eta_a)^2\,\tau_j^*\big),$$

where $(\tau_j f)(z) = f(z+e_j) - f(z)$, and $\nabla_j = \tau_j - 1$ is a discrete partial derivative.

(3) Actually, the above formula holds on the Cayley graph of any finitely generated group, as long as the generator does not contain an idempotent element. This is proven basically with the exact same proof as given below for $\mathbb{Z}$, except that one has to read the notation higher dimensional. To be more precise, let $\mathcal{S}$ be the generator corresponding to the Cayley graph. Since the group acts on itself, we get for each $s \in \mathcal{S}$ a translation $\tau_s f(z) := f(zs)$. We treat

$$\tau := (\tau_s)_{s\in\mathcal{S}}, \quad \nabla := (\nabla_s)_{s\in\mathcal{S}}$$

as columns and ∇^* as row and use matrix multiplication when interpreting

$$-\nabla^*\nabla = \sum_{s\in\mathcal{S}} \nabla_s^*\nabla_s = \Delta.$$

We also have to write sums whenever appropriate.

(4) The formulation of lemma 3.7 with the quadratic form instead of the operators has the advantage, that it is easily restricted to subgraphs, e. g., $G = \{1,\dots,L\}$. All we have to do is to note that $\ell^2(G)$ is embedded trivially into $\ell^2(\mathbb{Z})$. The corresponding operator to the restricted quadratic form is the restriction with simple boundary conditions.

(5) Thanks to the simple boundary conditions, the adjacency operator $A_{\mathbb{Z}} := \tau^{-1} + \tau = \Delta_{\mathbb{Z}} + 2$ is only a shift of the Laplacian $\Delta_{\mathbb{Z}}$. Lemma 3.7 transfers to $A_{\mathbb{Z}}$:

$$\begin{aligned}\big\langle A_{\mathbb{Z}} f, f\big\rangle &= \big\langle \Delta_{\mathbb{Z}} f, f\big\rangle + 2\|f\|_2^2 \\ &\geqslant \sum\nolimits_a \big\langle \Delta_{\mathbb{Z}}\eta_a f, \eta_a f\big\rangle - \Big\|\sum\nolimits_a (\nabla\eta_a)^2\Big\|_\infty \|f\|_2^2 + 2\|f\|_2^2 \\ &\geqslant \sum\nolimits_a \big\langle A_{\mathbb{Z}}\eta_a f, \eta_a f\big\rangle - \Big\|\sum\nolimits_a (\nabla\eta_a)^2\Big\|_\infty \|f\|_2^2,\end{aligned}$$

since $\sum_a \eta_a^2 = 1$.

(6) Another noteworthy generalization of lemma 3.7 is the following. Note that any multiplication operator commutes with the multiplication of η_a. Thus, lemma 3.7 holds for Schrödinger operators, i. e., $-\Delta + V$ with a potential $V\colon \mathbb{Z} \to \mathbb{R}$ acting via multiplication.

Proof of lemma 3.7. We follow the proof of [Sim83, Lemma 3.1], the analogous statement on $\mathbb{R}^d$. With the above definitions,

$$\Delta_{\mathbb{Z}} = -\nabla^*\nabla.$$

For $f, g \in \ell^2(\mathbb{Z})$, it is easy to check that

$$\nabla(fg) = (\nabla f)\,\tau g + f\,(\nabla g) \quad \text{and} \quad \nabla^*(fg) = (\nabla^* f)\,\tau^* g + f\,(\nabla^* g).$$

Using this and $-\nabla^*\tau = \nabla$ as well as $-\tau^*\nabla = \nabla^*$, we immediately calculate

$$\begin{aligned}\Delta_{\mathbb{Z}}(fg) &= -\nabla^*\nabla(fg) = -\nabla^*\big(\nabla f\,\tau g + f\,\nabla g\big)\\ &= \Delta_{\mathbb{Z}} f\,g - \nabla f\,\nabla^*\tau g - \nabla^* f\,\tau^*\nabla g + f\,\Delta_{\mathbb{Z}} g\\ &= \Delta_{\mathbb{Z}} f\,g + \nabla f\,\nabla g + \nabla^* f\,\nabla^* g + f\,\Delta_{\mathbb{Z}} g.\end{aligned}$$

Consequently,

$$[f, -\Delta_{\mathbb{Z}}] = (\Delta_{\mathbb{Z}} f) + (\nabla f)\,\nabla + (\nabla^* f)\,\nabla^*.$$

To compute $[f,[f,-\Delta_{\mathbb{Z}}]]$, consider, for $g \in \ell^2(\mathbb{Z})$,

$$\begin{aligned}[f,-\Delta_{\mathbb{Z}}](fg) &= (\Delta_{\mathbb{Z}} f)fg + \nabla f\,\nabla(fg) + \nabla^* f\,\nabla^*(fg)\\ &= f(\Delta_{\mathbb{Z}} f)g + \nabla f\,(\nabla f\,\tau g + f\,\nabla g) + \nabla^* f\,(\nabla^* f\,\tau^* g + f\,\nabla^* g)\\ &= f\big(\Delta_{\mathbb{Z}} f + (\nabla f)\,\nabla + (\nabla^* f)\,\nabla^*\big)g + (\nabla f)^2\,\tau g + (\nabla^* f)^2\,\tau^* g\\ &= f[f,-\Delta_{\mathbb{Z}}]g + (\nabla f)^2\,\tau g + (\nabla^* f)^2\,\tau^* g.\end{aligned}$$

Thus,

$$[f,[f,-\Delta_{\mathbb{Z}}]] = -(\nabla f)^2\,\tau - (\nabla^* f)^2\,\tau^*.$$

On the other hand, expanding the commutators yields

$$[f,[f,-\Delta_{\mathbb{Z}}]] = [f, f(-\Delta_{\mathbb{Z}}) + \Delta_{\mathbb{Z}} f] = -f^2\Delta_{\mathbb{Z}} + 2f\Delta_{\mathbb{Z}} f - \Delta_{\mathbb{Z}} f^2.$$

We combine the last two formuae for $f := \eta_a$, sum over a and use $\sum_a \eta_a^2 = 1$ to derive

$$-\Delta_{\mathbb{Z}} - \sum\nolimits_a \eta_a(-\Delta_{\mathbb{Z}})\eta_a = -\tfrac{1}{2}\sum\nolimits_a\big((\nabla\eta_a)^2\,\tau + (\nabla^*\eta_a)^2\,\tau^*\big).$$

For $f \in \ell^2(\mathbb{Z})$, we see

$$\begin{aligned}&\tfrac{1}{2}\Big|\Big\langle -\tfrac{1}{2}\sum\nolimits_a\big((\nabla\eta_a)^2\,\tau + (\nabla^*\eta_a)^2\,\tau^*\big)f, f\Big\rangle\Big|\\ &\qquad\leqslant \tfrac{1}{2}\Big(\Big\|\sum\nolimits_a(\nabla\eta_a)^2\Big\|_\infty + \Big\|\sum\nolimits_a(\nabla^*\eta_a)^2\Big\|_\infty\Big)\|f\|_2^2 = \Big\|\sum\nolimits_a(\nabla\eta_a)^2\Big\|_\infty\|f\|_2^2.\end{aligned}$$

Thus,

$$\Big|\langle -\Delta_{\mathbb{Z}} f, f\rangle - \sum\nolimits_a\langle -\Delta_{\mathbb{Z}}(\eta_a f), \eta_a f\rangle\Big| \leqslant \Big\|\sum\nolimits_a(\nabla\eta_a)^2\Big\|_\infty\|f\|_2^2.$$

The triangle inequality finishes the proof. ■

3.2.2. *The IMS localisation formula on the tree.* The discrete IMS formula is also valid on trees, in a very general setting. Indeed, points 3 and 4 of Remark 3.8 hint at the following way of proving the IMS localization formula on a tree of bounded degree. First note that the Cayley graph of the free group with s generators is a tree of degree $2s$. Then we can embed the bounded degree tree into the Cayley graph of a free group, and restrict to the subgraph again. It is enough for our purposes to consider a radially symmetric partition of unity, so that instead we will use in this section the spectral theory of the rooted trees to extend the IMS formula on $\mathbb{Z}$ to trees.

Proof of proposition 3.6. Step I. Fix $\eta_{\mathbb{R}} \in C^1(\mathbb{R}, [0,1])$ with support $\operatorname{supp}(\eta_{\mathbb{R}}) \subseteq [-1,1]$ such that, for any $x \in \mathbb{R}$,

$$\sum_{a\in\mathbb{Z}} \big(\eta_{\mathbb{R}}(x-a)\big)^2 = 1.$$

We define a partition of unity on $\mathbb{Z}$ as follows. For $r > 2$, let

$$\eta_{\mathbb{Z},r,a} \colon \mathbb{Z} \to \mathbb{R}, \quad \eta_{\mathbb{Z},r,a}(x) := \eta_{\mathbb{R}}\big(2r^{-1}x - a\big).$$

This gives a partition on $\mathbb{Z}$ satisfying $\#\operatorname{supp} \eta_{\mathbb{Z},r,a} \leqslant r$. Furthermore, by the mean value theorem and $\operatorname{supp} \eta_{\mathbb{R}} \subseteq [-1,1]$, we get

$$\begin{aligned} |\nabla \eta_{\mathbb{Z},r,a}(x)| &= \big|\eta_{\mathbb{R}}\big(2r^{-1}(x+1)-a\big) - \eta_{\mathbb{R}}\big(2r^{-1}x-a\big)\big| \\ &\leqslant 2r^{-1} \sup_{\xi\in[0,1]} \big|\eta'_{\mathbb{R}}\big(2r^{-1}(x+\xi)-a\big)\big| \\ &\leqslant 2r^{-1} \sup|\eta'_{\mathbb{R}}(\mathbb{R})| \cdot \mathbf{1}_{[2r^{-1}x-1, 2r^{-1}(x+1)+1]}(a). \end{aligned}$$

There are at most two values of $a \in \mathbb{Z}$ where the gradient is nonzero, since $r > 2$ and $2r^{-1}(x+1)+1-(2r^{-1}x-1) = 2+2r^{-1}$. We can thus bound the following sum by

$$\sum_{a\in\mathbb{Z}} (\nabla \eta_{\mathbb{Z},r,a}(x))^2 \leqslant 4 \sup|\eta'_{\mathbb{R}}(\mathbb{R})|^2 r^{-2} \sum_{a\in\mathbb{Z}} \mathbf{1}_{[2r^{-1}x-1, 2r^{-1}(x+1)+1]}(a) \leqslant C_3 r^{-2}$$

with $C_3 := 8 \sup|\tilde{\eta}'(\mathbb{R})|^2$.

Step II. We now define the partition on the tree. Let

$$\eta_{r,a} \colon \mathcal{T} \to [0,1], \quad \eta_{r,a}(v) := \eta_{\mathbb{Z},r,a}(|v|).$$

With this definition we have

$$\sum_{a\in\mathbb{N}} \eta_{r,a} = 1$$

on $\mathcal{T}$. The support of each $\eta_{r,a}$ is a disjoint union of rooted trees of length at most r, see fig. 5. For $\psi \in \ell^2(\mathcal{T}^L)$, we employ remark 3.8 and learn

$$\begin{aligned} \langle A^{(L)}\psi, \psi\rangle &= \big\langle \widehat{A^{(L)}\psi}, \widehat{\psi}\big\rangle = \sqrt{k}\big\langle \widehat{A}\widehat{\psi}, \widehat{\psi}\big\rangle \\ &= \sqrt{k}\sum\nolimits_{v,j} \big\langle (\widehat{A}\widehat{\psi})_{v,j}, \widehat{\psi}_{v,j}\big\rangle = \sqrt{k}\sum\nolimits_{v,j} \big\langle A_{\mathbb{Z}}\widehat{\psi}_{v,j}, \widehat{\psi}_{v,j}\big\rangle \\ &\geqslant \sqrt{k}\sum\nolimits_{v,j}\Big(\sum\nolimits_{a} \big\langle A_{\mathbb{Z}}(\eta_{\mathbb{Z},r,a}\widehat{\psi}_{v,j}), \eta_{\mathbb{Z},r,a}\widehat{\psi}_{v,j}\big\rangle - C_3 r^{-2}\|\widehat{\psi}_{v,j}\|_2^2\Big) \\ &= \sum\nolimits_a \sqrt{k}\sum\nolimits_{v,j} \big\langle A_{\mathbb{Z}}\widehat{(\eta_{r,a}\psi)}_{v,j}, \widehat{\eta_{r,a}\psi}_{v,j}\big\rangle - C_{\mathit{IMS}} r^{-2}\|\widehat{\psi}\|_2^2 \\ &= \sum\nolimits_a \big\langle A^{(L)}\widehat{(\eta_{r,a}\psi)}, \widehat{\eta_{r,a}\psi}\big\rangle - C_{\mathit{IMS}} r^{-2}\|\widehat{\psi}\|_2^2 \\ &= \sum\nolimits_a \big\langle A^{(L)}(\eta_{r,a}\psi), \eta_{r,a}\psi\big\rangle - C_{\mathit{IMS}} r^{-2}\|\widehat{\psi}\|_2^2, \end{aligned}$$

where $C_{\mathit{IMS}} := C_3\sqrt{k}$. ∎

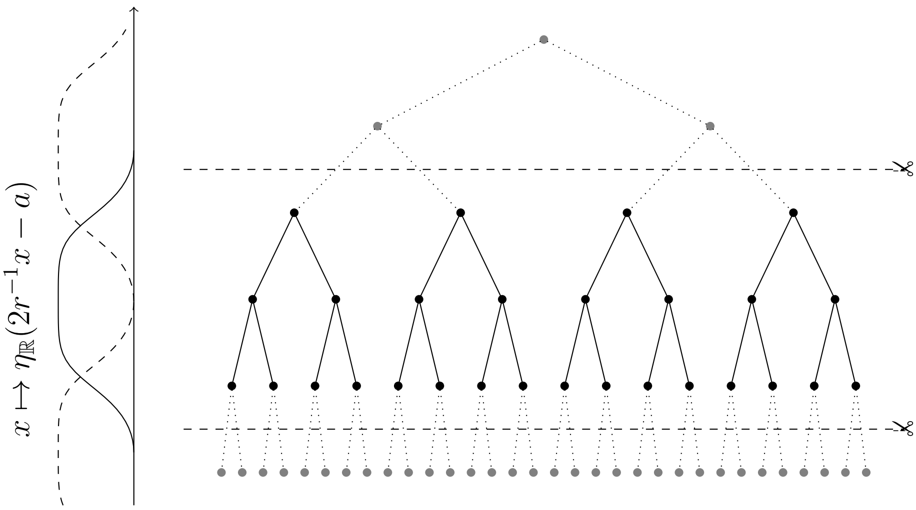

FIGURE 5. Shells of a tree split in trees

3.3. **An uncertainty principle.**

3.3.1. *On a finite segment of* $\mathbb{Z}$. Let us first prove a one-dimensional version of proposition 3.10. Afterwards, we transfer the result to the tree with lemma 3.5. To this end, let $L \in \mathbb{N}$ and $v \in \mathcal{T}_*^{L-1}$. Consider a function $\varphi \in \ell^2(\{|v|+1,\dots,L\})$, which can be written in the orthonormal basis of eigenfunctions of $-\Delta_{\mathbb{Z}}|\{|v|+1,\dots,L\}$ as

$$\varphi(z) = \sum_{1\leqslant m \leqslant L-|v|} \alpha_m \sqrt{\frac{2}{L+1-|v|}} \sin\Big(\frac{m\pi(z-|v|)}{L-|v|+1}\Big).$$

Given $\beta > 0$, we define the spectral projector $\hat{P}_\beta^{|v|,L}$ on $\ell^2(\{|v|+1,\dots,L\})$ via

$$\hat{P}_\beta^{|v|,L}\varphi(z) = \sum_{1\leqslant m \leqslant \beta+1} \alpha_m \sqrt{\frac{2}{L+1-|v|}} \sin\Big(\frac{m\pi(z-|v|)}{L-|v|+1}\Big),$$

$z \in \{|v|+1,\dots,L\}$.

Lemma 3.9. *Let $L \in \mathbb{N}$, $\beta > 0$, $0 < \delta < 1$ and $v \in \mathcal{T}_*^L$ be fixed, such that $|v| + 1 + \lceil \delta L \rceil \leqslant L$. Define, for $\varphi \in \ell^2(|v|+1,\dots,L\})$, the truncation*

$$T'_{|v|,\delta}\varphi := \mathbf{1}_{\{|v|+1+\lceil\delta L\rceil,\dots,L\}}\varphi.$$

Then, for $|v| \leqslant \big(1-\frac{1}{\beta+1}\big)(L+1)$, we have

$$\|\hat{P}_\beta^{|v|,L}\varphi - T'_{|v|,\delta}\hat{P}_\beta^{|v|,L}\varphi\|_2 \leqslant \sqrt{2}\pi\delta^{3/2}(\beta+1)^3\|\hat{P}_\beta^{|v|,L}\varphi\|_2.$$

Proof. We calculate, using Cauchy–Schwarz,

$$\begin{aligned}\|(\mathbf{1}_{\{|v|+1,\dots,L\}}-T'_\delta)\hat{P}^{(L)}_\beta\varphi\|_2^2 &= \frac{2}{L+1-|v|}\sum_{z=|v|+1}^{|v|+\lceil\delta L\rceil-1}\Big|\sum_{m=1}^{\beta+1}\alpha_m\sin\Big(\frac{m\pi(z-|v|)}{L+1-|v|}\Big)\Big|^2\\ &\leqslant \frac{2}{L+1-|v|}\sum_{z=1}^{\lceil\delta L\rceil-1}\sum_{m=1}^{\beta+1}|\alpha_m|^2\sum_{m=1}^{\beta+1}\Big(\sin\Big(\frac{m\pi z}{L+1-|v|}\Big)\Big)^2.\end{aligned}$$

Now using $\|\hat{P}^{(L)}_\beta\varphi\|_2^2=\sum_{m=1}^{\beta+1}|\alpha_m|^2$, and $\sin(t)\leqslant|t|$, valid for all $t\in\mathbb{R}$, the last line is smaller than

$$\begin{aligned}\frac{2\|\hat{P}^{(L)}_\beta\varphi\|_2^2}{L+1-|v|}\sum_{z=1}^{\lceil\delta L\rceil-1}\sum_{m=1}^{\beta+1}\Big(\frac{m\pi z}{L+1-|v|}\Big)^2 &= \frac{2\pi^2\|\hat{P}^{(L)}_\beta\varphi\|_2^2}{(L+1-|v|)^3}\sum_{z=1}^{\lceil\delta L\rceil-1}z^2\sum_{m=1}^{\beta+1}m^2\\ &\leqslant \frac{2\pi^2(\delta L)^3(\beta+1)^3}{(L+1-|v|)^3}\|\hat{P}^{(L)}_\beta\varphi\|_2^2.\end{aligned}$$

Now note that $|v|\leqslant\big(1-\frac{1}{\beta+1}\big)(L+1)$ implies

$$\frac{L}{L+1-|v|}\leqslant\frac{\beta+1}{L+1}L\leqslant\beta+1.$$

This bound and taking the square root yields the result. ■

3.3.2. *On a finite rooted tree.* For any $\beta>0$, we recall the definition of

$$E^{(L)}_\beta := 2\sqrt{k}\cos\Big(\frac{\beta+1}{L+1}\pi\Big) \tag{3.7}$$

from lemma 3.3. We want to study the neighbourhood $[E^{(L)}_\beta,E^{(L)}_0]$ of the principal eigenvalue $E^{(L)}_0$ of the adjacency matrix on the rooted tree $\mathcal{T}^L$. We define the spectral projector of $A^{(L)}$ on the energy interval $[E^{(L)}_\beta,\infty)$ as

$$\Pi^{(L)}_{E^{(L)}_\beta}\colon \ell^2(\mathcal{T}^L)\to\ell^2(\mathcal{T}^L),\quad \Pi^{(L)}_{E^{(L)}_\beta}\varphi:=\sum_{v,j,m\colon\lambda^L_{v,j,m}\geqslant E^{(L)}_\beta}\langle\varphi,\Psi^L_{v,j,m}\rangle\Psi^L_{v,j,m}.$$

We also define the space truncations

$$T_{|v|,\delta}\colon\ell^2(\mathcal{T}^L)\to\ell^2(\mathcal{T}^L),\quad T_{|v|,\delta}\varphi:=\varphi\mathbf{1}_{\{x\in\mathcal{T}^L:|x|>|v|+1+\delta L\}}$$

and a truncated version of $\Pi^{(L)}_{E^{(L)}_\beta}$

$$\tilde{\Pi}^{(L)}_{E^{(L)}_\beta}\colon \ell^2(\mathcal{T}^L)\to\ell^2(\mathcal{T}^L),\quad \tilde{\Pi}^{(L)}_{E^{(L)}_\beta}\varphi:=\sum_{v,j,m\colon\lambda^L_{v,j,m}\geqslant E^{(L)}_\beta}\langle\varphi,\Psi^L_{v,j,m}\rangle T_{|v|,\delta}\Psi^L_{v,j,m}.$$

Note that we can also write

$$\tilde{\Pi}^{(L)}_{E^{(L)}_\beta}\varphi=\sum_{v,j}T_{|v|,\delta}P_{v,j}\Pi^{(L)}_{E^{(L)}_\beta}\varphi. \tag{3.8}$$

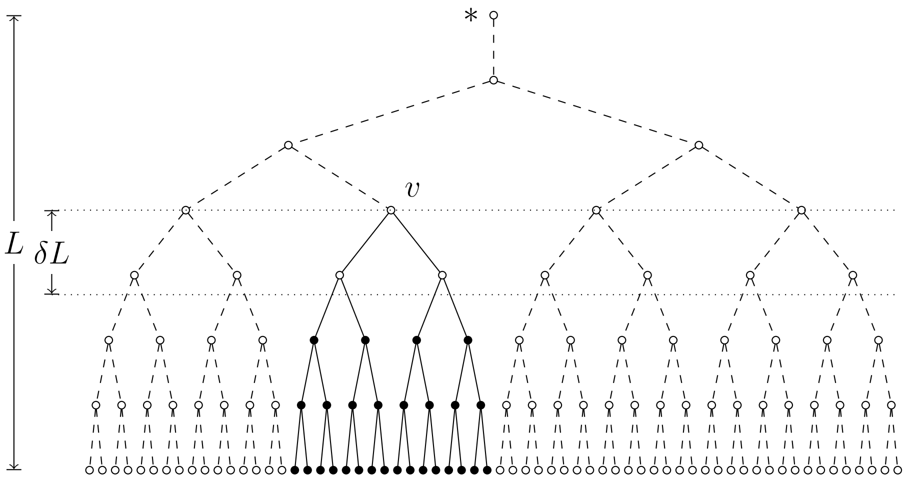

FIGURE 6. Illustration of (3.8). The subtree $\mathcal{T}_v^L$ is indicated with solid edges. Nodes in the support of functions truncated with $T_{|v|,\delta}$ in $\mathcal{T}_v^L$ are filled black.

Proposition 3.10. *Let $L \in \mathbb{N}$, $\beta > 0$ and $0 < \delta < 1$. Then, for any $\varphi \in \ell^2(\mathcal{T}^L)$,*

$$\|\Pi^{(L)}_{E^{(L)}_\beta}\varphi - \tilde{\Pi}^{(L)}_{E^{(L)}_\beta}\varphi\|_2 \leqslant \sqrt{2}\pi\delta^{3/2}(\beta+1)^3\|\Pi^{(L)}_{E^{(L)}_\beta}\varphi\|_2.$$

Proof. We will show equivalently that

$$\forall\varphi \in \Pi^{(L)}_{E^{(L)}_\beta}\ell^2(\mathcal{T}^L)\colon \|\Pi^{(L)}_{E^{(L)}_\beta}\varphi - \tilde{\Pi}^{(L)}_{E^{(L)}_\beta}\varphi\|_2 \leqslant \sqrt{2}\delta^{3/2}(\beta+1)^2\|\varphi\|_2.$$

Indeed, it follows from (3.8) that if $\Pi^{(L)}_{E^{(L)}_\beta}\varphi = 0$ then $\tilde{\Pi}^{(L)}_{E^{(L)}_\beta}\varphi = 0$. We assume thus from now on that $\varphi = \Pi^{(L)}_{E^{(L)}_\beta}\varphi$. For such φ, we see, by (3.8),

$$\big(\Pi^{(L)}_{E^{(L)}_\beta}\varphi - \tilde{\Pi}^{(L)}_{E^{(L)}_\beta}\big)\varphi = \sum_{v,j}(\mathbf{1}_{\mathcal{T}^L} - T_{|v|,\delta})P_{v,j}\varphi.$$

By lemma 3.5, we know that $P_{v,\delta}$ commutes with the radially symmetric truncation. Thus, by the orthogonality of the projections $P_{v,j}$,

$$\big\|\big(\Pi^{(L)}_{E^{(L)}_\beta}\varphi - \tilde{\Pi}^{(L)}_{E^{(L)}_\beta}\big)\varphi\big\|_2^2 = \Big\|\sum_{v,j} P_{v,j}(\mathbf{1}_{\mathcal{T}^L} - T_{|v|,\delta})\varphi\Big\|_2^2 = \sum_{v,j}\big\|P_{v,j}(\mathbf{1}_{\mathcal{T}^L} - T_{|v|,\delta})\varphi\big\|_2^2.$$

We study this norms via the unitary $\widehat{\cdot}$, see (3.5) and lemma 3.5. For each v, j, we have

$$\begin{aligned}\big\|P_{v,j}(\mathbf{1}_{\mathcal{T}^L} - T_{|v|,\delta})\varphi\big\|_2 &= \big\|\big(P_{v,j}(\mathbf{1}_{\mathcal{T}^L} - T_{|v|,\delta})\varphi\big)^\wedge\big\|_2 = \big\|\big((\mathbf{1}_{\mathcal{T}^L} - T_{|v|,\delta})\varphi\big)^\wedge_{v,j}\big\|_2\\ &= \big\|\big(\mathbf{1}_{\{|v|+1,\dots,L\}} - T'_{|v|,\delta}\big)\widehat{\varphi}_{v,j}\big\|_2.\end{aligned}$$

We learn from lemma 3.3 that the coefficients $(\alpha_{v,j,m})_{v,j,m}$ of

$$\varphi = \Pi^{(L)}_{E^{(L)}_\beta}\varphi = \sum_{v,j,m} \alpha_{v,j,m}\Psi^L_{v,j,m}$$

vanish as soon as $|v| > (L+1)(1-\frac{m}{\beta+1})$ or $m > \beta+1$. Therefore, we have $\widehat{\varphi}_{v,j} = \hat{P}^{|v|,L}_\beta \widehat{\varphi}_{v,j}$, and we can thus invoke lemma 3.9 to conclude

$$\big\|\big(\Pi^{(L)}_{E^{(L)}_\beta}\varphi - \tilde{\Pi}^{(L)}_{E^{(L)}_\beta}\big)\varphi\big\|_2^2 \leqslant 2\pi^2\delta^3(\beta+1)^6\sum_{v,j}\|\widehat{\varphi}_{v,j}\|_2^2 = 2\pi^2\delta^3(\beta+1)^6\|\varphi\|_2^2.$$

Since $\varphi = \Pi^{(L)}_{E^{(L)}_\beta}\varphi$, this is what we set out to prove. ■

4. Lifschitz tails: The upper bound

This section is devoted to the proof of the following theorem.

Theorem 4.1. *Let* $E_0 := (\sqrt{k}-1)^2$*. Then,*

$$\limsup_{E\to E_0} \frac{\log\log|\log\mathcal{N}(E)|}{\log(E-E_0)} \leqslant -\frac{1}{2}. \tag{4.1}$$

This theorem provides the converse to proposition 2.7. Note that no condition on the random variables is needed for the upper bound.

4.1. **Bound by a probability.** We remind that $\mathcal{N}(E)$ denotes the integrated density of states given by (1.6), $\mathrm{d}\mathcal{N} = \mathrm{d}\mathcal{N}/\mathrm{d}E$ its derivative and $\tilde{\mathcal{N}}(t)$ is the Laplace transform of $\mathrm{d}\mathcal{N}$. We start by proving the following Tauberian theorem, which links the long time behavior of $\tilde{\mathcal{N}}$ to the low energy asymptotic of $\mathcal{N}$.

Proposition 4.2. *Let* $\tilde{\mathcal{N}}(t)$ *be the Laplace transform of the density of states measure* $\mathrm{d}\mathcal{N}$*. Suppose that for some* $\eta > 0$*,*

$$\limsup_{t\to\infty} \mathrm{e}^{t(E_0+(\log t)^{-\eta})}\tilde{\mathcal{N}}(t) \leqslant 1 \tag{4.2}$$

with $E_0 := (\sqrt{k}-1)^2$*. Then,*

$$\limsup_{E\to E_0} \frac{\log\log|\log\mathcal{N}(E)|}{\log(E-E_0)} \leqslant -\frac{1}{\eta}. \tag{4.3}$$

Proof. Assume that inequality (4.2) holds. Then, there exists some t^* such that for all $t > t^*$,

$$\tilde{\mathcal{N}}(t) \leqslant 2\mathrm{e}^{-t(E_0+(\log t)^{-\eta})}. \tag{4.4}$$

Clearly, for $E \geqslant E_0$,

$$\begin{aligned}\mathcal{N}(E) &= \int_{E_0}^{E} \mathrm{d}\mathcal{N}(\lambda) \\ &\leqslant \mathrm{e}^{tE} \int_{E_0}^{E} \mathrm{e}^{-t\lambda} \mathrm{d}\mathcal{N}(\lambda) \\ &\leqslant \mathrm{e}^{tE} \int_{E_0}^{\infty} \mathrm{e}^{-t\lambda} \mathrm{d}\mathcal{N}(\lambda) = \mathrm{e}^{tE} \tilde{\mathcal{N}}(t),\end{aligned}$$

and by (4.4), for large t,

$$\mathcal{N}(E) \leqslant 2\mathrm{e}^{t(E-E_0)-t(\log t)^{-\eta}}$$

Now we choose t as follows

$$t = t(E) := \exp\Big((2E-2E_0)^{-1/\eta}\Big).$$

We see that, for small $E - E_0$,

$$\begin{aligned}\mathcal{N}(E) &\leqslant 2\exp\left(-(E-E_0)\exp\Big((2E-2E_0)^{-1/\eta}\Big)\right) \\ &\leqslant \exp\left(-\frac{1}{2}(E-E_0)\exp\Big((2E-2E_0)^{-1/\eta}\Big)\right).\end{aligned}$$

and for $E - E_0$ small enough

$$\log|\log \mathcal{N}(E)| \leqslant -\log(2) + \log(E-E_0) + (2E-2E_0)^{-1/\eta} \leqslant (E-E_0)^{-1/\eta}.$$

Now taking another logarithm, dividing by $\log(E - E_0)$ and taking the $\limsup$ proves the theorem. ■

The rest of this section will be devoted to prove that, as a consequence of theorem 4.10, condition (4.2) holds for any $\eta > 2$. This proves theorem 4.1.

Our next proposition compares $\tilde{\mathcal{N}}$ to a finite dimensional analog $\tilde{\mathcal{N}}^L$. For any $\Gamma \subset \mathcal{B}$, we denote by $H_\omega|\Gamma$ the operator H_ω with *simple* (sometimes called *Dirichlet*) boundary conditions, i.e. the operator defined by

$$H_\omega|\Gamma := \mathbf{1}_\Gamma H_\omega \mathbf{1}_\Gamma,$$

or equivalently, writing $H_\omega(v, w)$, $v, w \in \mathcal{B}$, for the matrix coefficients, it cand be defined by

$$(H_\omega|\Gamma)(v,w) := \begin{cases} H_\omega(v,w) & \text{if } v,w \in \Gamma \\ 0 & \text{elsewhere.} \end{cases} \tag{4.5}$$

Remember that $\mathcal{B}^L$ denotes the ball of radius L of the Bethe lattice. Let us define the averaged spectral density $\mathcal{N}^L$ of $H_\omega|\mathcal{B}^L$ by

$$\mathcal{N}^L(E) := \mathbb{E}\langle \delta_0, \mathbf{1}_{(-\infty,E]}(H_\omega|\mathcal{B}^L)\delta_0\rangle.$$

In particular, its Laplace transform can be written

$$\tilde{\mathcal{N}}^L(t) = \mathbb{E}\left[\langle \delta_0, \mathrm{e}^{-tH_\omega|\mathcal{B}^L}\delta_0\rangle\right].$$

Note also that, using functional calculus, we have

$$\tilde{\mathcal{N}}(t) = \mathbb{E}\left[\langle \delta_0, \mathrm{e}^{-tH_\omega}\delta_0\rangle\right].$$

In the following proposition we compare these two quantities. We define $\omega_+ := \|\omega_0\|_\infty$ for further use.

Proposition 4.3. *Let $\tilde{\mathcal{N}}^L$ be the Laplace transform of $\mathrm{d}\mathcal{N}^L$. Pick some positive constant $\zeta > \mathrm{e}^2\|H_\omega\| = \mathrm{e}^2((\sqrt{k}+1)^2 + \omega_+)$ and let $L = \lceil \zeta t\rceil$. Then, for any $t \geqslant 1$ the following holds:*

$$|\tilde{\mathcal{N}}(t) - \tilde{\mathcal{N}}^L(t)| \leqslant 4\mathrm{e}^{-\zeta t}.$$

Here, $\|H_\omega\| = \sup\Sigma$.

Proof. Assume $\zeta > \mathrm{e}^2\|H_\omega\|$ and $t \geqslant 1$. First let us note that H_ω is a bounded operator and (we actually have $\|H_\omega\| = (k+1+2\sqrt{k}) + \|V_\omega\|_\infty$). This allows us to expand the exponential as a sum like

$$\langle \delta_0, \mathrm{e}^{-tH_\omega}\delta_0\rangle = \sum_{n=0}^{L} \frac{(-t)^n}{n!}\langle \delta_0, H_\omega^n\delta_0\rangle + \sum_{n>L}\frac{(-t)^n}{n!}\langle \delta_0, H_\omega^n\delta_0\rangle,$$

which is also valid if we replace H_ω by $H_\omega|\mathcal{B}^L$. It is easy to see that the two first terms of this sum are 1 and $-tH_\omega(0,0) = -t(H_\omega|\mathcal{B}^L)(0,0)$ respectively. Expanding the matrix product, we see that, for $2 \leqslant n \leqslant L$

$$\langle \delta_0, H_\omega^n\delta_0\rangle = \sum_{x_1,\dots,x_{n-1}\in\mathcal{B}} H_\omega(0,x_1)H_\omega(x_2,x_3)\cdots H_\omega(x_{n-2},x_{n-1})H_\omega(x_{n-1},0).$$

Now, using that $H_\omega(v,w) = 0$ for $v, w \in \mathcal{B}^L$ satisfying $d(v,w) > 1$, the last sum reduces to

$$\sum_{(p_0,\dots,p_n):0\rightsquigarrow 0} H_\omega(p_0,p_1)H_\omega(p_2,p_3)\cdots H_\omega(p_{n-2},p_{n-1})H_\omega(p_{n-1},p_n), \tag{4.6}$$

where we have written $(p_0,\dots,p_n) : 0 \rightsquigarrow 0$ to denote a path $(p_0,\dots,p_n) \in (\mathcal{B})^{n+1}$ (which may include loops) starting at 0 and ending at 0. In particular, if $(p_0,\dots,p_n) : 0 \rightsquigarrow 0$,

$$d(p_0,p_i) \leqslant \sum_{0\leqslant j\leqslant n-1} d(p_j,p_{j+1}) \leqslant n \leqslant L \quad \text{for any } 0 \leqslant i \leqslant n, \tag{4.7}$$

i.e. the paths in the sum (4.6) are entirely contained in $\mathcal{B}^L$. Using (4.5), we see that for $2 \leqslant n \leqslant L$

$$\langle \delta_0, H_\omega^n\delta_0\rangle = \sum_{(p_0,\dots,p_n):0\rightsquigarrow 0}\ \prod_{1\leqslant i\leqslant n} (H_\omega|\mathcal{B}^L)(p_{i-1},p_i) = \langle \delta_0, (H_\omega|\mathcal{B}^L)^n\delta_0\rangle. \tag{4.8}$$

We conclude that the first $L+1$ terms of the expansions of $\langle\delta_0, \mathrm{e}^{-tH_\omega}\delta_0\rangle$ and $\langle\delta_0, \mathrm{e}^{-tH_\omega|\mathcal{B}^L}\delta_0\rangle$ coincide, and

$$\left|\langle\delta_0, \mathrm{e}^{-tH_\omega}\delta_0\rangle - \langle\delta_0, \mathrm{e}^{-tH_\omega|\mathcal{B}^L}\delta_0\rangle\right| = \sum_{n>L}\frac{t^n}{n!}\langle\delta_0, H_\omega^n\delta_0\rangle + \sum_{n>L}\frac{t^n}{n!}\langle\delta_0, (H_\omega|\mathcal{B}^L)^n\delta_0\rangle.$$

Let us estimate this error. We do it for for the first term, the second one is similar. By a simple calculation, we see that

$$\begin{aligned}\sum_{n>L}\frac{t^n}{n!}\|H_\omega\|^n &= \sum_{n\geqslant 1}\frac{t^{L+n}}{(L+n)!}\|H_\omega\|^{L+n}\\ &\leqslant \sum_{n\geqslant 1}\frac{t^{L+n}}{(L+n)^{(L+n)}\mathrm{e}^{-(L+n)}}\|H_\omega\|^{L+n}\\ &\leqslant \sum_{n\geqslant 1}\left(\frac{et\|H_\omega\|}{L}\right)^{L+n},\end{aligned}$$

where we have used $n! \geqslant n^n\mathrm{e}^{-n}$ and $(L+n)^{-1}\leqslant L^{-1}$. In particular, if $L=\lceil\zeta t\rceil$, we see that

$$\left(\frac{et\|H_\omega\|}{L}\right)^{L+n} \leqslant (e\|H_\omega\|/\zeta)^{\lceil\zeta t\rceil+n},$$

and as $\zeta > \mathrm{e}^2\|H_\omega\|$ and $t\geqslant 1$, we can bound the the error as

$$\sum_{n>L}\frac{t^n}{n!}\|H_\omega\|^n \leqslant \sum_{n\geqslant 1}\mathrm{e}^{-(\lceil\zeta t\rceil+n)} \leqslant 2\mathrm{e}^{-\zeta t}. \tag{4.9}$$

Noting that $\|H_\omega|\mathcal{B}^L\|\leqslant\|H_\omega\|$, a similar calculation leads to

$$\sum_{n>L}\frac{t^n}{n!}\langle\delta_0, (H_\omega|\mathcal{B}^L)^n\delta_0\rangle \leqslant 2\mathrm{e}^{-\zeta t}. \tag{4.10}$$

This ends the proof. ■

We will now study the large time behavior of $\tilde{\mathcal{N}}^L$.

Lemma 4.4. *Let $\Gamma\subset\mathcal{B}$ be finite and $H_\omega|\Gamma$ the restriction of H_ω with simple boundary conditions. Then*

$$\mathbb{E}\left[\langle\delta_0, \mathrm{e}^{-tH_\omega|\Gamma}\delta_0\rangle\right] \leqslant \mathbb{E}\left[\mathrm{e}^{-tE_{GS}(H_\omega|\Gamma)}\right].$$

Proof. Fix a realization ω and let $\{\lambda_i;\psi_i\}_{i\in\Gamma} = \{\lambda_i(\omega);\psi_i(\omega)\}_{i\in\Gamma}$ be a complete set of (eigenvalues, eigenfunctions) of $H_\omega|\Gamma$. Then, writing

$$\delta_0 = \sum \alpha_i(\omega)\psi_i(\omega)$$

we see that, as $E_{GS}(H_\omega|\Gamma) = \min_{i\in\Gamma}\lambda_i(\omega)$ and $\sum|\alpha_i|^2=1$,

$$\langle\delta_0, \mathrm{e}^{-tH_\omega|\Gamma}\delta_0\rangle = \sum_{i\in\Gamma}|\alpha_i(\omega)|^2\mathrm{e}^{-t\lambda_i(\omega)} \leqslant \sum_{i\in\Gamma}|\alpha_i(\omega)|^2\mathrm{e}^{-tE_{GS}(H_\omega|\Gamma)} = \mathrm{e}^{-tE_{GS}(H_\omega|\Gamma)}.$$

Taking the expectation in this inequality yields the desired result. ■

The following two lemmata link the behavior of the ground state energy of the Hamiltonian on a ball to the one on a finite rooted tree. This is needed in order to use the spectral theory developed in section 3.

Lemma 4.5. *Let $L \geqslant 1$ and $\mathcal{B}_v^L$ a ball of radius L centered at $v \in \mathcal{B}$, i.e.*

$$\mathcal{B}_v^L := \{w \in \mathcal{B} : d_{\mathcal{B}}(w, v) \leqslant L\}.$$

Then, for every $v \in \mathcal{B}$ with $|v| = L + 2$ there exists a rooted tree $\mathcal{T}^{3L} \subset \mathcal{B}$, of length $3L$, which contains $\mathcal{B}_v^L$.

Proof. Label the k branches of the Bethe lattice by the nodes $x \in \mathcal{T}^L$ satisfying $|x| = 1$ and assume that $d(0, v) = L + 1$. Then, there exists a unique minimal path $[0, v] = (0, v_1, v_2, \dots, v)$ of length $L + 1$. Because $d(v_1, v) = L$, we know that $v_1 \in \mathcal{B}^L$. In particular the whole ball is contained in the branch of the Bethe lattice v_1. Now choose $k - 1$ other branches to form the infinite rooted tree $\mathcal{T}$. The result is now clear because by definition $\mathcal{T}^L := \{v \in \mathcal{T} : |v| \leqslant L\}$ and for any $x \in \mathcal{B}_v^L$ we have $|x| \leqslant 2L + 1 \leqslant 3L$. ■

Conversely, it is easy to see that $\mathcal{T}^L \subset \mathcal{B}^L$, for all $L > 1$. This leads to the following lemma.

Lemma 4.6. *For any $L \geqslant 1$ and $|v| = L + 2$,*

$$E_{GS}(H_\omega | \mathcal{B}_v^L) \leqslant E_{GS}(H_\omega | \mathcal{T}_v^{3L}) \leqslant E_{GS}(H_\omega | \mathcal{B}_v^{3L})$$

Here $\mathcal{T}^L$ and $\mathcal{T}^{3L}$ are the trees satisfying $\mathcal{T}^L \subset \mathcal{B}^L \subset \mathcal{T}^{3L}$.

Proof. Let $v \in \mathcal{B}$ with $|v| = L + 2$ and $\mathcal{T}^{3L}$ be the rooted tree containing $\mathcal{B}_v^L$. Then

$$\begin{aligned} E_{GS}(H_\omega | \mathcal{B}_v^L) &= \inf_{\substack{\varphi \in \ell^2(\mathcal{B}_v^L) \\ \|\varphi\|_2 = 1}} \langle H_\omega \varphi, \varphi \rangle \\ &\leqslant \inf_{\substack{\varphi \in \ell^2(\mathcal{T}^{3L}) \\ \|\varphi\|_2 = 1}} \langle H_\omega \varphi, \varphi \rangle = E_{GS}(H_\omega | \mathcal{T}^{3L}) \leqslant E_{GS}(H_\omega | \mathcal{T}^{3L}). \end{aligned}$$

This means that

$$\mathbb{E}\left[\mathrm{e}^{-tE_{GS}(H_\omega | \mathcal{B}_v^L)}\right] \leqslant \mathbb{E}\left[\mathrm{e}^{-tE_{GS}(H_\omega | \mathcal{T}^{3L})}\right]$$

■

Using translation invariance, we can translate the point where we calculate the integrated densities of states $\mathcal{N}$ and $\mathcal{N}^L$. proposition 4.3 tells us then that it is enough to study, for some $v \in \mathcal{B}$ with $|v| = L + 2$,

$$\tilde{\mathcal{N}}^L(t) = \mathbb{E}\left[\langle \delta_v, \mathrm{e}^{-tH_\omega | \mathcal{B}_v^L} \delta_v \rangle\right].$$

We remind that $\mathcal{B}_v^L$ is the ball centered at v.

From now on we write $H^L_\omega := H_\omega|\mathcal{T}^L$. The next lemma is a simple bound on the expectation by a probability.

Lemma 4.7. *For any $\epsilon > 0$, $L > 1$ and $t > 1$, we have*

$$\mathbb{E}\big[\mathrm{e}^{-tE_{GS}(H^L_\omega)}\big] \leqslant \mathrm{e}^{-t(E_0+\epsilon(\log t)^{-2})} + \mathrm{e}^{-tE_0}\mathbb{P}\left(E_{GS}(H^L_\omega) < E_0 + \epsilon(\log t)^{-2)}\right). \tag{4.11}$$

Proof. We have indeed for all $E \geqslant E_0$

$$\begin{aligned}\mathbb{E}\big[\mathrm{e}^{-tE_{GS}(H^L_\omega)}\big] &= \mathbb{E}\big[(\mathbf{1}_{\{E_{GS}(H^L_\omega)\geqslant E\}} + \mathbf{1}_{\{E_{GS}(H^L_\omega)<E\}})\mathrm{e}^{-tE_{GS}(H^L_\omega)}\big] \\ &\leqslant \mathrm{e}^{-tE} + \mathrm{e}^{-tE_0}\mathbb{P}(E_{GS}(H^L_\omega) < E).\end{aligned}$$

■

We summarize the results of this section in the following proposition.

Proposition 4.8. *Assume $\epsilon > 0$ and $\zeta > \mathrm{e}^2\|H_\omega\|_2(\sqrt{k}+1)^2\omega_+$. If*

$$\limsup_{L\to\infty} \mathrm{e}^{\epsilon\zeta L}\mathbb{P}\left(E_{GS}(H^L_\omega) < E_0 + 4\epsilon(\log L)^{-2}\right) \leqslant 1 \tag{4.12}$$

then

$$\limsup_{t\to\infty} \mathrm{e}^{t(E_0+\epsilon(\log t)^{-2})}\tilde{\mathcal{N}}(t) \leqslant 1. \tag{4.13}$$

Proof. Let $t > 1$ and $L = \lceil\zeta t\rceil$. Then,

$$\begin{aligned}&\exp(t(E_0+\epsilon(\log t)^{-2})\tilde{\mathcal{N}}(t) \\ &\leqslant \exp(t(E_0+\epsilon(\log t)^{-2})(\tilde{\mathcal{N}}^L(t) + 4\mathrm{e}^{-\zeta t}) \quad \text{by proposition 4.3} \\ &\leqslant \exp(t(E_0+\epsilon(\log t)^{-2})(\mathbb{E}\mathrm{e}^{-tE_{GS}(H^{3L}_\omega)} + 4\mathrm{e}^{-\zeta t}) \quad \text{using lemmas 4.4 to 4.6} \\ &\leqslant \mathrm{e}^{-\epsilon(\log t)^{-2}} + \mathrm{e}^{t(E_0-\zeta+\epsilon(\log t)^{-2}))} + \mathrm{e}^{\epsilon t(\log t)^{-2}}\mathbb{P}\left[E_{GS}(H^{3L}_\omega) < E_0 + 2\epsilon(\log t)^{-2}\right],\end{aligned}$$

using lemma 4.7. For the first two terms in this sum we have

$$\lim_{t\to\infty} \mathrm{e}^{-\epsilon(\log t)^{-2}} + \mathrm{e}^{t(E_0-\zeta+(\log t)^{-2}))} = 0.$$

For the third term, noting that $\mathrm{e}^{\epsilon t(\log t)^{-2}} \leqslant \mathrm{e}^{\epsilon L/\zeta}$ and that for large $L = \lceil\zeta t\rceil$ we have

$$2\epsilon(\log t)^{-2} \leqslant 2\epsilon(\log L/\zeta)^{-2} \leqslant 4\epsilon(\log 3L)^{-2}$$

yields the result. ■

It is not hard to see that (4.13) implies (4.2) for every $\eta > 2$, so that theorem 4.1 is a consequence of condition (4.12).

4.2. **Reduction to a smaller scale.** In the following lemma we trade energy for probability. The *IMS localization formula* (proposition 3.6) furnishes a crucial ingredient of the proof.

Proposition 4.9. *For every $\epsilon > 0$ there exists $L^* > 1$ so that for any $L > L^*$ and $r = \epsilon^{-1/2} \log L$,*

$$\mathbb{P}\left(E_{GS}(H_\omega^L) \leqslant E_0 + \frac{4\epsilon}{(\log L)^2}\right) \leqslant \mathrm{e}^{\sqrt{\epsilon} r} k^{\mathrm{e}^{\sqrt{\epsilon} r}} \mathbb{P}\left(E_{GS}(H_\omega^{2r}) \leqslant E_0 + \frac{4 + C_{IMS}}{r^2}\right). \tag{4.14}$$

Proof. Assume both

$$\epsilon^{-1/2} \log(L) = r \tag{4.15}$$

and

$$E_{GS}(H_\omega^L) \leqslant E_0 + \frac{4\epsilon}{(\log L)^2} \leqslant E_0 + \frac{4}{r^2}. \tag{4.16}$$

Let $\{\eta_{a,r}^2\}$ be the family of spherically symmetric functions on $\mathcal{T}^L$ given by proposition 3.6. They satisfy

$$\sum_a \eta_{a,r}^2(v) = 1 \text{ on } \mathcal{T}^L$$

and

$$\mathcal{S}_{a,r} := \operatorname{supp} \eta_{a,r} \subset \{mr - r < |v| < mr + r\} \subset \mathcal{T}^L.$$

If φ_{GS}^L is the normalized the ground state of H_ω^L, then the IMS formula and $\sum \|\varphi_{GS}^L \eta_{l,m}\|_2^2 = 1$ yields

$$\begin{aligned}
\langle \varphi_{GS}^L, H_\omega^L \varphi_{GS}^L \rangle &\geqslant \sum_a E_{GS}(H_\omega^L | \mathcal{S}_{a,r}) \|\varphi_{GS}^L \eta_{a,r}\|_2^2 - \frac{C_{IMS}}{r^2} \|\varphi_{GS}^L\|_2^2 \\
&\geqslant \min_a E_{GS}(H_\omega^L | \mathcal{S}_{a,r}) - \frac{C_{IMS}}{r^2} \|\varphi_{GS}^L\|_2^2.
\end{aligned}$$

From (4.16) we deduce then

$$\min_a E_{GS}(H_\omega^L | \mathcal{S}_{a,r}) \leqslant E_0 + \frac{4 + C_{IMS}}{r^2}$$

and thus

$$\mathbb{P}\left(E_{GS}(H_\omega^L) \leqslant E_0 + \frac{4\epsilon}{(\log L)^2}\right) \leqslant \sum_a \mathbb{P}\left(E_{GS}(H_\omega^L | \mathcal{S}_{a,r}) \leqslant E_0 + \frac{4 + C_{IMS}}{r^2}\right).$$

Note that, using again proposition 3.6, the support $\mathcal{S}_{a,r}$ is a disjoint union of finite subtrees of length at most $2r$. We write this as

$$\mathcal{S}_{a,r} \subset \biguplus_{|v| = l(a,r)} \mathcal{T}_v^{2r},$$

for some $l(a,r)$ and where $\uplus$ denotes disjoint union, and thus

$$E_{GS}(H^L_\omega|_{\mathcal{S}_{a,r}}) \leqslant \min_{|v|=l(a,r)} E_{GS}(H^L_\omega|\mathcal{T}^{2r}_v).$$

The right hand side of this equation is the minimum of a collection of independent, identically distributed random variables. We deduce that

$$\begin{aligned} &\mathbb{P}\left(E_{GS}(H^L_\omega|\mathcal{S}_{a,r}) \leqslant E_0 + \frac{1+C_{IMS}}{r^2}\right) \\ &\leqslant \#\{v : |v| = l(a,r)\}\mathbb{P}\left(E_{GS}(H^L_\omega|\mathcal{T}^{2r}) \leqslant E_0 + \frac{4+C_{IMS}}{r^2}\right) \\ &\leqslant k^L \mathbb{P}\left(E_{GS}(H^{2r}_\omega) \leqslant E_0 + \frac{4+C_{IMS}}{r^2}\right). \end{aligned}$$

To end the proof, note from (4.15) that

$$L = \mathrm{e}^{\sqrt{\epsilon} r}.$$

and plugging this into

$$\mathbb{P}\left(E_{GS}(H^L_\omega) \leqslant E_0 + \frac{4\epsilon}{(\log L)^2}\right) \leqslant L k^L \mathbb{P}\left(E_{GS}(H^{2r}_\omega) \leqslant E_0 + \frac{4+C_{IMS}}{r^2}\right) \tag{4.17}$$

yields the result.

∎

We state the main probability estimate, which we will prove in the next section.

Theorem 4.10. *For every $\beta' > 0$ there exists some $\epsilon_{\beta'} > 0$ and $L^* > 1$ so that for any $L > L^*$,*

$$\mathbb{P}\left(E_{GS}(H^L_\omega) \leqslant E_0 + \beta' L^{-2}\right) \leqslant \exp(-\exp(\epsilon_{\beta'} L)). \tag{4.18}$$

We first state and prove the following important corollary.

Corollary 4.11. *For any $\epsilon > 0$ small enough and any $\zeta > 0$ there exists some $L^* > 1$ such that for all $L > L^*$*

$$\mathbb{P}\left(E_{GS}(H^L_\omega) \leqslant E_0 + \frac{4\epsilon}{(\log L)^2}\right) \leqslant \mathrm{e}^{-\zeta L}.$$

In particular, condition (4.12) of proposition 4.8 holds.

Proof. Let $\beta' > 4(4 + C_{IMS})$ and $r = \epsilon^{-1/2}\log(L)$ large enough. Then, using the bound given by theorem 4.10, which we assume to hold, we get that

$$\mathbb{P}\left(E_{GS}(H^{2r}_\omega) \leqslant E_0 + \frac{4+C_{IMS}}{r^2}\right) \leqslant \exp(-\exp(\epsilon_{\beta'} r)) \tag{4.19}$$

for some $\epsilon_{\beta'} > 0$, independent of r.

Let $0 < \epsilon < \epsilon_{\beta'}^2$. We estimate, using proposition 4.9 and (4.19),

$$\begin{aligned}\mathbb{P}\left(E_{GS}(H_\omega^L) \leqslant E_0 + \frac{4\epsilon}{(\log L)^2}\right) &\leqslant \mathrm{e}^{\sqrt{\epsilon}r}\exp(\log(k)\mathrm{e}^{\sqrt{\epsilon}r})\exp(-\exp(\epsilon_{\beta'}r))\\ &\leqslant \exp\bigl(\sqrt{\epsilon}r + \log(k)\mathrm{e}^{\sqrt{\epsilon}r} - \mathrm{e}^{\epsilon_{\beta'}r}\bigr)\\ &\leqslant \exp\bigl(-\mathrm{e}^{\epsilon_{\beta'}r}/2\bigr) = \exp(-L^{\epsilon_{\beta'}/\sqrt{\epsilon}}/2),\end{aligned}$$

for r large enough. This finishes the proof. ■

5. Main probability estimate

We remind the reader that

$$-\Delta_{\mathcal{B}} := k + 1 - A_{\mathcal{B}}$$

where $A_{\mathcal{B}}$ is the adjacency matrix of the infinite Bethe lattice $\mathcal{B}$ with symmetric spectrum $\sigma(A_{\mathcal{B}}) = [-2\sqrt{k}, 2\sqrt{k}]$. Thus, the Anderson Hamiltonian H_ω defined by (1.5) satisfies

$$H_\omega = k + 1 - A_{\mathcal{B}} + V_\omega.$$

We introduce the restriction of $A_{\mathcal{B}}$ to the finite rooted tree $\mathcal{T}^L$, which we denote by $A^{(L)}$. Note that it is also the adjacency matrix of $\mathcal{T}^L$. Any question about the ground state energy $E_{GS}(H_\omega^L)$ can be restated in terms of the principal eigenvalue $\Lambda_\omega^{(L)}$ of the operator $A_\omega^{(L)} = A^{(L)} - V_\omega^{(L)}$, which we define as

$$\Lambda_\omega^{(L)} := \sup_{\|\varphi\|_2=1} \langle \varphi, A_\omega^{(L)}\varphi\rangle = k + 1 - E_{GS}(H_\omega^L).$$

We have indeed for $L \in \mathbb{N}$ and $\beta > 0$ the equivalence

$$E_{GS}(H_\omega^L) \leqslant E_0 + \beta L^{-2} \iff \Lambda_\omega^{(L)} \geqslant 2\sqrt{k} - \beta L^{-2}.$$

If we take $\beta < \sqrt{k}\pi^2$ this inequality almost surely does not hold (trivial and obviously not very useful for our purposes). Then, we restate theorem 4.10 as follows.

Theorem 5.1. *For every $\beta > 0$ there exists some $\epsilon_\beta > 0$, $L^* > 1$ so that for any $L > L^*$,*

$$\Lambda_\omega^{(L)} < 2\sqrt{k} - \beta L^{-2}$$

with probability at least

$$1 - \mathrm{e}^{\mathrm{e}^{-\epsilon_\beta L}}.$$

This section will be devoted to the proof of theorem 5.1. Note that it furnishes the lower bound of theorem 1.5.

5.1. **Cutoffs in energy and space.** We claim first that, in order to attain an energy $E_0 + O(L^{-2})$ close to the bottom of the spectrum of H^L_ω (i.e. the top of the spectrum of $A^{(L)}_\omega$), a state must have both low kinetic energy and its potential energy close to the bottom of the spectrum. This will force the potential energy to deviate considerably from its mean, see proposition 5.4, which happens only with double exponentially small probability, see proposition 5.5.

To exploit the low energy of the states considered, we cut off all energies above a threshold. We implement this with the spectral projectors

$$\Pi^{(L)}_E \colon \ell^2(\mathcal{T}^L) \to \ell^2(\mathcal{T}^L)$$
$$\Pi^{(L)}_E \varphi = \mathbf{1}_{[E,+\infty)}(A^{(L)})\varphi = \sum_{\lambda^L_{v,j,m} \geqslant E} \langle \Psi^L_{v,j,m}, \varphi\rangle \Psi^L_{v,j,m},$$

where $E \in \mathbb{R}$ and the sum is taken over L-admissible indices (v, j, m) with eigenvalue bounded below by E, see lemma 3.2.

Recall that at the beginning of section 3 we introduced a vertex $*$ and the notation $\mathcal{T}^L_*$. We used them to index the eigenvalues and eigenfunctions on the tree, see lemma 3.2.

Definition 5.2. For every $v \in \mathcal{T}^{L-1}_*$, define the orthogonal spectral projectors

$$P_v := \sum_{j \in J_v} P_{v,j} \tag{5.1}$$

using the notation from (3.4).

Remark 5.3. Here are some properties of these projectors. Let $v \in \mathcal{T}^{L-1}_*$. Then

- If $\chi_v = \mathbf{1}_{\mathcal{T}^L_v}$ is the characteristic function of the subtree $\mathcal{T}^L_v$, then for any $w \in \mathcal{T}^{L-1}_v$,
$$P_w = P_w \chi_v = \chi_v P_w.$$
In particular $P_v = P_v \chi_v = \chi_v P_v$.
- If we denote by $\operatorname{supp}\varphi$ the support of $\varphi \in \ell^2(\mathcal{T}^L)$, then for any $w \in \mathcal{T}^{L-1}_* \setminus \mathcal{T}^{L-1}_v$
$$\operatorname{supp}\big(P_v\varphi\big) \cap \operatorname{supp}\big(P_w\varphi\big) = \varnothing.$$

Given $\delta \in (0, 1)$, the truncated spectral projector $\tilde\Pi^{(L)}_{E^{(L)}_\beta}$, see (3.8), can be written with this notation as

$$\tilde\Pi^{(L)}_E = \sum_{v \in \mathcal{T}^{L-1}_*} T_{|v|,\delta} P_v \Pi^{(L)}_E. \tag{5.2}$$

We note, for further use, that for any $v \in \mathcal{T}^{L-1}_*$,

$$\tilde\Pi^{(L)}_E P_v = T_{|v|,\delta} P_v \Pi^{(L)}_E. \tag{5.3}$$

This is easily seen using the commutativity and orthogonality of the spectral projectors. Using lemma 3.3, we also note that if $|v| > (1 - \frac{1}{\beta+1})(L+1)$ then

$$P_v \Pi_E^{(L)} = 0. \tag{5.4}$$

We finally introduce a notation for the centered potential:

$$\overline{V}_\omega^{(L)} := V_\omega^{(L)} - \bar{\omega}\mathbf{1}_{\mathcal{T}^L}, \tag{5.5}$$

where $\bar{\omega}$ is the expected value of the potential. We remind that the quantity $E_\beta^{(L)} := 2\sqrt{k}\cos\big(\frac{(\beta+1)\pi}{L+1}\big)$ was introduced in lemma 3.3. Let us now state the proposition.

Proposition 5.4. *Let $\beta' > 0$. For every $\beta \gg \beta'$ large enough, there exists some $\delta = \delta_\beta > 0$ and $L^* > 1$, so that for any $L > L^*$, then, the following inequality holds:*

$$\mathbb{P}\big(\Lambda_\omega^{(L)} \geqslant 2\sqrt{k} - \beta' L^{-2}\big) \leqslant \mathbb{P}\Big(\sup_{\|\varphi\|_2 \leqslant 1} \big|\langle \tilde{V}_\omega^{(L)}\varphi, \varphi\rangle\big| \geqslant \frac{\bar{\omega}}{16}\Big),$$

where we have introduced the notation

$$\tilde{V}_\omega^{(L)} := (\tilde{\Pi}^{(L)}_{E_\beta^{(L)}})^* \overline{V}_\omega^{(L)} \tilde{\Pi}^{(L)}_{E_\beta^{(L)}},$$

with $\tilde{\Pi}_E^{(L)}$ defined as in (5.2).

The key estimate is then given by the following proposition.

Proposition 5.5. *For any $\beta > 0$ large enough, let $\delta = \delta_\beta > 0$ given by proposition 5.4. Then, for L large enough,*

$$\mathbb{P}\Big(\sup_{\|\varphi\|_2 \leqslant 1} \big|\langle \tilde{V}_\omega^{(L)}\varphi, \varphi\rangle\big| \geqslant \frac{\bar{\omega}}{16}\Big) \leqslant \exp\big(-C_{k,\tilde{\omega}_+,\bar{\omega},\beta} k^{\delta_\beta L}\big).$$

Let us first prove proposition 5.4. We thereby reduce theorem 5.1 to proposition 5.5. The proof of proposition 5.5 is at the very end of this section. It hinges upon a series of lemmata and propositions which occupy the rest of this paper.

Proof of proposition 5.4. Fix a realisation ω of the random potential with the property

$$\Lambda_\omega^{(L)} \geqslant 2\sqrt{k} - \beta' L^{-2}.$$

Then, there exists a $\varphi \in \ell^2(\mathcal{T}^L)$ with $\|\varphi\|_2 = 1$ such that

$$\langle A_\omega^{(L)}\varphi, \varphi\rangle \geqslant 2\sqrt{k} - \beta' L^{-2},$$

or, equivalently,

$$\langle (2\sqrt{k} - A^{(L)})\varphi, \varphi\rangle + \langle V_\omega^{(L)}\varphi, \varphi\rangle \leqslant \beta' L^{-2},$$

using $\|\varphi\|_2 = 1$. Note that the principal eigenvalue of $A^{(L)}$ is smaller than $2\sqrt{k}$. Thus, both $2\sqrt{k} - A^{(L)}$ and $V_\omega^{(L)}$ are non-negative operators. This implies that we have both

$$\langle (2\sqrt{k} - A^{(L)})\varphi, \varphi\rangle \leqslant \beta' L^{-2} \tag{5.6}$$

and

$$\langle V_\omega^{(L)}\varphi, \varphi\rangle \leqslant \beta' L^{-2}. \tag{5.7}$$

We now proceed as follows. In a first step, we introduce the energy cutoff $\Pi^{(L)}_{E_\beta^{(L)}}$ into (5.7). Here, (5.6) tells us how to choose β in order to keep the truncated version of (5.7) powerful enough. In a second step, we bring the spatial cutoff in $\tilde{\Pi}^{(L)}_{E_\beta^{(L)}}$ into play. This time, we have to choose $\delta > 0$ small enough, depending on β.

For the first step, let us write

$$\amalg^{(L)}_{E_\beta^{(L)}} := \mathbf{1}_{\ell^2(\mathcal{T}^L)} - \Pi^{(L)}_{E_\beta^{(L)}} \tag{5.8}$$

and $\omega_+ := \|V_\omega\|_\infty$. Then, we find that

$$\begin{aligned}\langle V_\omega^{(L)}\varphi, \varphi\rangle &= \langle V_\omega^{(L)}\Pi^{(L)}_{E_\beta^{(L)}}\varphi, \Pi^{(L)}_{E_\beta^{(L)}}\varphi\rangle + 2\Re\langle V_\omega^{(L)}\Pi^{(L)}_{E_\beta^{(L)}}\varphi, \amalg^{(L)}_{E_\beta^{(L)}}\varphi\rangle + \\ &\quad + \langle V_\omega^{(L)}\,\amalg^{(L)}_{E_\beta^{(L)}}\,\varphi, \amalg^{(L)}_{E_\beta^{(L)}}\varphi\rangle \\ &\geqslant \langle V_\omega^{(L)}\Pi^{(L)}_{E_\beta^{(L)}}\varphi, \Pi^{(L)}_{E_\beta^{(L)}}\varphi\rangle - 2\omega_+\|\Pi^{(L)}_{E_\beta^{(L)}}\varphi\|_2\|\amalg^{(L)}_{E_\beta^{(L)}}\varphi\|_2.\end{aligned}$$

This, (5.7) and $\|\Pi^{(L)}_{E_\beta^{(L)}}\varphi\|_2 \leqslant 1$ imply that

$$\langle V_\omega^{(L)}\Pi^{(L)}_{E_\beta^{(L)}}\varphi, \Pi^{(L)}_{E_\beta^{(L)}}\varphi\rangle \leqslant \beta' L^{-2} + 2\omega_+\|\amalg^{(L)}_{E_\beta^{(L)}}\varphi\|_2.$$

We use now def. (5.5) in order to center the random variables so that their mean is zero. This gives,

$$\langle \overline{V}_\omega^{(L)}\Pi^{(L)}_{E_\beta^{(L)}}\varphi, \Pi^{(L)}_{E_\beta^{(L)}}\varphi\rangle \leqslant \beta' L^{-2} + 2\omega_+\|\amalg^{(L)}_{E_\beta^{(L)}}\varphi\|_2 - \bar{\omega}\|\Pi^{(L)}_{E_\beta^{(L)}}\varphi\|_2^2. \tag{5.9}$$

Using the non-negativity of the operator $2\sqrt{k} - A^{(L)}$, we see that

$$\begin{aligned}\left\langle (2\sqrt{k} - A^{(L)})\,\amalg^{(L)}_{E}\,\varphi, \amalg^{(L)}_{E}\varphi\right\rangle &= \sum_{\lambda^L_{v,j,m} \geqslant E} (2\sqrt{k} - \lambda^L_{v,j,m})|\langle \Psi^L_{v,j,m}, \varphi\rangle|^2 \\ &\leqslant \sum_{\lambda^L_{v,j,m} \geqslant -\infty} (2\sqrt{k} - \lambda^L_{v,j,m})|\langle \Psi^L_{v,j,m}, \varphi\rangle|^2 \\ &= \langle (2\sqrt{k} - A^{(L)})\varphi, \varphi\rangle.\end{aligned}$$

We use this with (5.6) to deduce that

$$\big\langle (2\sqrt{k} - A^{(L)})\, \Pi^{(L)}_{E^{(L)}_\beta}\, \varphi, \Pi^{(L)}_{E^{(L)}_\beta}\varphi \big\rangle \leqslant \beta' L^{-2}$$

and thus, using the definitions (5.8) and (3.7), this implies that

$$\big(2\sqrt{k} - E^{(L)}_\beta\big) \big\|\Pi^{(L)}_{E^{(L)}_\beta}\varphi\big\|^2 \leqslant \beta' L^{-2}.$$

Hence, using $\cos(x) \geqslant 1 - x^2/2$,

$$2\sqrt{k}\frac{(\beta+1)^2\pi^2}{(L+1)^2}\big\|\Pi^{(L)}_{E^{(L)}_\beta}\varphi\big\|^2 \leqslant \beta' L^{-2},$$

and thus

$$\big\|\Pi^{(L)}_{E^{(L)}_\beta}\varphi\big\| \leqslant \frac{L+1}{L(\beta+1)\pi}\sqrt{\frac{\beta'}{2\sqrt{k}}} \leqslant \frac{2}{(\beta+1)\pi}\sqrt{\frac{\beta'}{2\sqrt{k}}}.$$

From now on we assume we have chosen β so large that

$$\frac{1}{(\beta+1)\pi}\sqrt{\frac{2\beta'}{\sqrt{k}}} < \min\{1/\sqrt{2}, \bar{\omega}/(8\omega_+)\}.$$

This choice implies

$$1/2 \leqslant \|\Pi^{(L)}_{E^{(L)}_\beta}\varphi\|_2^2 \leqslant 1 \quad \text{and} \quad 2\omega_+ \|\Pi^{(L)}_{E^{(L)}_\beta}\varphi\|_2 \|\Pi^{(L)}_{E^{(L)}_\beta}\varphi\|_2 \leqslant \bar{\omega}/4.$$

We deduce from (5.9) that, for $L^2 > 8\beta'/\bar{\omega}$,

$$\langle \overline{V}^{(L)}_\omega \Pi^{(L)}_{E^{(L)}_\beta}\varphi, \Pi^{(L)}_{E^{(L)}_\beta}\varphi\rangle \leqslant -\frac{\bar{\omega}}{8}.$$

For the second step, let us now replace $\Pi^{(L)}_{E^{(L)}_\beta}$ by $\tilde{\Pi}^{(L)}_{E^{(L)}_\beta}$. We denote $\tilde{\omega}_+ := \|V_\omega - \bar{\omega}\|_\infty$. Choose $0 < \delta < 1$ satisfying

$$\sqrt{2}\pi\delta^{3/2}(\beta+1)^3 \leqslant \bar{\omega}/(32\tilde{\omega}_+).$$

Then, proposition 3.10 tells us that

$$2\tilde{\omega}_+ \|\Pi^{(L)}_{E^{(L)}_\beta}\varphi - \tilde{\Pi}^{(L)}_{E^{(L)}_\beta}\varphi\|_2 \leqslant \frac{\bar{\omega}}{16}\|\Pi^{(L)}_{E^{(L)}_\beta}\varphi\|_2 \leqslant \frac{\bar{\omega}}{16}.$$

Using this and $\|\tilde{\Pi}^{(L)}_{E^{(L)}_\beta}\varphi\|_2 \leqslant 1$, we deduce

$$\begin{aligned}
\langle \overline{V}^{(L)}_\omega \tilde{\Pi}^{(L)}_{E^{(L)}_\beta}\varphi, \tilde{\Pi}^{(L)}_{E^{(L)}_\beta}\varphi\rangle &= \langle \overline{V}^{(L)}_\omega \Pi^{(L)}_{E^{(L)}_\beta}\varphi, \Pi^{(L)}_{E^{(L)}_\beta}\varphi\rangle + \langle \overline{V}^{(L)}_\omega (\tilde{\Pi}^{(L)}_{E^{(L)}_\beta} - \Pi^{(L)}_{E^{(L)}_\beta}\varphi, \Pi^{(L)}_{E^{(L)}_\beta}\varphi\rangle + \\
&\quad + \langle \overline{V}^{(L)}_\omega \tilde{\Pi}^{(L)}_{E^{(L)}_\beta}\varphi, (\tilde{\Pi}^{(L)}_{E^{(L)}_\beta} - \Pi^{(L)}_{E^{(L)}_\beta}\varphi\rangle \\
&\leqslant -\frac{\bar{\omega}}{8} + 2\tilde{\omega}_+ \|\Pi^{(L)}_{E^{(L)}_\beta}\varphi - \tilde{\Pi}^{(L)}_{E^{(L)}_\beta}\varphi\|_2 \leqslant -\frac{\bar{\omega}}{16}.
\end{aligned}$$

We have thereby proven that, for L large enough,

$$\big\{\omega : \Lambda_\omega^{(L)} \geqslant 2\sqrt{k} - \beta' L^{-2}\big\} \subseteq \Big\{\omega : \sup_{\|\varphi\|_2 \leqslant 1} \big|\langle \overline{V}_\omega^{(L)} \tilde{\Pi}^{(L)}_{E_\beta^{(L)}} \varphi, \tilde{\Pi}^{(L)}_{E_\beta^{(L)}} \varphi\rangle\big| \geqslant \frac{\bar{\omega}}{16}\Big\}.$$

This proves proposition 5.4. ■

The spatial truncation we introduced into $\tilde{V}_\omega^{(L)}$ is adapted to the energy decomposition of the argument. More specifically, eigenfunctions with different anchors are treated differently. We therefore split the probability into different components depending on the anchors, see lemma 5.7.

We now prove a simple lemma.

Lemma 5.6. *Let $L \geqslant 1$ and $\varphi \in \ell^2(\mathcal{T}^L)$. Then:*

$$\sum_{v \in \mathcal{T}_*^{L-1}} \|\chi_v \varphi\|_2^2 \leqslant (L+1)\|\varphi\|_2^2.$$

Proof. We have $v \in \mathcal{T}_*^L$ and $w \in \mathcal{T}_v^L$ if and only if v lies in the shortest path from $*$ to w, which we write $v \in [*, w]$. Thus,

$$\sum_{v \in \mathcal{T}_*^{L-1}} \|\chi_v \varphi\|_2^2 \leqslant \sum_{v \in \mathcal{T}_*^{L}} \|\chi_v \varphi\|_2^2 = \sum_{v \in \mathcal{T}_*^{L}} \sum_{w \in \mathcal{T}_*^{L}} |\varphi(w)|^2 = \sum_{w \in \mathcal{T}_*^{L}} \sum_{v \in [*, w]} |\varphi(w)|^2.$$

Now it suffices to remark that the maximum length of any shortest path from $*$ to any point of the tree is smaller or equal to $L+1$. ■

We introduce the following quantity. For any given $L \geqslant 1$, $v \in \mathcal{T}^L$, and $w \in \mathcal{T}_v^L$, define

$$\Xi(L, v, w) := \frac{1}{2}(L+1)^{-1} k^{-(|w|-|v|)/2}. \tag{5.10}$$

We also adopt the convention $0/0 = 0$.

Lemma 5.7. *Let $L \geqslant 1$, $\kappa > 0$, $B^{(L)}$ the unit ball of $\ell^2(\mathcal{T}^L)$ and $\mathcal{E}, \mathcal{F} \subseteq B^{(L)}$. Then the following inequality holds true*

$$\mathbb{P}\Big(\sup_{\varphi \in \mathcal{E}, \psi \in \mathcal{F}} |\langle \tilde{V}_\omega^{(L)} \varphi, \psi\rangle| > \kappa\Big) \leqslant \sum_{v \in \mathcal{T}_*^{L-1}} \sum_{w \in \mathcal{T}_v^{L-1}} \mathbb{P}\Big(\sup_{\varphi \in \mathcal{E}, \psi \in \mathcal{F}} \frac{|\langle \tilde{V}_\omega^{(L)} P_v \varphi, P_w \psi\rangle|}{\|P_v \varphi\|_2 \|P_w \psi\|_2} > \kappa \Xi(L, v, w)\Big).$$

Proof. First let us remark that for any $\varphi \in \ell^2(\mathcal{T}^L)$ we have $\varphi = \sum_{v\in\mathcal{T}_*^{L-1}} P_v\varphi$ and thus, using remark 5.3, we see that

$$\sup_{\varphi\in\mathcal{E},\psi\in\mathcal{F}} |\langle \tilde{V}_\omega^{(L)}\varphi,\psi\rangle| \leqslant \sup_{\varphi\in\mathcal{E},\psi\in\mathcal{F}} \Big|\Big\langle \tilde{V}_\omega^{(L)} \sum_{v\in\mathcal{T}_*^{L-1}} P_v\varphi, \sum_{w\in\mathcal{T}_v^{L-1}} P_w\psi\Big\rangle\Big|$$

$$\leqslant 2 \sup_{\varphi\in\mathcal{E},\psi\in\mathcal{F}} \sum_{v\in\mathcal{T}_*^{L-1}} \sum_{w\in\mathcal{T}_v^{L-1}} |\langle \tilde{V}_\omega^{(L)} P_v\varphi, P_w\psi\rangle|. \tag{5.11}$$

The proof now proceeds as follows. In order to prove the inequality $\mathbb{P}(A) \leqslant \sum_j \mathbb{P}(B_j)$, we will show that $\bigcap_j B_j^c \subseteq A^c$. To do so, fix ω with the following property: for all $\varphi\in\mathcal{E}, \psi\in\mathcal{F}, v\in\mathcal{T}_*^{L-1}, w\in\mathcal{T}_v^{L-1}$,

$$|\langle \tilde{V}_\omega^{(L)} P_v\varphi, P_w\psi\rangle| \leqslant \kappa\Xi(L,v,w)\|P_v\varphi\|_2\|P_w\psi\|_2. \tag{5.12}$$

Then, for any $\varphi\in\mathcal{E},\psi\in\mathcal{F}$, we can use assumption (5.12) to bound

$$2\sum_{v\in\mathcal{T}_*^{L-1}} \sum_{w\in\mathcal{T}_v^{L-1}} |\langle \tilde{V}_\omega^{(L)} P_v\varphi, P_w\psi\rangle|$$

$$\leqslant \kappa(L+1)^{-1} \sum_{v\in\mathcal{T}_*^{L-1}} \|P_v\varphi\|_2 \sum_{w\in\mathcal{T}_v^{L-1}} k^{-(|w|-|v|)/2}\|P_w\psi\|_2$$

$$\leqslant \kappa(L+1)^{-1}\Big(\sum_{v\in\mathcal{T}_*^{L-1}} \|P_v\varphi\|_2^2\Big)^{1/2}\Big(\sum_{v\in\mathcal{T}_*^{L-1}} B_v^2\Big)^{1/2}, \tag{5.13}$$

where we have used Cauchy–Schwarz in the last line and furthermore defined

$$B_v := \sum_{w\in\mathcal{T}_v^{L-1}} k^{-(|w|-|v|)/2}\|P_w\psi\|_2.$$

Using Cauchy–Schwarz and $P_w = P_w\chi_v$ (remark 5.3) in this last quantity, we see that

$$\sum_{v\in\mathcal{T}_*^{L-1}} B_v^2 \leqslant \sum_{v\in\mathcal{T}_*^{L-1}} \sum_{w\in\mathcal{T}_v^{L-1}} k^{-(|w|-|v|)} \sum_{w\in\mathcal{T}_v^{L-1}} \|P_w\chi_v\psi\|_2^2.$$

We can use polar coordinates to estimate the first sum over w. Indeed, note that, for $n \geqslant |v|$, the number of elements of the sphere $\{w\in\mathcal{T}_v^{L-1} : |w| = n\}$ is bounded by $k^{n-|v|}$. Thus, $\sum_{w\in\mathcal{T}_v^{L-1}} k^{-(|w|-|v|)} \leqslant \sum_{n=|v|}^{L} 1 \leqslant L+1$. With the orthogonality of the P_w and lemma 5.6, we see that

$$\sum_{v\in\mathcal{T}_*^{L-1}} B_v^2 \leqslant (L+1)\sum_{v\in\mathcal{T}_*^{L-1}} \|\chi_v\psi\|_2^2 \leqslant (L+1)^2\|\psi\|_2^2.$$

We insert this bound into (5.13), apply $\sum_{v\in\mathcal{T}_*^{L-1}}\|P_v\varphi\|_2^2 = \|\varphi\|_2^2$ once more, and plug the result into (5.11), to see that assuming (5.12) for all $\varphi\in\mathcal{E},\psi\in\mathcal{F}, v\in\mathcal{T}_*^{L-1}, w\in\mathcal{T}_v^{L-1}$ leads to

$$\sup_{\varphi\in\mathcal{E},\psi\in\mathcal{F}} |\langle \tilde{V}_\omega^{(L)}\varphi,\psi\rangle| \leqslant \kappa.$$

This finishes the proof. ∎

5.2. **The epsilon-net argument.** The next problem we deal with is the fact that the ground state of H_ω is random. This is reflected in proposition 5.4 as follows. The supremum is inside the probability, so that φ and ψ are adapted to ω. In order to remove the supremum, we approximate the ball with a finite ϵ-net and show, with a union bound, that it suffices to consider only the elements of the net. The following two lemmas implement a classical ϵ-net argument.

Lemma 5.8. *Let $v \in \mathcal{T}_*^{L-1}$ and $B_v^{(L)}$ be the unit ball of $\operatorname{Im} P_v = P_v\big(\ell^2(\mathcal{T}^L)\big)$. Then there exists a finite set $\mathcal{M}_v \subseteq B_v^{(L)}$ so that for any $\varphi \in B_v^{(L)}$ there exists some $\tilde\varphi \in \mathcal{M}_v$ so that*

$$\|\varphi - \tilde\varphi\|_2 \leqslant 1/8$$

and furthermore

$$\#\mathcal{M}_v \leqslant 32^{k(L-|v|)}.$$

Proof. The existence of an ϵ-covering of the unit ball of a finite dimensional space having a cardinality smaller than $(4/\epsilon)^d$, where d is the dimension of the space, is a well-known fact, which can be established by scaling and volume counting, see for example [Pis99, formula (4.22)]. It suffices now to remark from the definition (5.1) that

$$\dim \operatorname{Im} P_v \leqslant k(L-|v|)$$

to establish the result. ∎

Lemma 5.9. *Let a scale $L \geqslant 1$, a constant $\kappa > 0$, $B^{(L)}$ the unit ball of $\ell^2(\mathcal{T}^L)$ and sets $\mathcal{E}, \mathcal{F} \subseteq B^{(L)}$ be given. Further, we fix, for each $v \in \mathcal{T}_*^{L-1}$, some set $\mathcal{M}_v$ given by lemma 5.8. Then, the following inequality*

$$\begin{aligned}\mathbb{P}\Big(\sup_{\varphi\in\mathcal{E},\psi\in\mathcal{F}} \frac{|\langle \tilde V_\omega^{(L)} P_v\varphi, P_w\psi\rangle|}{\|P_v\varphi\|_2\|P_w\psi\|_2} > \kappa\Xi(L,v,w)\Big) \\ \leqslant \sum_{i\in\mathbb{N}} \sum_{\tilde\varphi\in\mathcal{M}_v,\tilde\psi\in\mathcal{M}_w} \mathbb{P}\big(|\langle \tilde V_\omega^{(L)} P_v\tilde\varphi, P_w\tilde\psi\rangle| > 2^i\kappa\Xi(L,v,w)\|P_v\tilde\varphi\|_2\|P_w\tilde\psi\|_2\big)\end{aligned}$$

holds for all $v \in \mathcal{T}_*^{L-1}$, $w \in \mathcal{T}_v^{L-1}$.

Proof. Fix $v \in \mathcal{T}_*^{L-1}$, $w \in \mathcal{T}_v^{L-1}$. Using the fact that $P_v^2 = P_v$, we see that

$$\langle \tilde V_\omega^{(L)} P_v\varphi, P_w\psi\rangle = \Big\langle \tilde V_\omega^{(L)} P_v \frac{P_v\varphi}{\|P_v\varphi\|_2}, P_w\frac{P_w\psi}{\|P_w\psi\|_2}\Big\rangle \|P_v\varphi\|_2\|P_w\psi\|_2.$$

We deduce that

$$\sup_{\varphi\in\mathcal{E},\psi\in\mathcal{F}} \frac{|\langle \tilde V_\omega^{(L)} P_v\varphi, P_w\psi\rangle|}{\|P_v\varphi\|_2\|P_w\psi\|_2} \leqslant \sup_{\varphi\in B_v^{(L)},\psi\in B_w^{(L)}} |\langle \tilde V_\omega^{(L)} P_v\varphi, P_w\psi\rangle|.$$

We remind the reader that $0/0 = 0$.

Let $\mathcal{M}_v, \mathcal{M}_w$ be the $\frac{1}{8}$-coverings of $B_v^{(L)}, B_w^{(L)}$ given by lemma 5.8, respectively. Suppose that $\kappa' > 0$ and that

$$|\langle \tilde{V}_\omega^{(L)} P_v \tilde{\varphi}, P_w \tilde{\psi} \rangle| \leqslant \frac{\kappa'}{2} \|P_v \tilde{\varphi}\|_2 \|P_w \tilde{\psi}\|_2 \tag{5.14}$$

for all $\tilde{\varphi} \in \mathcal{M}_v$, $\tilde{\psi} \in \mathcal{M}_w$. Assume furthermore that

$$|\langle \tilde{V}_\omega^{(L)} P_v \varphi, P_w \psi \rangle| \leqslant 2\kappa' \|P_v \varphi\|_2 \|P_w \psi\|_2 \tag{5.15}$$

for every $\varphi \in B_v^{(L)}$, $\psi \in B_w^{(L)}$. Using the definition of $\mathcal{M}_v$, $\mathcal{M}_w$, we see that for any $\varphi \in B_v^{(L)}$, $\psi \in B_w^{(L)}$, there exists some $\tilde{\varphi} \in \mathcal{M}_v$, $\tilde{\psi} \in \mathcal{M}_w$ such that

$$\begin{aligned} |\langle \tilde{V}_\omega^{(L)} P_v \varphi, P_w \psi \rangle| &\leqslant |\langle \tilde{V}_\omega^{(L)} P_v \tilde{\varphi}, P_w \tilde{\psi} \rangle| + |\langle \tilde{V}_\omega^{(L)} P_v \big(\tilde{\varphi} - \varphi\big), P_w \tilde{\psi} \rangle| \\ &\quad + |\langle \tilde{V}_\omega^{(L)} P_v \varphi, P_w \big(\tilde{\psi} - \psi\big) \rangle| \\ &\leqslant \kappa'/2 + 2\kappa' \|P_v\big(\tilde{\varphi} - \varphi\big)\|_2 + 2\kappa' \|P_w(\tilde{\psi} - \psi)\|_2 \\ &\leqslant \kappa'/2 + \kappa'/4 + \kappa'/4 = \kappa'. \end{aligned}$$

We deduce that if $|\langle \tilde{V}_\omega^{(L)} P_v \varphi, P_w \psi \rangle| > \kappa'$ then we cannot have both (5.14) and (5.15). We use this below with $\kappa', 2\kappa', 4\kappa', \ldots$. Thus,

$$\begin{aligned} \mathbb{P}\Big(\sup_{\varphi \in \mathcal{E}, \psi \in \mathcal{F}} \frac{|\langle \tilde{V}_\omega^{(L)} P_v \varphi, P_w \psi \rangle|}{\|P_v \varphi\|_2 \|P_w \psi\|_2} > \kappa' \Big) &\leqslant \mathbb{P}\Big(\sup_{\varphi \in B_v^{(L)}, \psi \in B_w^{(L)}} |\langle \tilde{V}_\omega^{(L)} P_v \varphi, P_w \psi \rangle| > \kappa' \Big) \\ &\leqslant \sum_{\tilde{\varphi} \in \mathcal{M}_v, \psi \in \mathcal{M}_w} \mathbb{P}\big(|\langle \tilde{V}_\omega^{(L)} P_v \tilde{\varphi}, P_w \tilde{\psi} \rangle| > \kappa' \|P_v \tilde{\varphi}\|_2 \|P_w \tilde{\psi}\|_2 \big) \\ &\quad + \mathbb{P}\Big(\sup_{\varphi \in B_v^{(L)}, \psi \in B_w^{(L)}} |\langle \tilde{V}_\omega^{(L)} P_v \varphi, P_w \psi \rangle| > 2\kappa' \Big) \\ &\leqslant \sum_{i=1}^{\infty} \sum_{\tilde{\varphi} \in \mathcal{M}_v, \tilde{\psi} \in \mathcal{M}_w} \mathbb{P}\big(|\langle \tilde{V}_\omega^{(L)} P_v \tilde{\varphi}, P_w \tilde{\psi} \rangle| > 2^i \kappa' \|P_v \tilde{\varphi}\|_2 \|P_w \tilde{\psi}\|_2 \big). \end{aligned}$$

Now we choose $\kappa' := \kappa \Xi(L, v, w)$, and lemma 5.9 is proved. ∎

5.3. **Concentration inequalities.** With the lemmata we have up to now, we can attack the probability in proposition 5.5, but we will accumulate sums over $v \in \mathcal{T}_*^{L-1}$, $w \in \mathcal{T}_v^{L-1}$, $i \geqslant 1$, $\tilde{\phi} \in \mathcal{M}_v$ and $\tilde{\psi} \in \mathcal{M}_w$. The probabilities we sum over in the end should be very small in order to get a meaningful upper bound. We estimate these probabilities in proposition 5.10, which is the main probability estimate.

We remind the reader that $\Xi(L, v, w)$ was defined in (5.10) just before lemma 5.7, and that $\tilde{V}_\omega^{(L)} = (\Pi_{E_\beta^{(L)}}^{(L)})^* \overline{V}_\omega^{(L)} \Pi_{E_\beta^{(L)}}^{(L)}$. We further recall that $\overline{V}_\omega^{(L)}$ is the centered potential, see (5.5), and that the random variables ω_v are bounded almost surely, so $|\overline{V}_\omega^{(L)}| \leqslant \tilde{\omega}_+ := \|\omega_0 - \bar{\omega}\|_\infty$ almost surely.

Proposition 5.10. *Let $L \in \mathbb{N}$, $\beta \in (0, L]$, $v \in \mathcal{T}_*^{L-1}$, $w \in \mathcal{T}_v^{L-1}$, $\delta \in (0,1)$, and $\tilde{\varphi} \in \mathcal{M}_v$, $\tilde{\psi} \in \mathcal{M}_w$. Then*

$$|\langle \tilde{V}_\omega^{(L)} P_v \tilde{\varphi}, P_w \tilde{\psi} \rangle| > \kappa \Xi(L, v, w) \big\| P_v \Pi^{(L)}_{E_\beta^{(L)}} \tilde{\varphi} \big\|_2 \big\| P_w \Pi^{(L)}_{E_\beta^{(L)}} \tilde{\psi} \big\|_2$$

holds true with probability smaller than

$$2 \exp\big(-C_{k, \tilde{\omega}_+, \beta} \kappa^2 k^{\delta L} \big).$$

Here,

$$C_{k, \tilde{\omega}_+, \beta} := (64 \tilde{\omega}_+^2 k^6 (\beta + 1)^4)^{-1} > 0.$$

To prove proposition 5.10 we will need the following two lemmata, the proofs of which are just below the proof of proposition 5.10. The first one is just an application of a well-known subgaussian estimate.

Lemma 5.11. *For all $L \geqslant 1$, $\kappa > 0$ and any φ, $\psi \in \ell^2(\mathcal{T}_L)$, we have*

$$\mathbb{P}\big(|\langle \overline{V}_\omega^{(L)} \varphi, \psi \rangle| \geqslant \kappa \big) \leqslant 2 \exp\Big(-\frac{\kappa^2}{2 \tilde{\omega}_+^2 \|\varphi\|_4^2 \|\psi\|_4^2} \Big).$$

After applying lemma 5.11, we will be interested in certain ℓ^4-norms. The following estimate is taylored to our needs.

Lemma 5.12. *For all $L \in \mathbb{N}$, $\beta \in (0, L]$, $v \in \mathcal{T}_*^{L-1}$, $x \in \mathcal{T}_v^{L-1}$ satisfying $|x| > |v|$, and $\varphi \in \ell^2(\mathcal{T}^L)$,*

$$\big\| \chi_x P_v \Pi^{(L)}_{E_\beta^{(L)}} \varphi \big\|_4^4 \leqslant \frac{8 k^6 (\beta + 1)^4}{(L+1)^2} k^{-2(|x| - |v|)} \big\| P_v \Pi^{(L)}_{E_\beta^{(L)}} \varphi \big\|_2^4$$

holds true.

We now prove proposition 5.10.

Proof of proposition 5.10. First, recall (5.3). This allows us to write

$$\begin{aligned} P_w \tilde{V}_\omega^{(L)} P_v &= P_w (\tilde{\Pi}^{(L)}_{E_\beta^{(L)}})^* \overline{V}_\omega^{(L)} \tilde{\Pi}^{(L)}_{E_\beta^{(L)}} P_v \\ &= \Pi^{(L)}_{E_\beta^{(L)}} P_w T_{|w|, \delta} \overline{V}_\omega^{(L)} T_{|v|, \delta} P_v \Pi^{(L)}_{E_\beta^{(L)}}, \end{aligned}$$

since the operators $T_{|\bullet|, \delta}$, $P_\bullet$, and $\chi_\bullet$ are self-adjoint. Furthermore, note that $T_{|w|,\delta} = T^2_{|w|,\delta}$ and recall from remark 5.3 that $P_w = P_w \chi_w$. The diagonal operators $T_{|w|,\delta}$, χ_w and $\overline{V}_\omega^{(L)}$ commute, so

$$P_w \tilde{V}_\omega^{(L)} P_v = \Pi^{(L)}_{E_\beta^{(L)}} P_w T_{|w|,\delta} \overline{V}_\omega^{(L)} \chi_w T_{|w|,\delta} T_{|v|,\delta} P_v \Pi^{(L)}_{E_\beta^{(L)}}.$$

Finally, compute $T_{|w|,\delta} T_{|v|,\delta} = T_{|w|,\delta}$. This leads us to study the quantity

$$\begin{aligned} X(x, v, w) &:= \langle \tilde{V}_\omega^{(L)} P_v \tilde{\varphi}, P_w \tilde{\psi} \rangle \\ &= \langle \overline{V}_\omega^{(L)} \chi_w T_{|w|,\delta} P_v \Pi^{(L)}_{E_\beta^{(L)}} \tilde{\varphi}, T_{|w|,\delta} P_w \Pi^{(L)}_{E_\beta^{(L)}} \tilde{\psi} \rangle, \end{aligned}$$

which is a sum of independent, bounded random variables. Note that the number of nodes in $\{x \in \mathcal{T}_w^{L-1} : |x| = |w| + \lceil \delta L \rceil\}$ is smaller than or equal to $k^{\lceil \delta L \rceil}$. Use this and lemma 5.12 to calculate

$$\begin{aligned}\big\|T_{|w|,\delta} P_w \Pi^{(L)}_{E^{(L)}_\beta} \tilde{\psi}\big\|_4^4 &= \sum_{\substack{x \in \mathcal{T}_w^{L-1} \\ |x| = |w| + \lceil \delta L \rceil}} \big\|\chi_x P_w \Pi^{(L)}_{E^{(L)}_\beta} \tilde{\psi}\big\|_4^4 \leqslant k^{\lceil \delta L \rceil} \max_{\substack{x \in \mathcal{T}_w^{L-1} \\ |x| = l}} \big\|\chi_x P_w \Pi^{(L)}_{E^{(L)}_\beta} \tilde{\psi}\big\|_4^4 \\ &\leqslant 8k^6 \frac{(\beta+1)^4}{(L+1)^2} k^{-\lceil \delta L \rceil} \big\|P_w \Pi^{(L)}_{E^{(L)}_\beta} \tilde{\psi}\big\|_2^4\end{aligned}$$

and

$$\begin{aligned}\big\|\chi_w T_{|w|,\delta} P_v \Pi^{(L)}_{E^{(L)}_\beta} \tilde{\varphi}\big\|_4^4 &= \sum_{\substack{x \in \mathcal{T}_w^{L-1} \\ |x| = |w| + \lceil \delta L \rceil}} \big\|\chi_x P_v \Pi^{(L)}_{E^{(L)}_\beta} \tilde{\varphi}\big\|_4^4 \leqslant k^{\lceil \delta L \rceil} \max_{\substack{x \in \mathcal{T}_w^{L-1} \\ |x| = l}} \big\|\chi_x P_v \tilde{\varphi}\big\|_4^4 \\ &\leqslant 8k^6 \frac{(\beta+1)^4}{(L+1)^2} k^{-2|w| - \lceil \delta L \rceil + 2|v|} \big\|P_v \Pi^{(L)}_{E^{(L)}_\beta} \tilde{\varphi}\big\|_2^4.\end{aligned}$$

With these estimations, lemma 5.11, tells us that, if $\kappa' > 0$,

$$\begin{aligned}\log\Big(\mathbb{P}\big(|X(x,v,w)| \geqslant \kappa'\big)/2\Big) &\leqslant -\frac{\kappa'^2}{2\tilde{\omega}_+^2 \big\|\chi_x P_v \Pi^{(L)}_{E^{(L)}_\beta} \tilde{\varphi}\big\|_4^2 \big\|\chi_x P_w \Pi^{(L)}_{E^{(L)}_\beta} \tilde{\psi}\big\|_4^2} \\ &\leqslant -\frac{\kappa'^2 (L+1)^2 k^{|w| - |v| + \lceil \delta L \rceil}}{16 \tilde{\omega}_+^2 k^6 (\beta+1)^4 \big\|P_v \Pi^{(L)}_{E^{(L)}_\beta} \tilde{\varphi}\big\|_2^2 \big\|P_w \Pi^{(L)}_{E^{(L)}_\beta} \tilde{\psi}\big\|_2^2}.\end{aligned}$$

We plug in

$$\kappa' = \kappa \Xi(L,v,w) \big\|P_v \Pi^{(L)}_{E^{(L)}_\beta} \tilde{\varphi}\big\|_2 \big\|P_w \Pi^{(L)}_{E^{(L)}_\beta} \tilde{\psi}\big\|_2$$

and get

$$\log\Big(\frac{1}{2}\mathbb{P}\big(|X(l,v,w)| \geqslant \kappa \Xi(L,v,w) \big\|P_v \Pi^{(L)}_{E^{(L)}_\beta} \tilde{\varphi}\big\|_2 \big\|P_w \Pi^{(L)}_{E^{(L)}_\beta} \tilde{\psi}\big\|_2\big)\Big) \leqslant -\frac{\kappa^2 k^{\lceil \delta L \rceil}}{64 \tilde{\omega}_+^2 k^6 (\beta+1)^4}.$$

This finishes the proof. ■

Proof of lemma 5.11. Fix $\varphi, \psi \in \ell^2(\mathcal{T}^L)$. The expression

$$\langle \overline{V}^{(L)}_\omega \varphi, \psi \rangle = \sum_{v \in \mathcal{T}^L} (\omega_v - \bar{\omega}) \varphi(v) \psi(v)$$

is a sum of $\#\mathcal{T}^L$ independent random variables, namely $\{(\omega_v - \bar{\omega})\varphi(v)\psi(v)\}_{v \in \mathcal{T}^L}$, all of them having mean zero. For every $v \in \mathcal{T}^L$, we have almost surely

$$|(\omega_v - \bar{\omega})\varphi(v)\psi(v)| \leqslant \tilde{\omega}_+ |\varphi(v)\psi(v)|.$$

To bound the probability in question, we use Hoeffding's inequality ([Hoe63]) and Cauchy–Schwarz:

$$\begin{aligned}\mathbb{P}\big(|\langle (V_\omega^{(L)}-\bar\omega)\varphi,\psi\rangle|\geqslant\kappa\big)&\leqslant 2\exp\Big(\frac{-2\kappa^2}{\sum_{v\in\mathcal{T}^L}(2\tilde\omega_+|\varphi(v)||\psi(v)|)^2}\Big)\\&\leqslant 2\exp\Big(\frac{-\kappa^2}{2\tilde\omega_+^2\|\varphi\|_4^2\|\psi\|_4^2}\Big).\end{aligned}$$

■

Proof of lemma 5.12. Let $\varphi\in\ell^2(\mathcal{T}^L)$. To simplify notation, we assume $\varphi\in P_v\Pi^{(L)}_{E_\beta^{(L)}}\ell^2(\mathcal{T}^L)$. For any L-admissible (v,j,m), let $\alpha_{v,j,m}$ be defined by

$$\varphi=\sum_{m=1}^{\lfloor\beta+1\rfloor}\sum_{j\in J_v}\alpha_{v,j,m}\Psi^L_{v,j,m},$$

and thus $\sum_{j,m}|\alpha_{v,j,m}|^2=\|\varphi\|_2^2$.

Using Cauchy–Schwarz,

$$\begin{aligned}\|\chi_x\varphi\|_4^4&=\sum_{w\in\mathcal{T}_x^L}\Big|\sum_{m=1}^{\lfloor\beta+1\rfloor}\sum_{j\in J_v}\alpha_{v,j,m}\Psi^L_{v,j,m}(w)\Big|^4\\&\leqslant\Big(\sum_{m=1}^{\lfloor\beta+1\rfloor}\sum_{j\in J_v}|\alpha_{v,j,m}|^2\Big)^2\sum_{w\in\mathcal{T}_x^L}\Big(\sum_{m=1}^{\lfloor\beta+1\rfloor}\sum_{j\in J_v}\big|\Psi^L_{v,j,m}(w)\big|^2\Big)^2\\&=\|\varphi\|_2^4\sum_{w\in\mathcal{T}_x^L}\Big(\sum_{m=1}^{\lfloor\beta+1\rfloor}\sum_{j\in J_v}\Big|\sum_{u\in\mathcal{T}_v^L,u\sim v}\psi^\perp_{v,j}(u)\psi^{L-|v|}_{u,m}(w)\Big|^2\Big)^2.\end{aligned}$$

Again with Cauchy–Schwarz and then with the definition (3.1) of $\psi^{L-|v|}_{u,m}$ we see

$$\begin{aligned}\sum_{j\in J_v}\Big|\sum_{u\in\mathcal{T}_v^L,u\sim v}\psi^\perp_{v,j}(u)\psi^{L-|v|}_{u,m}(w)\Big|^2&\leqslant\sum_{j\in J_v}\sum_{u\in\mathcal{T}_v^L,u\sim v}|\psi^\perp_{v,j}(u)|^2\sum_{u\in\mathcal{T}_v^L,u\sim v}|\psi^{L-|v|}_{u,m}(w)|^2\\&\leqslant\frac{2k^2}{(L+|v|-1)k^{|w|-|v|-1}}.\end{aligned}$$

We use all this in the estimate above and derive

$$\begin{aligned}\|\chi_x\varphi\|_4^4&\leqslant\frac{4k^4}{(L+|v|-1)^2}\|\varphi\|_2^4\sum_{w\in\mathcal{T}_x^L}\Big(\sum_{m=1}^{\lfloor\beta+1\rfloor}k^{-(|w|-|v|-1)}\Big)^2\\&\leqslant\frac{4k^4\lfloor\beta+1\rfloor^2}{(L+|v|-1)^2}\|\varphi\|_2^4\sum_{w\in\mathcal{T}_x^L}k^{-2(|w|-|v|-1)}\end{aligned}$$

The remaining sum can be treated with radial coordinates:

$$\sum_{w\in\mathcal{T}_x^L} k^{-2(|w|-|v|-1)} \leqslant \sum_{l=|x|}^{L} k^{l-|x|}k^{-2(l-|v|-1)} = k^{-|x|-2|v|+2}\sum_{l=|x|}^{L} k^{-l}$$
$$\leqslant k^{-|x|-2|v|+2}\frac{k^{-|x|}}{1-k^{-1}} = \frac{k^3}{k-1}k^{-2(|x|-|v|)}.$$

Since $k \geqslant 2$, we can beautify $k^3/(k-1) \leqslant 2k^2$ and get

$$\|\chi_x\varphi\|_4^4 \leqslant \frac{8k^6\lfloor\beta+1\rfloor^2}{(L+|v|-1)^2}\|\varphi\|_2^4 k^{-2(|x|-|v|)}.$$

Because of (5.4), we can assume $|v| \leqslant (1-\frac{1}{\beta+1})(L+1)$. This is equivalent to $\frac{1}{L+1-|v|} \leqslant \frac{\beta+1}{L+1}$, and the claim follows. ■

We finally are in position to finish the proof of the key probability estimate.

Proof of proposition 5.5. We need to bound

$$p := \mathbb{P}\Big(\sup_{\|\varphi\|_2\leqslant 1}\big|\langle\tilde{V}_\omega^{(L)}\varphi,\varphi\rangle\big| \geqslant \frac{\bar{\omega}}{16}\Big) \leqslant \mathbb{P}\Big(\sup_{\|\varphi\|_2\leqslant 1}\big|\langle\tilde{V}_\omega^{(L)}\varphi,\varphi\rangle\big| > \frac{\bar{\omega}}{32}\Big)$$

from above. Let us define

$$\mathcal{E} := \mathcal{F} := \big\{\varphi \in \Pi^{(L)}_{E_\beta^{(L)}}\big(\ell^2(\mathcal{T}^L)\big) : \|\varphi\|_2 \leqslant 1\big\}.$$

By definition of $\tilde{V}_\omega^{(L)}$, see proposition 5.4, and by (5.3), we have

$$\sup_{\|\varphi\|_2\leqslant 1}\big|\langle\tilde{V}_\omega^{(L)}\varphi,\varphi\rangle\big| \leqslant \sup_{\varphi\in\mathcal{E}}\sup_{\psi\in\mathcal{F}}\big|\langle\tilde{V}_\omega^{(L)}\varphi,\psi\rangle\big|.$$

Using this, we can estimate with the help of lemma 5.7 and see

$$p \leqslant \sum_{v\in\mathcal{T}_*^{L-1}}\sum_{w\in\mathcal{T}_v^{L-1}}\mathbb{P}\bigg(\sup_{\varphi\in\mathcal{E},\psi\in\mathcal{F}}\frac{|\langle\tilde{V}_\omega^{(L)}P_v\varphi,P_w\psi\rangle|}{\|P_v\varphi\|_2\|P_w\psi\|_2} > \frac{\bar{\omega}}{32}\Xi(L,v,w)\bigg). \tag{5.16}$$

The terms in the sum (5.16) can be bounded using lemma 5.9,

$$\mathbb{P}\bigg(\sup_{\varphi\in\mathcal{E},\psi\in\mathcal{F}}\frac{|\langle\tilde{V}_\omega^{(L)}P_v\varphi,P_w\psi\rangle|}{\|P_v\varphi\|_2\|P_w\psi\|_2} > \frac{\bar{\omega}}{32}\Xi(L,v,w)\bigg)$$
$$\leqslant \sum_{i\in\mathbb{N}}\sum_{\tilde{\varphi}\in\mathcal{M}_v,\tilde{\psi}\in\mathcal{M}_w}\mathbb{P}\Big(|\langle\tilde{V}_\omega^{(L)}P_v\tilde{\varphi},P_w\tilde{\psi}\rangle| > 2^i\frac{\bar{\omega}}{32}\Xi(L,v,w)\|P_v\tilde{\varphi}\|_2\|P_w\tilde{\psi}\|_2\Big), \tag{5.17}$$

where, for $v \in \mathcal{T}^L$, $\mathcal{M}_v \subseteq \ell^2(\mathcal{T}_v^L)$ with $\#\mathcal{M}_v \leqslant 32^{kL}$, see lemma 5.8. Using $\|P_v\tilde{\varphi}\|_2 \geqslant \big\|P_v\Pi^{(L)}_{E_\beta^{(L)}}\tilde{\varphi}\big\|_2$, valid for all $\tilde{\varphi} \in \ell^2(\mathcal{T}^L)$, proposition 5.10 tells us that

$$\begin{aligned}
&\mathbb{P}\Big(|\langle \tilde{V}_\omega^{(L)} P_v\tilde{\varphi}, P_w\tilde{\psi}\rangle| > 2^i\frac{\bar{\omega}}{32}\Xi(L,v,w)\|P_v\tilde{\varphi}\|_2\|P_w\tilde{\psi}\|_2\Big)\\
\leqslant{}& \mathbb{P}\Big(|\langle \tilde{V}_\omega^{(L)} P_v\tilde{\varphi}, P_w\tilde{\psi}\rangle| > 2^i\frac{\bar{\omega}}{32}\Xi(L,v,w)\big\|P_v\Pi^{(L)}_{E_\beta^{(L)}}\tilde{\varphi}\big\|_2\big\|P_w\Pi^{(L)}_{E_\beta^{(L)}}\tilde{\psi}\big\|_2\Big)\\
\leqslant{}& 2\exp\Big(-C_{\kappa,\tilde{\omega}_+,\beta}2^{2i}\frac{\bar{\omega}^2}{1024}k^{\delta L}\Big).
\end{aligned}$$

Plugging back into (5.17) and using $\#\mathcal{M}_v \leqslant 32^{kL}$,

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{\varphi\in\mathcal{E},\psi\in\mathcal{F}}\frac{|\langle \tilde{V}_\omega^{(L)} P_v\varphi, P_w\psi\rangle|}{\|P_v\varphi\|_2\|P_w\psi\|_2} > \frac{\bar{\omega}}{16}\Xi(L,v,w)\Big)\\
\leqslant{}& \sum_{i\in\mathbb{N}}\sum_{\tilde{\varphi}\in\mathcal{M}_v,\tilde{\psi}\in\mathcal{M}_w} 2\exp\Big(-C_{\kappa,\tilde{\omega}_+,\beta}\frac{\bar{\omega}^2}{1024}2^{2i}k^{\delta L}\Big)\\
\leqslant{}& 2\cdot 32^{2kL}\sum_{i\in\mathbb{N}}\exp\Big(-C_{\kappa,\tilde{\omega}_+,\beta}2^{2i}\frac{\bar{\omega}^2}{1024}k^{\delta L}\Big).
\end{aligned}$$

The remaining sum can be bounded with a geometric series, since for all $x > \log 2$, we have

$$\sum_{i\in\mathbb{N}}\exp(-x2^{2i}) \leqslant \sum_{i=1}^{\infty}(\mathrm{e}^{-x})^i = \frac{\mathrm{e}^{-x}}{1-\mathrm{e}^{-x}} \leqslant 2\mathrm{e}^{-x}.$$

Finally, put this back into (5.16), to get, for all $L \in \mathbb{N}$ large enough,

$$\begin{aligned}
p &\leqslant \sum_{v\in\mathcal{T}_*^{L-1}}\sum_{w\in\mathcal{T}_v^{L-1}} 4\cdot 32^{2kL}\exp\Big(-C_{\kappa,\tilde{\omega}_+,\beta}\frac{\bar{\omega}^2}{1024}k^{\delta L}\Big)\\
&\leqslant 4k^{2L}32^{2kL}\exp\Big(-C_{\kappa,\tilde{\omega}_+,\beta}\frac{\bar{\omega}^2}{1024}k^{\delta L}\Big).
\end{aligned}$$

Taking L large enough, we get

$$p \leqslant \exp\Big(-C_{\kappa,\tilde{\omega}_+,\beta}\frac{\bar{\omega}^2}{2024}k^{\delta L}\Big).$$

The end. ∎

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