SCALE-FREE UNCERTAINTY PRINCIPLES AND WEGNER ESTIMATES FOR RANDOM BREATHER POTENTIALS

PRINCIPES D'INCERTITUDE INDÉPENDANTS DE L'ÉCHELLE ET ESTIMÉES DE WEGNER POUR DES POTENTIELS "RANDOM BREATHER"

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Abstract. We present new scale-free quantitative unique continuation principles for Schrödinger operators. They apply to linear combinations of eigenfunctions corresponding to eigenvalues below a prescribed energy, and can be formulated as an uncertainty principle for spectral projectors. This extends recent results of Rojas-Molina & Veselić [12], and Klein [9]. We apply the scale-free unique continuation principle to obtain a Wegner estimate for a random Schrödinger operator of breather type. It holds for arbitrarily high energies. Schrödinger operators with random breather potentials have a non-linear dependence on random variables. We explain the challenges arising from this non-linear dependence.

Resumé. Nous présentons de nouveaux principes de continuation unique indépendants de l'échelle pour des opérateurs de Schrödinger. Nos résultats concernent des combinaisons linéaires de fonctions propres correspondantes aux valeurs propres au-dessous d'une énergie prescrite et ils peuvent être formulés en forme de principes d'incertitude pour des projecteurs spectraux. Ceci généralise des résultats récents de Rojas-Molina & Veselić [12], et Klein [9]. Nous utilisons des estimées de continuation unique indépendantes de l'échelle et obtenons ainsi une estimée de Wegner pour un opérateur de Schrödinger aléatoire de type "breather". De tels opérateurs dépendent des variables aléatoires d'une façon non-linéaire et nous expliquons les difficultés liées à cette non-linéarité.

1. Introduction

A Wegner estimate is an upper bound on the expected number of eigenvalues in a prescribed energy interval of a finite box Schrödinger operator. The expectation here refers to the potential which is random. The most studied example in this situation is the so-called alloy-type potential, sometimes

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also called continuum Anderson model, cf. Remark 2.6 below. A particular feature of this model is that randomness enters the model via a countable number of random variables, and these random variables influence the potential in a linear way. In the random breather model we study here, this dependence is no longer linear but becomes non-linear. What remains is the monotone dependence of the potential on the random variables. The topic of the present note is to explain how to effectively use this monotonicity in order to derive a Wegner estimate. This only works if it is possible to cast the monotonicity in a quantitative form.

In order to achieve this, we prove a scale-free uncertainty relation or unique continuation principle for spectral projectors of Schrödinger operators. Our Theorem 2.1 answers positively a question raised in [12]. A partial answer (for short energy intervals) was shortly after given in [9]. Previously, there has been in the literature on random operators a plethora of related results, applicable in specialised situations, see e.g. [12] for a discussion. However, the lack of a result like Theorem 2.1 was a bottleneck for further progress.

Estimates as in Theorem 2.1 have been developed and applied in a different area of mathematics, namely control theory for partial differential equations, starting with the seminal paper [10]. In this context they are called *spectral inequalities*. In fact, our proof of Theorem 2.1 highlights how ideas from the theory of random Schrödinger operators and control theory complement each other in an efficient way.

2. Results

Let $d \in \mathbb{N}$, $\delta > 0$, $L \in \mathbb{N}$ and $V\colon \mathbb{R}^d \to \mathbb{R}$ measurable. Denote by $\Lambda_L = (-L/2, L/2)^d$ a cube in $\mathbb{R}^d$, by

$$S_{L,\delta} = \Lambda_L \cap \Big(\bigcup_{j\in\mathbb{Z}^d} B(z_j, \delta) \Big)$$

the union of δ-balls centered at the points z_j and contained in the corresponding unit cubes $\Lambda_1 + j$, and by H_L one of the self-adjoint restrictions of the Schrödinger operator $-\Delta + V$ to Λ_L with either Dirichlet, Neumann, or periodic boundary conditions.

We formulate a scale-free quantitative unique continuation property for the operator H_L.

Theorem 2.1. *Let $\delta \in (0, 1/2)$, $K_V \geq 0$ and $E \in \mathbb{R}$. Then there is a constant $C_{\mathrm{sfuc}} = C_{\mathrm{sfuc}}(\delta, K_V, E) \in (0, \infty)$, such that for all measurable potentials $V\colon \mathbb{R}^d \to [-K_V, K_V]$, all scales $L \in \mathbb{N}$ with $L \geq 18\mathrm{e}\sqrt{d}$, all sequences $(z_j)_{j\in\mathbb{Z}^d} \subset \mathbb{R}^d$ such that $\forall j \in \mathbb{Z}^d : B(z_j, \delta) \subset \Lambda_1 + j$ and all linear combinations of eigenfunctions*

$$\psi = \sum_{n\in\mathbb{N}\colon E_n \leq E} \alpha_n \psi_n$$

(where $\psi_n \in W^{2,2}(\Lambda_L; \mathbb{R})$ satisfies $H_L\psi_k = E_n\psi_n$ and $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{C}$) we have

$$\int_{S_{L,\delta}} |\psi|^2 \geq C_{\mathrm{sfuc}} \int_{\Lambda_L} |\psi|^2.$$

Moreover, there is a constant K_0 depending only on the dimension d such that $C_{\mathrm{sfuc}}(\delta, K_V, E) \geq \delta^{K_0(1+K_V^{2/3}+|E|^{1/2})}$.

We can reformulate this statement as an uncertainty principle. For this purpose, denote by $\chi_I(H_L)$ the spectral projector of H_L onto an interval I and by $W_{L,\delta}$ the characteristic function of the set $S_{L,\delta}$.

Corollary. *Under the same assumptions as in the above Theorem we have*

$$\chi_{(-\infty,E]}(H_L)\; W_{L,\delta}\; \chi_{(-\infty,E]}(H_L) \geq \delta^{K_0(1+K_V^{2/3}+|E|^{1/2})}\chi_{(-\infty,E]}(H_L). \tag{1}$$

Inequality (1) is to be understood in the sense of quadratic forms.

Remark 2.2. For our purposes, the explicit quantitative dependence of the constant $C_{\mathrm{sfuc}}(\delta, K_V, E)$ is essential. In particular, C_{sfuc} *does not* depend on the scale $L \in \mathbb{N}$. It depends on the radius δ in a polynomial way, and on K_V and $|E|$ in an exponential way. Note also that the constant is unaffected by a translation of a ball $B(z_j, \delta)$ as long as it stays in the corresponding unit cube. The bound is uniform in the ensemble of potentials $\{V\colon \mathbb{R}^d \to [-K_V, K_V] \text{ measurable}\}$. This is important, because we want to apply the theorem to random Schrödinger operators. There the constant must not depend on the particular configuration of randomness. Since the operator is lower bounded, we have $\chi_{(-\infty,E]}(H_L) = \chi_{[-K_V,E]}(H_L)$.

We apply Theorem 2.1 to *random breather models*, a class of random Schrödinger operators where the randomness enters the potential in a non-linear way. Consider a sequence $\omega = (\omega_j)_{j\in\mathbb{Z}^d}$ of (almost surely) positive, bounded, independent and identically distributed random variables with distribution measure μ, as well as a compactly supported, measurable function $u\colon \mathbb{R}^d \to \mathbb{R}$. The *random breather potential* is the function

$$V_\omega(x) := \sum_{j\in\mathbb{Z}^d} u\Big(\frac{x-j}{\omega_j}\Big),$$

while the family $(H_\omega)_\omega$ with $H_\omega := -\Delta + V_\omega$ is called *random breather model*.

Random breather potentials have been introduced in [3], and studied in [4] and [8]. However, all these papers assumed unnatural regularity conditions, excluding the most basic and standard type of single site potential, where u equals the characteristic function of a ball or a cube. This was not a coincidence but a consequence of the linearization method used in the proofs. Our proof does not rely on linearization, but merely on monotonicity. While we have results for a broad class of random breather models, we restrict ourselves in this note for the purpose of clarity to the two mentioned cases, i.e.

$$u = \chi_{B_1}, \qquad \text{thus} \qquad V_\omega(x) = \sum_{j\in\mathbb{Z}^d} \chi_{B_{\omega_j}}(x-j), \tag{2a}$$

$$u = \chi_{\Lambda_2}, \qquad \text{thus} \qquad V_\omega(x) = \sum_{j\in\mathbb{Z}^d} \chi_{\Lambda_{2\omega_j}}(x-j) \tag{2b}$$

In fact, since our proofs are based on the analysis of level sets of random potentials, they work also for other types of stochastic fields with non-linear,

monotone randomness, not just for random breather potentials. Specifically, the function $\omega_j \mapsto \langle \phi, V_\omega \phi \rangle$ merely needs to be polynomially increasing.

Note also that the random potential is uniformly bounded and non-negative, and thus the operator H_ω is self-adjoint.

Theorem 2.3 (Wegner estimate for the random breather model)**.** *Let H_ω be as in (2). Assume that μ has a bounded density ν supported in $[\omega_-, \omega_+]$ with $0 \leq \omega_- < \omega_+ < 1/2$. Fix $E_0 \in \mathbb{R}$. Then there are $C = C(d, E_0)$ and $\epsilon_{\max} = \epsilon_{\max}(d, E_0, \omega_+) \in (0, \infty)$ such that for all $\epsilon \in (0, \epsilon_{\max}]$ and $E \geq 0$ with $[E - \epsilon, E + \epsilon] \subset (-\infty, E_0]$, we have*

$$\mathbb{E}\left[\operatorname{Tr}\left[\chi_{[E-\epsilon, E+\epsilon]}(H_{\omega,L})\right]\right] \leq C \|\nu\|_\infty \epsilon^{[K_0(2+|E_0+1|^{1/2})]^{-1}} |\ln \epsilon|^d L^d.$$

The constant $\epsilon_{\max}$ can be chosen as

$$\epsilon_{\max} = \frac{1}{4}\left(\frac{1/2 - \omega_+}{2}\right)^{K_0(2+|E_0+1|^{1/2})},$$

where K_0 is the constant from Theorem 2.1.

Here $\mathbb{E}$ denotes the expectation w.r.t. the random variables $\omega_j, j \in \mathbb{Z}^d$, and $H_{\omega,L}$ the restriction of H_ω to the cube Λ_L with Dirichlet boundary conditions. Theorem 2.3 implies local Hölder continuity of the integrated density of states (IDS) and is sufficient for the multiscale-analysis proof of spectral localization. This will be elaborated in detail elsewhere.

Remark 2.4. The proof of Theorem 2.1 relies on Carleman estimates with and without boundary term, see e.g. [10] and [5, 1], on interpolation inequalities and an auxiliary Cauchy problem in $d+1$ dimensions as discussed in [10, 11, 7], and finally on geometric covering arguments developed in the theory of random Schrödinger operators, e.g. [12].

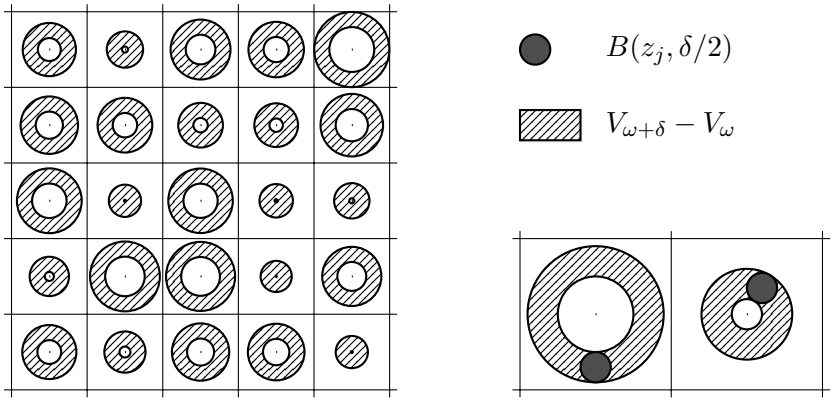
The proof of Theorem 2.3 relies on the method outlined in [6] and [12]. It can be traced back to Wegner's original work [14]. Additional steps are necessary, since the breather model has a non-linear dependence on the random variables, unlike the well-studied Anderson model. We also do not have the differentiability of the map $\omega_j \mapsto \langle \phi, H_\omega \phi \rangle$ in the usual sense. Thus, for instance the proofs of [2, 9] do not apply. However, the strategy of [6, 12] is quite versatile and can be adapted to our setting. The key idea is not to rely on differentiability of quadratic forms but rather directly on the Courant-Hilbert variational principle for eigenvalues.

In particular, the following lemma is crucial for the proof of Theorem 2.3. It relies on the quantitative version of the uncertainty principle from Theorem 2.1. Denote the eigenvalues of $H_{\omega,L}$ by $\{E_n(\omega)\}_{n\in\mathbb{N}}$, enumerated increasingly and counting multiplicities. For $\delta \in \mathbb{R}$ we define $\omega + \delta$ by $(\omega + \delta)_j := \omega_j + \delta$ for all $j \in \mathbb{Z}^d$.

Lemma 2.5. *Let $H_{\omega,L}$ be as above and assume that $\omega \in [\omega_-, \omega_+]^{\mathbb{Z}^d}$, $\delta \leq 1/2 - \omega_+$. Then, for all $n \in \mathbb{N}$ with $E_n(\omega) \in (-\infty, E_0]$ we have*

$$E_n(\omega + \delta) \geq E_n(\omega) + \left(\frac{\delta}{2}\right)^{\left[K_0\left(2+|E_0+1|^{1/2}\right)\right]},$$

where K_0 is the constant from Theorem 2.1.



(a) Support of $V_{\omega+\delta} - V_\omega$ (Support de $V_{\omega+\delta} - V_\omega$)

(b) Choice of $B(z_j, \delta/2)$ (Choix des $B(z_j, \delta/2)$)

FIGURE 1. Illustration of the increments $V_{\omega+\delta} - V_\omega$ and the choice of the balls $B(z_j, \delta/2)$ (Illustration des incréments $V_{\omega+\delta} - V_\omega$ et du choix des boules $B(z_j, \delta/2)$)

Thus, we obtain a lifting estimate on the eigenvalues which is independent on the length scale. Details of the proof of Theorem 2.3 can be found in [13].

Remark 2.6 (Challenges due to non-linearity). The challenges are best understood by comparing the breather model with the alloy-type potential $V_\omega(x) = \sum_{j\in\mathbb{Z}^d} \omega_j u(x-j)$ (for simplicity consider $u = \chi_{B(r)}$). The latter depends in a linear way on the random coupling constants constituting the configuration $\omega = (\omega_j)_j$. In particular, the derivatives of eigenvalues $E_n(\omega)$ (of finite box restrictions on $-\Delta + V_\omega$) w.r.t. each ω_j are easily calculated via the Hellman-Feynman formula. In contrast, for the breather model the derivatives $\frac{\partial}{\partial\omega_j} E_n(\omega)$ are only defined in distributional sense. Thus, one is lead to implement eigenvalue perturbation theory using increments $V_{\omega+\delta} - V_\omega$, with positive δ. Note that in the case of the alloy-type model, for any fixed δ, the increment $V_{\omega+\delta} - V_\omega$ is independent of the configuration ω and a $\mathbb{Z}^d$-periodic function. Therefore, it is not needed to know the explicit dependence of C_{sfuc} on δ. For the breather model this is not the case. In particular, $V_{\omega+\delta} - V_\omega$ is a non-periodic function and its support depends both on δ and ω. Specifically, it is a union of annuli of width δ and ω-dependent radii, cf. Fig. 1(a). Technically, one has to estimate the mass of the square of an eigenfunction in this support set as a function of ω and δ. For the application of Theorem 2.1 one has to chose balls $B(z_j, \delta/2)$ lying inside the annuli, see Fig. 1(b). To obtain Hölder continuity of the IDS one has to control the behaviour of C_{sfuc} as $\delta \searrow 0$.

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