Multiscale unique continuation properties of eigenfunctions

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Multiscale unique continuation properties of eigenfunctions

Denis Borisov, Ivica Nakić, Christian Rose, Martin Tautenhahn and Ivan Veselić

Abstract. Quantitative unique continuation principles for multiscale structures are an important ingredient in a number applications, e.g. random Schrödinger operators and control theory.

We review recent results and announce new ones regarding quantitative unique continuation principles for partial differential equations with an underlying multiscale structure. They concern Schrödinger and second order elliptic operators. An important feature is that the estimates are scale free and with quantitative dependence on parameters. These unique continuation principles apply to functions satisfying certain 'rigidity' conditions, namely that they are solutions of the corresponding elliptic equations, or projections on spectral subspaces. Carleman estimates play an important role in the proofs of these results.

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1. Introduction

Motivation: Retrival of global properties from local data

In several branches of mathematics, as well as in applications, one often encounters problems of the following type: Given a region in space $\Lambda \subset \mathbb{R}^d$, a subset $S \subset \Lambda$, and a function $f\colon \Lambda \to \mathbb{R}$, what can be said about certain properties of $f\colon \Lambda \to \mathbb{R}$ given

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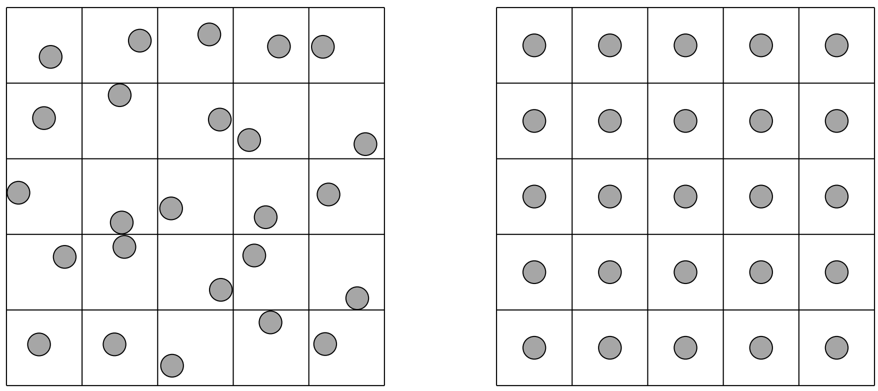
certain properties of $f|_S : S \to \mathbb{R}$? In specific cases one may want to reconstruct f as accurately as possible based on knowledge of $f|_S$, in others it may be sufficient to estimate some features of f.

It is clear that for this task additional global information on f is needed. Indeed, if f is one of the indicator functions χ_S or $\chi_{\Lambda\setminus S}$, an estimate based on $f|_S$ would yield wrong results. The first helpful property which comes to one's mind is some regularity or smoothness property of f. However, since there are C^∞-functions supported inside S (or inside $\Lambda \setminus S$) this is not quite the right condition. The required property of f is more adequately described as *rigidity*, as we will see in specific theorems formulated below.

In this paper we are mainly concerned with problems with a multiscale structure. For this reason it is natural to require that the set S is in some sense equidistributed within Λ. At this point we will not give a precise definition of such sets. It will become clear that such a set S should be relatively dense in $\mathbb{R}^d$ or Λ, and should have positive density. A particularly nice set S would be a periodic arrangement of balls, and we want to include small perturbations of such a configuration. Thus, equidistributed sets could be seen as a generalization of such a situation, cf. Fig. 1.



(a) non-periodic (b) periodic arrangement of balls

FIGURE 1. Examples of equidistributed sets S within region $\Lambda \subset \mathbb{R}^2$.

Example: Shannon sampling theorem

We recall a well known theorem as an example or benchmark, see e.g. [4]. This way we will see what is the best we can hope for in the task of reconstructing a function. Moreover, we will encounter one possible interpretation what the term *rigidity* means, and see major differences between the reconstruction problem in dimension one and higher dimensions.

The Shannon sampling theorem states: Let $f \in C(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that the Fourier transform

$$\hat{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\,x\,p} \, f(x) \, \mathrm{d}x$$

vanishes outside $[-\pi K, \pi K]$. Then the series

$$(S_K f)(x) = \sum_{j \in \mathbb{Z}} f\left(\frac{j}{K}\right) \frac{\sin \pi (K\,x - j)}{\pi (K\,x - j)} \tag{1.1}$$

converges absolutely and uniformly for $x \in \mathbb{R}$ and

$$S_K f = f \text{ on } \mathbb{R}.$$

Thus we can reconstruct the original function f from the sample values $f(j/K)$, which are multiplied with weights depending on the distance to the point $x \in \mathbb{R}$ and summed up. Here the rigidity condition is implemented by the requirement $\operatorname{supp} \hat{f} \subset [-\pi K, \pi K]$, which implies that f is entire. A remarkable feature of this exact result is, that it is stable under perturbations: If the nodes j deviate slightly from the integers, or if the measurement data $f(\frac{j}{K})$ are inaccurate, the error $f - S_K f$ can still be controlled. If the support condition $\operatorname{supp} \hat{f} \subset [-\pi K, \pi K]$ is violated, the *aliasing error* is estimated as

$$\sup_{x \in \mathbb{R}} |f(x) - S_K f(x)| \le \sqrt{\frac{2}{\pi}} \int_{|p| > \pi K} |\hat{f}(p)| \, \mathrm{d}p. \tag{1.2}$$

This will give, for instance, good results for centered Gaussians with appropriate variance.

Statements (1.1) and (1.2) are strong with respect to the sampling set $S = \mathbb{Z}$, which is very thin. It has zero Lebesgue measure, in fact, it is discrete. Albeit, it is relatively dense in $\mathbb{R}$, so it has some of the properties we associated with an equidistributed set. Compared to Shannon's theorem, the results we present below appear much weaker. This is, among others, due to two features: we consider functions on multidimensional space, which, in addition, have low regularity, in fact are defined as equivalence classes in some L^2 or Sobolev space. In this situation evaluation of a function at a point may not have a proper meaning. This is one of the reasons why we have to consider samples S which are composed of small balls, rather than single points. A second aspect where dimensionality comes into play is the following: A polynomial of one variable of degree N vanishes identically if it has $N+1$ zeros. A non-trivial polynomial in two variables may vanish on an uncountable set (albeit not on one of positive measure). This illustrates that reconstruction estimates for functions of several variables are more subtle than Shannon's theorem. Consequently, one has to settle for more modest goals than the full reconstruction of the function f. We want to derive an equidistribution property for functions satisfying some rigidity property. As will be detailed later this result is called — depending on the context and scientific environment — scale free unique continuation property, observability estimate, or uncertainty relation. A first result of this type is formulated in the next section.

2. Equidistribution property of Schrödinger eigenfunctions

The following result [14] was motivated by questions arising in the spectral theory of random Schrödinger operators. Later, it turned out that similar estimates are of relevance in the control theory of the heat equation.

We fix some notation. For $L > 0$ we denote by $\Lambda_L = (-L/2, L/2)^d$ a cube in $\mathbb{R}^d$. For $\delta > 0$ the open ball centered at $x \in \mathbb{R}$ with radius δ is denoted by $B(x, \delta)$. For a sequence of points $(x_j)_j$ indexed by $j \in \mathbb{Z}^d$ we denote the collection of balls $\cup_{j\in\mathbb{Z}^d} B(x_j, \delta)$ by S and its intersection with Λ_L by S_L. We will be dealing with certain self-adjoint operators on subsets of $\mathbb{R}^d$. Let Δ be the d-dimensional Laplacian, $V : \mathbb{R}^d \to \mathbb{R}$ a bounded measurable function, and $H_L = (-\Delta + V)_{\Lambda_L}$ a Schrödinger operator on the cube Λ_L with Dirichlet or periodic boundary conditions. The corresponding domains are denoted by $\mathcal{C}(\Delta_{\Lambda,0}) \subset W^{2,2}(\Lambda_L)$ and $\mathcal{C}(\Delta_{\Lambda,\mathrm{per}})$, respectively. Note that we denote a multiplication operator by the same symbol as the corresponding function.

Theorem 2.1 ([14]). *Let $\delta, K > 0$. Then there exists $C \in (0, \infty)$ such that for all $L \in 2\mathbb{N} + 1$, all measurable $V : \mathbb{R}^d \to [-K, K]$, all real-valued $\psi \in \mathcal{C}(\Delta_{\Lambda,0}) \cup \mathcal{C}(\Delta_{\Lambda,\mathrm{per}})$ with $(-\Delta + V)\psi = 0$ almost everywhere on Λ_L, and all sequences $(x_j)_{j\in\mathbb{Z}^d} \subset \mathbb{R}^d$, such that $\forall j \in \mathbb{Z}^d : B(x_j, \delta) \subset \Lambda_1 + j$ we have*

$$\int_{S_L} \psi^2 \geq C \int_{\Lambda_L} \psi^2. \tag{2.1}$$

To appreciate the result properly, the quantitative dependence of the constant C on model parameters is crucial. The very formulation of the theorem states that C is independent of position of the balls $B(x_j, \delta)$ within $\Lambda_1 + j$, and independent of the scale $L \in 2\mathbb{N} + 1$. The estimates given in Section 2 of [14] show moreover, that C depends on the potential V only through the norm $\|V\|_\infty$ (on an exponential scale), and it depends on the small radius $\delta > 0$ polynomially, i.e. $C \gtrsim \delta^N$, for some $N \in \mathbb{N}$ which depends on the dimension d and $\|V\|_\infty$. This shows that we are not able to control the integral $\int_{S_L} \psi^2$ by evaluating ψ at the midpoints $j \in \mathbb{Z}^d$ of the unit cubes. One sees with what rate the estimate diverges, as the balls become smaller and approximate a single point. The polynomial behavior $C \gtrsim \delta^N$ can be readily understood when looking at monomials $\psi_n(x) = x^n$ on the unit interval $(0, 1)$. There we have

$$\int_{(0,\delta)} \psi_n^2 = \frac{\delta^{2n+1}}{2n+1} = \delta^{2n+1} \int_{(0,1)} \psi_n^2.$$

We formulated the theorem only for the eigenvalue zero, but it is easily applied to other eigenfunctions as well since

$$H_L\psi = E\psi \Leftrightarrow (H_L - E)\psi = 0.$$

Consequently the constant $K = K_V$ has to be replaced with the possibly larger $K = K_{V-E}$.

There is a very natural question, which was spelled out in [14], namely does the same estimate (2.1) hold true for linear combinations $\psi \in \operatorname{Ran} \chi_{(-\infty,E]}(H_L)$ of eigenfunctions as well? The property in question can be equivalently stated as: Given $\delta > 0, K \geq 0, E \in \mathbb{R}$ there is a constant $C > 0$ such that for all measurable $V \colon \mathbb{R}^d \to [-K, K]$, all $L \in 2\mathbb{N}+1$, and all sequences $(x_j)_{j\in\mathbb{Z}^d} \subset \mathbb{R}^d$ with $B(x_j, \delta) \subset \Lambda_1 + j$ for all $j \in \mathbb{Z}^d$ we have

$$\chi_{(-\infty,E]}(H_L)\, W_L\, \chi_{(-\infty,E]}(H_L) \geq C\ \chi_{(-\infty,E]}(H_L), \tag{2.2}$$

where $W_L = \chi_{S_L}$ is the indicator function of S_L and $\chi_I(H_L)$ denotes the spectral projector of H_L onto the interval I. Here $C = C_{\delta,K,E}$ is determined by δ, K, E alone.

Note that all considered operators are lower bounded by $-K$ in the sense of quadratic forms. Thus the spectral projection on the energy interval $(-\infty, E]$ is the same as the spectral projection on the energy interval $[-K, E]$. The upper bound E in the energy parameter is crucial for preventing the corresponding eigenfunctions to oscillate too much.

One can pose a modified version of the question: Given $\delta > 0, K \geq 0, a < b \in \mathbb{R}$ is there is a constant $\tilde{C} > 0$ such that for all measurable $V \colon \mathbb{R}^d \to [-K, K]$, all $L \in 2\mathbb{N}+1$, and all sequences $(x_j)_{j\in\mathbb{Z}^d} \subset \mathbb{R}^d$ with $B(x_j, \delta) \subset \Lambda_1 + j$ for all $j \in \mathbb{Z}^d$ we have

$$\chi_{[a,b]}(H_L)\, W_L\, \chi_{[a,b]}(H_L) \geq \tilde{C}\ \chi_{[a,b]}(H_L). \tag{2.3}$$

Here $\tilde{C} = \tilde{C}_{\delta,K,a,b}$ depends (only) on δ, K, a, b. Note that inequality (2.2) implies (2.3) since

$$\begin{aligned}
&\chi_{[a,b]}(H_L)\, W_L\, \chi_{[a,b]}(H_L) \\
&\qquad = \chi_{[a,b]}(H_L)\, \chi_{(-\infty,b]}(H_L)\, W_L\, \chi_{(-\infty,b]}(H_L)\, \chi_{[a,b]}(H_L) \\
&\qquad \geq C_{\delta,K,b}\ \chi_{[a,b]}(H_L)\, \chi_{(-\infty,b]}(H_L)\, \chi_{[a,b]}(H_L) \\
&\qquad = C_{\delta,K,b} \chi_{[a,b]}(H_L).
\end{aligned}$$

However, $C_{\delta,K,b}$ may be substantially smaller than $\tilde{C}_{\delta,K,a,b}$ due to the enlarged energy interval.

Klein obtained a positive answer to the question for sufficiently short intervals.

Theorem 2.2 ([8]). *Let $d \in \mathbb{N}$, $E \in \mathbb{R}$, $\delta \in (0, 1/2]$ and $V : \mathbb{R}^d \to \mathbb{R}$ be measurable and bounded. There is a constant $M_d > 0$ such that if we set*

$$\gamma = \frac{1}{2} \delta^{M_d \left(1 + (2\|V\|_\infty + E)^{2/3}\right)},$$

then for all energy intervals $I \subset (-\infty, E]$ *with length bounded by* 2γ*, all* $L \in 2\mathbb{N}+1$*,* $L \geq 72\sqrt{d}$ *and all sequences* $(x_j)_{j\in\mathbb{Z}^d} \subset \mathbb{R}^d$ *with* $B(x_j, \delta) \subset \Lambda_1 + j$ *for all* $j \in \mathbb{Z}^d$

$$\chi_I(H_L)\, W_L\, \chi_I(H_L) \geq \gamma^2 \chi_I(H_L). \tag{2.4}$$

Although this does not answer the above posed question for arbitrary compact intervals, the result is sufficient for many questions in spectral theory of random Schrödinger operators. A generalization of Theorem 2.2 to intervals of arbitrary length is given in Section 4. This answers completely the question posed in [14].

Depending on the context and the area of mathematics the above described estimates carry various names. If one speaks of an *equidistribution property of eigenfunctions*, one is interested in the comparison of the measure $|\psi(x)|^2 \mathrm{d}x$ with the uniform distribution on the cube Λ_L. The term *scale free unique continuation principle* is used in works concerning random Schrödinger operators. It refers to a quantitative version of the classical unique continuation principle, which is uniform on all large length scales. One can interpret Theorem 2.1 as an *uncertainty relation*: the condition $H_L\psi = E\psi$ corresponds to a restriction in momentum/Fourier-space and enforces a delocalization/flatness property in direct space. Similarly, the spectral projector $\chi_{(-\infty,E]}$ in Ineq. (2.2) corresponds to a restriction in momentum space. Here we see a direct analogy to Shannon's theorem discussed above: If the Fourier transform of a function is sufficiently concentrated, the function itself cannot vary too much over short distances. Inequality (2.3) can also be interpreted as a *gain of positive definiteness*. It says that for a general self-adjoint operator $A \geq 0$, which may have a kernel, and an appropriately chosen spectral projector P of the Hamiltonian, the restriction $PAP \geq cP$ is strictly positive. In control theory results as we discuss them are sometimes called *observability* estimates. This term is more common for time-dependent partial differential equations, but sometimes used for stationary ones as well.

In the literature on random Schrödinger operators related results have been derived before in a number of papers. For more details we refer to Section 1 of [14].

3. Methods and background

A paradigmatic result for the *weak unique continuation principle* is the following. A solution of $\Delta f \equiv 0$ on $\mathbb{R}^d$ satisfying $f \equiv 0$ on $B(0,\delta)$ for arbitrary small, but positive δ, must vanish on all of $\mathbb{R}^d$. The restrictive conditions can be relaxed. First of all, the condition $f \equiv 0$ on $B(0,\delta)$ can be replaced by

$$\forall N \in \mathbb{N} \qquad \lim_{\delta \searrow 0} \delta^{-N} \int\limits_{B(0,\delta)} |f(x)| \mathrm{d}x = 0.$$

In this form the implication is called *strong unique continuation principle*. Moreover, the Laplacian Δ can be replaced by a rather general second order elliptic operator. We will discuss related results in Sections 5 and 6. A powerful method to prove unique continuation statements, as well as quantitative versions thereof, are Carleman estimates. Originally, Carleman [5] derived them for functions of two variables. Later Müller [11] extended the estimates to higher dimensions. By now, there are hundreds of papers dealing with Carleman estimates. We will describe one explicit version in Section 5, which is an important tool for the quantitative unique continuation estimates discussed shortly for Schrödinger operators. In Section 6 we will present new results in this direction which deal with elliptic second order operators with variable coefficients.

Quantitative unique continuation principle

In [1] Bourgain and Kenig derived the following pointwise quantitative unique continuation principle.

Theorem 3.1. *Assume* $(-\Delta+V)u = 0$ *on* $\mathbb{R}^d$ *and* $u(0) = 1$, $\|u\|_\infty \leq C$, $\|V\|_\infty \leq C$. *Let* $x_0 \in \mathbb{R}^d$, $|x_0| = R > 1$. *Then there exists a constant* $C' > 0$ *such that*

$$\max_{|x-x_0|\leq 1} |u(x)| > C' \exp\left(-C'(\log R) R^{4/3}\right).$$

In our context a version of this result with local L^2-averages is more appropriate. Various estimates of this type have been given in [7, 2, 14]. We quote here the version from the last mentioned paper.

Theorem 3.2. *Let* $K, R, \beta \in [0,\infty), \delta \in (0,1]$. *There exists a constant* $C_{\mathrm{qUC}} = C_{\mathrm{qUC}}(d, K_V, R, \delta, \beta) > 0$ *such that, for any* $G \subset \mathbb{R}^d$ *open, any* $\Theta \subset G$ *measurable, satisfying the geometric conditions*

$$\operatorname{diam}\Theta + \operatorname{dist}(0,\Theta) \leq 2R \leq 2\operatorname{dist}(0,\Theta), \quad \delta < 4R, \quad B(0,14R) \subset G,$$

and any measurable $V\colon G \to [-K,K]$ *and real-valued* $\psi \in W^{2,2}(G)$ *satisfying the differential inequality*

$$|\Delta\psi| \leq |V\psi| \quad \text{a.e. on } G \quad \text{as well as} \quad \int_G |\psi|^2 \leq \beta \int_\Theta |\psi|^2,$$

we have

$$\int_{B(0,\delta)} |\psi|^2 \geq C_{\mathrm{qUC}} \int_\Theta |\psi|^2.$$

4. Equidistribution property of linear combinations of eigenfunctions

In this section we announce a result from an ongoing project of I. Nakić, M. Tautenhahn and I. Veselić [13], namely which gives Ineq. (2.1) also for linear combinations of eigenfunctions $\psi \in \operatorname{Ran} \chi_{(-\infty,E]}$ for arbitrary $E \in \mathbb{R}$. As shown above, this implies Ineq. (2.2) for arbitrary $E \in \mathbb{R}$ and hence Ineq. (2.3). Indeed, our result gives a full answer to the open question in [14] whether Theorem 3.2 holds also

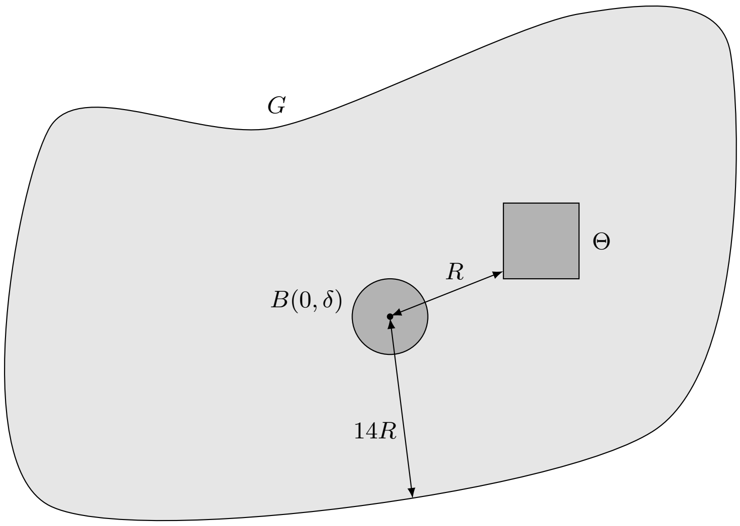


FIGURE 2. Assumptions in Theorem 3.2 on the geometric constellation of G, Θ, and $B(0,\delta)$

for linear combinations of eigenfunctions, which was partially answered in [8], cf. Theorem 2.2.

Since we first show Ineq. (2.1) for arbitrary $E \in \mathbb{R}$, the constant $\tilde{C}$ in Ineq (2.3) will not be optimal, since it does not depend on the lower bound a of the interval $[a, b]$.

We denote by E_k and ψ_k, $k \in \mathbb{N}$, the eigenvalues and eigenfunctions of the Schrödinger operator H_L with Dirichlet or Neumann boundary conditions. Without loss of generality one may assume that the eigenfunctions ψ_k are real-valued. Let $E \in \mathbb{R}$, $\alpha_k \in \mathbb{C}$ and let

$$\psi = \sum_{k\in\mathbb{N}\colon\, E_k\le E} \alpha_k \psi_k$$

be a linear combination of the eigenfunctions of the operator H_L. Then we have the following result.

Theorem 4.1 **(**[13]**).** *Let $\delta \in (0, 1/2]$, $E \in \mathbb{R}$, and $K \geq 0$. Then there exists $C > 0$ such that for all $L \in 2\mathbb{N}+1$ large enough, all measurable $V\colon \mathbb{R}^d \to [-K, K]$, all sequences $(x_j)_{j\in\mathbb{Z}^d} \subset \mathbb{R}^d$, such that for all $j \in \mathbb{Z}^d$ $B(x_j, \delta) \subset \Lambda_1 + j$, we have*

$$\int_{S_L} |\psi|^2 \geq C \int_{\Lambda_L} |\psi|^2. \tag{4.1}$$

The constant C in (4.1) depends only on the dimension d, the radius of the balls δ, and constants K and E. Hence, as in Theorem 2.1, the constant is independent on the position of the balls $B(x_j, \delta)$, the scale L, and it depends on the potential V only through the norm $\|V\|_\infty$. The constant C depends exponentially on the parameters d and E. The dependence on δ is more complicated and will not be reproduced here.

Here we give a sketch of the proof. We use two different Carleman inequalities in $\mathbb{R}^{d+1}$, one with a boundary term in $\mathbb{R}^d \times \{0\}$ and the other without boundary terms. From these Carleman estimates we deduce two interpolation inequalities for a solution of a Schrödinger equation in $\mathbb{R}^{d+1}$. In the final step we apply these interpolation inequalities to the function $F : \Lambda_L \times \mathbb{R} \to \mathbb{C}$ defined by

$$F(x) = \sum_{\substack{k\in\mathbb{N} \\ E_k \leq \mu}} \alpha_k \psi_k(x') s_k(x_{d+1}),$$

where $\mathbb{R}^{d+1} \ni x = (x', x_{d+1})$, $x' \in \mathbb{R}^d$, $x_{d+1} \in \mathbb{R}$ and

$$s_k(x) = \begin{cases} \sinh(\omega_k x)/\omega_k, & E_k > 0, \\ x, & E_k = 0, \\ \sin(\omega_k x)/\omega_k, & E_k < 0. \end{cases}$$

This function F satisfies $\Delta F = VF$ on $\Lambda_L \times \mathbb{R}$ and $\partial_{d+1} F(x', 0) = \psi(x')$ on Λ_L, and one can obtain upper and lower estimates for the H^1-norm of the function F in terms of the parameters K, E, d and $\sum_{E_k \leq E} |\alpha_k|^2$.

Full proofs and complete references will be given in [13].

5. Explicit Carleman estimates for elliptic operators

As mentioned above, Carleman estimates play a significant role in the results about unique continuation principles. In the case of quantitative unique continuation principles on multiscale structures, it is important to have a Carleman estimate with a precise dependence on various parameters as possible.

We consider the second order elliptic partial differential operator

$$L = -\sum_{i,j=1}^{d} \partial_i \left(a^{ij} \partial_j \right),$$

acting on functions in $\mathbb{R}^d$. We introduce the following assumption on the coefficient functions a^{ij}.

Assumption (A). Let $r, \vartheta_1, \vartheta_2 > 0$. The operator L satisfies $A(r, \vartheta_1, \vartheta_2)$, if and only if $a^{ij} = a^{ji}$ for all $i, j \in \{1, \ldots, d\}$ and for almost all $x, y \in B(r)$ and all $\xi \in \mathbb{R}^d$ we have

$$\vartheta_1^{-1} |\xi|^2 \leq \sum_{i,j=1}^{d} a^{ij}(x) \xi_i \xi_j \leq \vartheta_1 |\xi|^2 \quad \text{and} \quad \sum_{i,j=1}^{d} |a^{ij}(x) - a^{ij}(y)| \leq \vartheta_2 |x - y|.$$

Here $B(r) \subset \mathbb{R}^d$ denotes the open ball in $\mathbb{R}^d$ with radius r and center zero. Let the entries of the inverse of the matrix $(a^{ij}(x))_{i,j=1}^d$ be denoted by $a_{ij}(x)$.

We present the result for the ball $B(1)$, but by scaling arguments this result can be generalized to arbitrary large balls $B(R)$, now with a different weight function which depends also on R.

In the following theorem we quote a Carleman estimate from [6]. In particular, we treat the simpler elliptic case and remark that the estimate is valid on the whole domain, i.e. $\delta = 1$ holds in the notation of [6]. In the case of the pure Laplacian this has already implemented in [1]. We plan to give quantitative estimates for all the parameters. This is part of an ongoing work of I. Nakić, C. Rose and M. Tautenhahn [12].

For $\mu > 0$ let $\sigma : \mathbb{R}^d \to [0, \infty)$ and $\psi \colon [0, \infty) \to [0, \infty)$ be given by

$$\sigma(x) = \left(\sum_{i,j=1}^{d} a_{ij}(0) x_i x_j \right)^{1/2} \quad \text{and} \quad \psi(s) = s \cdot \exp\left[-\int_0^s \frac{1 - \mathrm{e}^{-\mu t}}{t} \mathrm{d}t \right].$$

We define the weight function $w \colon \mathbb{R}^d \to [0, \infty)$ by $w(x) = \psi(\sigma(x))$. Note that the weight function satisfies the bounds

$$\forall x \in B(1)\colon \quad \frac{|x|}{C_3 \sqrt{\vartheta_1}} \leq w(x) \leq \sqrt{\vartheta_1} |x| \quad \text{with} \quad C_3 = \mathrm{e}\mu. \tag{5.1}$$

Theorem 5.1 **([6]).** *Let $\vartheta_1, \vartheta_2 > 0$ and Assumption $A(1, \vartheta_1, \vartheta_2)$ be satisfied. Then there exist constants $\mu, C_1, C_2 > 0$ depending only on ϑ_1, ϑ_2 and the dimension d such that for all $f \in C_0^\infty(B(0,1) \setminus \{0\})$ and all $\alpha > C_1$ we have*

$$\int \alpha w^{1-2\alpha} |\nabla f|^2 + \alpha^3 w^{-1-2\alpha} f^2 \leq C_2 \int w^{2-2\alpha} (Lf)^2.$$

Explicit bounds on $\mu = \mu(\vartheta_1, \vartheta_2)$ are planned to be given in [12]. In particular,

$$\forall T > 0: \quad \mu_T = \sup\{\mu(\vartheta_1, \vartheta_2) \colon 0 < \vartheta_1, \vartheta_2 \leq T\} < \infty. \tag{5.2}$$

With a regularization procedure (see, for example, [17, Theorem 1.6.1]) this result can be extended to the functions in $H_0^2(B(0,1))$ which are compactly supported away from the origin.

6. Quantitative unique continuation estimates for elliptic operators

In this section we announce a result from an ongoing work of D. I. Boris, M. Tautenhahn and I. Veselić [3]. It concerns a quantitative unique continuation principle for elliptic second order partial differential operators with slowly varying coefficients.

As in the previous section we denote by L the second order partial differential operator

$$Lu = -\sum_{i,j=1}^{d} \partial_i \left(a^{ij} \partial_j u \right),$$

acting on functions u on $\mathbb{R}^d$.

Theorem 6.1 ([3])**.** *Let* $R, \vartheta_1, \vartheta_2 \in (0, \infty)$, $D_0 < 6R$, $K_V, \beta \in [0, \infty)$, $\delta \in (0, 4R]$, *let* $C_3 = C_3(d, \vartheta_1, \vartheta_2)$ *be the constant from Eq.* (5.1), *and assume that*

$$A(12R + 2D_0, \vartheta_1, \vartheta_2) \quad \text{and} \quad \vartheta_1 C_3 < \frac{1}{4R}$$

are satisfied. Then there exists $C_{\mathrm{qUC}} = C_{\mathrm{qUC}}(d, \vartheta_1, \vartheta_2, R, D_0, K_V, \delta, \beta) > 0$, *such that, for any* $G \subset \mathbb{R}^d$ *open,* $x \in G$ *and* $\Theta \subset G$ *measurable, satisfying*

$$\operatorname{diam}\Theta + \operatorname{dist}(x, \Theta) \leq 2R \leq 2\operatorname{dist}(x, \Theta) \quad \text{and} \quad B(x, 12R + 2D_0) \subset G,$$

and any measurable $V : G \to [-K_V, K_V]$ *and real-valued* $\psi \in W^{2,2}(G)$ *satisfying the differential inequality*

$$|L\psi| \leq |V\psi| \quad \text{a.e. on } G \quad \text{as well as} \quad \int_G |\psi|^2 \leq \beta \int_\Theta |\psi|^2,$$

we have

$$\int_{B(x,\delta)} |\psi|^2 \geq C_{\mathrm{qUC}} \int_\Theta |\psi|^2.$$

Theorem 6.1 generalizes Theorem 2.1 to second order elliptic operators with slowly varying coefficient functions. This is explicitly given by the assumption $\vartheta_1 C_3 < 1/(4R)$. Indeed, for fixed $R > 0$ the last inequality is satisfied for ϑ_1 sufficiently small, since (5.2) implies $\lim_{\vartheta_1 \to 0} \vartheta_1 \mu(\vartheta_1, \vartheta_2) = 0$. Furthermore, once one has a quantitative estimate on the dependence $(\vartheta_1, \vartheta_2) \mapsto \mu$, the assumption $4R\vartheta_1 C_3 < 1$ can be formulated as a condition involving ϑ_1, ϑ_2 and R only.

The proof of Theorem 6.1 is based on ideas developed in [14] for the pure Laplacian. The key tool for the proof is a Carleman estimate. For second order elliptic operators there exist plenty of them in the literature, see e.g. [9, 10, 15]. However, since we are interested in quantitative estimates, the Carleman estimate from Theorem 5.1 proved to be useful in this context.

References

[1] J. Bourgain and C. E. Kenig. On localization in the continuous Anderson-Bernoulli model in higher dimension. *Invent. Math.*, 161(2):389–426, 2005.

[2] J. Bourgain and A. Klein. Bounds on the density of states for Schrödinger operators. *Invent. Math.*, 194(1):41–72, 2013.

[3] D. I. Borisov, M. Tautenhahn, and I. Veselić. Equidistribution properties of eigenfunctions of divergence form operators. in preparation.

[4] P. L. Butzer, W. Splettstoesser, and R.L. Stens. The sampling theorem and linear prediction in signal analysis. Jahresber. Deutsch. Math.-Verein. 90:1-70, 1988.

[5] T. Carleman. Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes. *Ark. Mat. Astr. Fys.*, 26(17):1–9, 1939.

[6] L. Escauriaza and S. Vessella. *Optimal Three Cylinder Inequalities for Solutions to Parabolic Equations with Lipschitz Leading Coefficients*, volume 333 of *Contemp. Math.*, pages 79–87. American Mathematical Society, 2003.

[7] F. Germinet and A. Klein. A comprehensive proof of localization for continuous Anderson models with singular random potentials. *J. Eur. Math. Soc.*, 15(1):53–143, 2013.

[8] A. Klein. Unique continuation principle for spectral projections of Schrödinger operators and optimal Wegner estimates for non-ergodic random Schrödinger operators. *Commun. Math. Phys.*, 323(3):1229–1246, 2013.

[9] H. Koch and D. Tataru. Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients. *Commun. Pur. Appl. Math.*, 54(3):339–360, 2001.

[10] J. Le Rousseau and G. Lebeau. On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations. *ESAIM Contr. Optim. Ca.*, 18(3):712–747, 2012.

[11] C. Müller. On the behavior of the solutions of the differential equation $\delta u = f(x,u)$ in the neighborhood of a point. *Commun. Pur. Appl. Math.*, 7(3):505–515, 1954.

[12] I. Nakić, C. Rose, and M. Tautenhahn. A Carleman estimate for second order elliptic operators. in preparation.

[13] I. Nakić, M. Tautenhahn, and I. Veselić. Equidistribution properties of linear combinations of eigenfunctions. in preparation.

[14] C. Rojas-Molina and I. Veselić. Scale-free unique continuation estimates and applications to random Schrödinger operators. *Commun. Math. Phys.*, 320(1):245–274, 2013.

[15] J. Le Rousseau. Carleman estimates and some applications to control theory. In *Control of Partial Differential Equations*, Lecture Notes in Mathematics, pages 207–243. 2012.

[16] S. Vessella. Quantitative estimates of unique continuation for parabolic equations, determination of unknown time-varying boundaries and optimal stability estimates. *Inverse Probl.*, 24(2):023001, 2008.

[17] W. P. Ziemer. *Weakly differentiable functions.* Springer, New York, 1989.

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