# EXPANSION OF THE ALMOST SURE SPECTRUM IN A WEAK DISORDER REGIME.

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ABSTRACT. The spectrum of random ergodic Schrödinger-type operators is almost surely a deterministic subset of the real line. As soon as the disorder is switched on via a global coupling constant, the spectrum expands. We estimate how much the spectrum expands at its bottom in a discrete setting.

## CONTENTS

## 1. INTRODUCTION

Due to the self-averaging property of ergodic Schrödinger operators the resulting spectrum is almost surely a fixed subset of the real line. If a random operator is a perturbation of a periodic operator, it is of interest to know how the spectrum expands once we switch on the disorder via a global coupling constant. Apart from the genuine interest to identify the location of the spectrum, this is also of central importance when identifying energy regions where the spectrum is localised. Otherwise it may happen that one proves a Wegner estimate, a Lifschits tail bound or a similar statement related to localisation, and then later

discovers that the considered energy regime belongs to the resolvent set.

In this paper we consider an $\epsilon$-small random perturbation of a discrete translation-invariant operator and we study how the bottom of its spectrum behaves. By symmetry, similar estimates apply to the location of the maximum of the spectrum, in a weak disorder regime. To fix the ideas, let us introduce a prototypical example. Let $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ and $\Delta_{\mathbb{Z}^d} : \mathcal{H} \to \mathcal{H}$ the (negative definite) discrete Laplacian on $\mathbb{Z}^d$, i.e.

$$\left(\Delta_{\mathbb{Z}^d} u\right)(n) := \sum_{|n-m|_\infty = 1} \left(u(m) - u(n)\right).$$

We define the operator $H_0 : \mathcal{H} \to \mathcal{H}$ by

$$H_0 := -\Delta_{\mathbb{Z}^d} + W,$$

where $W$ is the operator multiplication by a real-valued function, which we also write $W$ and which we assume periodic with respect to the subgroup $\gamma := N\mathbb{Z}^d$.

Let $\square := [0, N-1]^d \subset \mathbb{Z}^d$ and $V^\square \in \ell^\infty(\mathbb{Z}^d)$ be a non-trivial, compactly supported single-site potential satisfying

$$\mathrm{supp}(V^\square) \subset \square.$$

Let $(\omega_k)_{k\in\gamma}$ be a sequence of non-trivial, bounded, independent, identically distributed random variables. For the sake of the introduction, assume that $\{-1, 1\} \in \mathrm{supp}\,\omega_0 \subset [-1, 1]$. From now on we denote by $V_\omega : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ the diagonal operator defined, for $f \in \ell^2(\mathbb{Z}^d)$, as

$$(V_\omega f)(x) = \sum_{k \in N\mathbb{Z}^d} \omega_k V^\square(x-k) f(x).$$

To motivate our results, let us consider the following discrete, alloy-type random Schrödinger operator defined by

$$(1) \qquad\qquad H_{\omega,\epsilon} := H_0 + \epsilon V_\omega.$$

Under the stated assumptions, this operator is ergodic, and thus there exists a set $\Sigma_\epsilon \subset \mathbb{R}$ such that

$$\sigma(H_{\omega,\epsilon}) = \Sigma_\epsilon$$

with probability 1 (see e.g. [18]). From now on we refer to $\Sigma_\epsilon$ as the *almost-sure spectrum* of $H_{\omega,\epsilon}$. The best known example of this kind of operators is the celebrated *Anderson model*, where $H_0$ is the discrete Laplacian on $\mathbb{Z}^d$ (i.e. $W \equiv 0$), $V^\square = \delta_0$ and $N = 1$. In this case, it is not hard to see ([18]) that the bottom of the spectrum of the perturbed operator $E_\epsilon := \inf(\Sigma_\epsilon)$ moves away from the bottom of the spectrum of the free operator $E_0 := \inf(\Sigma_0)$ as

$$E_\epsilon = E_0 - \epsilon.$$

If one considers instead, for example, the dipole model, i.e. $V^\square = \delta_0 - \delta_{e_1}$, it is proven in [4] that

$$E_\epsilon \leqslant E_0 - C\epsilon^2.$$

In this note we study this question for a very general, wide class of operators (see assumptions in section 2). More precisely, we prove some upper bounds of the quantity $E_\epsilon - E_0$, which in turns gives us information on the location of the spectrum of the perturbed operator. We also discuss some partial results on the lower bound.

In order to state the result in this setting, we need to consider the operator $H_0$ with $N\mathbb{Z}^d$-periodic boundary conditions. Because of the translation invariance, the subspace of $N\mathbb{Z}^d$-periodic functions in $\ell^\infty(\mathbb{Z}^d)$ is invariant under the action of $H_0$. This subspace is $N^d$-dimensional, so that the action of the operator corresponds to a matrix we denote by

$$(2) \qquad\qquad H_0^\square : \ell^2(\square) \to \ell^2(\square).$$

We now state the result.

**Theorem 1.1.** *Let $H_{\omega,\epsilon}$ be the alloy-type random Schrödinger operator defined by (1) and $E_\epsilon$ the bottom of its corresponding almost-sure spectrum. To the $N\mathbb{Z}^d$-periodic operator $H_0$ we associate a Hermitian matrix $H_0^\square \in \mathbb{C}^{N^d \times N^d}$, defined as in (2), and we let $\psi_1 \in \ell^2(\square)$ be the positive ground state of $H_0^\square$. Define*

$$A_1 := -\left\langle \psi_1, V^\square \psi_1 \right\rangle_{\ell^2(\square)}.$$

*There exists $A_2 \leqslant 0$ such that for $\epsilon > 0$ small enough*

$$E_\epsilon \leqslant E_0 + \epsilon A_1 + \epsilon^2 A_2.$$

*Furthermore, if $A_1 = 0$ then $|A_2|$ is non-zero and larger than the spectral gap of $H_0^\square$, i.e. the difference between its two smallest eigenvalues.*

We provide an explicit formula for the constant $A_2$ in the next section as it requires the introduction of many additional notations.

We have an analogous estimate for (fibers of) periodic operators, see Theorems 5.1 and 5.8. In fact, the estimate for periodic operators is one step in the proof of Theorem 1.1. In the context of periodic operators we have a related, complementary lower bound, see Lemmas 5.7 and 5.9.

We would like to make some remarks on the relevance of this result. First, the location of the bottom of the spectrum with respect to the coupling constant has been the subject of several papers : with periodic potentials in dimension one [21] and in arbitrary dimension [12], [6], as well as with random positive potentials [13] and under some generic assumptions on $W$ [15]. Understanding the spectrum provides valuable information on the solutions of partial differential equations. In particular, if one considers the Schrödinger equation for the Hamiltonian

$H_{\omega,\epsilon}$, the spectral type of the Anderson model characterizes the transport properties of the underlying disordered medium. For this model, the spectrum is expected to exhibit a transition from localized states at the bottom of the spectrum (pure point spectrum with exponentially localized eigenfunctions) to extended states (absolutely continuous spectrum) in the bulk of the spectrum. This *Anderson transition* is still a conjecture in the setting of this article. The existence of localized states at the bottom of the spectrum has been studied in many papers. We invite the reader to consult the monographs [5], [18], [20], [9] and their extensive bibliography. The perturbative regime $\epsilon \ll 1$ has attracted much attention [1], [22], [14], [15], [7], [4], [10], [8], [2], [3]. In this regime one can prove very precise estimates of the interval of localization, namely that states with energies in

$$I_\eta(\epsilon) := (-\infty, -C_0\epsilon^\eta] \cap \Sigma_\epsilon = (-\infty, -C_0\epsilon^\eta] \cap [E_\epsilon, +\infty)$$

are localized. In [7] it was proved that in dimension $d = 3$ one may take $\eta$ to be as large as 2. This result is meaningful, as for the Anderson model $E_\epsilon = -C_1\epsilon$. If we now consider different potentials, we may have a quadratic expansion of the bottom of the spectrum $E_\epsilon$, and understanding where the bottom of the spectrum lies appears to be crucial, so that the interval of localization is non-trivial. Some of the issues addressed in this note were already explored in [15] where it is assumed that the single-site potential has a non-zero mean and the Floquet eigenvalues of the underlying periodic potential $W$ are assumed to be non-degenerated, as well as in [4] for the dipole potential. These are special cases of our models. The general operator we study corresponds roughly to tridiagonal block matrices of the form

(3)

$$\begin{pmatrix} \ddots & \ddots & \ddots & & & \\ B^* & A & B & 0 & \ddots & \\ 0 & B^* & A & B & 0 & \\ \ddots & 0 & B^* & A & B & \\ & & \ddots & \ddots & \ddots & \end{pmatrix} + \begin{pmatrix} \ddots & & \ddots & & \ddots & \\ 0 & \omega_{n-1}V^\square & 0 & & \ddots & \\ \ddots & 0 & \omega_n V^\square & 0 & & \ddots \\ & \ddots & & 0 & \omega_{n+1}V^\square & 0 \\ & & \ddots & & \ddots & \ddots \end{pmatrix},$$

where $A$ and $V^\square$ are Hermitian and $\{\omega_n\}$ i.i.d. random variables. We introduce in section 2 the general framework in which our results are obtained.

To finish the introduction, let us briefly address the question of the optimality of the lower bound (or at least its exponent). As far as the authors know, there is no result in the literature in this direction in a discrete setting (apart from the Anderson model, where the bottom of the spectrum is known explicitly). One may naively expect, from perturbation theory, that the behavior should be linear or quadratic. The question turns out to be more subtle as the behavior may depend

on the speed at which the Floquet eigenvalues associated to the bottom of the spectrum approach their minimum, as the following example shows.

**Theorem 1.2.** *Let $H_0 := (-\Delta_{\mathbb{Z}})^2$ defined on $\ell^2(\mathbb{Z})$ and $V^{\square}$ the multiplication operator given by the following single-site potential:*

$$V^{\square}(n) := -\frac{1}{2}\delta_{-1}(n_1) + \delta_0(n_1) - \frac{1}{2}\delta_1(n_1).$$

*Pick some $\xi > 1/4$. Then for $\epsilon > 0$ small enough we have*

$$(4) \qquad\qquad E_\epsilon := \inf \sigma(H_0 + \epsilon V_\omega) \leqslant -\frac{1}{6}\epsilon^{1+2\xi}.$$

For this example (which is of the form (3)), the coefficient $A_1$ corresponding to the linear term vanishes. The bound in Theorem 1.2 is nevertheless better than quadratic thanks to the quartic behavior of the Floquet eigenvalues in a neighbourhood of their minimum. We believe that in the situation of Theorem 1.1, the infimum of the spectrum should expand linearly or quadratically, due to the quadratic behavior of the Floquet eigenvalues at the bottom of the spectrum. Unfortunately, apart from the trivial linear bound, we have no corresponding lower bound, although some results in this article provide a first step in this direction.

This work can be extended in several directions. It would be very interesting to find the corresponding lower bounds, or at least conditions under which the infimum of the spectrum does not expand linearly. A related question concerns the expansion of the spectrum near a band edge, where one can also prove Anderson localisation. Note that if one studies the expansion of the spectrum from a band edge instead of the bottom of the spectrum, the Floquet eigenvalues may vanish faster than quadratically when approaching the edge, even for the operator defined by (1). Rather than a pathological example, Theorem 1.2 provides a model for this situation. A last question of interest is the study of overlapping single-site potentials. Under some non-degeneracy condition (see Remark 4.5) the results stated here can be extended to this situation, but a full understanding needs to consider periodic approximations of every order, something we also believe necessary to prove the corresponding lower bounds.

In a forthcoming project we consider the same questions for operators of Schrödinger type in the continuum setting, i.e. for operators acting on (dense subspaces) of $L^2(\mathbb{R}^d)$. Most of our findings are similar. In the continuum, it is more natural to define the operators via quadratic forms, and then formulate appropriate regularity conditions. Also, certain additional compactness arguments are necessary, due to the infinite dimensionality of the Hilbert space liperiodicity cell of the

. On the other hand, in specific situations, better results are possible in the continuum setting, due to unique continuation principles for solutions of partial differential equations.

## 2. General model

Let $d \geqslant 1$ be the space dimension, $D = \mathbb{Z}^d$ be the physical space and $\gamma = N\mathbb{Z}^d$ a sub-lattice of $D$. We denote by $\square$ its periodicity cell, i.e. $\square := [0, N-1]^d \cap \mathbb{Z}^d$. Note that $D = \bigcup_{k \in \gamma} \{x \in D : x - k \in \square\}$. We also denote the reciprocal periodicity cell as $\square^* := [0, \frac{2\pi}{N})^d$. From now on we assume the following hypotheses to hold.

(HA) Let $H_0 : \ell^2(D) \to \ell^2(D)$ be a bounded, non-negative symmetric operator defined by the matrix

$$H_0 := (H_0(k, k'))_{k,k' \in D},$$

satisfying the following properties:
  - for all $k, k' \in D$, we have $H_0(k, k') = \overline{H_0(k, k')}$;
  - there exists $k_0 \neq 0$ such that $H_0(0, k_0) \neq 0$;
  - the associated operator is $\gamma$-invariant, i.e. for every $k \in \gamma$

$$\langle \tau_k u, H_0 \tau_k v \rangle = \langle u, H_0 v \rangle,$$

    where $u, v \in \ell^2(D)$ and $\tau_k$ is the translation by $k \in \gamma$ operator; and
  - the associated operator is of *finite hopping range* with hopping range $N$, i.e. if $|k - k'| \geqslant N$ then

$$H_0(k, k') = 0.$$

  - Through a global energy shift we may assume, with no loss of generality, that $E_0 := \inf \sigma(H_0) = 0$.

  Note that if an operator is of finite hopping range with hopping range $R$, for some $R > 0$, then it also is of finite hopping range with hopping range $R'$ for any $R' > R$. On the other hand, any $\gamma$-invariant operator is also $n\gamma$-invariant, $n \in \mathbb{N}$. This means that we can always assume that $R = N$, without loss of generality.

(HB) Let $V^\square : \square \to \mathbb{R}$ be a non-trivial Hermitian matrix (we call it the *single-cell potential*, even when $V^\square$ is not diagonal). For any bounded sequence $(\omega_k)_{k \in \gamma}$ of real numbers, we define the block diagonal operator

$$V_\omega : \ell^2(D) \to \ell^2(D)$$

$$V_\omega := \sum_{k \in \gamma} \omega_k \tau_{-k} V^\square \tau_k.$$

For any real number $q \in \mathbb{R}$, we denote also by $q$ the constant sequence indexed by $\gamma$, equal to $q$ on every site. We thus have, for example, that

$$(5) \qquad V_q := q \sum_{k \in \gamma} \tau_{-k} V^\square \tau_k$$

and $V_q$ is $\gamma$-invariant.

From now on, the values of $\omega$ will be drawn from a sequence of bounded, non-trivial, independent and identically distributed random variables with distribution measure $\mu$. We will write $S_\mu := \operatorname{supp} \mu$ and we assume that

$$\{s_-, s_+\} \in S_\mu \subset [s_-, s_+],$$

where $s_-$ and $s_+$ satisfy one of the following alternatives:

(HC) The random variables change sign, i.e. $s_- < 0 < s_+$.
(HC') The random variables are positive, i.e. $0 \leqslant s_- < s_+$.

The methods in this paper may also be adapted to negative random variables.

*Remark* 2.1. It looks tempting to renormalize the random variables by adding and substracting some periodic potential, but in this case the underlying non-random operator depends on $\epsilon$. On the other hand, it is indeed allowed to rescale the random variables by absorbing the scaling factor in the single site potential $V^\square$.

Let us now define our object of study. For each $\epsilon > 0$, we let

$$H_{\omega,\epsilon} := H_0 + \epsilon V_\omega$$

which is a self-adjoint, ergodic operator. We denote its almost-sure spectrum by $\Sigma_\epsilon$ and by

$$(6) \qquad E_\epsilon := \inf \Sigma_\epsilon$$

the bottom of the spectrum. We also write $H_{q,\epsilon} := H_0 + \epsilon V_q$ the corresponding operator with $V_\omega$ replaced by the periodic potential $V_q$ (defined as in (5)) and $E_{q,\epsilon} := \inf \sigma(H_{q,\epsilon})$. In the following we will study the bottom of the spectrum $E_\epsilon$ of the random operator for small $\epsilon$.

We define a finite dimensional matrix associated to the above objects. Define the $(|\square| \times |\square|)$-matrix $H_0^\square(\theta)$ by its coefficients

$$(7) \qquad \left(H_0^\square(\theta)\right)(k, k') := \sum_{m \in \gamma} e^{i\theta \cdot m} H_0(k, k' - m)$$

$$= \sum_{\substack{|m| \leqslant N \\ m \in N\mathbb{Z}^d}} e^{i\theta \cdot m} H_0(k, k' - m),$$

where $k, k' \in \square$. Note that the second line is a consequence of the finite hopping range and the sum in (7) is thus finite. Now define the matrix $H_{q,\epsilon}^{\square}(\theta)$ by

$$H_{q,\epsilon}^{\square}(\theta) := H_0^{\square}(\theta) + \epsilon q V^{\square}.$$

*Remark* 2.2. The matrix $H_{q,\epsilon}^{\square}$ represents the action of $H_{q,\epsilon}$ on the fiber of $\theta$-quasiperiodic functions in the Floquet–Bloch direct integral decomposition. More precisely, let (abusing notation) $\varphi \in \ell^2(\square) \subset \ell^2(\mathbb{Z}^d)$. Then, regarding $H_{q,\epsilon}$ as an operator $\ell^{\infty}(\mathbb{Z}^d) \to \ell^{\infty}(\mathbb{Z}^d)$,

$$(8) \quad H_{\epsilon,q}^{\square}(\theta)\varphi = \chi_{\square} H_{\epsilon,q} \sum_{m \in \gamma} e^{i\theta \cdot m} \tau_m \varphi \in \ell^2(\square), \quad \text{(abusing notation)}$$

where $\chi_{\square}$ is the indicator function of $\square \subset \mathbb{Z}^d$.

## 3. Main results

Recall that, by the continuity of the Floquet–Bloch eigenvalues ([16], [19]), there exists some $\theta$ such that

$$E_0 := \inf \sigma(H_0) = \inf \sigma(H_0^{\square}(\theta)) = 0.$$

We denote by $\Theta \subset \square^*$ the compact set of $\theta$ for which the last equality holds. From now on we fix some $\theta \in \Theta$, so the quantities below will depend on $\theta$. Let $\mathcal{V}_0$ be the eigenspace of $H_0(\theta)$ associated to the eigenvalue $E_0 = 0$, $p$ its multiplicity and choose an orthonormal basis $\psi_j$, $j = 1, \dots, p$ spanning $\mathcal{V}_0$ and diagonalizing the Hermitian matrix $A \in \mathbb{C}^{p \times p}$, given by the coefficients

$$A_{ij} := \langle \psi_i, V^{\square} \psi_j \rangle.$$

We take the eigenvalues of matrix $A$ in the ascending order counting multiplicities so that $P_1 := A_{11} = \langle \psi_1, V^{\square} \psi_1 \rangle$ is the minimal eigenvalue and $P_p := A_{pp} = \langle \psi_p, V^{\square} \psi_p \rangle$ is the maximal eigenvalue of $A$.

Our result for sign-changing random variables read as follows.

**Theorem 3.1.** *Assume* (HA), (HB) *and* (HC). *Fix* $\theta \in \Theta$ *and define*

$$(9) \quad A_1 := \inf_{q \in S_{\mu}} \inf_{\substack{\psi \in \mathcal{V}_0 \\ \|\psi\|_{\ell^2(\square)}=1}} q \langle \psi, V^{\square} \psi \rangle = \min(s_+ P_1, s_- P_p) \leqslant 0,$$

*and*

$$(10) \quad A_2 := -\max(s_-^2, s_+^2) \sup_{\substack{\psi \in \mathcal{V}_0 \\ \|\psi\|_{\ell^2(\square)}=1}} \sup_{\substack{\varphi \in \mathcal{V}_0^{\perp} \\ \|\varphi\|_{\ell^2(\square)}=1}} \frac{\left| \langle \psi, V^{\square} \varphi \rangle \right|^2}{\langle H_0^{\square}(\theta)\varphi, \varphi \rangle} \leqslant 0.$$

*For any* $\epsilon > 0$ *small enough the following holds: if* $A_1 \neq 0$,

$$E_{\epsilon} \leqslant \epsilon A_1,$$

*whereas if $A_1 = 0$, then*

$$E_\epsilon \leqslant \epsilon^2 A_2 + O(\epsilon^3).$$

*Finally, if $A_1 = A_2 = 0$, then*

$$E_\epsilon \leqslant 0.$$

Our result for positive random variables read as follows.

**Theorem 3.2.** *Assume* (HA), (HB) *and* (HC'). *Fix $\theta \in \Theta$. Let us define the subspace $\mathcal{V}_{01} \subset \mathcal{V}_0$ as*

$$\mathcal{V}_{01} := \operatorname*{span}_{\{i: P_i = P_1\}} \langle \psi_i \rangle,$$

*i.e. the eigenspace of $A$ associated to its minimal eigenvalue $P_1$. Define*

$$(11) \qquad A_1' := \inf_{q \in S_\mu} \inf_{\substack{\psi \in \mathcal{V}_{01} \\ \|\psi\|_{\ell^2(\square)} = 1}} q \langle \psi, V^\square \psi \rangle = \min(s_+ P_1, s_- P_1) \in \mathbb{R},$$

*and*

$$(12) \qquad A_2' := -s_+^2 \sup_{\substack{\psi \in \mathcal{V}_{01} \\ \|\psi\|_{\ell^2(\square)} = 1}} \sup_{\substack{\varphi \in \mathcal{V}_0^\perp \\ \|\varphi\|_{\ell^2(\square)} = 1}} \frac{\left| \langle \psi, V^\square \varphi \rangle \right|^2}{\langle H_0^\square(\theta) \varphi, \varphi \rangle} \leqslant 0.$$

*For any $\epsilon > 0$ small enough the following holds: if $P_1 \neq 0$,*

$$E_\epsilon \leqslant \epsilon A_1',$$

*whereas if $P_1 = 0$, then*

$$E_\epsilon \leqslant \epsilon^2 A_2' + O(\epsilon^3).$$

*Finally, if $P_1 = A_2' = 0$, then*

$$E_\epsilon \leqslant 0.$$

In the following section we will reduce the problem of studying $E_\epsilon$ to that of understand

## 4. Periodic comparison operators

Define

$$\square_n := \bigcup_{\substack{m \in \gamma \\ |m| \leqslant nN}} \square + m$$

and $\chi_n := \chi_{\square_n}$, i.e.

$$\chi_n(x) = \begin{cases} 1 & \text{for } x \in \square_n \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\square_0 = \square$ and that $\square_n$ is just the collection of $(2n+1)^d$ disjoint translates of $\square$. Let us start by stating the following lemma.

**Lemma 4.1.** *Let $u$ be a $\theta$-quasi-$\gamma$-periodic function, i.e. such that for all $n \in \mathbb{Z}^d$ and $k \in \gamma$ we have*

$$u(n+k) = e^{-i\theta \cdot k} u(n).$$

*Define,*

$$u_n := \chi_n u, \quad n = 0, 1, 2, \ldots.$$

*Then*

$$\lim_{n \to \infty} \frac{\langle u_n, H_{q,\epsilon} u_n \rangle_{\ell^2(D)}}{\|u_n\|_{\ell^2(D)}^2} = \frac{\langle u_0, H_{q,\epsilon}^{\square}(\theta) u_0 \rangle_{\ell^2(\square)}}{\|u_0\|_{\ell^2(\square)}^2}.$$

The proof of this lemma is found in the appendix.

Let us now define

$$\Omega_{\mathrm{per}}^n := \{\omega \in \Omega : \omega \text{ is periodic w.r.t. } n\gamma\}.$$

We now state the first comparison theorem.

**Theorem 4.2.** *Assume* (HA), (HB) *and either* (HC) *or* (HC'). *Let $\epsilon \geqslant 0$, $n \in \mathbb{N}$ and let $\omega \in \Omega_{\mathrm{per}}^n$ be a $n\gamma$-periodic sequence of real numbers satisfying $\omega \in (S_\mu)^\gamma$, i.e. $\omega_k \in S_\mu$ for all $k \in \gamma$. Then, we have*

$$\sigma(H_{\omega,\epsilon}) \subset \Sigma_\epsilon.$$

We immediately deduce the following upper bound on the minimum of the spectrum.

**Corollary 4.3.** *Assume* (HA), (HB) *and either* (HC) *or* (HC'). *Let $\epsilon \geqslant 0$, then*

$$E_\epsilon \leqslant \inf_{q \in S_\mu} E_{q,\epsilon}.$$

*Proof of Theorem 4.2.* For the calculation below, we need a Weyl sequence of compactly supported functions. This can indeed be done, since we only deal with bounded operators. Fix $\omega \in \Omega_{\mathrm{per}}^n$ and $E \in \sigma(H_{\omega,\epsilon})$. By Floquet–Bloch theory, there exists some $\theta$ and some normalized state $f \in \ell^2(\square)$ for which

$$E = \langle H_{q,\epsilon}^{\square}(\theta) f, f \rangle.$$

We extend $f$ as a $\theta$-quasi-$\gamma$-periodic function, i.e. for any $x \in \mathbb{Z}^d$ let $k \in \gamma$ such that $x - k \in \square$ and let

$$f(x) := e^{i\theta \cdot k} f(x - k).$$

Using Lemma 4.1, extract a sub-sequence $\{f_n\}$ from the sequence of functions $\left\{ \dfrac{\chi_n f}{\|\chi_n f\|_2} \right\}$, such that

$$|\langle (H_{\omega,\epsilon} - E) f_n, f_n \rangle| \leqslant 1/n$$

and satisfying, for a sequence $l_n \in \mathbb{N}$,

$$\mathrm{supp}\, f_n \subset \Lambda_{l_n},$$

where $\Lambda_{l_n}$ is a cube centered at zero and sidelength $l_n$. For $x \in \gamma$ we define

$$\Omega(x, n) := \{\omega' \in \Omega : \forall k \in (x + \Lambda_{l_n}) \cap \gamma : |\epsilon(\omega_k' - \omega_k)| \leqslant 1/n\} .$$

Now, since $\omega \in (S_\mu)^\gamma$,

$$\mathbb{P}\left[\Omega(x, n)\right] > 0,$$

and for $x, y \in \gamma$ satisfying $|x - y| > l_n$, the events $\Omega(x, n)$ and $\Omega(y, n)$ are independent (and identically distributed). Using Borel–Cantelli lemma, we see that the event

$$\Omega' := \bigcap_{n \in \mathbb{N}} \bigcup_{x \in \gamma} \Omega(x, n)$$

has probability one.

From the definition of $\Omega(x, n)$, we have that given $\omega' \in \Omega'$ and $n \in \mathbb{N}$, there exists a $x(n, \omega')$ such that $\omega' \in \Omega(x, n)$. We write from now on $\tau_{x(n,\omega')} f_n$ for the translated function $f_n(\cdot - x(n, \omega'))$. Let $\omega' \in \Omega'$ and $n \in \mathbb{N}$, and calculate

$$\left\langle (H_{\omega',\epsilon} - E)\tau_{x(n,\omega')} f_n, \tau_{x(n,\omega')} f_n \right\rangle$$
$$= \left\langle (H_0 - E)\tau_{x(n,\omega')} f_n, \tau_{x(n,\omega')} f_n \right\rangle + \epsilon \left\langle V_{\omega'} \tau_{x(n,\omega')} f_n, \tau_{x(n,\omega')} f_n \right\rangle$$
$$= \left\langle (H_0 - E) f_n, f_n \right\rangle + \epsilon \left\langle V_\omega \tau_{x(n,\omega')} f_n, \tau_{x(n,\omega')} f_n \right\rangle$$
$$\qquad\qquad + \epsilon \left\langle V_{\omega'-\omega} \tau_{x(n,\omega')} f_n, \tau_{x(n,\omega')} f_n \right\rangle$$
$$= \left\langle (H_{\omega,\epsilon} - E) f_n, f_n \right\rangle + \epsilon \left\langle V_{\omega'-\omega} \tau_{x(n,\omega')} f_n, \tau_{x(n,\omega')} f_n \right\rangle$$

Note that $|\epsilon V_{\omega'-\omega}(x)| \leqslant \|V^\square\|_\infty/n$ if $x \in \operatorname{supp} f_n(\cdot - x(n, \omega'))$, so that

$$\left| \left\langle (H_{\omega',\epsilon} - E)\tau_{x(n,\omega')} f_n, \tau_{x(n,\omega')} f_n \right\rangle - \left\langle (H_{\omega,\epsilon} - E) f_n, f_n \right\rangle \right| \leqslant \frac{1}{n}\|V^\square\|_\infty.$$

In particular, we see that $\tau_{x(n,\omega')} f_n$ is a Weyl sequence. $\qquad\square$

*Remark* 4.4. This is an adaptation of a well known argument of Kirsch and Martinelli [11] in the continuous setting, with $S_\mu$ connected and $V^\square$ a multiplication operator.

*Remark* 4.5. When the random potential is diagonal (as in the introduction), the proof above can be adapted to *overlapping*, but compactly supported single-site potentials $V^\square \in \ell^\infty(D)$ as long as

$$\sum_{n \in \gamma} V^\square(\cdot - n) \not\equiv 0.$$

Note that if this condition does not hold then $H_q = H_0$ for all $q$. One way around this problem would be to consider periodic (non-constant) sequences of coupling constants $\omega_n$ such that the resulting periodic potential is not zero.

We prove the following converse to Theorem 4.2. We define

$$\Omega_{\mathrm{per}} := \{\omega \in \Omega : \exists n \in \mathbb{N} \text{ such that } \omega \text{ is periodic w.r.t. } n\gamma\} = \overline{\bigcup_{n \in \mathbb{N}} \Omega_{\mathrm{per}}^n}.$$

**Lemma 4.6.** *Denote by* $\Sigma_\epsilon$ *the almost sure spectrum of* $H_{\omega,\epsilon}$. *Then:*

$$\Sigma_\epsilon \subset \overline{\bigcup_{\omega \in \Omega_{per}} \sigma(H_{\omega,\epsilon})}.$$

*Proof.* Let $n \in \mathbb{N}$ and set

$$\omega_k^{(n)} = \omega_k \quad \text{for } f \in \square_N$$
$$\omega_k^{(n)} = \omega_j \quad \text{if } j - k \in N\gamma.$$

Let $C_0(\mathbb{Z}^d)$ be the set of compactly supported functions in $\ell^2(\mathbb{Z}^d)$. Choose any $\varphi \in C_0(\mathbb{Z}^d)$. Then

$$\lim_{n \to \infty} \|H_{\omega,\epsilon}\varphi - H_{\omega^{(n)},\epsilon}\varphi\| = 0,$$

i.e. we have strong convergence $H_{\omega^{(n)}} \to H_\omega$. Since the operators $H_\omega$ are bounded, the set $C_0$ is an operator core for $H_\omega$. This implies that we have strong convergence on the whole $\ell^2(\mathbb{Z}^d)$.

By the resolvent equation, for any $E \in \mathbb{R} \backslash \Sigma$,

$$(H_{\omega,\epsilon} - E)^{-1} - (H_{\omega^{(n)},\epsilon} - E)^{-1}$$
$$= (H_{\omega,\epsilon} - E)^{-1}(V_\omega - V_{\omega^{(n)}})(H_{\omega^{(n)},\epsilon} - E)^{-1}$$
$$= (H_{\omega^{(n)},\epsilon} - E)^{-1}(V_\omega - V_{\omega^{(n)}})(H_{\omega,\epsilon} - E)^{-1},$$

which converges strongly to 0. We know that if $E \in \mathbb{R} \backslash \Sigma$, then $(H_{\omega,\epsilon} - E)^{-1}\varphi \in \ell^2(D)$ for any $\varphi \in \ell^2(D)$ and that, using Theorem 4.2, the inclusion $\sigma(H_{\omega,\epsilon}(n)) \subset \Sigma$ holds for any $\omega$ in the support of the product measure $\bigotimes_D \mu$. To conclude, we apply Theorem VIII.24 in [19] which tells us that

$$\sigma(H_{\omega,\epsilon}) \subset \overline{\bigcup_{n \in \mathbb{N}} \sigma(H_{\omega^{(n)}})}.$$

This finishes the proof.                                                      □

In particular we obtain the following corollary.

**Corollary 4.7.** *Let*

$$\Omega_{per} := \{\omega \in \Omega : \exists N \in \mathbb{N} \text{ such that } \omega \text{ is periodic w.r.t. } N\gamma\}$$

*and denote by* $\Sigma_\epsilon$ *the almost sure spectrum of* $H_{\omega,\epsilon}$. *Then:*

$$\inf \Sigma_\epsilon = \inf \bigcup_{\omega \in \Omega_{per}} \sigma(H_{\omega,\epsilon}).$$

## 5. Perturbation calculation

For the readers convenience we recall the definition of the constants $A_1$ and $A_2$, the notation and the statement of the theorems before the proofs. By the continuity of the Floquet–Bloch eigenvalues there exists some $\theta$ such that

$$E_0 := \inf \sigma(H_0) = \inf \sigma(H_0^\square(\theta)) = 0.$$

We denote by $\Theta \subset \square^*$ the compact set of $\theta$ for which the last equality holds. From now on we fix some $\theta \in \Theta$, so the quantities below will depend on $\theta$. Let $\mathcal{V}_0$ be the eigenspace of $H_0(\theta)$ associated to the eigenvalue $E_0 = 0$, $p$ its multiplicity and choose an orthonormal basis $\psi_j$, $j = 1, \ldots, p$ spanning $\mathcal{V}_0$ and diagonalizing the Hermitian matrix $A \in \mathbb{C}^{p \times p}$, given by the coefficients

$$A_{ij} := \left\langle \psi_i, V^\square \psi_j \right\rangle.$$

We take the eigenvalues of the matrix $A$ in the ascending order counting multiplicities so that $P_1 := A_{11} = \left\langle \psi_1, V^\square \psi_1 \right\rangle$ is the minimal eigenvalue and $P_p := A_{pp} = \left\langle \psi_p, V^\square \psi_p \right\rangle$ is the maximal eigenvalue of $A$.

5.1. **Sign-changing random variables.** In this subsection we assume (HC) to hold. We will only treat this case in detail as the calculation for positive random variables is very similar. Recall from (HC) that $s_- < 0 < s_+$. We define the following quantities :

$$(13) \qquad A_1 := \inf_{\substack{q \in S_\mu}} \inf_{\substack{\psi \in \mathcal{V}_0 \\ \|\psi\|_{\ell^2(\square)} = 1}} q \left\langle \psi, V^\square \psi \right\rangle = \min(s_+ P_1, s_- P_p) \leqslant 0,$$

and

$$(14) \qquad A_2 := -\max(s_-^2, s_+^2) \sup_{\substack{\psi \in \mathcal{V}_0 \\ \|\psi\|_{\ell^2(\square)} = 1}} \sup_{\substack{\varphi \in \mathcal{V}_0^\perp \\ \|\varphi\|_{\ell^2(\square)} = 1}} \frac{\left| \left\langle \psi, V^\square \varphi \right\rangle \right|^2}{\langle H_0^\square(\theta)\varphi, \varphi \rangle} \leqslant 0.$$

We will prove the following theorem, which is only a restatement of Theorem 3.1.

**Theorem 5.1.** *Assume* (HA), (HB) *and* (HC). *Fix $\theta \in \Theta$. Then, if $A_1 \neq 0$,*

$$E_\epsilon \leqslant \epsilon A_1,$$

*whereas if $A_1 = 0$, then*

$$E_\epsilon \leqslant \epsilon^2 A_2 + O(\epsilon^3).$$

*Finally, if $A_1 = A_2 = 0$, then*

$$E_\epsilon \leqslant 0.$$

*Remark* 5.2.

- We remind that we have fixed $\theta$ to simplify notations, but $A_1$ and $A_2$ depend on $\theta$. The best bound for the behavior of the bottom of the spectrum is obtained by looking at each $\theta \in \Theta$ and taking the minimum.
- We see that our bound on the bottom of the spectrum behaves linearly, quadratically or it doesn't move with $\epsilon$. In the continuous analogous setting, if the unique continuation principle is not violated, then the analogous result does not allow the third case $A_1 = A_2 = 0$. This yields only a linear or quadratic bound.
- The definition of the quantities $A_1$, $A_2$ may seem complicated at first sight, but these choices are optimal, in the sense of Lemma 5.7 below, which is a converse of Lemma 5.6 in the regime $\epsilon \ll 1$.

Before proving the theorem, let us provide a much simpler, non-optimal upper bound for $A_2$ as well as a condition ensuring that $|A_1| + |A_2| \neq 0$.

5.2. **A simple non-degeneracy condition.** Theorem 5.1 tells us that if $A_2 \neq 0$, then the expansion of the bottom of the spectrum is at least quadratically, but if $A_1 = A_2 = 0$ we can only say that the spectrum starts at zero. When $V^\square$ is diagonal this only happens if the support of the single-cell potential and the eigenfunctions $\psi_1, \ldots, \psi_p$ are disjoint (the $\psi_i$ were defined at the beginning of section 5). In a continuous setting this can only happen if the potential violates the unique continuation principle. For a discussion on the validity of the unique continuation principle see for instance [23].

Let us discuss the condition in our general setting. First let us remark that if $A_1 = 0$, then the matrix $A \in \mathbb{C}^{p \times p}$ vanishes identically, i.e.

$$(15) \qquad A_1 = 0 \implies \sup_{\substack{\psi \in \mathcal{V}_0 \\ \|\psi\|_{\ell^2(\square)} = 1}} \left| \left\langle \psi, V^\square \psi \right\rangle \right| = 0.$$

Thus, $A_1 = 0$ implies

$$(\forall \psi \in \mathcal{V}_0) \quad V^\square \psi \in \mathcal{V}_0^\perp$$

because of (15). The operator $H_0^\square$ is invertible on $\mathcal{V}_0^\perp$ and thus there exists thus some $\varphi \in \mathcal{V}_0^\perp$ such that

$$(16) \qquad H_0^\square(\theta)\varphi = V^\square \psi^*.$$

Hence, we have that

$$\left\langle V^\square \varphi, \psi^* \right\rangle = \left\langle H_0^\square(\theta)\varphi, \varphi \right\rangle.$$

Now, assume there exists some $\psi^* \in \mathcal{V}_0$ such that

$$(17) \qquad V^\square \psi^* \neq 0.$$

Then $\varphi$ in (16) does not vanish and

$$A_2 \leqslant -\max(s_-^2, s_+^2) \frac{\left| \left\langle V^\square \varphi, \psi^* \right\rangle \right|^2}{\left\langle H_0^\square(\theta)\varphi, \varphi \right\rangle} = -\max(s_-^2, s_+^2) \left\langle H_0^\square(\theta)\varphi, \varphi \right\rangle < 0,$$

because $\varphi \notin \ker H_0^{\square}(\theta)$.

*Remark* 5.3. Formally, we have

$$A_2 \leqslant -\max(s_-^2, s_+^2) \left\langle \psi^*, V^{\square} H_0^{\square}(\theta)^{-1} V^{\square} \psi^* \right\rangle$$

when $A_1 = 0$.

In the converse direction, $A_1 = 0$ together with $A_2 = 0$ implies that

$$(\forall \psi \in \mathcal{V}_0)(\forall \varphi \in \ell^2(\square)) \quad \left\langle V^{\square} \psi, \varphi \right\rangle = 0,$$

i.e. that

$$(\forall \psi \in \mathcal{V}_0) \quad V^{\square} \psi = 0.$$

We summarize the above discussion as follows.

**Lemma 5.4.** *Under the assumptions of Theorem 5.1 we have that*

$$A_1 = 0 \text{ and } A_2 = 0 \quad \text{if and only if} \quad (\forall \psi^* \in \mathcal{V}_0) \ V^{\square} \psi^* = 0.$$

5.3. **Proof of Theorem 5.1.** We subdivide the proof of Theorem 5.1 into two lemmas.

**Lemma 5.5.** *Assume (HA), (HB) and either (HC) or (HC'). Let $u \in \ell^2(\square)$ and $E_\epsilon$ as in (6). Then,*

$$E_\epsilon \leqslant \inf_{q \in S_\mu} \inf_{u \in \ell^2(\square)} \frac{\left\langle H_{\epsilon,q}^{\square}(\theta)u, u \right\rangle}{\|u\|_{\ell^2(\square)}} \qquad \text{for any } \theta \in \square^*$$

*Proof.* By Corollary 4.3 it is enough to consider the periodic realizations of the potential. By the Courant–Weyl–Fischer min–max principle,

$$(18) \qquad E_\epsilon \leqslant E_{q,\epsilon} = \min \sigma(H_{\epsilon,q}) = \inf_{\substack{a \in \ell^2(\mathbb{Z}^d) \\ \|a\|_2 = 1}} \left\langle H_{\epsilon,q} a, a \right\rangle.$$

Finally, by Lemma 4.1,

$$(19) \qquad \inf_{\substack{a \in \ell^2(\mathbb{Z}^d) \\ \|a\| = 1}} \left\langle H_{\epsilon,q} a, a \right\rangle \leqslant \inf_{u \in \ell^2(\square)} \frac{\left\langle H_{\epsilon,q}^{\square}(\theta)u, u \right\rangle}{\|u\|_{\ell^2(\square)}}.$$

This proves the lemma.

∎

We state now the second lemma.

**Lemma 5.6.** *Let $A_1$, $A_2$ as in (9), (10), assume (HA)-(HC) and fix $\theta \in \Theta$. Then, for $\epsilon > 0$ small enough, if $A_1 \neq 0$,*

$$\inf_{q \in S_\mu} \inf_{\|u\|_{\ell^2(\square)} = 1} \left\langle H_{\epsilon,q}^{\square}(\theta)u, u \right\rangle \leqslant \epsilon A_1,$$

*whereas if $A_1 = 0$, then*

$$\inf_{q \in S_\mu} \inf_{\|u\|_{\ell^2(\square)} = 1} \left\langle H_{\epsilon,q}^{\square}(\theta)u, u \right\rangle \leqslant \epsilon^2 A_2 + O(\epsilon^3)$$

*Finally, if $A_1 = A_2 = 0$, then*

$$\inf_{q \in S_\mu} \inf_{\|u\|_{\ell^2(\square)}=1} \left\langle H_{\epsilon,q}^\square(\theta)u, u \right\rangle \leqslant 0.$$

*Proof.* It is enough to show that for some $q \in S_\mu$, there is some normalized state $u \in \ell^2(\square)$ satisfying

$$\left\langle H_{\epsilon,q}^\square(\theta)u, u \right\rangle \leqslant \epsilon A_1 \quad \text{or} \quad \epsilon^2 A_2 + O(\epsilon^3) \quad \text{or} \quad 0 \quad \text{resp.}.$$

Let $\psi \in \mathcal{V}_0$ and $\varphi \in \mathcal{V}_0^\perp$, to be chosen later, and $u = \psi + \epsilon q \varphi$. We assume furthermore $\|\psi\| = 1$. We expand

$$\|u\|^2 = \|\psi\|^2 + \epsilon^2 q^2 \|\varphi\|^2$$

and thus

$$(20) \qquad 1/\|u\|^2 = 1 - \epsilon^2 q^2 \|\varphi\|^2 + O(\epsilon^4 \|\varphi\|^4).$$

We calculate the kinetic energy of this state, i.e.
$$(21)$$
$$\left\langle H_0^\square(\theta)u, u \right\rangle = \left\langle H_0^\square(\theta)\psi, \psi \right\rangle + 2\epsilon q \operatorname{Re} \left\langle H_0^\square(\theta)\psi, \varphi \right\rangle + \epsilon^2 q^2 \left\langle H_0^\square(\theta)\varphi, \varphi \right\rangle.$$

Because $\psi \in \mathcal{V}_0$ and $E_0 = 0$, we see that (21) becomes

$$\left\langle H_0^\square(\theta)u, u \right\rangle = \epsilon^2 q^2 \left\langle H_0^\square(\theta)\varphi, \varphi \right\rangle.$$

We expand the potential energy as

$$\epsilon q \left\langle V^\square u, u \right\rangle = \epsilon q \left\langle V^\square \psi, \psi \right\rangle + 2\epsilon^2 q^2 \operatorname{Re} \left\langle V^\square \varphi, \psi \right\rangle + \epsilon^3 q^3 \left\langle V^\square \varphi, \varphi \right\rangle.$$

Thus,

$$\left\langle H_{\epsilon,q}^\square(\theta)u, u \right\rangle = \epsilon q \left\langle V^\square \psi, \psi \right\rangle + \epsilon^2 q^2 \left( \left\langle H_0^\square(\theta)\varphi, \varphi \right\rangle + 2 \operatorname{Re} \left\langle \psi, V^\square \varphi \right\rangle \right)$$
$$(22) \qquad\qquad + \epsilon^3 q^3 \left\langle V^\square \varphi, \varphi \right\rangle.$$

*Case $A_1 \neq 0$.* Note that in this case $P_1 P_p \neq 0$. From now on we assume that $s_+ P_1 \leqslant s_- P_p$. If this is not the case, we can always replace $V^\square \mapsto -V^\square$ and $\omega_n \mapsto -\omega_n$ to get an equivalent model. In this case, we take $\psi = \psi_1$, $\varphi = 0$ and $q = s_+$. Then, (22) becomes

$$\left\langle H_{\epsilon,q}^\square(\theta)u, u \right\rangle = \epsilon q \left\langle V^\square \psi_1, \psi_1 \right\rangle = \epsilon s_+ P_1,$$

which proves the result in this case, as $u$ is normalized.

*Case $A_1 = 0$ and $A_2 \neq 0$.* First let us remark that if $A_1 = 0$ then the matrix $A \in \mathbb{C}^{p \times p}$ vanishes identically, i.e.

$$(23) \qquad A_1 = 0 \implies \sup_{\substack{\psi \in \mathcal{V}_0 \\ \|\psi\|_{\ell^2(\square)}=1}} \left| \left\langle \psi, V^\square \psi \right\rangle \right| = 0.$$

In this case we have that, for any $\psi \in \mathcal{V}_0$ and $\varphi \in \mathcal{V}_0^\perp$, the expansion (22) becomes
$$(24)$$
$$\left\langle H_{\epsilon,q}^\square(\theta)u, u \right\rangle = \epsilon^2 q^2 \left( \left\langle H_0^\square(\theta)\varphi, \varphi \right\rangle + 2 \operatorname{Re} \left\langle \psi, V^\square \varphi \right\rangle \right) + \epsilon^3 q^3 \left\langle V^\square \varphi, \varphi \right\rangle.$$

Note that, for $\psi \in \mathcal{V}_0$ and $\varphi \in \mathcal{V}_0^\perp$ such that

$$\|\psi\|_2 = \|\varphi\|_2 = 1$$

the map

$$(\varphi, \psi) \mapsto \frac{\left|\left\langle \psi, V^\square \varphi \right\rangle\right|^2}{\left\langle H_0^\square(\theta)\varphi, \varphi \right\rangle}$$

is continuous. Given that the spaces involved are finite-dimensional and their respective unit balls thus compact, we know that there exists a couple $(\psi^*, \varphi^*)$ maximizing this quantity, i.e.

$$\frac{\left|\left\langle \psi^*, V^\square \varphi^* \right\rangle\right|^2}{\left\langle H_0^\square(\theta)\varphi^*, \varphi^* \right\rangle} = \sup_{\substack{\psi \in \mathcal{V}_0 \\ \|\psi\|_{\ell^2(\square)}=1}} \sup_{\substack{\varphi \in \mathcal{V}_0^\perp \\ \|\varphi\|_{\ell^2(\square)}=1}} \frac{\left|\left\langle \psi, V^\square \varphi \right\rangle\right|^2}{\left\langle H_0^\square(\theta)\varphi, \varphi \right\rangle}.$$

Let $\psi = \psi^*$ and $\varphi = \lambda\varphi^*$ in the definition of $u$, where

$$\lambda = -\frac{\left\langle \psi^*, V^\square \varphi^* \right\rangle}{\left\langle H_0^\square(\theta)\varphi^*, \varphi^* \right\rangle} \in \mathbb{C}.$$

Replacing, we see that

$$\left\langle H_0^\square(\theta)\varphi, \varphi \right\rangle + 2\mathrm{Re}\left\langle \psi, V^\square \varphi \right\rangle = |\lambda|^2 \left\langle H_0^\square(\theta)\varphi^*, \varphi^* \right\rangle + 2\mathrm{Re}\,\overline{\lambda}\left\langle \psi^*, V^\square \varphi^* \right\rangle$$

$$= \frac{\left|\left\langle \psi^*, V^\square \varphi^* \right\rangle\right|^2}{\left\langle H_0^\square(\theta)\varphi^*, \varphi^* \right\rangle} - 2\frac{\left|\left\langle \psi^*, V^\square \varphi^* \right\rangle\right|^2}{\left\langle H_0^\square(\theta)\varphi^*, \varphi^* \right\rangle}$$

$$= -\frac{\left|\left\langle \psi^*, V^\square \varphi^* \right\rangle\right|^2}{\left\langle H_0^\square(\theta)\varphi^*, \varphi^* \right\rangle}.$$

Using this in (24) and letting $q^2 = \max(s_-^2, s_+^2)$, we obtain

$$\left\langle H_{\epsilon,q}^\square(\theta)u, u \right\rangle = -\epsilon^2 \max(s_-^2, s_+^2)\frac{\left|\left\langle \psi^*, V^\square \varphi^* \right\rangle\right|^2}{\left\langle H_0^\square(\theta)\varphi^*, \varphi^* \right\rangle} + O(\epsilon^3 q^3 \|\varphi^*\|^2)$$

$$= \epsilon^2 A_2 + O(\epsilon^3 q^3 \|\varphi\|^2).$$

Normalizing $u$ by multiplying by (20) gives the result.

*Case $A_1 = 0$ and $A_2 = 0$.* Choose $\varphi = 0$ and any normalized $\psi \in \mathcal{V}_0$. The development using $u$ in this case gives

$$\left\langle H_{\epsilon,q}^\square(\theta)u, u \right\rangle = \epsilon^3 q^3 \left\langle V^\square \varphi, \varphi \right\rangle = 0$$

and this yields the desired result.

∎

We prove the following converse lemma.

**Lemma 5.7.** *Let $A_1$, $A_2$ as in (9), (10), assume (HA), (HB) and (HC), and fix $\theta \in \Theta$. Then, if $A_1 \neq 0$,*

$$\inf_{q \in S_\mu} \inf_{\|u\|_{\ell^2(\square)}=1} \left\langle H_{\epsilon,q}^\square(\theta)u, u \right\rangle \geqslant \epsilon A_1 + O(\epsilon^{3/2}),$$

*whereas if $A_1 = 0$, then*

$$\inf_{q \in S_\mu} \inf_{\|u\|_{\ell^2(\square)} = 1} \left\langle H_{\epsilon,q}^\square(\theta) u, u \right\rangle \geqslant \epsilon^2 A_2 + O(\epsilon^3)$$

*Finally, if $A_1 = A_2 = 0$, then*

$$\inf_{q \in S_\mu} \inf_{\|u\|_{\ell^2(\square)} = 1} \left\langle H_{\epsilon,q}^\square(\theta) u, u \right\rangle \geqslant 0.$$

*Proof.* Fix $\epsilon > 0$ and let $q \in S_\mu$ be the value which minimizes the map

$$q \mapsto \inf_{\|u\|_{\ell^2(\square)} = 1} \left\langle H_{\epsilon,q}^\square(\theta) u, u \right\rangle,$$

i.e. $q \in \{s_-, s_+\}$. We lower bound this quantity by minimizing over a larger set by writing

$$\inf_{\|u\|_{\ell^2(\square)} = 1} \left\langle H_{\epsilon,q}^\square(\theta) u, u \right\rangle \geqslant \inf_{\substack{\psi \in \mathcal{V}_0 \\ \|\psi\|_{\ell^2(\square)} \leqslant 1}} \inf_{\substack{\varphi \in \mathcal{V}_0^\perp \\ \|\varphi\|_{\ell^2(\square)} \leqslant 1}} \left\langle H_{\epsilon,q}^\square(\theta)(\psi + \varphi), (\psi + \varphi) \right\rangle.$$

By continuity and compactness, there exists some pair $(\psi^*, \varphi^*)$ in $\mathcal{V}_0 \times \mathcal{V}_0^\perp$ realizing the infimum on the right hand side. We see that

$$(25) \quad \left\langle H_0^\square(\theta)(\psi^* + \varphi^*), (\psi^* + \varphi^*) \right\rangle = \left\langle H_0^\square(\theta)\varphi^*, \varphi^* \right\rangle \geqslant g \|\varphi^*\|_{\ell^2(\square)}^2,$$

where the constant $g$ is the spectral gap $H_0^\square$, which is also its (positive) second eigenvalue. We study the different cases.

*Case $A_1 \neq 0$.* From Lemma 5.6, we know already that

$$(26) \quad |A_1| + |A_2| \neq 0 \implies \left\langle H_{\epsilon,q}^\square(\theta)(\psi^* + \varphi^*), (\psi^* + \varphi^*) \right\rangle < 0.$$

Using (25) and (26) we get that

$$\|\varphi^*\|_{\ell^2(\square)}^2 \leqslant -g^{-1} \epsilon q \left\langle V^\square(\psi^* + \varphi^*), (\psi^* + \varphi^*) \right\rangle \leqslant 4g^{-1} \|V^\square\|_\infty \epsilon q.$$

We deduce then that

$$\inf_{\|u\|_{\ell^2(\square)} = 1} \left\langle H_{\epsilon,q}^\square(\theta) u, u \right\rangle$$

$$\geqslant \epsilon q \left\langle V^\square \psi^*, \psi^* \right\rangle + 2\epsilon q \operatorname{Re} \left\langle V^\square \varphi^*, \psi^* \right\rangle + \epsilon q \left\langle V^\square \varphi^*, \varphi^* \right\rangle$$

$$\geqslant \epsilon A_1 - 4g^{-1/2} \epsilon^{3/2} q^{3/2} \|V^\square\|_\infty^{3/2} - 4g^{-1} \epsilon^2 q^2 \|V^\square\|_\infty^2.$$

*Case $A_1 = 0$ and $A_2 \neq 0$.* In this case

$$\left\langle H_{\epsilon,q}^\square(\theta)(\psi^* + \varphi^*), (\psi^* + \varphi^*) \right\rangle$$

$$= \left\langle H_0^\square(\theta)\varphi^*, \varphi^* \right\rangle + 2\epsilon q \operatorname{Re} \left\langle V^\square \varphi^*, \psi^* \right\rangle + \epsilon q \left\langle V^\square \varphi^*, \varphi^* \right\rangle.$$

Using (26) we see that $\varphi^* \neq 0$. Furthermore, (25) and (26) together imply that

$$\|\varphi^*\|_{\ell^2(\square)}^2 \leqslant \epsilon q g^{-1} \|V^\square\|_\infty (2\|\varphi^*\|_{\ell^2(\square)} + \|\varphi^*\|_{\ell^2(\square)}^2)$$

$$\leqslant 3\epsilon q g^{-1} \|V^\square\|_\infty \|\varphi^*\|_{\ell^2(\square)}.$$

Note that $\|\varphi^*\|_{\ell^2(\square)}$ is on both sides of the inequality. Simplifying,

$$\|\varphi^*\|_{\ell^2(\square)} \leqslant 3\epsilon q g^{-1}\|V^\square\|_\infty. \tag{27}$$

Expanding as $\epsilon \to 0$, employing (27) and then simply multiplying by one, we write

$$\inf_{\|u\|_{\ell^2(\square)}=1} \left\langle H_{\epsilon,q}^\square(\theta)u, u \right\rangle$$

$$\geqslant \left\langle H_0^\square(\theta)\varphi^*, \varphi^* \right\rangle + 2\operatorname{Re}\epsilon q \left\langle V^\square\psi^*, \varphi^* \right\rangle + \epsilon q \left\langle V^\square\varphi^*, \varphi^* \right\rangle$$

$$= |\lambda|^2 \frac{\left\langle H_0^\square(\theta)\varphi^*, \varphi^* \right\rangle}{|\lambda|^2} + 2\operatorname{Re}\overline{\lambda}\epsilon q \frac{\left\langle V^\square\psi^*, \varphi^* \right\rangle}{\overline{\lambda}} + O(\epsilon^3)$$

We choose $\lambda$ as

$$\lambda = -\frac{\left\langle H_0^\square(\theta)\varphi^*, \varphi^* \right\rangle}{\left\langle V^\square\psi^*, \varphi^* \right\rangle}.$$

It is well defined for small $\epsilon$. Indeed, using (25) and (26) we see that

$$-2\epsilon q \operatorname{Re}\left\langle V^\square\varphi^*, \psi^* \right\rangle \geqslant \left\langle H_0^\square(\theta)\varphi^*, \varphi^* \right\rangle + \epsilon q \left\langle V^\square\varphi^*, \varphi^* \right\rangle$$

$$\geqslant c\|\varphi^*\|_{\ell^2(\square)}^2 - \epsilon q\|V^\square\|_\infty\|\varphi^*\|_{\ell^2(\square)}^2$$

and we know that $\varphi^* \neq 0$. We see that $\lambda \neq 0$ for $\epsilon$ small enough.

Using our choice of $\lambda$ gives

$$c\|\varphi^*\|_{\ell^2(\square)}^2 - \epsilon q\|V^\square\|_\infty\|\varphi^*\|_{\ell^2(\square)}^2$$

$$= (|\lambda|^2 - 2\operatorname{Re}\overline{\lambda}\epsilon q)\frac{\left|\left\langle V^\square\psi^*, \varphi^* \right\rangle\right|^2}{\left\langle H_0^\square(\theta)\varphi^*, \varphi^* \right\rangle} + O(\epsilon^3)$$

$$\geqslant -\epsilon^2 A_2 + O(\epsilon^3),$$

where we have used that $|\lambda|^2 - 2\operatorname{Re}\overline{\lambda}\epsilon q \geqslant |\lambda|^2 - 2|\lambda|\epsilon q \geqslant -\epsilon^2 q^2$.

*Case $A_1 = 0$ and $A_2 = 0$.* In this case

$$0 \geqslant \inf_{\|u\|_{\ell^2(\square)}} \left\langle H_{\epsilon,q}^\square(\theta)u, u \right\rangle = \left\langle H_0^\square(\theta)\varphi^*, \varphi^* \right\rangle + \epsilon q \left\langle V^\square\varphi^*, \varphi^* \right\rangle \tag{28}$$

$$\geqslant c\|\varphi^*\|^2 - O(\epsilon)\|\varphi^*\|^2, \tag{29}$$

where the first inequality comes from Lemma 5.6. It is now clear that $\psi^* = 0$ and this finishes the proof. ∎

5.4. **Positive random variables.** We study in this subsection the case involving positive random variables. We remind the reader the definition of the constants involved, for which we use the functions $\psi_i$, the matrix $A$, its eigenvalues $P_i$ and the linear space $\mathcal{V}_0$, which can be found at the beginning of this section. We define the subspace $\mathcal{V}_{01} \subset \mathcal{V}_0$ as

$$\mathcal{V}_{01} := \operatorname{span}_{\{i:P_i=P_1\}} \langle \psi_i \rangle,$$

i.e. the eigenspace of $A$ associated to its minimal eigenvalue $P_1$.

We recall the following quantities :

$$(30) \qquad A_1' := \inf_{q \in S_\mu} \inf_{\substack{\psi \in \mathcal{V}_{01} \\ \|\psi\|_{\ell^2(\square)} = 1}} q \left\langle \psi, V^\square \psi \right\rangle = \min(s_+ P_1, s_- P_1) \in \mathbb{R},$$

and

$$(31) \qquad A_2' := -s_+^2 \sup_{\substack{\psi \in \mathcal{V}_{01} \\ \|\psi\|_{\ell^2(\square)} = 1}} \sup_{\substack{\varphi \in \mathcal{V}_0^\perp \\ \|\varphi\|_{\ell^2(\square)} = 1}} \frac{\left| \left\langle \psi, V^\square \varphi \right\rangle \right|^2}{\left\langle H_0^\square(\theta) \varphi, \varphi \right\rangle} \leqslant 0.$$

We also restate Theorem 3.2 for the reader's convenience.

**Theorem 5.8.** *Assume* (HA), (HB) *and* (HC'). *Fix* $\theta \in \Theta$. *Then, for* $\epsilon > 0$ *small enough, if* $P_1 \neq 0$,

$$E_\epsilon \leqslant \epsilon A_1',$$

*whereas if* $P_1 = 0$, *then*

$$E_\epsilon \leqslant \epsilon^2 A_2' + O(\epsilon^3).$$

*Finally, if* $P_1 = A_2' = 0$, *then*

$$E_\epsilon \leqslant 0.$$

*Sketch of proof.* The proof of this theorem is very similar to the proof of Theorem 5.1. Indeed, Lemma 5.5 is also valid in this setting. We proceed then as in Lemma 5.6 up to equation (22). If $A_1' \neq 0$ we let

$$u = \psi_1, \quad \varphi = 0, \quad q = \begin{cases} s_+ & \text{if} \quad P_1 < 0 \\ s_- & \text{if} \quad P_1 \geqslant 0 \end{cases}$$

in (22). If $A_1' = 0$ but $A_2' \neq 0$, then we find $\psi^* \in \mathcal{V}_{01}$ and $\varphi \in \mathcal{V}_0$ realizing the supremum in the definition of $A_2'$ and then we proceed as in Lemma 5.6. Finally, if $A_1' = A_2' = 0$ we take $u = \psi_1$ in (22). ∎

We prove the following converse lemma.

**Lemma 5.9.** *Let* $A_1'$, $A_2'$ *as in* (11), (12), *assume* (HA), (HB) *and* (HC'), *and fix* $\theta \in \Theta$. *Then, if* $A_1' \neq 0$,

$$\inf_{q \in S_\mu} \inf_{\|u\|_{\ell^2(\square)} = 1} \left\langle H_{\epsilon,q}^\square(\theta) u, u \right\rangle \geqslant \epsilon A_1' + O(\epsilon^{3/2}),$$

*whereas if* $P_1 = 0$, *then*

$$\inf_{q \in S_\mu} \inf_{\|u\|_{\ell^2(\square)} = 1} \left\langle H_{\epsilon,q}^\square(\theta) u, u \right\rangle \geqslant \epsilon^2 A_2' + O(\epsilon^3)$$

*Finally, if* $P_1 = A_2' = 0$, *then*

$$\inf_{q \in S_\mu} \inf_{\|u\|_{\ell^2(\square)} = 1} \left\langle H_{\epsilon,q}^\square(\theta) u, u \right\rangle \geqslant 0.$$

*Proof.* We adapt here the proof of Lemma 5.7.

Fix $\epsilon > 0$ and let $q \in S_\mu$ be the value which minimizes the map

$$q \mapsto \inf_{\|u\|_{\ell^2(\square)}=1} \left\langle (H_{\epsilon,q}^\square(\theta) - \epsilon A_1')u, u \right\rangle,$$

i.e. $q \in \{s_-, s_+\}$. We lower bound this quantity by minimizing over a larger set by writing

$$\inf_{\|u\|_{\ell^2(\square)}=1} \left\langle (H_{\epsilon,q}^\square(\theta) - \epsilon A_1')u, u \right\rangle$$

$$\geqslant \inf_{\substack{\psi \in \mathcal{V}_0 \\ \|\psi\|_{\ell^2(\square)} \leqslant 1}} \inf_{\substack{\varphi \in \mathcal{V}_0^\perp \\ \|\varphi\|_{\ell^2(\square)} \leqslant 1}} \left\langle (H_{\epsilon,q}^\square(\theta) - \epsilon A_1')(\psi + \varphi), (\psi + \varphi) \right\rangle.$$

By continuity and compactness, there exists some pair $(\psi^*, \varphi^*)$ in $\mathcal{V}_0 \times \mathcal{V}_0^\perp$ realizing the infimum on the right hand side. We see that

$$(32) \quad \left\langle H_0^\square(\theta)(\psi^* + \varphi^*), (\psi^* + \varphi^*) \right\rangle = \left\langle H_0^\square(\theta)\varphi^*, \varphi^* \right\rangle \geqslant g\|\varphi^*\|_{\ell^2(\square)}^2,$$

where the constant $g$ is the spectral gap of $H_0^\square$, which is also its (positive) second eigenvalue.

We study the different cases.

*Case $A_1' \neq 0$.* We know already that

$$(33) \qquad \left\langle (H_{\epsilon,q}^\square(\theta) - \epsilon A_1')(\psi^* + \varphi^*), (\psi^* + \varphi^*) \right\rangle \leqslant 0.$$

Using (32), (33) and $|A_1'| \leqslant q\|V^\square\|$ we get

$$\|\varphi^*\|_{\ell^2(\square)}^2 \leqslant -g^{-1}\epsilon \left\langle (qV^\square - A_1')(\psi^* + \varphi^*), (\psi^* + \varphi^*) \right\rangle \leqslant 4g^{-1}\|V^\square\|_\infty \epsilon q.$$

Note that

$$q \left\langle V^\square \psi^*, \psi^* \right\rangle \geqslant q \inf_{\substack{\psi \in \mathcal{V}_0 \\ \|\psi\|_{\ell^2(\square)} \leqslant 1}} \left\langle V^\square \psi, \psi \right\rangle = q \inf_{\substack{\psi \in \mathcal{V}_{00} \\ \|\psi\|_{\ell^2(\square)} \leqslant 1}} \left\langle V^\square \psi, \psi \right\rangle \geqslant E_1'.$$

It implies

$$\inf_{\|u\|_{\ell^2(\square)}=1} \left\langle H_{\epsilon,q}^\square(\theta)u, u \right\rangle$$

$$\geqslant \epsilon q \left\langle V^\square \psi^*, \psi^* \right\rangle + 2\epsilon q \operatorname{Re} \left\langle V^\square \varphi^*, \psi^* \right\rangle + \epsilon q \left\langle V^\square \varphi^*, \varphi^* \right\rangle$$

$$\geqslant \epsilon A_1' - 4g^{-1/2}\epsilon^{3/2}q^{3/2}\|V^\square\|_\infty^{3/2} - 4g^{-1}\epsilon^2 q^2 \|V^\square\|_\infty^2.$$

*Case $A_1' = 0$ and $A_2' \neq 0$.* We know that

$$(34) \qquad \left\langle H_{\epsilon,q}^\square(\theta)(\psi^* + \varphi^*), (\psi^* + \varphi^*) \right\rangle < 0.$$

We will decompose further $\psi^* = \psi_{01}^* + \psi_{0\perp}^* \in \mathcal{V}_0$, with $\psi_{01}^* \in \mathcal{V}_{01}$ and $\psi_{0\perp}^* \in \mathcal{V}_{01}^\perp$. We have

$$(35) \qquad \left\langle V^\square u, u \right\rangle \geqslant 2 \operatorname{Re} \left\langle V^\square \psi_{01}^*, \varphi^* \right\rangle + O\left(\|\varphi^*\|^2\right)$$

Indeed, using the definition of $u$, we see that,

$$\langle V^\square u, u\rangle = \langle V^\square \psi_{01}^*, \psi_{01}^*\rangle + 2\operatorname{Re}\langle V^\square \psi_{01}^*, \psi_{0\perp}^*\rangle + \langle V^\square \psi_{0\perp}^*, \psi_{0\perp}^*\rangle$$

$$+ 2\operatorname{Re}\langle V^\square \psi_{01}^*, \varphi^*\rangle + 2\operatorname{Re}\langle V^\square \psi_{0\perp}^*, \varphi^*\rangle + \langle V^\square \varphi^*, \varphi^*\rangle$$

$$(36) \qquad = \langle V^\square \psi_{0\perp}^*, \psi_{0\perp}^*\rangle + 2\operatorname{Re}\langle V^\square \psi_{01}^*, \varphi^*\rangle$$

$$+ 2\operatorname{Re}\langle V^\square \psi_{0\perp}^*, \varphi^*\rangle + \langle V^\square \varphi^*, \varphi^*\rangle.$$

Let us also note that

$$A_1 = 0 \implies P_1 \geqslant 0,$$

and, in particular

$$\langle V^\square \psi_{0\perp}^*, \psi_{0\perp}^*\rangle \geqslant 0.$$

Hence if $\langle V^\square \psi_{0\perp}^*, \varphi^*\rangle = 0$, we immediately get (35) from (36).

Assume that $\langle V^\square \psi_{0\perp}^*, \varphi^*\rangle \neq 0$. For each $\mu \in \mathbb{C}$,

$$(36) = |\mu|^2 \frac{\langle V^\square \psi_{0\perp}^*, \psi_{0\perp}^*\rangle}{|\mu|^2} + 2\operatorname{Re}\overline{\mu}\frac{\langle V^\square \psi_{0\perp}^*, \varphi^*\rangle}{\overline{\mu}}$$

$$+ 2\operatorname{Re}\langle V^\square \psi_{01}^*, \varphi^*\rangle + O\big(\|\varphi^*\|^2\big).$$

We choose $\mu$ as

$$\mu := -\frac{\langle V^\square \psi_{0\perp}^*, \psi_{0\perp}^*\rangle}{\langle V^\square \psi_{0\perp}^*, \varphi^*\rangle}.$$

We obtain

$$(36) = (|\mu|^2 - 2\operatorname{Re}\overline{\mu})\frac{\big|\langle V^\square \psi_{0\perp}^*, \varphi^*\rangle\big|^2}{\langle V^\square \psi_{0\perp}^*, \psi_{0\perp}^*\rangle} + 2\operatorname{Re}\langle V^\square \psi_{01}^*, \varphi^*\rangle + O\big(\|\varphi^*\|^2\big)$$

$$\geqslant -\frac{1}{2}\frac{\big|\langle V^\square \psi_{0\perp}^*, \varphi^*\rangle\big|^2}{\langle V^\square \psi_{0\perp}^*, \psi_{0\perp}^*\rangle} + 2\operatorname{Re}\langle V^\square \psi_{01}^*, \varphi^*\rangle + O\big(\|\varphi^*\|^2\big)$$

$$= 2\operatorname{Re}\langle V^\square \psi_{01}^*, \varphi^*\rangle + O\big(\|\varphi^*\|^2\big),$$

which is also (35).

Using this result, we obtain

$$\langle H_{\epsilon,q}^\square(\theta)(\psi^* + \varphi^*), (\psi^* + \varphi^*)\rangle$$

$$\geqslant \langle H_0^\square(\theta)\varphi^*, \varphi^*\rangle + 2\epsilon q \operatorname{Re}\langle V^\square \psi_{01}^*, \varphi^*\rangle + O\big(\epsilon q \|\varphi^*\|^2\big)$$

Equality (36) and inequalities (32) and (34) together imply that

$$\|\varphi^*\|_{\ell^2(\square)}^2 \leqslant \epsilon q g^{-1}\|V^\square\|_\infty (2\|\varphi^*\|_{\ell^2(\square)} + \|\varphi^*\|_{\ell^2(\square)}^2)$$

$$\leqslant 3\epsilon q g^{-1}\|V^\square\|_\infty \|\varphi^*\|_{\ell^2(\square)}.$$

Note that $\|\varphi^*\|_{\ell^2(\square)}$ is involved in both sides of the inequality. Simplifying,

$$(37) \qquad \|\varphi^*\|_{\ell^2(\square)} \leqslant 3\epsilon q g^{-1}\|V^\square\|_\infty.$$

Using (37) and then simply multiplying by $1 = |\lambda|^2/|\lambda|^2 = \overline{\lambda}/\lambda$, we get

$$\inf_{\|u\|_{\ell^2(\square)}=1} \left\langle H_{\epsilon,q}^{\square}(\theta)u, u \right\rangle$$

$$\geqslant |\lambda|^2 \frac{\left\langle H_0^{\square}(\theta)\varphi^*, \varphi^* \right\rangle}{|\lambda|^2} + 2\operatorname{Re}\overline{\lambda}\epsilon q \frac{\left\langle V^{\square}\psi_{01}^*, \varphi^* \right\rangle}{\overline{\lambda}} + O(\epsilon^3)$$

In view of this inequality and (34), we see that $\left\langle V^{\square}\psi_{01}^*, \varphi^* \right\rangle \neq 0$ for small $\epsilon$. We choose $\lambda$ as

$$\lambda = -\frac{\left\langle H_0^{\square}(\theta)\varphi^*, \varphi^* \right\rangle}{\left\langle V^{\square}\psi_{01}^*, \varphi^* \right\rangle}.$$

It is well defined for small $\epsilon$. Using our choice of $\lambda$, it gives

$$= (|\lambda|^2 - 2\operatorname{Re}\overline{\lambda}\epsilon q) \frac{\left|\left\langle V^{\square}\psi_{01}^*, \varphi^* \right\rangle\right|^2}{\left\langle H_0^{\square}(\theta)\varphi^*, \varphi^* \right\rangle} + O(\epsilon^3)$$

$$\geqslant -\epsilon^2 A_2' + O(\epsilon^3),$$

where we have used that $|\lambda|^2 - 2\operatorname{Re}\overline{\lambda}\epsilon q \geqslant |\lambda|^2 - 2|\lambda|\epsilon q \geqslant -\epsilon^2 q^2$.

*Case $A_1 = 0$ and $A_2 = 0$.* In this case, re-using (35) and employing $A_2' = 0$,

$$0 \geqslant \inf_{\|u\|_{\ell^2(\square)}=1} \left\langle H_{\epsilon,q}^{\square}(\theta)u, u \right\rangle$$

$$= \left\langle H_0^{\square}(\theta)\varphi^*, \varphi^* \right\rangle + 2\epsilon q \operatorname{Re}\left\langle V^{\square}\psi_{01}^*, \varphi^* \right\rangle + \epsilon q O\left(\|\varphi^*\|^2\right)$$

$$= \left\langle H_0^{\square}(\theta)\varphi^*, \varphi^* \right\rangle + \epsilon q O\left(\|\varphi^*\|^2\right)$$

$$\geqslant c\|\varphi^*\|^2 - |O(\epsilon)|\,\|\varphi^*\|^2,$$

where the first inequality comes from Lemma 5.6. It is now clear that $\psi^* = 0$ and this finishes the proof. ∎

## 5.5. **Application to the generalized Anderson model.**

*Proof of Theorem 1.1.* It is enough to verify that the assumptions of Theorem 5.1 are satisfied. Let $C_W := \inf\sigma(-\Delta_{\mathbb{Z}^d} + W)$. It is clear that the operator $H_0 := -\Delta_{\mathbb{Z}^d} + W - C_W$ satisfies hypothesis (HA). In this case the set $\Theta$ consists of the single point $\theta = 0$ (see Theorem 5.10 below).

Let us check property (17) for the operator

$$H_0(0) := -\Delta_{\square} + W_{\square} + C_W,$$

where $\Delta_{\square}$ is the Laplacian on $\square$ with periodic boundary conditions and $W_{\square}$ is the restriction of $W$ to $\square$. To check the property we use Perron–Frobenius theorem [17]. For $m > |\square|$, we verify that

$$\langle \delta_x, (\Delta_{\square} - W_{\square} + \|W\|_{\infty} + 2d + 1)^m \delta_y \rangle \geqslant \langle \delta_x, (\Delta_{\square} + 2d + 1)^m \delta_y \rangle \geqslant 1.$$

This implies that the largest eigenvalue of the matrix $\Delta_\square - W_\square + \|W\|_\infty + 2d + 1$ is simple and its corresponding eigenfunction $\psi_1$ is positive (i.e. $(\forall n \in \square) \, \psi_1(n) > 0$). Because of this strict positivity, condition (17) is satisfied as soon as $V^\square \not\equiv 0$. The subspace $\mathcal{V}_0$ is thus one-dimensional and contains only $\psi_1$. The theorem is now proven, by simply stating the consequences of Theorem 5.1.

∎

We know recall Theorem 2.4 in [12], with our notations. It implies that 0 is the unique $\theta \in \square^*$ realizing the minimum of the spectrum.

**Theorem 5.10.** *Let $H_0 = -\triangle_{\mathbb{Z}^d} + W$ with $W$ a periodic potential with respect to $\gamma = N\mathbb{Z}^d$, and $E_0(\theta)$ be the smallest eigenvalue of $H_0(\theta)$. Then*

$$(a_-/a_+)^2 \left( 2d - \sum_{i=1}^d \cos(\theta_i) \right) \leqslant E_0(\theta) - E_0(0) \leqslant \left( 2d - \sum_{i=1}^d \cos(\theta_i) \right).$$

*Here, $a_\pm = \pm \max \pm\psi_1$ and $\psi_1$ is the positive ground state of $H_0(0)$.*

## 6. APPENDIX

6.1. **An interesting example: Proof of Theorem 1.2.** Let $H_0 := \Delta_{\mathbb{Z}}^2$ defined on $\ell^2(\mathbb{Z})$. This operator has hopping range $N = 3$ (see (HA)) and thus $\square = \{0, 1, 2\}$. We define $V^\square$ as the multiplication operator given by the following single-site potential:

$$V^\square : \ell^2(\square) \to \mathbb{R}$$

$$V^\square(n) := -\frac{1}{2}\delta_{-1}(n_1) + \delta_0(n_1) - \frac{1}{2}\delta_1(n_1).$$

With these definitions, we see that, for $\theta \in [-\pi/3, \pi/3)^d$,

$$H_0^\square(\theta) = \begin{pmatrix} 6 & -4 + e^{-3i\theta} & 1 - 4e^{-3i\theta} \\ -4 + e^{3i\theta} & 6 & -4 + e^{-3i\theta} \\ 1 - 4e^{-3i\theta} & -4 + e^{3i\theta} & 6 \end{pmatrix},$$

after (7). This matrix has a simple ground state

$$\psi_0(\theta) := (e^{-i\theta}, 1, e^{i\theta})/\sqrt{3}$$

with eigenvalues $E_0(\theta) = (2 - 2\cos(\theta))^2$. Let now $\tilde{f}_n(\theta) = \chi_n \tilde{\psi}_0(\theta) \in \ell^2(\mathbb{Z}^d)$ where $\tilde{\psi}_0(\theta)$ is the $\theta$-quasi-$\gamma$-periodic extension of $\psi_0(\theta)$. Finally, for $\xi > 1/4$, let

$$u_n := f_n(0) + \epsilon^\xi f_n(\epsilon^\xi).$$

Let us calculate the kinetic energy. We see that

(38) $\langle H_0 u_n, u_n \rangle$

$= \langle H_0 f_n(0), f_n(0) \rangle + 2\epsilon^\xi \operatorname{Re}\langle H_0 f_n(0), f_n(\epsilon^\xi) \rangle + \epsilon^{2\xi} \langle H_0 f_n(\epsilon^\xi), f_n(\epsilon^\xi) \rangle.$

Let $\delta > 0$ and pick $n$ so large so that

$$\left| \frac{\langle f_n(0), H_0 f_n(0)\rangle_{\ell^2(D)}}{\|f_n(0)\|^2_{\ell^2(D)}} - \frac{\langle \psi_0(0), H_0^\square(\theta)\psi_0(0)\rangle_{\ell^2(\square)}}{\|\psi_0(0)\|^2_{\ell^2(\square)}} \right| \leqslant \delta,$$

$$\left| \frac{\langle f_n(\epsilon^\xi), H_0 f_n(\epsilon^\xi)\rangle_{\ell^2(D)}}{\|f_n(\epsilon^\xi)\|^2_{\ell^2(D)}} - \frac{\langle \psi_0(\epsilon^\xi), H_0^\square(\theta)\psi_0(\epsilon^\xi)\rangle_{\ell^2(\square)}}{\|\psi_0(\epsilon^\xi)\|^2_{\ell^2(\square)}} \right| \leqslant \delta,$$

and

$$\left| \frac{\|H_0 f_n(0)\|^2_{\ell^2(D)}}{\|f_n(0)\|^2_{\ell^2(D)}} \right| = \left| \frac{\langle f_n(0), H_0^2 f_n(0)\rangle_{\ell^2(D)}}{\|f_n\|^2_{\ell^2(D)}} \right| \leqslant \delta.$$

Then, from (38) we see that

$$\langle H_0 u_n, u_n\rangle$$
$$\leqslant \delta \|f_n(0)\|^2 + 2\epsilon^\xi \delta \|f_n(\epsilon^\xi)\| + \epsilon^{2\xi} \langle \psi_0(\epsilon^\xi), H_0^\square(\theta)\psi_0(\epsilon^\xi)\rangle_{\ell^2(\square)} \|f_n(\theta)\|^2$$
$$\leqslant 3\delta + \epsilon^{2\xi} E_0(\epsilon^\xi)\|f_n(\theta)\|^2.$$

Letting $n \to \infty$ and $\delta \to 0$ we see that

$$\langle H_0 u_n, u_n\rangle \leqslant \epsilon^{2\xi} E_0(\epsilon^\xi) \leqslant C\epsilon^{6\xi}.$$

Now let us calculate the potential energy.

$$\epsilon \langle V_q u_n, u_n\rangle$$
$$= \epsilon\langle V_q f_n(0), f_n(0)\rangle + 2\epsilon^{1+\xi} \mathrm{Re}\langle V_q f_n(0), f_n(\epsilon^\xi)\rangle + \epsilon^{1+2\xi}\langle V_q f_n(\epsilon^\xi), f_n(\epsilon^\xi)\rangle$$
$$= 2\epsilon^{1+\xi} \mathrm{Re}\langle V_q f_n(0), f_n(\epsilon^\xi)\rangle.$$

Now we can calculate explicitly

$$\langle V_q f_n(\epsilon^\xi), f_n(\epsilon^\xi)\rangle = \frac{1}{6}\left(-e^{-i\epsilon^\xi} + 2 - e^{i\epsilon^\xi}\right) = -\frac{1}{3}\epsilon^\xi + O(\epsilon^{2\xi}).$$

This shows that, for small $\epsilon$,

$$\langle H_{\epsilon,q} u_n, u_n\rangle \leqslant C\xi^{6\xi} - \frac{1}{3}\epsilon^{1+2\xi} + O(\epsilon^{1+2\xi}) \leqslant -\frac{1}{6}\epsilon^{1+2\xi},$$

where we have used that $6\xi > 1 + 2\xi$.

6.2. **Proof of Lemma 4.1.** As the $V_q$ is block-diagonal, it is enough to do the calculation for the free operator $H_0$. Let us first calculate some norms. Because of the quasi-periodicity, we easily see that

$$\|u_n\|^2_{\ell^2(D)} = (2n+1)^d \|u_0\|^2_{\ell^2(\square)}. \tag{39}$$

and

$$\|u_n - u_{n-1}\|^2_{\ell^2(D)} \leqslant Cn^{d-1} \|u_0\|^2_{\ell^2(\square)}.$$

So we have that

$$\langle H_0 u_n, u_n\rangle = \langle H_0 u_n, u_{n-1}\rangle + O(n^{d-1})\|u_0\|^2_{\ell^2(\square)}. \tag{40}$$

For any $k \in \square_{n-1}$ and $k' \in \mathbb{Z}^d \setminus \square_n$, we have that $|k - k'| \geqslant N$ and thus, because of the finite hopping range (assumption (HB)),

$$\langle H_0 u_n, u_{n-1} \rangle = \langle H_0 u, u_{n-1} \rangle.$$

Now, we develop

$$\big\langle H_0 u, u_{n-1} \big\rangle$$
(41)
$$= \sum_{k \in \mathbb{Z}^d} \sum_{k' \in \mathbb{Z}^d} H_0(k, k') u(k') \overline{u_{n-1}(k)}$$
$$= \sum_{\substack{m \in \gamma \\ |m| \leqslant (n-1)N}} \sum_{m' \in \gamma} \sum_{k \in \square + m} \sum_{k' \in \square + m'} H_0(k, k') u(k') \overline{u_{n-1}(k)}$$
$$= \sum_{\substack{m \in \gamma \\ |m| \leqslant (n-1)N}} \sum_{m' \in \gamma} \sum_{k \in \square} \sum_{k' \in \square} H_0(k + m, k' + m') u(k' + m') \overline{u_{n-1}(k + m)}$$

Using the translation invariance (assumption (HB)), the last quantity is equal to

(42)

$$\sum_{\substack{m \in \gamma \\ |m| \leqslant (n-1)N}} \sum_{m' \in \gamma} \sum_{k \in \square} \sum_{k' \in \square} H_0(k, k' + m' - m) u(k' + m' - m) \overline{u_{n-1}(k)}$$
$$= \sum_{\substack{m \in \gamma \\ |m| \leqslant (n-1)N}} \sum_{m' \in \gamma} \sum_{k \in \square} \sum_{k' \in \square} e^{i\theta \cdot (m - m')} H_0(k, k' - m' + m) u_0(k') \overline{u_0(k)}$$
$$= \sum_{\substack{m \in \gamma \\ |m| \leqslant (n-1)N}} \sum_{m'' \in \gamma} \sum_{k \in \square} \sum_{k' \in \square} e^{i\theta \cdot m''} H_0(k, k' - m'') u_0(k') \overline{u_0(k)}$$
$$= (2n - 3)^d \langle H_0(\theta) u_0, u_0 \rangle.$$

We see from this calculation and (40) thus that

$$\left| \langle H_0 u_n, u_n \rangle - (2n - 3)^d \langle H_0(\theta) u_0, u_0 \rangle \right| \leqslant C n^{d-1} \| u_0 \|_{\ell^2(\square)}^2.$$

As $(2n - 3)/(2n - 1) \to 1$, dividing by $\| u_n \|_{\ell^2(D)}^2$, using (39) and taking the limit proves the lemma.

∎

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## References

[1] M. Aizenman. Localization at weak disorder: some elementary bounds. *Rev. Math. Phys.*, 06:1163, 1994.

[2] D. Borisov and I. Veselić. Low lying spectrum of weak-disorder quantum waveguides. *J. Stat. Phys.*, 142(1), 2011.

[3] D. Borisov and I. Veselić. Low lying eigenvalues of randomly curved quantum waveguides. *J. Funct. Anal.*, 265(11), 2013.

[4] Z. Cao and A. Elgart. The weak localization for the alloy-type Anderson model on a cubic lattice. *J. Stat. Phys.*, 148(6):1006–1039, 2012.

[5] R. Carmona and J. Lacroix. *Spectral theory of random Schrödinger operators.* Probability and its Applications. Birkhäuser Boston, 1990.

[6] Y. Colin de Verdière. Sur les singularités de Van Hove génériques. (On generic Van Hove singularities). *Mém. Soc. Math. Fr., Nouv. Sér.*, 46:99–109, 1991.

[7] A. Elgart. Lifshitz tails and localization in the three-dimensional Anderson model. *Duke Math. J.*, 146(2):331–360, 2009.

[8] A. Elgart, M. Tautenhahn, and I. Veselić. Anderson localization for a class of models with a sign-indefinite single-site potential via fractional moment method. *Ann. Henri Poincaré*, 12(8):1571–1599, 2011.

[9] P. D. Hislop. Lectures on Random Schrödinger Operators. *Fourth Summer School in Analysis and Mathematical Physics: topics in spectral theory and quantum mechanics, May 2005, Universidad Nacional Autónoma de México, Cuernavaca, México*, 476:41, 2008.

[10] F. Hoecker-Escuti. *Le modèle d'Anderson en régime de faible désordre.* PhD thesis, Université Paris 13, June 2013.

[11] W. Kirsch and F. Martinelli. On the spectrum of Schrödinger operators with a random potential. *Commun. Math. Phys.*, 85:329–350, 1982.

[12] W. Kirsch and B. Simon. Comparison theorems for the gap of Schrödinger operators. *J. Funct. Anal.*, 75:396–410, 1987.

[13] W. Kirsch, P. Stollmann, and G. Stolz. Localization for random perturbations of periodic Schrödinger operators. *Random Oper. Stoch. Equ.*, 6(3):241–268, 1998.

[14] F. Klopp. Weak disorder localization and Lifshitz tails. *Commun. Math. Phys.*, 232(1):125–155, 2002.

[15] F. Klopp. Weak disorder localization and Lifshitz tails: continuous Hamiltonians. In *Annales Henri Poincare*, volume 3, pages 711–737. Springer, 2002.

[16] H. Krueger. Periodic and limit-periodic discrete Schrödinger operators. *arXiv:1108.1584*, 2011.

[17] C. D. Meyer. *Matrix analysis and applied linear algebra.* Society for Industrial and Applied Mathematics, 2000.

[18] L. Pastur and A. Figotin. *Spectra of random and almost-periodic operators*, volume 297. 1992.

[19] M. Reed and B. Simon. *Analysis of Operators*, volume 1 : Functional Analysis of *Methods of Modern Mathematical Physics*. Academic Press, Inc., 1981.

[20] P. Stollmann. *Caught by disorder. Bound states in random media.*, volume 20 of *Progress in Mathematical Physics*. Birkhäuser, 2001.

[21] E. C. Titchmarsh. *Eigenfunction expansions associated with second-order differential equations.* Oxford University Press, 1946.

[22] W.-M. Wang. Localization and universality of Poisson statistics for the multi-dimensional Anderson model at weak disorder. *Invent. Math.*, 146(2):365–398, 2001.

[23] T. H. Wolff. Recent work on sharp estimates in second-order elliptic unique continuation problems. *J. Geom. Anal.*, 3(6):621–650, 1993.

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