INVERSION OF CENTROSKEWSYMMETRIC TOEPLITZ-PLUS-HANKEL BEZOUTIANS *

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Abstract. In this paper we compute the inverse of a nonsingular, centroskewsymmetric Toeplitz-plus-Hankel Bezoutian B of (even) order n and find a representation of B^{-1} as a sum of a Toeplitz and a Hankel matrix. Two possibilities are discussed. In the first one, the problem is reduced to the inversion of two skewsymmetric Toeplitz Bezoutians of order n. In the second one, the problem is tackled via the inversion of two Hankel Bezoutians of half the order $\frac{n}{2}$. The inversion of Toeplitz or Hankel Bezoutians was the subject of a previous paper. Both approaches lead to fast $O(n^2)$ inversion algorithms.

Key words. Bezoutian matrix, Toeplitz matrix, Hankel matrix, Toeplitz-plus-Hankel matrix, matrix inversion

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1. Introduction. The present paper is devoted to the inversion of special types of structured matrices, so-called Toeplitz-plus-Hankel Bezoutians (shortly, $T+H$-*Bezoutians*). We assume that the matrix entries are taken from a field $\mathbb{F}$ with characteristic not equal to 2. In a previous paper [2], we investigated centrosymmetric $T+H$-Bezoutians. The focus of this paper are centroskewsymmetric (briefly, centroskew) $T+H$-Bezoutians. Recall that an $n\times n$ matrix A is called *centroymmetric* or *centroskew*, if $J_nAJ_n = A$ or $J_nAJ_n = -A$, respectively, where J_n denotes the flip matrix of order n,

$$J_n := \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}. \tag{1.1}$$

Before we start to explain the content of the paper in more detail, let us give a very short historical account on Bezoutians. Bezoutians were introduced in connection with elimination theory (see [18]). Their importance for the inversion of Hankel and Toeplitz matrices was discovered by Lander [14] much later in 1974. In particular, he observed that the inverse of a nonsingular Hankel (Toeplitz) matrix is a Hankel (Toeplitz) Bezoutian and vice versa.

The inversion of Toeplitz and Hankel matrices has been the subject of a large amount of literature. The starting point were the papers of Trench [17] and Gohberg/Semencul [5]. Later, in [8], it was discovered that the inverse of a nonsingular matrix which is the sum of a Toeplitz and a Hankel matrix possess a generalized Bezoutian structure. These especially structured matrices $B=[b_{ij}]_{i,j=0}^{n-1}$ were called *Toeplitz-plus-Hankel Bezoutians* and are characterized by the property that there exists eight polynomials $\mathbf{u}_i(t), \mathbf{v}_i(t)$ $(i=1,2,3,4)$ with coefficients in $\mathbb{F}$ and of degree at most $n+1$ such that, in polynomial language,

$$\sum_{i,j=0}^{n-1} b_{ij}s^it^j = \frac{\sum_{i=1}^{4}\mathbf{u}_i(t)\mathbf{v}_i(s)}{(t-s)(1-ts)}.$$

Again, there is a large number of papers dealing with the inversion of $T+H$ matrices (see e.g. [15],[4], [9], [10], [16], [13], and the references therein).

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The converse problem - the inversion of Bezoutians - has been devoted little attention up to now (see [7], [6]). A general approach to the inversion problem for Hankel and Toeplitz Bezoutians was given in [3]. As far as we know, the only paper dedicated to the inversion of $T + H$-Bezoutians is our paper [2]. In this paper, using results of [13] and [1], $T + H$-Bezoutians which are centrosymmetric were considered. Fast inversions algorithms as well as matrix representations of their inverses as $T + H$ matrices were presented.

In the present paper we discuss two possibilities how to compute the inverse of a centroskew $T + H$-Bezoutian B and how to represent the inverse as a $T + H$ matrix. Both possibilities are based on a splitting property, which was discovered in Section 8 of [11] (see also [13]), and holds for both centrosymmetric and centroskew $T + H$-Bezoutians. If B is a nonsingular, centroskew $T + H$-Bezoutian, necessarily of even size $n = 2\ell$, then B can be represented in the form $B = B_{+-} + B_{-+}$, where $B_{\pm\mp}$ have additional symmetries and, more importantly, a particular and simpler Bezoutian structure. These matrices are called *split-Bezoutians*.

In our first approach it is proved that both splitting parts of B are directly related to nonsingular skewsymmetric Toeplitz Bezoutians. It remains to use the results of [3] to compute the inverses of these Toeplitz Bezoutians and to represent them as Toeplitz matrices. From there the representation of B^{-1} as a $T + H$-matrix is obtained.

The second approach is analogous to the method of inversion which we used in our paper [2] for the inverting centrosymmetric $T + H$-Bezoutains. Starting again with the splitting we use now a result of [13] to transform B_{+-} and B_{-+} into nonsingular Hankel Bezoutians of half the order $\ell = \frac{n}{2}$. Then we take advantage of formulas and algorithms established in [3] in order to compute the inverses of these Hankel Bezoutians, which are Hankel matrices H_1 and H_2 the parameters of which are given by the solutions of corresponding Bezout equations (as described in [3]). At this point the formula for the inverse of the $T + H$-Bezoutian B is of the form

$$B^{-1} = W^{-T} \begin{bmatrix} \mathbf{0} & H_2 \\ H_1 & \mathbf{0} \end{bmatrix} W^{-1} ,$$

where W is a certain explicit transformation (involving triangular matrices). It remains to discover the Toeplitz-plus-Hankel structure behind this representation, i.e., we want to find a Toeplitz matrix T and a Hankel matrix H such that

$$B^{-1} = T + H .$$

This goal can be achieved utilizing finite versions of results given in [1].

The paper is structured as follows. After some preliminaries in Section 2 we recall in Sections 3 and 4 some basic facts on (centroskew) Toeplitz-plus-Hankel matrices respective Toeplitz and Hankel Bezoutians. Section 5 is dedicated to the splitting of centroskew $T+H$-Bezoutians. Moreover, an algorithm is discussed to decide whether a centroskew matrix B is a nonsingular $T + H$-Bezoutian. In Sections 6 and 7 the two possibilities for the inversion of centroskew $T + H$-Bezoutians are deduced. At the end of both sections a corresponding fast algorithm is presented. Fast means here that the complexity of the algorithms is $O(n^2)$, where n is the order of B. In Section 8 we discuss the connections, the advantages, and disadvantages of both approaches.

2. Preliminaries. In what follows we consider vectors or matrices the entries of which are taken from a field $\mathbb{F}$ with a characteristic not equal to 2. By $\mathbb{F}^n$ we denote the linear space of all vectors of length n, by $\mathbb{F}^{m\times n}$ the linear space of all $m \times n$ matrices, and I_n denotes the identity matrix in $\mathbb{F}^{n\times n}$.

It will often be convenient to use polynomial language. Let $\mathbb{F}^n[t]$ denote the linear space of all polynomials in t of degree less than n with coefficients in $\mathbb{F}$. To each $\mathbf{x} = (x_j)_{j=0}^{n-1} \in \mathbb{F}^n$ we associate the polynomial

$$\mathbf{x}(t) := \sum_{j=0}^{n-1} x_j t^j \in \mathbb{F}^n[t]. \tag{2.1}$$

Occasionally, when using a different indexing, $\mathbf{x} = (x_j)_{j=-n+1}^{n-1} \in \mathbb{F}^{2n-1}$, we associate the polynomial

$$\mathbf{x}(t) := t^{n-1} \sum_{j=-n+1}^{n-1} x_j t^j \in \mathbb{F}^{2n-1}[t]. \tag{2.2}$$

Moreover, we associate to a matrix $A = [\, a_{ij} \,]_{i,j=0}^{n-1}$ the bivariate polynomial

$$A(t,s) := \sum_{i,j=0}^{n-1} a_{ij}\, t^i s^j \tag{2.3}$$

and call it the *generating polynomial of* A.

Given a vector $\mathbf{x} \in \mathbb{F}^n$ we denote

$$\mathbf{x}^J := J_n \mathbf{x}\,,$$

where J_n was introduced in (1.1), which in polynomial language means $\mathbf{x}^J(t) = \mathbf{x}(t^{-1})t^{n-1}$. With this abbreviation a vector $\mathbf{x} \in \mathbb{F}^n$ (or its corresponding polynomial) is said to be *symmetric* if $\mathbf{x} = \mathbf{x}^J$ and *skewsymmetric* if $\mathbf{x} = -\mathbf{x}^J$. The matrices

$$P_\pm := \tfrac{1}{2}(I_n \pm J_n) \tag{2.4}$$

are the projections from $\mathbb{F}^n$ onto the subspaces $\mathbb{F}^n_\pm$ consisting of all symmetric, respective skewsymmetric vectors, i.e.,

$$\mathbb{F}^n_\pm := \left\{ \, \mathbf{x} \in \mathbb{F}^n \;:\; \mathbf{x}^J = \pm \mathbf{x} \, \right\}. \tag{2.5}$$

The various spaces $\mathbb{F}^n_\pm$ for n even or odd are related to each other. This can be easily expressed in polynomial language as follows,

$$\begin{aligned} \mathbb{F}^{2\ell}_+[t] &= \left\{ \, (t+1)\,\mathbf{x}(t) \;:\; \mathbf{x}(t) \in \mathbb{F}^{2\ell-1}_+[t] \, \right\}, \\ \mathbb{F}^{2\ell}_-[t] &= \left\{ \, (t-1)\,\mathbf{x}(t) \;:\; \mathbf{x}(t) \in \mathbb{F}^{2\ell-1}_+[t] \, \right\}, \\ \mathbb{F}^{2\ell+1}_-[t] &= \left\{ \, (t^2-1)\,\mathbf{x}(t) \;:\; \mathbf{x}(t) \in \mathbb{F}^{2\ell-1}_+[t] \, \right\}. \end{aligned} \tag{2.6}$$

Recall that a matrix A of order n is called *centroskew* if $A = -J_n A J_n$. Since $(\det J_n)^2 = 1$ the order n of a nonsingular, centroskew matrix is even, $n = 2\ell$.

It is easy to see that a matrix A is centroskew if and only if

$$P_- A P_- = P_+ A P_+ = \mathbf{0}. \tag{2.7}$$

In particular, a centroskew matrix A maps $\mathbb{F}^n_\pm$ to $\mathbb{F}^n_\mp$, i.e., $AP_\pm = P_\mp A P_\pm$.

Let us recall the definition of Toeplitz and Hankel matrices. An $n \times n$ *Toeplitz matrix* generated by the vector $\mathbf{a} = (a_i)_{i=-n+1}^{n-1} \in \mathbb{F}^{2n-1}$ is the matrix

$$T_n(\mathbf{a}) = [\, a_{i-j} \,]_{i,j=0}^{n-1} \,.$$

We will use (2.2) in order to assign its (polynomial) symbol, in slight deviation from standard notation. An $n \times n$ *Hankel matrix* generated by $\mathbf{s} = (s_i)_{i=0}^{2n-2} \in \mathbb{F}^{2n-1}$ is the matrix

$$H_n(\mathbf{s}) = [\, s_{i+j} \,]_{i,j=0}^{n-1} \,,$$

where (2.1) is used to denote its symbol.

For Toeplitz matrices we have

$$T_n(\mathbf{a})^T = J_n T_n(\mathbf{a}) J_n = T_n(\mathbf{a}^J). \tag{2.8}$$

In particular, a Toeplitz matrix is skewsymmetric if and only if it is centroskew, or, equivalently, if its symbol is a skewsymmetric vector.

3. Centroskew Toeplitz-plus-Hankel matrices. Toeplitz-plus-Hankel matrices (shortly, $T+H$ matrices) are matrices which are a sum of a Toeplitz and a Hankel matrix. Since $T_n(\mathbf{b})J_n$ is a Hankel matrix it is possible to represent any $T+H$ matrix by means of two Toeplitz matrices,

$$R_n = T_n(\mathbf{a}) + T_n(\mathbf{b}) J_n \qquad (\mathbf{a}, \mathbf{b} \in \mathbb{F}^{2n-1}). \tag{3.1}$$

Related to this representation there is another one, using the projections (2.4) and the symbols $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{a} - \mathbf{b}$,

$$R_n = T_n(\mathbf{c}) P_+ + T_n(\mathbf{d}) P_- \,. \tag{3.2}$$

Restricting our attention to centroskew $T+H$ matrices we have the the following result regarding the underlying symbols (compare [13]).

PROPOSITION 3.1. *The $T+H$ matrix R_n is centroskew if and only if the symbols $\mathbf{a}, \mathbf{b}$ as well as $\mathbf{c}, \mathbf{d}$ of the Toeplitz matrices in (3.1) respective in (3.2) can be chosen as skewsymmetric vectors. This choice is unique.*

Proof. Let R_n be given by (3.1). Using (2.8), the centroskewsymmetry of R_n is equivalent to

$$T_n(\mathbf{a} + \mathbf{a}^J) + T_n(\mathbf{b} + \mathbf{b}^J) J_n = 0 \,,$$

which implies

$$\mathbf{a} + \mathbf{a}^J = \mathbf{e}_{\alpha,\beta} := (\alpha, \beta, \alpha, \beta, \ldots, \beta, \alpha)^T \in \mathbb{F}_+^{2n-1}$$

for some $\alpha, \beta \in \mathbb{F}$ and

$$\mathbf{b} + \mathbf{b}^J = \mathbf{f}_{\alpha,\beta} := \begin{cases} -\mathbf{e}_{\alpha,\beta} & \text{if } n \text{ is odd} \\ -\mathbf{e}_{\beta,\alpha} & \text{if } n \text{ is even.} \end{cases}$$

If we define $\hat{\mathbf{a}} = \mathbf{a} - \frac{1}{2}\mathbf{e}_{\alpha,\beta}$ and $\hat{\mathbf{b}} = \mathbf{b} - \frac{1}{2}\mathbf{f}_{\alpha,\beta}$, then $\hat{\mathbf{a}}, \hat{\mathbf{b}} \in \mathbb{F}_-^{2n-1}$ and

$$T_n(\mathbf{a}) + T_n(\mathbf{b}) J_n = T_n(\hat{\mathbf{a}}) + T_n(\hat{\mathbf{b}}) J_n.$$

Hence we can choose skewsymmetric vectors as symbols, and it is also easy to see that these choices are unique. Obviously, the same is true for the symbols $\mathbf{c}$ and $\mathbf{d}$ of the representation (3.2). □

From now on we will assume that the symbols $\mathbf{a}, \mathbf{b}$ $(\mathbf{c}, \mathbf{d})$ of a centroskew $T+H$ matrix R_n are chosen as skewsymmetric vectors. Moreover, in this case we can also write

$$R_n = P_- T_n(\mathbf{c}) P_+ + P_+ T_n(\mathbf{d}) P_-$$

instead of (3.2) (see (2.7)).

PROPOSITION 3.2. *The centroskew $T+H$ matrix R_n is nonsingular if and only if*

$$R_n^- := T_n(\mathbf{a}) - T_n(\mathbf{b}) J_n = T_n(\mathbf{c}) P_- + T_n(\mathbf{d}) P_+$$

is nonsingular.

Proof. Using (2.8) for both $T_n(\mathbf{a})$ and $T_n(\mathbf{b})$ it is immediately clear that the transpose R_n^T is equal to $-T_n(\mathbf{a}) + T_n(\mathbf{b}) J_n$. □ The following two facts are also known from [13], Corollary 3.7, but we present a simpler proof here.

THEOREM 3.3. *The centroskew $T+H$ matrix $R_n = T_n(\mathbf{c})P_+ + T_n(\mathbf{d})P_-$ is nonsingular if and only if $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$ are both nonsingular.*

Proof. Since the vector $\mathbf{c}$ and $\mathbf{d}$ are skewsymmetric, the Toeplitz matrices $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$ are skewsymmetric and centroskew. Now, using (2.7), it is easy to see that

$$\begin{bmatrix} R_n & \mathbf{0} \\ \mathbf{0} & R_n^- \end{bmatrix} = \begin{bmatrix} P_+ & P_- \\ P_- & P_+ \end{bmatrix} \begin{bmatrix} \mathbf{0} & T_n(\mathbf{d}) \\ T_n(\mathbf{c}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} P_+ & P_- \\ P_- & P_+ \end{bmatrix}, \tag{3.3}$$

where

$$\begin{bmatrix} P_+ & P_- \\ P_- & P_+ \end{bmatrix} \begin{bmatrix} P_+ & P_- \\ P_- & P_+ \end{bmatrix} = \begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & I_n \end{bmatrix}.$$

□

The following theorem gives some information about the inverse of a centroskew $T+H$ matrix.

THEOREM 3.4. *Let the centroskew $T+H$ matrix $R_n = T_n(\mathbf{c})P_+ + T_n(\mathbf{d})P_-$ be nonsingular. Then its inverse is given by*

$$R_n^{-1} = T_n(\mathbf{c})^{-1} P_- + T_n(\mathbf{d})^{-1} P_+. \tag{3.4}$$

Proof. We can use (3.3) and pass to the inverse,

$$\begin{bmatrix} R_n^{-1} & \mathbf{0} \\ \mathbf{0} & (R_n^-)^{-1} \end{bmatrix} = \begin{bmatrix} P_+ & P_- \\ P_- & P_+ \end{bmatrix} \begin{bmatrix} \mathbf{0} & T_n(\mathbf{c})^{-1} \\ T_n(\mathbf{d})^{-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} P_+ & P_- \\ P_- & P_+ \end{bmatrix}.$$

Noting that $T_n(\mathbf{c})^{-1}$ and $T_n(\mathbf{d})^{-1}$ are centroskew, the proof is easy to complete by using (2.7). □ The inverses $T_n(\mathbf{c})^{-1}$ and $T_n(\mathbf{d})^{-1}$ of the Toeplitz matrices are so-called Toeplitz Bezoutians, which together with their Hankel counterparts are analyzed next.

4. Toeplitz and Hankel Bezoutians. For later use, we are going to introduce the notions of Toeplitz Bezoutians (shortly, *T-Bezoutians*) and Hankel Bezoutians (shortly, *H-Bezoutians*).

A matrix $B \in \mathbb{F}^{n\times n}$ is called a T-Bezoutian if there exists vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}^{n+1}$ such that, in polynomial language,

$$B(t,s) = \frac{\mathbf{u}(t)\mathbf{v}^J(s) - \mathbf{v}(t)\mathbf{u}^J(s)}{1-ts}.$$

In this case we write $B = \mathrm{Bez}_T(\mathbf{u},\mathbf{v})$. Analogously, a matrix $B \in \mathbb{F}^{n\times n}$ is called an H-Bezoutian if there exists vectors $\mathbf{u},\mathbf{v} \in \mathbb{F}^{n+1}$ such that

$$B(t,s) = \frac{\mathbf{u}(t)\mathbf{v}(s) - \mathbf{v}(t)\mathbf{u}(s)}{t-s}.$$

Then we write $B = \mathrm{Bez}_H(\mathbf{u},\mathbf{v})$. It is also possible to define T- and H-Bezoutians via suitable displacement transformations. However, we will not make use of it.

H-Bezoutians are always symmetric, while T-Bezoutians B are always persymmetric, i.e., $J_n B J_n = B^T$. The two kinds of Bezoutians are related to each other by $\mathrm{Bez}_H(\mathbf{u},\mathbf{v}) = -\mathrm{Bez}_T(\mathbf{u},\mathbf{v})J_n$.

It is well known (see, e.g., [7]) that $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ as well as $\mathrm{Bez}_T(\mathbf{u},\mathbf{v})$ are nonsingular if and only if $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are *generalized coprime*, which means that the polynomials $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are coprime in the usual sense and that $\deg \mathbf{u}(t) = n$ or $\deg \mathbf{v}(t) = n$.

The following connection between Toeplitz matrices (Hankel matrices) and T-Bezoutians (H-Bezoutians) is a classical result discovered by Lander in 1974 [14].

THEOREM 4.1. *A nonsingular matrix is a T-Bezoutian (H-Bezoutian) if and only if its inverse is a Toeplitz matrix (Hankel matrix).*

Let us consider for a moment the Hankel case and discuss the question: Given the H-Bezoutian $B = \mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ with generalized coprime polynomials $\mathbf{u}(t),\mathbf{v}(t)$, how can we compute the symbol $\mathbf{s}$ of its inverse, a Hankel matrix $H_n(\mathbf{s}) = B^{-1}$? The answer was given in [3].

THEOREM 4.2. *Assume $\mathbf{u}(t),\mathbf{v}(t) \in \mathbb{F}^{n+1}[t]$ to be generalized coprime polynomials, and let $B = \mathrm{Bez}_H(\mathbf{u},\mathbf{v})$. Then B is nonsingular, the Bezout equations*

$$\mathbf{u}(t)\boldsymbol{\alpha}(t) + \mathbf{v}(t)\boldsymbol{\beta}(t) = 1\,, \tag{4.1}$$

$$\mathbf{u}^J(t)\boldsymbol{\gamma}^J(t) + \mathbf{v}^J(t)\boldsymbol{\delta}^J(t) = 1 \tag{4.2}$$

have unique solutions $\boldsymbol{\alpha}(t),\boldsymbol{\beta}(t),\boldsymbol{\gamma}(t),\boldsymbol{\delta}(t) \in \mathbb{F}^n[t]$, and $\mathbf{s} = (s_i)_{i=0}^{2n-2} \in \mathbb{F}^{2n-1}$ given by

$$\mathbf{s}^J(t) = -\boldsymbol{\alpha}(t)\boldsymbol{\delta}(t) + \boldsymbol{\beta}(t)\boldsymbol{\gamma}(t)$$

is the symbol of the inverse of B, $B^{-1} = H_n(\mathbf{s}) = [\, s_{i+j}\,]_{i,j=0}^{n-1}$. For T-Bezoutians the analogous result reads as follows [3].

THEOREM 4.3. *Assume $\mathbf{u}(t),\mathbf{v}(t) \in \mathbb{F}^{n+1}[t]$ to be generalized coprime polynomials, and let $B = \mathrm{Bez}_T(\mathbf{u},\mathbf{v})$. Then B is nonsingular, the Bezout equations (4.1) and (4.2) have unique solutions $\boldsymbol{\alpha}(t),\boldsymbol{\beta}(t),\boldsymbol{\gamma}(t),\boldsymbol{\delta}(t) \in \mathbb{F}^n[t]$, and $\mathbf{c} = (c_i)_{i=-n+1}^{n-1} \in \mathbb{F}^{2n-1}$ given by*

$$\mathbf{c}(t) = t^{n-1}\sum_{i=-n+1}^{n-1} c_i t^i = \boldsymbol{\alpha}(t)\boldsymbol{\delta}(t) - \boldsymbol{\beta}(t)\boldsymbol{\gamma}(t).$$

is the symbol of the inverse of B, $B^{-1} = T_n(\mathbf{c}) = [\, c_{i-j}\,]_{i,j=0}^{n-1}$. For our purposes it is important to specialize the previous result to the case of centroskew T-Bezoutians. As shown in [12], Section 5, if the T-Bezoutian $\mathrm{Bez}_T(\mathbf{u},\mathbf{v})$ is nonsingular and centroskew, then $\mathbf{u},\mathbf{v}$ are symmetric vectors, i.e., $\mathbf{u},\mathbf{v} \in \mathbb{F}_+^{n+1}$ (with n, of course, being even). Thus we have $\boldsymbol{\alpha} = \boldsymbol{\gamma}^J$ and $\boldsymbol{\beta} = \boldsymbol{\delta}^J$ for the (unique) solutions of (4.1) and (4.2). This implies that

$$\mathbf{c}(t) = t^{n-1}\sum_{i=-n+1}^{n-1} c_i t^i = \boldsymbol{\alpha}(t)\boldsymbol{\beta}^J(t) - \boldsymbol{\beta}(t)\boldsymbol{\alpha}^J(t). \tag{4.3}$$

Remark that $\mathbf{c} \in \mathbb{F}_-^{2n-1}$ is a skewsymmetric vector and that $T_n(\mathbf{c}) = B^{-1}$ is a skewsymmetric and centroskew matrix.

5. Splitting of centroskew $T+H$-Bezoutians. To define Toeplitz-plus-Hankel Bezoutians ($T+H$-Bezoutians) let us consider the following transformation

$$\nabla_{T+H} : \mathbb{F}^{n\times n} \to \mathbb{F}^{(n+2)\times(n+2)}$$

defined by

$$\nabla_{T+H}(B) = [b_{i-1,j} + b_{i-1,j-2} - b_{i,j-1} - b_{i-2,j-1}]_{i,j=0}^{n+1},$$

where $B = [b_{ij}]_{i,j=0}^{n-1}$ stipulating $b_{ij} = 0$ whenever i or j is not in the set $\{0,\dots,n-1\}$. Equivalently, in polynomial language,

$$(\nabla_{T+H}(B))(t,s) = (t-s)(1-ts)B(t,s).$$

A matrix $B \in \mathbb{F}^{n\times n}$ is called a $T+H$-*Bezoutian* if

$$\operatorname{rank} \nabla_{T+H}(B) \le 4.$$

This condition is equivalent to the existence of eight vectors $\mathbf{u}_i, \mathbf{v}_i$ $(i = 1,2,3,4)$ in $\mathbb{F}^{n+2}$ such that

$$(t-s)(1-ts)B(t,s) = \sum_{i=1}^{4} \mathbf{u}_i(t)\mathbf{v}_i(s).$$

For the $T+H$ case we know from [8] the following important fact.

THEOREM 5.1. *A nonsingular matrix is a $T+H$-Bezoutian if and only if its inverse is a $T+H$ matrix.*

The focus of this paper are centroskew $T+H$-Bezoutians B, i.e., those which satisfy $J_nBJ_n = -B$. As we will see in Theorem 5.3 below, nonsingular, centroskew $T+H$-Bezoutians admit a certain splitting. Let us start with the following trivial facts concerning splitting properties of an arbitrary centroskew matrix A (see [13], Section 5).

LEMMA 5.2. *Let A be a centroskew matrix of order n. Then A allows the splitting*

$$A = A_{+-} + A_{-+},$$

where $A_{+-} := AP_- = P_+A$ is a matrix the columns of which are symmetric vectors and the rows are skewsymmetric, $A_{-+} := AP_+ = P_-A$ is a matrix the columns of which are skewsymmetric vectors and the rows are symmetric. Furthermore,

$$\textit{rank}\, A = \textit{rank}\, A_{+-} + \textit{rank}\, A_{-+}.$$

In the case of a centroskew $T+H$-Bezoutian B, the result below will tell us that the splitting parts B_{+-} and B_{-+} can be represented as a product of three matrices. The middle factor is a so-called *split-Bezoutian of* $(+)$*type*. This is a $T+H$-Bezoutian involving two symmetric vectors $\mathbf{u}_+, \mathbf{v}_+$ the generating polynomial of which is given by

$$(\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_+,\mathbf{v}_+))(t,s) = \frac{\mathbf{u}_+(t)\mathbf{v}_+(s) - \mathbf{v}_+(t)\mathbf{u}_+(s)}{(t-s)(1-ts)}.$$

The matrix $\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_+,\mathbf{v}_+)$ is centrosymmetric and all rows and columns are symmetric vectors. Moreover, introduce the following $n\times(n-1)$ matrices

$$M_{n-1}^{\pm} := \begin{bmatrix} \pm1 & 0 & \cdots & 0 \\ 1 & \pm1 & \ddots & \vdots \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \pm1 \\ 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{5.1}$$

THEOREM 5.3. [13] *Let n be even. Then $B \in \mathbb{F}^{n\times n}$ is a nonsingular, centroskew $T+H$-Bezoutian if and only if it can be represented in the form*

$$B = M_{n-1}^{+}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{f}_+,\mathbf{g}_+)(M_{n-1}^{-})^T + M_{n-1}^{-}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{y}_+,\mathbf{z}_+)(M_{n-1}^{+})^T \tag{5.2}$$

with $\mathbf{f}_+,\mathbf{g}_+,\mathbf{y}_+,\mathbf{z}_+ \in \mathbb{F}_+^{n+1}$ such that $\{\mathbf{f}_+(t),\mathbf{g}_+(t)\}$ and $\{\mathbf{y}_+(t),\mathbf{z}_+(t)\}$ are pairs of coprime polynomials.

Note that the terms in the sum (5.2) are equal to the splitting parts B_{+-} and B_{-+}. The split-Bezoutians occuring therein are matrices of order $n-1$. In polynomial language this formula reads as

$$\begin{aligned} B(t,s) = (t+1)&\frac{\mathbf{f}_+(t)\mathbf{g}_+(s)-\mathbf{g}_+(t)\mathbf{f}_+(s)}{(t-s)(1-ts)}(s-1) \\ &+(t-1)\frac{\mathbf{y}_+(t)\mathbf{z}_+(s)-\mathbf{z}_+(t)\mathbf{y}_+(s)}{(t-s)(1-ts)}(s+1). \end{aligned} \tag{5.3}$$

To see this notice that $M_{n-1}^{\pm}$ is the matrix of the operator of multiplication by $t\pm 1$ in the corresponding polynomial spaces (with respect to the canonical bases).

REMARK 5.4. Different pairs of linearly independent vectors $(\mathbf{u}_+,\mathbf{v}_+)$ and $(\hat{\mathbf{u}}_+,\hat{\mathbf{v}}_+)$ produce the same split-Bezoutian of (+)type,

$$\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_+,\mathbf{v}_+) = \mathrm{Bez}_{\mathrm{sp}}(\hat{\mathbf{u}}_+,\hat{\mathbf{v}}_+)\,,$$

if and only if there is $\Phi \in \mathbb{F}^{2\times 2}$ with $\det\Phi = 1$ such that

$$[\,\hat{\mathbf{u}}_+,\hat{\mathbf{v}}_+\,] = [\,\mathbf{u}_+,\mathbf{v}_+\,]\,\Phi.$$

REMARK 5.5. Given a centroskew matrix B of even order n, one can ask how to decide whether B is a nonsingular $T+H$-Bezoutian and how to determine the vectors $\mathbf{f}_+,\mathbf{g}_+,\mathbf{y}_+,\mathbf{z}_+$ occurring in (5.2). This can be done by the following procedure:

1. Compute $B_{+-} := P_+B$ and $B_{-+} := P_-B$.
2. Verify whether $\operatorname{rank}\nabla_{T+H}(B_{+-}) = \operatorname{rank}\nabla_{T+H}(B_{-+}) = 2$.
   (If this is not fulfilled, stop: B is singular or B is not a $T+H$-Bezoutian.)
3. Determine bases $\{\mathbf{u}_\pm,\mathbf{v}_\pm\}$ in the image of $\nabla_{T+H}(B_{\pm\mp})$.
   (Due to the properties of $B_{\pm\mp}$, we have $\mathbf{u}_\pm,\mathbf{v}_\pm \in \mathbb{F}_\pm^{n+2}$.)
4. Compute

   $$\mathbf{f}_+(t) = \mathbf{u}_+(t)/(t+1), \qquad \mathbf{g}'_+(t) = \mathbf{v}_+(t)/(t+1)$$

   and

   $$\mathbf{y}_+(t) = \mathbf{u}_-(t)/(t-1), \qquad \mathbf{z}'_+(t) = \mathbf{v}_-(t)/(t-1).$$

   (Recall (2.6) and note that $\mathbf{f}_+,\mathbf{g}'_+,\mathbf{y}_+,\mathbf{z}'_+ \in \mathbb{F}_+^{n+1}$.)
5. Determine whether $\{\mathbf{f}_+(t),\mathbf{g}'_+(t)\}$ and $\{\mathbf{y}_+(t),\mathbf{z}'_+(t)\}$ are pairs of coprime polynomials.
   (If this is not fulfilled, stop: B is singular.)

6. Compute the unique vectors $\mathbf{f}'_+, \mathbf{g}_+, \mathbf{y}'_+, \mathbf{z}_+ \in \mathbb{F}_+^{n+1}$ such that

$$\nabla_{T+H}(B_{+-})(t,s) = (t+1)\big(\mathbf{f}_+(t)\mathbf{g}_+(s) - \mathbf{g}'_+(t)\mathbf{f}'_+(s)\big)(s-1)$$

and

$$\nabla_{T+H}(B_{-+})(t,s) = (t-1)\big(\mathbf{y}_+(t)\mathbf{z}_+(s) - \mathbf{z}'_+(t)\mathbf{y}'_+(s)\big)(s+1).$$

Note: In fact, there exists $\lambda, \mu \in \mathbb{F}\setminus\{0\}$ such that

$$\mathbf{f}_+(t) = \lambda \mathbf{f}'_+(t),\ \mathbf{g}_+(t) = \lambda^{-1}\mathbf{g}'_+(t),\ \mathbf{y}_+(t) = \mu \mathbf{y}'_+(t),\ \mathbf{z}_+(t) = \mu^{-1}\mathbf{z}'_+(t).$$

Therefore,

$$\nabla_{T+H}(B_{+-})(t,s) = (t+1)\left(\mathbf{f}_+(t)\mathbf{g}_+(s) - \mathbf{g}_+(t)\mathbf{f}_+(s)\right)(s-1)$$

and

$$\nabla_{T+H}(B_{-+})(t,s) = (t-1)\left(\mathbf{y}_+(t)\mathbf{z}_+(s) - \mathbf{z}_+(t)\mathbf{y}_+(s)\right)(s+1).$$

Hence, to compute λ it suffices to compare a nonzero entry of $\nabla_{T+H}(B_{+-})$ with the corresponding entry in the polynomial

$$(t+1)\big(\mathbf{f}_+(t)\mathbf{g}'_+(s) - \mathbf{g}'_+(t)\mathbf{f}_+(s)\big)(s-1).$$

The same applies to μ.

7. Now, $B = B_{+-} + B_{-+}$ is a nonsingular $T+H$-Bezoutian with

$$\begin{aligned} B_{+-} &= M_{n-1}^{+}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{f}_+, \mathbf{g}_+)(M_{n-1}^{-})^T,\\ B_{-+} &= M_{n-1}^{-}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{y}_+, \mathbf{z}_+)(M_{n-1}^{+})^T, \end{aligned}$$

where the two pairs $\{\mathbf{f}_+(t), \mathbf{g}_+(t)\}$ and $\{\mathbf{y}_+(t), \mathbf{z}_+(t)\}$ are unique up to transformations discussed in Remark 5.4.

6. Inversion of $T+H$-Bezoutians via skewsymmetric T-Bezoutians. In this section we present our first approach to invert centroskew $T+H$-Bezoutians. It is done via reduction to certain T-Bezoutians, which can be inverted using the result of Section 4. The following key result is based on the representation obtained in Theorem 5.3.

THEOREM 6.1. *Let $B \in \mathbb{F}^{n\times n}$ be a centroskew $T+H$-Bezoutian given in the form* (5.2) *with symmetric $\mathbf{f}_+, \mathbf{g}_+, \mathbf{y}_+, \mathbf{z}_+ \in \mathbb{F}_+^{n+1}$. Then*

$$B = 2\,\mathrm{Bez}_T(\mathbf{f}_+, \mathbf{g}_+)P_- - 2\,\mathrm{Bez}_T(\mathbf{y}_+, \mathbf{z}_+)P_+\,. \tag{6.1}$$

Proof. Recall that (5.2) reads in polynomial language as (5.3). Obviously, the generating polynomial of $2\,\mathrm{Bez}_T(\mathbf{f}_+, \mathbf{g}_+)P_-$ is equal to

$$\left(\mathbf{f}_+(t)\mathbf{g}_+(s) - \mathbf{g}_+(t)\mathbf{f}_+(s)\right)\left(\frac{1}{1-ts} + \frac{1}{t-s}\right).$$

Since

$$\frac{1}{1-ts} + \frac{1}{t-s} = \frac{(1+t)(1-s)}{(1-ts)(t-s)}$$

we obtain

$$2\,\mathrm{Bez}_T(\mathbf{f}_+,\mathbf{g}_+)P_- = M_{n-1}^+\mathrm{Bez}_{\mathrm{sp}}(\mathbf{f}_+,\mathbf{g}_+)(M_{n-1}^-)^T\,.$$

Analogously, using

$$\frac{1}{1-ts}-\frac{1}{t-s} = -\frac{(1-t)(1+s)}{(1-ts)(t-s)}\,,$$

it follows that

$$-2\,\mathrm{Bez}_T(\mathbf{y}_+,\mathbf{z}_+)P_+ = M_{n-1}^-\mathrm{Bez}_{\mathrm{sp}}(\mathbf{y}_+,\mathbf{z}_+)(M_{n-1}^+)^T\,.$$

This concludes the proof. □

It follows from the definition of T-Bezoutians that for symmetric vectors $\mathbf{f}_+,\mathbf{g}_+,\mathbf{y}_+,\mathbf{z}_+$

$$B_1 := \mathrm{Bez}_T(\mathbf{f}_+,\mathbf{g}_+) \quad \text{and} \quad B_2 := \mathrm{Bez}_T(\mathbf{y}_+,\mathbf{z}_+)$$

are centroskew and skewsymmetric, i.e., $B_i^T = J_nB_iJ_n = -B_i$.

PROPOSITION 6.2. *Let $B\in\mathbb{F}^{n\times n}$ be a nonsingular, centroskew $T+H$-Bezoutian given by* (5.2) *or* (6.1) *with pairs $\{\mathbf{f}_+(t),\mathbf{g}_+(t)\}$ and $\{\mathbf{y}_+(t),\mathbf{z}_+(t)\}$ of symmetric coprime polynomials in $\mathbb{F}_+^{n+1}[t]$. Then*

$$B_1 = \mathrm{Bez}_T(\mathbf{f}_+,\mathbf{g}_+) \quad \textit{and} \quad B_2 = \mathrm{Bez}_T(\mathbf{y}_+,\mathbf{z}_+)$$

are invertible, and

$$B^{-1} = \frac{1}{2}(B_1^{-1}P_+ - B_2^{-1}P_-).$$

Proof. Since the polynomials are symmetric, coprimeness implies generalized coprimeness, and hence the T-Bezoutians are invertible. We write (6.1) as

$$\tfrac{1}{2}B = B_1P_- - B_2P_+\,,$$

and take its tranpose,

$$\tfrac{1}{2}B^T = -P_-B_1 + P_+B_2\,.$$

Both equations can be written, in analogy to (3.3), in the following form:

$$\frac{1}{2}\begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & -B^T \end{bmatrix} = \begin{bmatrix} P_+ & P_- \\ P_- & P_+ \end{bmatrix}\begin{bmatrix} \mathbf{0} & B_1 \\ -B_2 & \mathbf{0} \end{bmatrix}\begin{bmatrix} P_+ & P_- \\ P_- & P_+ \end{bmatrix}. \tag{6.2}$$

Here one has to use that B_1 and B_2 are centroskew (see (2.7)). Notice that also this identity implies the invertibility of B_1 and B_2. Now one can pass to the inverse of this equation and obtain the desired expression for B^{-1} in terms of B_1^{-1} and B_2^{-1}. □

The inverses of the above T-Bezoutians are Toeplitz matrices

$$T_n(\mathbf{c}) = B_1^{-1} \text{ and } T_n(\mathbf{d}) = B_2^{-1}\,.$$

From Theorem 4.3 and the remarks made afterwards we know how to obtain the symbols $\mathbf{c},\mathbf{d}$ of these (skewsymmetric) Toeplitz matrices (see also (4.1) and (4.3)). Indeed, $\mathbf{c},\mathbf{d}\in\mathbb{F}_-^{2n-1}$ are given by

$$\mathbf{c}(t) = \boldsymbol{\alpha}(t)\boldsymbol{\beta}^J(t) - \boldsymbol{\beta}(t)\boldsymbol{\alpha}^J(t), \tag{6.3}$$

$$\mathbf{d}(t) = \boldsymbol{\gamma}(t)\boldsymbol{\delta}^J(t) - \boldsymbol{\delta}(t)\boldsymbol{\gamma}^J(t), \tag{6.4}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta} \in \mathbb{F}^n$ are the solutions of the Bezout equations

$$\mathbf{g}_+(t)\boldsymbol{\alpha}(t) + \mathbf{f}_+(t)\boldsymbol{\beta}(t) = 1, \tag{6.5}$$

$$\mathbf{z}_+(t)\boldsymbol{\gamma}(t) + \mathbf{y}_+(t)\boldsymbol{\delta}(t) = 1. \tag{6.6}$$

We can now summarize this as follows.

THEOREM 6.3. *Let $B \in \mathbb{F}^{n\times n}$ be a centroskew $T+H$-Bezoutian given by (5.2) or (6.1) with pairs $\{\mathbf{g}_+(t), \mathbf{f}_+(t)\}$ and $\{\mathbf{y}_+(t), \mathbf{z}_+(t)\}$ of symmetric coprime polynomials in $\mathbb{F}_+^{n+1}[t]$. Then n is even, B is nonsingular and*

$$B^{-1} = \frac{1}{2}(T_n(\mathbf{c})P_+ - T_n(\mathbf{d})P_-), \tag{6.7}$$

where $\mathbf{c}, \mathbf{d} \in \mathbb{F}_-^{2n-1}$ are given by (6.3) and (6.4).

Note that (6.7) reads as

$$B^{-1} = \frac{1}{4}\left[c_{j-k} + c_{j+k+1-n}\right]_{j,k=0}^{n-1} - \frac{1}{4}\left[d_{j-k} - d_{j+k+1-n}\right]_{j,k=0}^{n-1},$$

which is a (centroskew) sum of a Toeplitz and a Hankel matrix.

REMARK 6.4. Since $\mathbf{c} = (c_i)_{i=-n+1}^{n-1}$ and $\mathbf{d} = (d_i)_{i=-n+1}^{n-1}$ are skewsymmetric vectors it suffices to compute only their last $n-1$ components $(c_i)_{i=1}^{n-1}$ and $(d_i)_{i=1}^{n-1}$. To that aim introduce for a given vector $\mathbf{x} = (x_i)_{i=0}^{n-1}$ the following upper triangular Toeplitz matrix of order n,

$$U_n(\mathbf{x}) = \begin{bmatrix} x_0 & x_1 & \cdots & \cdots & x_{n-1} \\ & x_0 & x_1 & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & x_0 & x_1 \\ 0 & & & & x_0 \end{bmatrix}. \tag{6.8}$$

Now, as is easy to see and has already been stated in Section 6 of [3], equations (6.3) and (6.4) become

$$(c_i)_{i=0}^{n-1} = U_n(\boldsymbol{\beta})\boldsymbol{\alpha} - U_n(\boldsymbol{\alpha})\boldsymbol{\beta}, \quad (d_i)_{i=0}^{n-1} = U_n(\boldsymbol{\delta})\boldsymbol{\gamma} - U_n(\boldsymbol{\gamma})\boldsymbol{\delta}, \tag{6.9}$$

where $c_0 = d_0 = 0$.

Let us now present the steps of a corresponding inversion algorithm.

ALGORITHM 6.1. *We are given a centroskew $T+H$-Bezoutian B of even order n in the form (5.2) with pairs $\{\mathbf{f}_+(t), \mathbf{g}_+(t)\}$ and $\{\mathbf{y}_+(t), \mathbf{z}_+(t)\}$ of symmetric coprime polynomials in $\mathbb{F}_+^{n+1}[t]$.*

1. *Solve the Bezout equations (6.5) and (6.6) by Euclid's algorithm.*
2. *Determine the (skewsymmetric) symbols $\mathbf{c}$ and $\mathbf{d}$ by either*
   (i) computing their last components according to (6.9), or,
   (ii) computing them from (6.3) and (6.4).
3. *Compute the matrices*

$$A_{\mathbf{c}} := T_n(\mathbf{c})P_+ \text{ and } A_{\mathbf{d}} := T_n(\mathbf{d})P_-.$$

4. *Then the inverse of B is given by*

$$B^{-1} = \frac{1}{2}(A_{\mathbf{c}} - A_{\mathbf{d}}).$$

7. Inversion of $T+H$-Bezoutians via H-Bezoutians of half order. In our second approach we start again from the representation (5.2) of a centroskew $T+H$-Bezoutian B of order $n=2\ell$, i.e.,

$$B = M_{n-1}^{+}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{f}_+,\mathbf{g}_+)(M_{n-1}^{-})^T + M_{n-1}^{-}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{y}_+,\mathbf{z}_+)(M_{n-1}^{+})^T\,.$$

Recall that both $\mathrm{Bez}_{\mathrm{sp}}(\mathbf{f}_+,\mathbf{g}_+)$ and $\mathrm{Bez}_{\mathrm{sp}}(\mathbf{y}_+,\mathbf{z}_+)$ are split-Bezoutians of odd order $n-1$ and of $(+)$type since the vectors $\mathbf{f}_+,\mathbf{g}_+,\mathbf{y}_+,\mathbf{z}_+ \in \mathbb{F}_+^{n+1}$ are symmetric. Such split-Bezoutians are can be reduced to H-Bezoutians of half order ℓ as established in the following theorem. Introduce a matrix S_ℓ of size $(2\ell-1)\times\ell$ as the isomorphism defined by

$$S_\ell : \mathbb{F}^\ell \to \mathbb{F}_+^{2\ell-1}, \qquad (S_\ell \mathbf{x})(t) = \mathbf{x}(t+t^{-1})t^{\ell-1}, \quad \mathbf{x}\in\mathbb{F}^\ell.$$

Notice that

$$(S_\ell)^T = \begin{bmatrix} 0 & & & & \binom{0}{0} & & & & 0 \\ & & & \binom{1}{0} & 0 & \binom{1}{1} & & & \\ & & \binom{2}{0} & 0 & \binom{2}{1} & 0 & \binom{2}{2} & & \\ & \iddots & & & & & & \ddots & \\ \binom{\ell-1}{0} & 0 & \binom{\ell-1}{1} & 0 & \cdots & 0 & \binom{\ell-1}{\ell-2} & 0 & \binom{\ell-1}{\ell-1} \end{bmatrix}.$$

THEOREM 7.1. [13] *Let $\mathbf{u}_+,\mathbf{v}_+ \in \mathbb{F}_+^{n+1}$, $n=2\ell$, and let $\mathbf{u},\mathbf{v}\in\mathbb{F}^{\ell+1}$ be such that $\mathbf{u}_+ = S_{\ell+1}\mathbf{u}$, $\mathbf{v}_+ = S_{\ell+1}\mathbf{v}$. Then*

$$\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_+,\mathbf{v}_+) = -S_\ell\,\mathrm{Bez}_H(\mathbf{u},\mathbf{v})\,S_\ell^T\,. \tag{7.1}$$

Notice that the pair $\mathbf{u}(t)$ and $\mathbf{v}(t)$ is generalized coprime if and only if the pair $\mathbf{u}_+(t)$ and $\mathbf{v}_+(t)$ is coprime.

Combining this theorem with Theorem 5.3 we conclude the following.

THEOREM 7.2. *Let $n=2\ell$. Then $B\in\mathbb{F}^{n\times n}$ is a nonsingular, centroskew $T+H$-Bezoutian if and only if it can be represented in the form*

$$B = M_{n-1}^{+}S_\ell\mathrm{Bez}_H(\mathbf{g},\mathbf{f})S_\ell^T(M_{n-1}^{-})^T + M_{n-1}^{-}S_\ell\mathrm{Bez}_H(\mathbf{z},\mathbf{y})S_\ell^T(M_{n-1}^{+})^T \tag{7.2}$$

with generalized coprime pairs $\{\mathbf{f}(t),\mathbf{g}(t)\}$ and $\{\mathbf{z}(t),\mathbf{y}(t)\}$.

The vectors $\mathbf{f},\mathbf{g},\mathbf{z},\mathbf{y}\in\mathbb{F}^{\ell+1}$ are given by

$$\mathbf{f}_+ = S_{\ell+1}\mathbf{f}, \quad \mathbf{g}_+ = S_{\ell+1}\mathbf{g}, \quad \mathbf{y}_+ = S_{\ell+1}\mathbf{y}, \quad \mathbf{z}_+ = S_{\ell+1}\mathbf{z}, \tag{7.3}$$

or, equivalently, by

$$\mathbf{f}_+(t) = t^\ell\mathbf{f}(t+t^{-1}), \quad \mathbf{g}_+(t) = t^\ell\mathbf{g}(t+t^{-1}), \text{ etc.}$$

Let us introduce the shift matrix of order m

$$V_m = \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 0 & & & & 0 \end{bmatrix}, \tag{7.4}$$

as well as the matrices

$$T_m^{\pm} = I_m \pm V_m\,, \quad T_m = I_m - V_m^2. \tag{7.5}$$

Moreover, we need the following matrices of order m,

$$Q_m = \begin{bmatrix} \binom{0}{0} & 0 & \binom{2}{1} & 0 & \cdots & \\ & \binom{1}{0} & 0 & \binom{3}{1} & & \vdots \\ & & \binom{2}{0} & 0 & \ddots & 0 \\ & & & \binom{3}{0} & \ddots & \binom{m-1}{1} \\ & & & & \ddots & 0 \\ 0 & & & & & \binom{m-1}{0} \end{bmatrix}, \tag{7.6}$$

i.e.,

$$Q_m := [\, q_{ij}\,]_{i,j=0}^{m-1} \quad \text{with} \quad q_{ij} = \begin{cases} \binom{j}{\frac{j-i}{2}} & \text{if } j \geq i \text{ and } j-i \text{ is even} \\ 0 & \text{otherwise,} \end{cases}$$

as well as

$$U_m := [\, u_{ij}\,]_{i,j=0}^{m-1} \quad \text{with} \quad u_{ij} = \begin{cases} \binom{-i-1}{\frac{j-i}{2}} & \text{if } j \geq i \text{ and } j-i \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Noting that $\binom{-i-1}{k} = (-1)^k \binom{i+k}{k}$, we see that

$$U_m = \begin{bmatrix} \binom{0}{0} & 0 & -\binom{1}{1} & 0 & \cdots & \\ & \binom{1}{0} & 0 & -\binom{2}{1} & & \vdots \\ & & \binom{2}{0} & 0 & \ddots & 0 \\ & & & \binom{3}{0} & \ddots & -\binom{m-2}{1} \\ & & & & \ddots & 0 \\ 0 & & & & & \binom{m-1}{0} \end{bmatrix}. \tag{7.7}$$

It can be proved straightforwardly (see also Lemma 5.1 in [2]) that

$$U_{\ell+1} T_{\ell+1} Q_{\ell+1} = I_{\ell+1}. \tag{7.8}$$

Observe that $Q_{\ell+1}$ is the lower part of $S_{\ell+1}$. The upper part, i.e., the first ℓ rows of $S_{\ell+1}$, is the $\ell \times (\ell+1)$ matrix $J_{\ell+1} Q_{\ell+1}$ after cancelling its last row.

Denoting by $\mathbf{f}_+^l, \mathbf{g}_+^l, \ldots$ the last $\ell+1$ components of $\mathbf{f}_+, \mathbf{g}_+, \ldots$, respectively, (7.3) can be written as

$$\mathbf{f} = Q_{\ell+1}^{-1} \mathbf{f}_+^l, \quad \mathbf{g} = Q_{\ell+1}^{-1} \mathbf{g}_+^l\,, \quad \text{etc.,} \tag{7.9}$$

where $Q_{\ell+1}^{-1} = U_{\ell+1} T_{\ell+1}$.

Let us continue with discussing what follows from Theorem 7.2. The representation (7.2) can be written in the form

$$B = W_n \begin{bmatrix} \mathbf{0} & \mathrm{Bez}_H(\mathbf{g}, \mathbf{f}) \\ \mathrm{Bez}_H(\mathbf{z}, \mathbf{y}) & \mathbf{0} \end{bmatrix} W_n^T ,$$

where $W_n := [M_{n-1}^+ S_\ell \mid M_{n-1}^- S_\ell] \in \mathbb{F}^{n\times n}$. We are going to rewrite W_n in a suitable way. A straightforward computation yields

$$M_{n-1}^{\pm} S_\ell = \begin{bmatrix} \pm J_\ell T_\ell^{\pm} Q_\ell \\ T_\ell^{\pm} Q_\ell \end{bmatrix}.$$

Thus

$$W_n = \begin{bmatrix} J_\ell & -J_\ell \\ I_\ell & I_\ell \end{bmatrix} \begin{bmatrix} T_\ell^+ Q_\ell & \mathbf{0} \\ \mathbf{0} & T_\ell^- Q_\ell \end{bmatrix}.$$

This matrix is invertible, and

$$W_n^{-1} = \frac{1}{2} \begin{bmatrix} (T_\ell^+ Q_\ell)^{-1} & \mathbf{0} \\ \mathbf{0} & (T_\ell^- Q_\ell)^{-1} \end{bmatrix} \begin{bmatrix} J_\ell & I_\ell \\ -J_\ell & I_\ell \end{bmatrix}.$$

From Theorem 4.2 we obtain an inversion formula of the form

$$B^{-1} = W_n^{-T} \begin{bmatrix} \mathbf{0} & H_\ell(\mathbf{s}_2) \\ H_\ell(\mathbf{s}_1) & \mathbf{0} \end{bmatrix} W_n^{-1}. \tag{7.10}$$

Here $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{F}^{2\ell-1}$ are obtained by solving the Bezout equations

$$\begin{aligned} \mathbf{g}(t)\boldsymbol{\alpha}_1(t) + \mathbf{f}(t)\boldsymbol{\beta}_1(t) &= 1, \\ \mathbf{g}^J(t)\boldsymbol{\gamma}_1^J(t) + \mathbf{f}^J(t)\boldsymbol{\delta}_1^J(t) &= 1, \end{aligned} \tag{7.11}$$

and

$$\begin{aligned} \mathbf{z}(t)\boldsymbol{\alpha}_2(t) + \mathbf{y}(t)\boldsymbol{\beta}_2(t) &= 1, \\ \mathbf{z}^J(t)\boldsymbol{\gamma}_2^J(t) + \mathbf{y}^J(t)\boldsymbol{\delta}_2^J(t) &= 1. \end{aligned} \tag{7.12}$$

and computing for $i = 1, 2$,

$$\mathbf{s}_i^J(t) = -\boldsymbol{\alpha}_i(t)\boldsymbol{\delta}_i(t) + \boldsymbol{\beta}_i(t)\boldsymbol{\gamma}_i(t)\,. \tag{7.13}$$

Now the inversion formula (7.10) can be written as stated in the following result.

PROPOSITION 7.3. *Let $B \in \mathbb{F}^{n\times n}$, $n = 2\ell$, be a nonsingular, centroskew $T + H$-Bezoutian given in the from* (7.2)*. Then*

$$B^{-1} = \frac{1}{4} \begin{bmatrix} -J_\ell \\ I_\ell \end{bmatrix} A_\ell^{(1)} \Big[J_\ell ,\, I_\ell \Big] + \frac{1}{4} \begin{bmatrix} J_\ell \\ I_\ell \end{bmatrix} A_\ell^{(2)} \Big[-J_\ell ,\, I_\ell \Big] \tag{7.14}$$

with

$$A_\ell^{(1)} := (T_\ell^- Q_\ell)^{-T} H_\ell(\mathbf{s}_1)(T_\ell^+ Q_\ell)^{-1}, \quad A_\ell^{(2)} := (T_\ell^+ Q_\ell)^{-T} H_\ell(\mathbf{s}_2)(T_\ell^- Q_\ell)^{-1}.$$

Here $\mathbf{s}_1$ and $\mathbf{s}_2$ are obtained from (7.11)–(7.13).

It now remains to identify $A_\ell^{(i)}$ as centroskew $T+H$ matrices, for which we need further results. We will make use of the following three kinds of $T+H$ matrices of order ℓ, which we introduce for a skewsymmetric vector $\mathbf{c} = (c_k)_{k=-2\ell+1}^{2\ell-1}$ and a symmetric vector $\mathbf{a}^\# = (a_k^\#)_{k=-2\ell+2}^{2\ell-2}$,

$$\mathrm{TH}_\ell^\pm(\mathbf{c}) = \left(c_{i-j} \pm c_{i+j+1}\right)_{i,j=0}^{\ell-1}$$

and

$$\mathrm{TH}_\ell^\#(\mathbf{a}^\#) = \left(a_{i-j}^\# + a_{i+j}^\#\right)_{i,j=0}^{\ell-1}.$$

The proof of the following lemma is straightforward.

LEMMA 7.4. *Let a skewsymmetric vector* $\mathbf{c} = (c_k)_{k=-2\ell+1}^{2\ell-1} \in \mathbb{F}_-^{4\ell-1}$, *and a symmetric vector* $\mathbf{a}^\# = (a_k^\#)_{k=-2\ell+2}^{2\ell-2} \in \mathbb{F}_+^{4\ell-3}$ *be related via*

$$a_k^\# = c_{k+1} - c_{k-1}.$$

Then

$$D_\ell \mathrm{TH}_\ell^\#(\mathbf{a}^\#) D_\ell = -(T_\ell^+)^T \mathrm{TH}_\ell^-(\mathbf{c}) T_\ell^- = (T_\ell^-)^T \mathrm{TH}_\ell^+(\mathbf{c}) T_\ell^+ \tag{7.15}$$

with $D_\ell := \mathrm{diag}(\frac{1}{2}, 1, 1, \ldots, 1)$ *and* $T_\ell^\pm$ *given by* (7.5).

Notice that the relationship between $\mathbf{a}^\#$ and $\mathbf{c}$ can be expressed by

$$\begin{bmatrix} 2 & & & & 0 \\ 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{2\ell-1} \end{bmatrix} = \begin{bmatrix} a_0^\# \\ a_1^\# \\ \vdots \\ a_{2\ell-2}^\# \end{bmatrix}.$$

THEOREM 7.5. *Given a skewsymmetric vector* $\mathbf{c} = (c_k)_{k=-2\ell+1}^{2\ell-1} \in \mathbb{F}_-^{4\ell-1}$, *define*

$$\mathbf{s} = Q_{2\ell-1}^T T_{2\ell-1}^T (c_k)_{k=1}^{2\ell-1} \in \mathbb{F}^{2\ell-2}, \tag{7.16}$$

with $T_{2\ell-1}$ *introduced by* (7.5). *Then*

$$H_\ell(\mathbf{s}) = -Q_\ell^T (T_\ell^+)^T \mathrm{TH}_\ell^-(\mathbf{c}) T_\ell^- Q_\ell = Q_\ell^T (T_\ell^-)^T \mathrm{TH}_\ell^+(\mathbf{c}) T_\ell^+ Q_\ell.$$

Proof. Introduce $\mathbf{a}^\# = (a_k^\#)_{k=-2\ell+2}^{2\ell-2} \in \mathbb{F}_+^{4\ell-3}$ as in the previous lemma, i.e.,

$$D_{2\ell-1}^{-1} T_{2\ell-1}^T (c_k)_{k=1}^{2\ell-1} = (a_k^\#)_{k=0}^{2\ell-2}.$$

In [1], Theorem 5, it was shown that

$$H_\ell(\mathbf{s}) = Q_\ell^T D_\ell \mathrm{TH}_\ell^\#(\mathbf{a}^\#) D_\ell Q_\ell$$

if $\mathbf{a}^\#$ is symmetric and $\mathbf{s} = Q_{2\ell-1}^T D_{2\ell-1} (a_k^\#)_{k=0}^{2\ell-2}$. Combining this with the lemma we arrive at the stated formula. □

Notice that using (7.8), equation (7.16) can be written as

$$(c_k)_{k=1}^{2\ell-1} = U_{2\ell-1}^T \mathbf{s} \tag{7.17}$$

with U_m given in (7.7).

Applying Theorem 7.5 to $H_\ell(\mathbf{s}_i)$ with $\mathbf{s}_i$ given in (7.13) we see that

$$A_\ell^{(1)} = \mathrm{TH}_\ell^+(\mathbf{c}^{(1)}), \qquad A_\ell^{(2)} = -\mathrm{TH}_\ell^-(\mathbf{c}^{(2)})$$

where

$$\mathbf{c}^{(i)} = \begin{bmatrix} -J_{2\ell-1}U_{2\ell-1}^T\mathbf{s}_i \\ 0 \\ U_{2\ell-1}^T\mathbf{s}_i \end{bmatrix}. \tag{7.18}$$

Combining this with the above formula (7.14), it follows that

$$B^{-1} = \frac{1}{4}\begin{bmatrix} -J_\ell \\ I_\ell \end{bmatrix} \mathrm{TH}_\ell^+(\mathbf{c}^{(1)})\,[\, J_\ell \;\; I_\ell \,] - \frac{1}{4}\begin{bmatrix} J_\ell \\ I_\ell \end{bmatrix} \mathrm{TH}_\ell^-(\mathbf{c}^{(2)})\,[\, -J_\ell \;\; I_\ell \,], \tag{7.19}$$

which equals

$$\frac{1}{4}\left[c_{j-k}^{(1)} + c_{j+k+1}^{(1)}\right]_{j,k=-\ell}^{\ell-1} - \frac{1}{4}\left[c_{j-k}^{(2)} - c_{j+k+1}^{(2)}\right]_{j,k=-\ell}^{\ell-1},$$

i.e.,

$$\frac{1}{4}\left[c_{j-k}^{(1)} + c_{j+k+1-n}^{(1)}\right]_{j,k=0}^{n-1} - \frac{1}{4}\left[c_{j-k}^{(2)} - c_{j+k+1-n}^{(2)}\right]_{j,k=0}^{n-1}.$$

Summarizing we arrive at the following result.

THEOREM 7.6. *The inverse of a nonsingular, centroskew $T+H$-Bezoutian B of order $n=2\ell$ given by* (7.2) *admits the representation*

$$B^{-1} = \frac{1}{2}(T_n(\mathbf{c}^{(1)})P_+ - T_n(\mathbf{c}^{(2)})P_-), \tag{7.20}$$

where $\mathbf{c}^{(i)}$ is given in (7.18) *and* (7.11)–(7.13).

Finally, let us present the steps of a corresponding algorithm for the inversion of a centroskew $T+H$-Bezoutian.

ALGORITHM 7.1.

We are given a centroskew $T+H$-Bezoutian of order $n=2\ell$ in the form (5.2) *with pairs $\{\mathbf{f}_+(t), \mathbf{g}_+(t)\}$ and $\{\mathbf{y}_+(t), \mathbf{z}_+(t)\}$ of symmetric coprime polynomials in $\mathbb{F}_+^{n+1}[t]$.*

1. *Compute the vectors $\mathbf{f}, \mathbf{g}, \mathbf{y}, \mathbf{z} \in \mathbb{F}^{\ell+1}$ according to* (7.9), *where $Q_{\ell+1}^{-1} = U_{\ell+1}T_{\ell+1}$ with $U_{\ell+1}, T_{\ell+1}$ defined in* (7.7), (7.5).
2. *Solve the Bezout equations* (7.11) *and* (7.12) *by Euclid's algorithm.*
3. *Compute the vectors $\mathbf{s}_i$ by polynomial multiplication according to* (7.13).
4. *Compute the symbols $\mathbf{c}^{(i)}$ as in* (7.18).
5. *Compute the matrices*

$$A_1 := T_n(\mathbf{c}^{(1)})P_+ \;\text{ and }\; A_2 := T_n(\mathbf{c}^{(2)})P_-.$$

6. *Then the inverse of B is given by*

$$B^{-1} = \frac{1}{2}(A_1 - A_2).$$

8. Final remarks. Comparing the inversion formulas (6.7) and (7.20) we obtain for the symbols of the Toeplitz matrices the following equalities

$$\mathbf{c}^{(1)} = \mathbf{c} \quad \text{and} \quad \mathbf{c}^{(2)} = \mathbf{d}\,.$$

Indeed this is a consequence of the uniqueness of the representation (see Proposition 3.1). To compute these symbols we have discussed two different possibilities, based on representations of the splitting parts of B,

$$B_{+-} = M_{n-1}^{+}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{f}_+, \mathbf{g}_+)(M_{n-1}^{-})^T\,, \quad B_{-+} = M_{n-1}^{-}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{y}_+, \mathbf{z}_+)(M_{n-1}^{+})^T.$$

Indeed, the representations

$$B_{+-} = 2\,\mathrm{Bez}_T(\mathbf{f}_+, \mathbf{g}_+)P_- \quad \text{and} \quad B_{-+} = -2\,\mathrm{Bez}_T(\mathbf{y}_+, \mathbf{z}_+)P_+\,.$$

led to the first algorithm, whereas

$$B_{+-} = M_{n-1}^{+}S_\ell \mathrm{Bez}_H(\mathbf{g}, \mathbf{f})S_\ell^T(M_{n-1}^{-})^T$$

and

$$B_{-+} = M_{n-1}^{-}S_\ell \mathrm{Bez}_H(\mathbf{z}, \mathbf{y})S_\ell^T(M_{n-1}^{+})^T$$

was the basis of the second algorithm.

The advantage of the first approach is that it is simpler and straightforward. One has to invert two T-Bezoutians of order n, but the symbols $\mathbf{c}, \mathbf{d}$ of the corresponding two skewsymmetric Toeplitz matrices are obtained directly after solving corresponding Bezout equations.

The second approach has the benefit that one has to invert two H-Bezoutians of half the order $\ell = \frac{n}{2}$, which involves solving corresponing Bezout equations of half the size. On the other hand, one has to perform additional matrix-vector multiplication before and after this step, where the matrices are

$$Q_{\ell+1}^{-1} = U_{\ell+1}T_{\ell+1}, \qquad \text{and} \qquad U_{2\ell-1}^T\,.$$

Let us finally remark that the second approach is the analogue of the method for inverting centrosymmetric $T + H$-Bezoutians, which we discuss in a previous paper [2]. Thus we have shown here that the method of [2] also works in the case of centroskew $T + H$-Bezoutians. But, we have not (yet) found how our first approach could be modified in order to be applicable to centrosymmetric $T + H$-Bezoutians. (Note that in this case R_n given in (3.2) nonsingular has not the consequence that $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$ are nonsingular.) It seems, somewhat surprisingly, as if the centroskewsymmetric case is easier to deal with than the centrosymmetric case.

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