Absolute algebraic connectivity of double brooms

Sebastian Richter[1], Israel Rocha[2]

Abstract

We use a geometric technique based on embeddings of graphs to provide an explicit formula for the absolute algebraic connectivity and its eigenvectors of double brooms.

Keywords: Absolute algebraic connectivity; Laplacian Matrix; Fiedler Vector

1. Introduction and Preliminaries

Ordering graphs is a classic problem in combinatorics that many authors have been studying throughout the years. One common approach to this problem is the use of spectral parameters of graphs. There is a great deal of papers using different sorts of spectral parameters, for instance the largest eigenvalues [15, 10, 17], index [25], Laplacian index [23, 13, 18], graph energy [20, 12, 21], Laplacian energy [19, 8, 9] and the algebraic connectivity [11, 24, 16, 22].

The algebraic connectivity, defined as the second smallest eigenvalue of the Laplacian matrix is one of the most popular spectral invariants. First studied by Fiedler in [5], he named this eigenvalue algebraic connectivity due to its significance on connectivity properties of graphs. For instance a graph is connected if and only if the algebraic connectivity is different from zero. After that, many applications for this eigenvalue were found, for example: graph partitioning, expanding properties of graphs, isoperimetric number, genus and the traveling salesman problem. For a survey of such results we refer the reader to [1].

Despite all the effords, a complete order of the full class of graphs by the algebraic connectivity is not known and it seems to be a very difficult problem. However, there are many studies providing a partial order in particular classes. One of them, named the double broom trees, is the subject of this paper.

First, we consider a graph $G = (V, E)$ with vertex set V and edge set E. In this paper, P_n denotes the path with n vertices. A double broom tree with n vertices is defined as follows: take a path P_{d-1} and label its vertices as $s_1, \dots, s_{d-1}$, then for $0 \leq k \leq \left\lfloor \frac{n-d+1}{2} \right\rfloor - 1$ attach $\left\lceil \frac{n-d+1}{2} \right\rceil + k$ leaves at s_1 and $\left\lfloor \frac{n-d+1}{2} \right\rfloor - k$ leaves at s_{d-1}. Clearly the diameter of such double broom is d. We denote the

[1]Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany. sebastian.richter@mathematik.tu-chemnitz.de

[2]Instituto de Matemática, Universidade Federal do Rio Grande do Sul, CEP 91509-900, Porto Alegre, RS, Brazil. israel.rocha@ufrgs.br

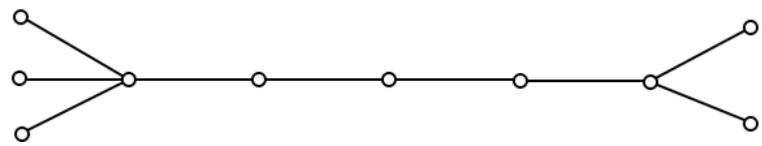

Figure 1: A double broom $T(10, 6, 0)$.

double broom with n vertices, diameter d and parameter k by $T(n, d, k)$. See Figure 1.

The first study ordering double brooms by the algebraic connectivity is due to Grone and Merris [11]. They completely ordered the class of double brooms with diameter three. Later, Fallat and Kirkland [4] generalized these results for the whole set of double brooms. The authors proved that for fixed n and d the algebraic connectivity $a(T(n, d, k))$ is a strictly increasing function of k. In order to prove it, the authors used algebraic techniques together with an application of the Perron-Frobenius theory. We notice that such techniques do not provide the value of $a(G)$, which is a rather difficult problem. We refer the reader to the survey of results about ordering graphs by algebraic connectivity [2].

In [6], Fiedler studied the absolute algebraic connectivity $\hat{a}(G)$ of a graph G. It is defined as the maximum of algebraic connectivities for all nonnegative valuations on edges of G whose values sum up to $|E|$. A sharp upper bound for the absolute algebraic connectivity in terms of vertex connectivity is proved in [14]. We notice that up to now there is no partial order for graphs by the absolute algebraic connectivity.

Since the absolute algebraic connectivity is not a well known parameter we proceed with its study using a different approach. In this paper, we provide an explicit formula for the absolute algebraic connectivity of double brooms. Therefore, an order by the absolute algebraic connectivity in this class is straightforward. In order to compute the formulas, we use a geometric technique that is not well known and we would like to share with the spectral graph theory community.

Our approach relies on results of [7], where the authors used semidefinite programming techniques to describe the absolute algebraic connectivity as a problem of finding an embedding of vertices of the graph in n-space. More precisely, they showed that

$$\begin{array}{rl} \dfrac{|E|}{\hat{a}(G)} & = \max \sum_{i \in V} \|v_i\|^2 \\ s.t. & \left(\sum_{i \in V} v_i\right)^2 = 0 \\ & \|v_i - v_j\| \leq 1, ij \in E, \\ & v_i \in \mathbb{R}^n \text{ for } i \in V \end{array} \tag{1}$$

By solving problem (1) for the class of double brooms we calculate expressions for the absolute algebraic connectivity. Then it is possible to get a better understanding of the behaviour of the absolute algebraic connectivity in this class. For instance, for fixed n and d the absolute algebraic connectivity $\hat{a}(T(n,d,k))$ as a function of k increases, but with a different function from a certain value of k. This phenomenon can be summarized in the following Theorem.

Theorem 1. *For a double broom $T(n,d,k)$, let $p=-d$ whenever $T(n,d,k)$ has an odd number of leaves and $p=0$ otherwise. Define*

$$M=\begin{cases} n(\frac{1}{2}d^2-d+1)-d(\frac{1}{6}d^2+\frac{1}{2}d-\frac{7}{6})-k(d^2-2d) & k\geq\frac{n(d-2)+p}{2d} \text{ and } p\neq 0\\ n(\frac{1}{2}d^2-d+1)-\frac{1}{6}d(d^2-1)-k(d^2-2d) & k\geq\frac{n(d-2)+p}{2d} \text{ and } p=0\\ \frac{1}{4}nd^2-\frac{1}{6}d(d^2-1)-\frac{d^2}{n}(k^2+k+\frac{1}{4}) & k<\frac{n(d-2)+p}{2d} \text{ and } p\neq 0\\ \frac{1}{4}nd^2-\frac{1}{6}d(d^2-1)-\frac{k^2d^2}{n} & k<\frac{n(d-2)+p}{2d} \text{ and } p=0.\end{cases}$$

Then $\hat{a}(T(n,d,k))=\frac{n-1}{M}$.

Furthermore, it is possible to provide a full description of the eigenspace associated with $\hat{a}(T(n,d,k))$. For this purpose we use a result of [7] that we summarize as follows.

Lemma 2. *Let $v_1,\ldots,v_n$ be an optimal solution of (1) and define the matrix $V=[v_1,v_2,\ldots,v_n]$. Let $z\in\mathbb{R}^n\setminus\{0\}$ and define the vector $u=V^Tz$. Then u is an eigenvector to $\hat{a}(G)$.*

Since we provide a full description of the embedding, we use projections of the embedding to obtain the following result.

Theorem 3. *Let $T(n,d,k)$ be a double broom and define $p=-d$ whenever $T(n,d,k)$ has an odd number of leaves and $p=0$ otherwise, let $m=\left(\left\lfloor\frac{n-d+1}{2}\right\rfloor-k\right)d^2+\frac{2d^3-3\,d^2+d}{6}-1$ and $q=\left\lfloor\frac{n-d+1}{2}\right\rfloor+k$ (the number of leaves on the left side). Let $f=[u_1,u_2,\ldots,u_n]^T$ be an eigenvector to $\hat{a}(T(n,d,k))$.*

If $k<\frac{n(d-2)+p}{2d}$, then $\hat{a}(T(n,d,k))$ has one eigenvector given by

$$u_i=\begin{cases}-(\lfloor\frac{n-d+1}{2}\rfloor-k+\frac{d-1}{2})\frac{d}{n} & i=1,\ldots,q\\ u_1+i & i=q+1,\ldots,q+d\,.\\ u_1+q+d+1 & i=q+d+1,\ldots,n\end{cases}$$

Otherwise $\hat{a}(T(n,d,k))$ has two eigenvectors: the first one is given by

$$u_i=\begin{cases} i & i=q+1,\ldots,q+d\\ q+d+1 & i=q+d+1,\ldots,n\\ -\frac{m}{q} & i=1,\ldots,q \text{ and } q \text{ even}\\ -\frac{m+1}{q-1} & i=1,\ldots,q-1 \text{ and } q \text{ odd}\\ -1 & i=q \text{ and } q \text{ odd.}\end{cases}$$

The second one for even q is given by

$$u_i = \begin{cases} \sqrt{1-(\frac{m}{q})^2} & i = 1, \dots, \frac{q}{2} \\ -\sqrt{1-(\frac{m}{q})^2} & i = \frac{q}{2}+1, \dots, q \\ 0 & otherwise. \end{cases}$$

If q is odd then we have

$$u_i = \begin{cases} \sqrt{1-(\frac{m-1}{q-1})^2} & i = 1, \dots, \frac{q-1}{2} \\ -\sqrt{1-(\frac{m-1}{q-1})^2} & i = \frac{q+1}{2}, \dots, q-1 \\ 0 & otherwise. \end{cases}$$

The paper is organized as follows. In Section 2 we prove that there are two possible structures for an optimal embedding of double brooms. In Section 3 we compute the values of these two embeddings and we provide conditions for either one or the other to be the optimal.

2. Embeddings Description

In this section we rule out the embeddings that cannot be an optimal solution of (1). Clearly the first constraint of problem (1) requires that the embedding has the barycenter at the origin, which we call equilibrium constraint, and the second constraint requires that the distances of adjacent vertices are bounded by one, which we call distance constraints.

In general solving problem (1) is difficult. However, the authors of [7] showed a property for an optimal embedding which is described in the so called Separator-Shadow Theorem.

Theorem 4. *(Separator-Shadow) Let $v_i \in R^n$ $(i \in N)$ be an optimal solution of (1) for a connected graph $G = (V, E)$ and let S be a separator in G giving rise to disconnected sets C_1 and C_2. For at least one C_j,*

$$conv\,\{0, v_i\} \cap conv\,\{v_s : s \in S\} \neq \emptyset, \forall i \in C_j.$$

In words, the straight line segments $conv\,\{0, v_i\}$ of all nodes $i \in C_j$ intersect the convex hull of the points in S.

We remark some additional properties of an optimal embedding that follow from results proven in [7].

Remark 5. Optimal embeddings of a tree satisfy $\|v_i - v_j\| = 1$ for all $ij \in E$.

Remark 6. The barycenter of an optimal embedding of a tree is either on an edge or on a vertex.

After applying the Separator-Shadow Theorem, we can discard many structures as candidates for an optimal embedding of a double broom.

Furthermore, it is possible to rule out embeddings of dimensions higher than two. In order to do this, we need the definitions of the tree-decomposition and the tree-width from [3].

Definition 7. For a graph $G = (N, E)$ a tree-decomposition of G is a tree $T = (\mathcal{N}, \mathcal{E})$ with $\mathcal{N} \subseteq 2^N$ and $\mathcal{E} \subseteq \binom{\mathcal{N}}{2}$ satisfying the following requirements:

(i) $N = \cup_{U \in \mathcal{N}} U$

(ii) For every $e \in E$ there is an $U \in \mathcal{N}$ with $e \in U$.

(iii) If $U_1, U_2, U_3 \in \mathcal{N}$ with U_2 on the T-path from U_1 to U_2, then $U_1 \cap U_3 \subseteq U_2$.

The width of T is the number $max\,\{|U| - 1 : U \in \mathcal{N}\}$. The tree-width $tw(G)$ is the least width of any tree-decomposition of G.

In [7] the authors showed the following relation between the dimension of an optimal embedding and the tree-width of a graph.

Theorem 8. *For each connected graph G there exists an optimal embedding of (1) of dimension at most at most $tw(G) + 1$.*

To proceed with the analysis, we refer to leaves on the left side and on the right side as the leaves attached to s_1 and s_{d-1}, respectively. By definition, the number of leaves on the left side is greater than or equal to the number of leaves on the right side.

Proposition 9. *An optimal embedding of a double broom in respect to (1) has one of the following forms:*

1. 1-dimensional: all nodes are on a straight line and the leaves are stretched out. The barycenter is on an edge that does not contain a leaf or a vertex different from s_1;
2. 2-dimensional: the barycenter is on s_1 and all vertices are over a line, with the exception of the leaves attached to s_1. The leaves attached to s_1 are embedded on the unit circle with center s_1.

Proof. Since the tree-width of a tree is one, we can apply Theorem 8 to see that there is an optimal embedding of a double broom with dimension at most two.

In reference to Remark 6, we can suppose that the barycenter is on an edge or a vertex.

First, we consider an optimal embedding with the barycenter on an edge $s_k s_{k+1} \in \{s_1 s_2, \ldots, s_{d-2} s_{d-1}\}$. Applying the Separator-Shadow Theorem for $v_s = s_k$, $s_{k+1} \in C_1$, we have $conv\,\{0, s_i\} \cap s_k = \emptyset, \forall s_i \in C_1$. Therefore, $conv\{0, s_i\} \cap s_k \neq \emptyset$ holds for all $s_i \in C_2$. Hence, every $v \in C_2$ is embedded over the line containing $conv\,\{0, v_i\}$, $v_i \in C_2$. Again by the Separator-Shadow Theorem, for the same embedding, by choosing $v_s = s_{k+1}$, $s_k \in C_1$, we conclude that all vertices must be embedded over a line containing the barycenter. By the maximality of the embedding and Remark 5, every vertex must be embedded as far as possible from the barycenter. Therefore, all leaves are stretched out on this line and it is an embedding of the form (1).

Now suppose an optimal embedding where the barycenter is on a vertex $v \in \{s_2, \ldots, s_{d-2}\}$. Again, by the maximality of the embedding and Remark 5 all leaves must be stretched out over a line and it is an embedding of the form (1).

If the barycenter is on a leaf, then the embedding cannot be maximal and satisfy the equilibrium constraint at the same time. Note, that since the left side has more leaves than the right side, the barycenter cannot be on s_{d-1} nor on an edge on the right side.

Suppose the barycenter is on an edge containing s_1 and a leaf from the left side, say $v_1 s_1$. Then we can choose $v_s = s_1$ as a separator, set $C_1 = v_1$ and $C_2 = V \backslash C_1$. Since $conv\{0, v_1\} \cap s_1 = \emptyset$, all vertices in C_2 have to be embedded over a line containing the barycenter. By the maximality of the embedding and Remark 5, every vertex in C_2 must be embedded as far as possible from the barycenter. This embedding is not feasible, because the barycenter is on $v_1 s_1$ and the equilibrium constraint will not be satisfied.

Eventually, we consider an embedding where the barycenter is on s_1. In this case we get an embedding of the form (2). Applying the Separator-Shadow Theorem for $v_s = s_1$ it is easy to see that every component must be over a line segment containing the origin. Furthermore, the nodes must be embedded as far as possible from the origin in order to attain the maximum. By Remark 6, we can embed the leaves on the left on the unit circle with center s_1, and with a suitable choice of the coordinates it is possible to satisfy the equilibrium constraint. $\square$

3. Embedding calculations

In this section we will calculate the quantity $\sum_{i=1}^{n} \|v_i\|^2$ for the two described possible forms of optimal embeddings of double brooms. We start with embeddings of the form (2).

Lemma 10. *Let $u_1, \ldots, u_n$ be an embedding of $T(n,d,k)$ of the form (2). If $T(n,d,k)$ has an even number of leaves, then $\sum_{i=1}^{n} \|u_i\|^2 = n + \frac{1}{6}d - \frac{1}{6}d^3 + \frac{1}{2}d^2 n - dn - kd^2 + 2\,kd$. Otherwise, $\sum_{i=1}^{n} \|u_i\|^2 = n + \frac{7}{6}d - \frac{1}{6}d^3 - \frac{1}{2}d^2 + \frac{1}{2}d^2 n - dn - kd^2 + 2kd$.*

Proof. Suppose $T(n,d,k)$ has an even number of leaves. This embedding has $\frac{n-d+1}{2} + k$ and $\frac{n-d+1}{2} - k$ leaves on the left side and right side, respectively. Since each edge has length one, we get

$$\begin{aligned} \sum_{i=1}^{n} \|u_i\|^2 &= \frac{n-d+1}{2} + k + \sum_{i=1}^{d-2} i^2 + \left(\frac{n-d+1}{2} - k\right)(d-1)^2 \\ &= n + \frac{1}{6}d - \frac{1}{6}d^3 + \frac{1}{2}d^2 n - dn - kd^2 + 2kd. \end{aligned}$$

Suppose $T(n,d,k)$ has an odd number of leaves. For this embedding, there are $\frac{n-d+2}{2} + k$ and $\frac{n-d}{2} - k$ leaves on the left side and right side, respectively. Similarly, we have

$$
\begin{aligned}
\sum_{i=1}^{n} \|u_i\|^2 &= \frac{n-d+2}{2} + k + \sum_{i=1}^{d-2} i^2 + \left(\frac{n-d}{2} - k\right)(d-1)^2 \\
&= n + \frac{7}{6}d - \frac{1}{6}d^3 - \frac{1}{2}d^2 + \frac{1}{2}d^2 n - dn - kd^2 + 2kd.
\end{aligned}
$$

This shows the lemma. □

We can calculate simple expressions for embeddings of the form (1) in the same fashion. Since the coordinates of the embedded vertices are not that obvious as before we start by giving expressions for these. In order to simplify the calculations we summarize the leaves on the left and the right to a weighted node, respectively. The weights on these two nodes are set to the corresponding number of leaves. The weights of all other nodes are set to one. Therefore, $\sum_{i=1}^{n} u_i = \sum_{i=1}^{\bar{n}} w_i \overline{u}_i$ and $\sum_{i=1}^{n} u_i^2 = \sum_{i=1}^{\bar{n}} w_i \overline{u}_i^2$ with $\bar{n} = d+1$. Here $\overline{u}_1, \overline{u}_{\bar{n}}$ represent the coordinates of leaves on the left and the right, respectively, $w_1 = \lceil \frac{n-d+1}{2} \rceil + k$, $w_{\bar{n}} = \lfloor \frac{n-d+1}{2} \rfloor - k$ and $w_i = 1$ for $i = 2, \ldots, \bar{n}-1$.

Proposition 11. *Let $\overline{u}_1, \ldots \overline{u}_{\bar{n}}$ be the coordinates of an optimal embedding of $T(n,d,k)$ of the form (1). Then $\overline{u}_1 = -(\lfloor \frac{n-d+1}{2} \rfloor - k + \frac{d-1}{2})\frac{d}{n}$ and $\overline{u}_i = u_1 + i$ for $i = 2, \ldots, \bar{n}$.*

Proof. Let $\overline{u}_1, \ldots \overline{u}_{\bar{n}}$ be the coordinates of an optimal embedding of $T(n,d,k)$. To satisfy the equilibrium constraint we need $0 = \sum_{i=1}^{n} u_i = \sum_{i=1}^{\bar{n}} w_i \overline{u}_i$, hence

$$
\begin{aligned}
\sum_{i=1}^{\bar{n}} w_i \overline{u_i} &= \left(\left\lceil \frac{n-d+1}{2} \right\rceil + k\right)\overline{u}_1 + \left(\left\lfloor \frac{n-d+1}{2} \right\rfloor - k\right)(\overline{u}_1 + d) \\
&\quad + (d+1)\overline{u}_1 + \sum_{i=1}^{d-1} i \\
&= n\overline{u}_1 + d\left(\left\lfloor \frac{n-d+1}{2} \right\rfloor - k + \frac{d-1}{2}\right).
\end{aligned}
$$

Therefore, $\overline{u}_1 = -\frac{d}{n}\left(\lfloor \frac{n-d+1}{2} \rfloor - k + \frac{d-1}{2}\right)$ and $\overline{u}_i = \overline{u}_1 + i$ because each edge has length one. □

Lemma 12. *Let $u_1, \ldots, u_n$ be the coordinates of an embedding of $T(n,d,k)$ of the form (1). If $T(n,d,k)$ has an even number of leaves, then $\sum_{i=1}^{n} u_i^2 = \frac{1}{4}d^2 n - \frac{1}{6}d^3 + \frac{1}{6}d - \frac{k^2 d^2}{n}$ and $\sum_{i=1}^{n} u_i^2 = \frac{1}{4}d^2 n - \frac{1}{6}d^3 + \frac{1}{6}d - \frac{k^2 d^2}{n} - \frac{kd^2}{n} - \frac{1}{4}\frac{d^2}{n}$ otherwise.*

Proof. Let $\overline{u}_1, \ldots, \overline{u}_{\bar{n}}$ be the coordinates of the embedding of $T(n,d,k)$ as described above. First, we consider double brooms with an even number of leaves. By Proposition 11 we have $\overline{u}_1 = -\frac{d}{n}\left(\frac{n}{2} - k\right)$ and $\overline{u}_i = \overline{u}_1 + i$ for $i = 2, \ldots, d+1$ and therefore

$$
\begin{aligned}
\sum_{i=1}^{n} u_i^2 &= \sum_{i=1}^{\bar{n}} w_i \overline{u}_i^2 \\
&= \left(\frac{n-d+1}{2}+k\right)\left(\frac{d}{n}\left(\frac{n}{2}-k\right)\right)^2 \\
&\quad + \left(\frac{n-d+1}{2}-k\right)\left(-\frac{d}{n}\left(\frac{n}{2}-k\right)+d\right)^2 + \sum_{i=1}^{d-1}\left(-\frac{d}{n}\left(\frac{n}{2}-k\right)+i\right)^2 \\
&= \frac{1}{4}d^2 n - \frac{1}{6}d^3 + \frac{1}{6}d - \frac{k^2 d^2}{n}.
\end{aligned}
$$

Eventually, let $T(n,d,k)$ be a double broom with an odd number of leaves. Again by Proposition 11 we get $\overline{u}_1 = -\frac{d}{n}\left(\frac{n-1}{2}-k\right)$ and $\overline{u}_i = \overline{u}_1 + i$ for $i = 2,\ldots,d+1$, hence,

$$
\begin{aligned}
\sum_{i=1}^{n} u_i^2 &= \sum_{i=1}^{\bar{n}} w_i \overline{u}_i^2 \\
&= \left(\frac{n-d+2}{2}+k\right)\left(\frac{d}{n}\left(\frac{n-1}{2}-k\right)\right)^2 \\
&\quad + \left(\frac{n-d}{2}-k\right)\left(-\frac{d}{n}\left(\frac{n-1}{2}-k\right)+d\right)^2 \\
&\quad + \sum_{i=1}^{d-1}\left(-\frac{d}{n}\left(\frac{n-1}{2}-k\right)+i\right)^2 \\
&= \frac{1}{4}d^2 n - \frac{1}{6}d^3 + \frac{1}{6}d - \frac{k^2 d^2}{n} - \frac{kd^2}{n} - \frac{1}{4}\frac{d^2}{n}.
\end{aligned}
$$

□

Finally, we compare embeddings of the form (1) and (2) and describe which one has a larger value.

Proposition 13. *Let $T(n,d,k)$ be a double broom, $u_1,\ldots,u_n$ be the coordinates of its embedding of the form (1) and $v_1,\ldots,v_n$ be the coordinates of its embedding of the form (2). Define $p=-d$ whenever $T(n,d,k)$ has an odd number of leaves and $p=0$ otherwise. If $k \geq \frac{n(d-2)+p}{2d}$, then $\sum_{i=1}^{n} u_i^2 < \sum_{i=1}^{n} \|v_i\|^2$, otherwise $\sum_{i=1}^{n} \|v_i\|^2 < \sum_{i=1}^{n} u_i^2$.*

Proof. Suppose $T(n,d,k)$ has an odd number of leaves. First, we claim that an embedding of the form (2) can only occur if $k \geq \frac{n(d-2)-d}{2d}$. As we described in Proposition 11 the coordinates of the vertices of an one dimensional embedding are $\overline{u}_1 = -\frac{d}{n}(\lfloor\frac{n-d+1}{2}\rfloor - k + \frac{d-1}{2})$ and $\overline{u}_i = \overline{u}_1 + i$ for $i = 2,\ldots,d+1$. By Proposition 9 we conclude $\overline{u}_2 = 0$ as a necessary condition for an embedding of the form (2), hence, $-\frac{d}{n}(\frac{n-1}{2}-k)+1 = 0$ and therefore $k = \frac{n(d-2)-d}{2d}$. Now, it is easy to see that for an embedding of the form (2) we need $k \geq \frac{n(d-2)-d}{2d}$ in order to satisfy the equilibrium constraint.

Using Lemmas 10 and 12, we get

$$\begin{aligned}
\sum u_i^2 - \sum_{i=1}^{n} \|v_i\|^2 &= \frac{d^2}{4}(n-2d+2) + \frac{1}{6}d(d-1)(2d-1) \\
&\quad - d^2\frac{(2k+1)^2}{2n} + \frac{d^2}{4n} + \frac{d^2}{n}(k^2+k) \\
&\quad - \left(n + \frac{7}{6}d - \frac{1}{6}d^3 - \frac{1}{2}d^2 + \frac{1}{2}d^2n - dn - kd^2 + 2kd\right) \\
&= -\frac{d^2}{n}k^2 + \left(\frac{(4n-4)d^2}{4n} - 2d\right)k \\
&\quad - \frac{(n^2-2n+1)d - 4n^2 + 4n}{4n}d - n.
\end{aligned}$$

Now consider this difference as a quadratic function in k. Since the roots of this function are both $k_0 = \frac{dn-d-2n}{2d}$, the difference is negative for all $k \neq k_0$. Hence, the result follows for an odd number of leaves.

Suppose $T(n,d,k)$ has an even number of leaves. Using the same procedure as before, we can see that for an embedding of the form (2) we need $k \geq \frac{n(d-2)}{2d}$ in order to satisfy the equilibrium constraint.

Using Lemmas 10 and 12, we have

$$\begin{aligned}
\sum u_i^2 - \sum_{i=1}^{n} \|v_i\|^2 &= \frac{1}{4}d^2n - \frac{1}{6}d^3 + \frac{1}{6}d - \frac{k^2d^2}{n} \\
&\quad - \left(n + \frac{1}{6}d - \frac{1}{6}d^3 + \frac{1}{2}d^2n - dn - kd^2 + 2kd\right) \\
&= -\frac{k^2d^2}{n} + \left(d^2 - 2d\right)k - \frac{1}{12}\frac{d\left(2d^2n - 2n - 3dn^2\right)}{n} \\
&\quad - \frac{1}{2}d^2n - n - \frac{1}{6}d + \frac{1}{6}d^3 + dn.
\end{aligned}$$

We calculate the roots of this quadratic function again which are both $k_0 = \frac{dn-2n}{2d}$. Therefore, the difference is negative for all $k \neq k_0$. Hence, the result follows for the even case and the proof is complete. $\square$

Now we are ready to prove the main result of this section.

Theorem 14. *Let $T(n,d,k)$ be a double broom and define $p = -d$ whenever $T(n,d,k)$ has an odd number of leaves and $p = 0$ otherwise. If $k \geq \frac{n(d-2)+p}{2d}$, then the optimal solution of the problem (1) is given by an embedding of the form (2) and by an embedding of the form (1) otherwise.*

Proof. Since we can restrict our attention to embeddings of the form (1) and (2), the statement is just a consequence of Proposition 13. $\square$

Now in view of the problem (1), Theorem 14, Lemma 10 and Lemma 12 the proof of Theorem 1 is straightforward.

Eventually, we give a proof for Lemma 3.

Proof. (Theorem 3) To see that it holds, we apply Theorem 2 and Proposition 11 using the 1-dimensional coordinates as a projection for the eigenvector. Therefore, using Theorem 14 we are done if $k < \frac{n(d-2)+p}{2d}$.

Note, that for $k \geq \frac{n(d-2)+p}{2d}$ we have two eigenvectors that are projections of the following 2-dimensional embeddings onto the x and y axis

If q is even then

$$v_i = \begin{cases} (-\frac{m}{q}, \sqrt{1-(\frac{m}{q})^2}) & i = 1, \dots, \frac{q}{2} \\ (-\frac{m}{1}, -\sqrt{1-(\frac{m}{q})}) & i = \frac{q}{2}+1, \dots, q \\ (i, 0) & i = q+1, \dots, q+d \\ (q+d+1, 0) & \text{otherwise.} \end{cases}$$

If q is odd we have

$$v_i = \begin{cases} (-\frac{m+1}{q-1}, \sqrt{1-\left(\frac{m-1}{q-1}\right)^2}) & i = 1, \dots, \frac{q-1}{2} \\ (-\frac{m+1}{q-1}, -\sqrt{1-\left(\frac{m-1}{q-1}\right)^2}) & i = \frac{q+1}{2}, \dots, q-1 \\ (-1, 0) & i = q \\ (i, 0) & i = q+1, \dots, q+d \\ (q+d+1, 0) & \text{otherwise.} \end{cases}$$

$\square$

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