A new eigenvalue bound for independent sets

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Abstract. Let G be a simple, undirected, and connected graph on n vertices with eigenvalues $\lambda_1 \leq ... \leq \lambda_n$. Moreover, let m, δ, and α denote the size, the minimum degree, and the independence number of G, respectively. W.H. Haemers proved $\alpha \leq \frac{-\lambda_1\lambda_n}{\delta^2-\lambda_1\lambda_n}n$ and, if η is the largest Laplacian eigenvalue of G, then $\alpha \leq \frac{\eta-\delta}{\eta}n$ is shown by C.D. Godsil and M.W. Newman. We prove $\alpha \leq \frac{2\sigma-2}{\sigma\delta}m$ for the largest normalized eigenvalue σ of G, if $\delta \geq 1$. For $\varepsilon > 0$, an infinite family $\mathcal{F}_\varepsilon$ of graphs is constructed such that $\frac{2\sigma-2}{\sigma\delta}m = \alpha < (\frac{2}{3}+\varepsilon)\min\{\frac{-\lambda_1\lambda_n}{\delta^2-\lambda_1\lambda_n}n, \frac{\eta-\delta}{\eta}n\}$ for all $G \in \mathcal{F}_\varepsilon$. Moreover, a sequence of graphs is presented showing that the difference between $\frac{2\sigma-2}{\sigma\delta}m$ and D.M. Cvetković's upper bound $\min\{|\{i \in \{1,...,n\}|\lambda_i \leq 0\}|, |\{i \in \{1,...,n\}|\lambda_i \geq 0\}|\}$ on α can be arbitrarily small.

Keywords. independence number, eigenvalues

1 Introduction and Result

We use standard notation and terminology of graph theory and consider a finite, simple, and undirected graph G with vertex set $V = \{1, ..., n\}$ and edge set E, where $m = |E|$. Let d_i and δ denote the degree of $i \in V$ in G and the minimum degree of G, respectively. Furthermore, we assume that G has no isolated vertices, i.e. $\delta \geq 1$. A set of vertices $I \subseteq V$ in G is *independent*, if no two vertices in I are adjacent. The *independence number* α of G is the maximum cardinality of an independent set of G.

The independence number is one of the most fundamental and well-studied graph parameters [14]. In view of its computational hardness [11] various bounds on the independence number have been proposed, for a survey see [12].

In this paper, we are interested in upper bounds on α involving eigenvalues of matrices assigned to G (lower bounds on α in terms of eigenvalues can be found in [16]). Let $\lambda_1 \leq ... \leq \lambda_n$ denote the eigenvalues of the adjacency matrix A of G. Our starting point is the following Delsarte-Hoffman-bound [4, 8, 10, 13]. If G is an r-regular graph, then

$$\alpha \leq \frac{-\lambda_1}{r-\lambda_1}n.$$

Note that $\lambda_n = r$ if G is r-regular [4]. In [9, 10], W.H. Haemers proved the following extension of the Delsarte-Hoffman-bound for arbitrary graphs.

$$\alpha \leq \frac{-\lambda_1\lambda_n}{\delta^2 - \lambda_1\lambda_n} n.$$

If all eigenvalues of G are taken into consideration, then D.M. Cvetković [4, 6, 7] proved

$$\alpha \leq \min\{|\{i \in \{1, ..., n\} | \lambda_i \leq 0\}|, |\{i \in \{1, ..., n\} | \lambda_i \geq 0\}|\}.$$

Let D be the *degree matrix* of G, that is an $(n \times n)$ diagonal matrix, where d_i is the i-th element of the main diagonal. Moreover, let $0 = \eta_1 \leq ... \leq \eta_n$ be the eigenvalues of the *Laplacian matrix* $L = D - A$ of G [1].

In [8], C.D. Godsil and M.W. Newman established the following inequality, which is also a consequence of a result in [2] concerning the size of a cut in a graph.

$$\alpha \leq \frac{\eta_n - \delta}{\eta_n} n.$$

For G without isolated vertices, the *normalized Laplacian* is the $(n \times n)$ matrix $\mathcal{L} = (l_{ij})$ with $l_{ij} = 1$ if $i = j$, $l_{ij} = -\frac{1}{\sqrt{d_i d_j}}$ if $ij \in E$, and $l_{ij} = 0$ otherwise. The eigenvalues $\sigma_1 \leq ... \leq \sigma_n$ of $\mathcal{L}$ are the *normalized eigenvalues* of G [3, 5]. It is known that $\sigma_1 = 0$ and $1 < \sigma_n \leq 2$ [3, 5].

Our result is the following inequality.

$$\alpha \leq \frac{2\sigma_n - 2}{\sigma_n \delta} m.$$

For its proof, let $\{\underline{u}_1, ..., \underline{u}_n\}$ be an orthonormal basis of R^n consisting of eigenvectors of the symmetric matrix $\mathcal{L}$ such that $\underline{u}_i$ is an eigenvector of σ_i for $i = 1, ..., n$.
Moreover, let $\underline{y} = (y_1, ..., y_n) \in R^n$ and $\underline{y} = \mu_1\underline{u}_1 + ... + \mu_n\underline{u}_n$ for suitable $\mu_1, ..., \mu_n \in R$.
It follows
$\underline{y}^T\mathcal{L}\underline{y} = \sigma_2\mu_2^2 + ... + \sigma_n\mu_n^2 = -\sigma_n\mu_1^2 + (\sigma_2 - \sigma_n)\mu_2^2 + ... + (\sigma_{n-1} - \sigma_n)\mu_{n-1}^2 + \sigma_n(\mu_1^2 + ... + \mu_n^2)$ $\leq -\sigma_n\mu_1^2 + \sigma_n(\mu_1^2 + ... + \mu_n^2) = -\sigma_n(\underline{y}^T\underline{u}_1)^2 + \sigma_n(\underline{y}^T\underline{y})$. Let M be an $(n \times n)$ diagonal matrix, where $\frac{1}{\sqrt{d_i}}$ is the i-th element of the main diagonal, and I be the $(n \times n)$ identity matrix. With $M^T = M$ and $\mathcal{L} = I - MAM$, we obtain $\underline{y}^T MAM\underline{y} \geq \sigma_n(\underline{y}^T\underline{u}_1)^2 + (1 - \sigma_n)\underline{y}^T\underline{y}$. We may choose $\underline{u}_1^T = \frac{1}{\sqrt{2m}}(\sqrt{d_1}, ..., \sqrt{d_n})$ and, substituting $y_i = x_i\sqrt{d_i}$ for $i = 1, ..., n$, it follows that

$$\sigma_n(\sum_{i=1}^{n} d_i x_i)^2 + 2m(1 - \sigma_n)\sum_{i=1}^{n} d_i x_i^2 \leq 4m \sum_{ij \in E} x_i x_j \quad (1)$$

for arbitrary real numbers $x_1, ..., x_n$. Let I be a maximum independent set of G and $\underline{x} = (x_1, ..., x_n)$ with $x_i = 1$ if $i \in I$ and $x_i = 0$, otherwise. By inequality (1),
$\sigma_n(\sum_{i \in I} d_i)^2 + 2m(1 - \sigma_n)\sum_{i \in I} d_i = \sigma_n(\sum_{i=1}^{n} d_i x_i)^2 + 2m(1 - \sigma_n)\sum_{i=1}^{n} d_i x_i^2 \leq 4m \sum_{ij \in E} x_i x_j$,
hence, with $\sum_{ij \in E} x_i x_j = 0$ and $\sum_{i \in I} d_i \geq \delta\alpha$, it follows $\alpha \leq \frac{2\sigma_n - 2}{\sigma_n\delta} m$.

Let us remark that $\sum\limits_{i=1}^{n} d_i x_i = \sum\limits_{ij\in E}(x_i + x_j)$ and $\sum\limits_{i=1}^{n} d_i x_i^2 = \sum\limits_{ij\in E}(x_i^2 + x_j^2)$. Hence, if G is bipartite, then $\sigma_n = 2$ [5, 15] and (1) is equivalent to

$$(\sum_{ij\in E}(x_i + x_j))^2 \leq m \sum_{ij\in E}(x_i + x_j)^2. \tag{2}$$

Note that inequality (2) is a consequence of the Cauchy-Schwarz inequality and, therefore, (2) is valid also for an arbitrary (not necessarily bipartite) graph G.
Using (2), $2m\sum\limits_{i=1}^{n} d_i x_i^2 - (\sum\limits_{i=1}^{n} d_i x_i)^2 = 2m\sum\limits_{ij\in E}(x_i^2 + x_j^2) - (\sum\limits_{ij\in E}(x_i + x_j))^2$
$\geq m(2\sum\limits_{ij\in E}(x_i^2+x_j^2) - \sum\limits_{ij\in E}(x_i+x_j)^2) = m\sum\limits_{ij\in E}(x_i - x_j)^2 \geq 0$ and it follows that the coefficient $c(\sigma_n)$ of σ_n in inequality (1) is not positive. Hence, the left side of (1) is a non-increasing function in σ_n and, if $\sigma_n < 2$ and $c(\sigma_n) < 0$, then (1) is stronger than (2).

Next, for given $\varepsilon > 0$, we present the infinite family $\mathcal{F}_\varepsilon$ mentioned in the abstract.
For a positive integer k, consider the graph G_k on $n = 3k + 1$ vertices obtained from k pairwise disjoint paths each on 3 vertices, where the vertices of these paths are numbered arbitrarily from 1 up to $3k$, an additional vertex $n = 3k + 1$, and additional $2k$ edges connecting n with the $2k$ endvertices of these paths. Obviously, G_k is bipartite, n has degree $2k$, each other vertex has degree 2, $m = 4k$, $\delta = 2$, and, since G_k is bipartite, $\sigma_n = 2$ and $\lambda_1 = -\lambda_n$ [5, 15]. Moreover, $\frac{2\sigma_n - 2}{\sigma_n\delta}m = \alpha = 2k = \frac{2}{3}(n-1)$. Let $\underline{x} \in R^n$ be defined by $x_i = 1$ if i is a neighbour of n, $x_n = \sqrt{2k}$, and $x_i = 0$ otherwise. It follows $\lambda_n \geq \frac{\underline{x}^T A \underline{x}}{\underline{x}^T \underline{x}} = \sqrt{2k}$ by Rayleigh's principle [4], hence, $\frac{-\lambda_1\lambda_n}{\delta^2 - \lambda_1\lambda_n}n = \frac{\lambda_n^2}{4+\lambda_n^2}n > \frac{k}{k+2}3k$. If $\underline{x}$ is defined by $x_i = 1$ if $i = n$ and $x_i = 0$ otherwise, then $\eta_n \geq \frac{\underline{x}^T L \underline{x}}{\underline{x}^T \underline{x}} = \frac{(\underline{x}^T D \underline{x} - \underline{x}^T A \underline{x})}{\underline{x}^T \underline{x}} = 2k$, hence, $\frac{\eta_n - \delta}{\eta_n}n > \frac{k-1}{k}3k$. Let the integer $l = l_\varepsilon \geq 2$ be chosen large enough such that $\frac{l}{l+2} > 1 - \frac{3\varepsilon}{2+3\varepsilon}$. It follows $\frac{k-1}{k} \geq \frac{k}{k+2}$ and $(\frac{2}{3} + \varepsilon)\min\{\frac{-\lambda_1\lambda_n}{\delta^2 - \lambda_1\lambda_n}n, \frac{\eta_n - \delta}{\eta_n}n\} > (\frac{2}{3} + \varepsilon)(1 - \frac{3\varepsilon}{2+3\varepsilon})3k = 2k$ for all $k \geq l$, hence, we are done with $\mathcal{F}_\varepsilon = \{G_k \mid k \geq l_\varepsilon\}$.

Eventually, we show that the difference between $\frac{2\sigma_n - 2}{\sigma_n\delta}m$ and D.M. Cvetković's bound $\min\{|\{i \in \{1,...,n\}|\lambda_i \leq 0\}|, |\{i \in \{1,...,n\}|\lambda_i \geq 0\}|\}$ can be arbitrarily small. For two graphs G on $V(G)$ and G' on $V(G')$, the *cartesian product* $G \times G'$ is the graph on $V(G) \times V(G')$ and two vertices (v, v') and (w, w') of $G \times G'$ are adjacent in $G \times G'$ if and only if $v = w$ and $v'w' \in E(G')$ or $v' = w'$ and $vw \in E(G)$. For a positive integer k, consider the bipartite and 4-regular graph $H_k = C_{4k} \times C_{4k}$ on $n = 16k^2$ vertices, where C_{4k} is the cycle on $4k$ vertices, and it follows $\frac{2\sigma_n - 2}{\sigma_n\delta}m = \alpha = \frac{n}{2}$ for H_k.
If λ and λ' are eigenvalues of G and G', respectively, then $\lambda + \lambda'$ is an eigenvalue of $G \times G'$ [4]. If the graph G is bipartite, then the set of its eigenvalues is symmetric w.r.t. 0 [4].
Since C_{4k} has the eigenvalue 0 with multiplicity 2 [4], H_k has the eigenvalue 0 with multiplicity $4k + 2$, and, consequently,
$\min\{|\{i \in \{1,...,n\}|\lambda_i \leq 0\}|, |\{i \in \{1,...,n\}|\lambda_i \geq 0\}|\} = \frac{n}{2} + 2k + 1$ for G_k.

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