CONDITIONAL WEGNER ESTIMATE FOR THE STANDARD RANDOM BREATHER POTENTIAL

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Abstract. We prove a conditional Wegner estimate for Schrödinger operators with random potentials of breather type. More precisely, we reduce the proof of the Wegner estimate to a scale free unique continuation principle. The relevance of such unique continuation principles has been emphasized in previous papers, in particular in recent years.

We consider the *standard* breather model, meaning that the single site potential is the characteristic function of a ball or a cube. While our methods work for a substantially larger class of random breather potentials, we discuss in this particular paper only the standard model in order to make the arguments and ideas easily accessible.

1. Introduction

A Wegner estimate is an upper bound on the expected number of eigenvalues in a prescribed energy interval, of a finite box Hamiltonian. The expectation here refers to the potential which is random. This research direction is pursued by many authors. In this technical report, we cannot discuss the history of the subject appropriately, but refer e.g. to [6] and the references therein.

Wegner estimates have been derived for Hamiltonians living on $\mathbb{Z}^d$ or $\mathbb{R}^d$, more precisely on bounded subsets of rectangular shape of these spaces. Here we will be only interested in models on continuum space $\mathbb{R}^d$. The most studied example in this situation is the so called alloy-type potential, sometimes also called continuum Anderson model. A particular feature of this model is that randomness enters the model via a countable number of random variables, and these r.v. influence the potential in a linear way. In the model we study here this dependence is no longer linear, but becomes non-linear. What remains, is the monotone dependence of the potential on the r.v. The topic of the present note is to explain, how to effectively use this monotonicity. This only works if it is possible to cast the monotonicity in a quantitative form.

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With this respect we consider the study of the random breather model as paradigmatic for a better understanding of random Schrödinger operators with non-linear randomness.

To our best knowledge random breather potentials have been first considered in the mathematical physics literature in the work [2].

1.1. Wegner estimate for the random breather model.

We prove a Wegner estimate, Theorem 1.2, for the Random Breather model.
In the following, $\Lambda_s(x) := x + [-s/2, s/2]^d$ is a d-dimensional cube of side length $s > 0$, centered at the point $x \in \mathbb{R}^d$. $B_r(x)$ denotes the ball of radius $r \geq 0$ around $x \in \mathbb{R}^d$. If $x = 0$, we omit the x and write Λ_s or B_r. If the side length s is fixed, we simply write Λ.

Let $0 \leq \omega_- < \omega_+ < 1/2$ and let μ be a probability measure on $\mathbb{R}$ with bounded density ν_μ and support in $[\omega_-, \omega_+]$. We define the probability space

$$(\Omega, \mathcal{A}, \mathbb{P}) = (\times_{i \in \mathbb{Z}^d} \mathbb{R}, \otimes_{i \in \mathbb{Z}^d} \mathcal{B}(\mathbb{R}), \otimes_{i \in \mathbb{Z}^d} \mu) .$$

Here, $\mathcal{B}$ is the Borel σ-algebra. For $\omega \in \Omega$ and $j \in \mathbb{Z}^d$ we denote the projection onto the j-th coordinate of Ω by ω_j. The $\{\omega_j\}_{j \in \mathbb{Z}^d}$ form a process of $[\omega_-, \omega_+]$-valued independent and identically distributed random variables on $\mathbb{Z}^d$. For $i, j \in \mathbb{Z}^d$ and $\delta \in \mathbb{R}$ we define $\omega + \delta$ and $\omega + \delta e_i \in \Omega$ by

$$(\omega + \delta)_j := \omega_j + \delta \text{ for all } j \in \mathbb{Z}^d$$

$$(\omega + \delta e_i)_j := \begin{cases} \omega_j + \delta & \text{if } j = i \\ \omega_j & \text{if } j \neq i. \end{cases}$$

Definition 1.1. We define families of random potentials $\{V_\omega\}_{\omega \in \Omega}$

$$V_\omega(x) := \sum_{j \in \mathbb{Z}^d} \chi_{B_{\omega_j}}(x - j) \quad \text{or} \tag{1}$$

$$V_\omega(x) := \sum_{j \in \mathbb{Z}^d} \chi_{\Lambda_{2\omega_j}}(x - j). \tag{2}$$

The (standard) **Random Breather model** for **balls** in case (1) or **cubes** in case (2), respectively is the family of operators $\{H_\omega\}_{\omega \in \Omega}$ where

$$H_\omega := -\Delta + V_\omega.$$

For each ω the operator H_ω is nonnegative and self-adjoint on a dense subspace of $L^2(\mathbb{R}^d)$. We also define for a cube $\Lambda \subset \mathbb{R}^d$ the restriction $H_{\omega,\Lambda}$ of H_ω to Λ with Dirichlet boundary conditions and the restriction of the potential $V_{\omega,\Lambda} : \Lambda \to \mathbb{R}$.

We formulate a scale free quantitative unique continuation property, which the random operator $H_{\omega,\Lambda}$ may or may not have. Denote by $\chi_I(H_{\omega,\Lambda})$ the spectral projector of $H_{\omega,\Lambda}$ onto an interval I.

Hypothesis (SFUCP)**.** Let $\{z_k\}_{k \in \mathbb{Z}^d}$ be a sequence of points in $\mathbb{R}^d$. Denote by W_δ the characteristic function of the set $\bigcup_{k \in \mathbb{Z}^d} B_\delta(z_k)$. Given $b \in \mathbb{R}$ there is $M \geq 1$ such that for all $0 \leq \delta \leq 1/2$

$$\chi_{(-\infty,b]}(H_{\omega,\Lambda})\, W_\delta|_\Lambda\, \chi_{(-\infty,b]}(H_{\omega,\Lambda}) \geq \delta^M \chi_{(-\infty,b]}(H_{\omega,\Lambda}) \tag{3}$$

if the balls satisfy $B_\delta(z_k) \subset \Lambda_1(k)$ for all $k \in \mathbb{Z}^d$.

Inequality (3) is understood in the sense of quadratic forms.

Theorem 1.2 (Wegner estimate for the Random Breather model)**.** *Let $\{H_\omega\}$ be the (standard) Random Breather model as defined in* (1) *or* (2) *and $\Lambda = \Lambda_L$ a sufficiently large cube of side length $L \in 2\mathbb{N}+1$, centered at the origin. Assume that the hypothesis SFUCP holds. Let $b \in \mathbb{R}$. Then there are constants*

$$0 < C < \infty, \text{ depending only on } d, b,$$
$$0 < \varepsilon_{\max} < \infty, \text{ depending only on } M, \omega_+,$$

such that for all $0 < \varepsilon \leq \varepsilon_{\max}$, $E \in \mathbb{R}$ with $[E-\varepsilon, E+\varepsilon] \subseteq (-\infty, b-1]$ we have

$$\mathbb{E}\left[\operatorname{Tr}\left[\chi_{[E-\varepsilon,E+\varepsilon]}(H_{\omega,\Lambda})\right]\right] \leq C \sqrt[M]{4}\|\nu_\mu\|_\infty \varepsilon^{1/M} \left|\ln \varepsilon\right|^d L^d. \tag{4}$$

$\varepsilon_{\max}$ can be chosen as

$$\varepsilon_{\max} = \frac{1}{4}\left(\frac{1/2-\omega_+}{2}\right)^M.$$

Remark 1.3. (a) We can give an explicit bound on the constant C, namely

$$C \leq 2 \cdot 32^d (e^b \cdot 2(\sqrt[4]{d}+1)! + 2^d).$$

(b) The $|\ln \varepsilon|^d$ term can be hidden by choosing a slightly smaller $\tilde{M} < M$ and a different constant $\tilde{C} > 0$. Hiding also the factor $\sqrt[M]{4}\|\nu_\mu\|_\infty$ in this constant, we obtain a bound of the form

$$\mathbb{E}\left[\operatorname{Tr}\left[\chi_{[E-\varepsilon,E+\varepsilon]}(H_{\omega,\Lambda})\right]\right] \leq \tilde{C} \cdot \varepsilon^{1/\tilde{M}} L^d.$$

(c) The factor $1/2-\omega_+$ in $\varepsilon_{\max}$ is due to technical reasons. It can be replaced by other positive values with the tradeoff that the dependency of M becomes more involved.

Remark 1.4. Since this note is of a technical nature, we will not discuss here the following issues, but will do this in a subsequent paper.

- It is possible to use Neumann or periodic boundary conditions instead of Dirichlet ones as well.
- The Wegner estimate holds for a much larger class of single site potentials.
- Previous results on Wegnes estimates for such models will be also discussed there. This includes a discussion of the classes of single site potentials covered (and not covered) in pervious papers. In any case, so far no Wegner estimates for the standard random breather potential have been proven.
- A discussion of the scale free unique continuation principle (SFUCP) which is the basis of the main Theorem.

2. Estimates on the spectral shift function

We reproduce here the estimates on the singular values of seminggroup differences and the spectral shift function of two Schrödinger operators differing by a compactly supported potential obtained in [4]. We reduce the estimates to the particular, simple situation we are dealing with. The constants are calculated explicitly and more accurately than in [4].

We will be dealing here with a pair of operators: $H_0 = -\Delta + V_0$ with V_0 non-negative and bounded, and $H_1 := H_0 + V$, with V non-negative, bounded, and supported in a ball of radius $1/2$. Let $\Lambda = \Lambda_L$ be a cube of sidelength $L \in 2\mathbb{N}+1$ and call H_0^Λ and H_1^Λ the restrictions of H_0 and H_1 onto Λ with Dirichlet boundary conditions. Let

$$V_{\text{eff}}^\Lambda := e^{-H_1^\Lambda} - e^{-H_0^\Lambda}.$$

This is a compact operator and we will enumerate its singular values decreasingly by $\mu_1 \geq \mu_2 \geq \dots$. Then we have the following

Theorem 2.1 (Theorem 1 from [4])**.** *For $n > N_0 := 4^d$, the singular values of V_{eff}^Λ obey*

$$\mu_n(V_{\text{eff}}^\Lambda) \leq (\sqrt[4]{d}+1)\exp\left(-\frac{n^{1/d}}{16}\right)$$

We start the proof with a lemma

Lemma 2.2. *Let $H = H_0$ or H_1 be as above and let $H^{\mathcal{U}}$ be the Dirichlet restriction of H to an open set $\mathcal{U}$ with finite volume $|\mathcal{U}|$. Then the n^{th} eigenvalue E_n of $H^{\mathcal{U}}$ satisfies*

$$E_n \geq \frac{2\pi d}{e}\left(\frac{n}{|\mathcal{U}|}\right)^{2/d}. \tag{5}$$

Proof. We have $H^U \geq -\Delta^{\mathcal{U}}$ where $-\Delta^{\mathcal{U}}$ is the Dirichlet restriction of $-\Delta$ to $\mathcal{U}$. Hence

$$\mathrm{Tr}\left(e^{-2tH^{\mathcal{U}}}\right) \leq \mathrm{Tr}\left(e^{-2t\Delta^{\mathcal{U}}}\right) = \left\|e^{-t\Delta^{\mathcal{U}}}\right\|_{\mathrm{HS}}^2 = \int\int_{\mathcal{U}\times\mathcal{U}} |e^{-t\Delta^{\mathcal{U}}}(x,y)|^2 dx\, dy$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm. Using that the kernel of the Dirichlet semigroup is bounded by the free kernel, that is

$$\int_{\mathcal{U}} |e^{-t\Delta^{\mathcal{U}}}(x,y)|^2 dy \leq \int_{\mathcal{U}} |e^{-t\Delta}(x,y)|^2 dy = \int_{\mathcal{U}} (4\pi t)^{-d} \exp\left(-\frac{|x-y|^2}{2t}\right) dy$$
$$\leq (8\pi t)^{-d/2} \int_{\mathbb{R}} (2\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{2t}\right) dy = (8\pi t)^{-d/2},$$

see [1] for the boundedness by the free kernel and [3] Theorem 2.3.1. for the free kernel, we estimate

$$\|e^{-t\Delta^U}\|_{\mathrm{HS}}^2 \leq |\mathcal{U}|(8\pi t)^{-d/2}.$$

Thus

$$\mathrm{Tr}(e^{-2tH^{\mathcal{U}}}) \leq |\mathcal{U}|(8\pi t)^{-d/2}.$$

Denote by $\mathcal{N}^{\mathcal{U}}(E)$ the number of eigenvalues of $H^{\mathcal{U}}$ smaller or equal to E. By Čebyšev's inequality and the above bound

$$\mathcal{N}^{\mathcal{U}}(E) \le e^{2tE}\int_{-\infty}^{E} e^{-2ts}d\mathcal{N}^{\mathcal{U}}(s) \le e^{2tE}\mathrm{Tr}(e^{-2tH^{\mathcal{U}}})$$
$$\le |\mathcal{U}|\cdot(8\pi t)^{-d/2}e^{2tE} = |\mathcal{U}|\left(\frac{eE}{2\pi d}\right)^{d/2},$$

where in the last inequality $t := \frac{d}{4E}$ has been chosen. Since $n \le \mathcal{N}^{\mathcal{U}}(E_n)$, this implies (5). □

Proof of Theorem 2.1. The n^{th} singular value will be estimated by Dirichlet decoupling at a scale R_n which monotonously depends on n. Let without loss of generality $\mathrm{supp}(V) \subseteq B_{1/2}(0)$. Choose a large $R = R_n > 2$ to be specified later. Call H_j^R ($j = 0$ or 1) the Dirichlet restriction of H_j onto $\Lambda_{2R} = [-R;R]^{2d}$ and let

$$A_R := e^{-H_1^R} - e^{-H_0^R} \quad \text{and} \quad D_R := V_{\text{eff}}^{\Lambda} - A_R.$$

We apply Lemma 2 and find

$$\mu_n(e^{-H_j^R}) \le \exp(-\frac{\pi d}{2e}n^{2/d}R^{-2}) \le \exp\left(-\frac{1}{16}n^{2/d}R^{-2}\right)$$

for $j = 1, 2$. Since A_R is the difference of two nonnegative operators, its singular values obey the same bound:

$$\mu_n(A_R) \le \mu_n(e^{-H_2^R}) \le \exp\left(-\frac{1}{16}n^{2/d}R^{-2}\right).$$

If the operator D_R is bounded, then $\mu_n(V_{\text{eff}}^{\Lambda}) \le \mu_n(A_R) + \|D_R\|$. We will now estimate the norm of D_R by using the Feynman-Kac formula for Schrödinger semigroups with Dirichlet boundary conditions, see [1]. Let $\mathbb{E}_x$ and $\mathbb{P}_x$ denote the expectation and probability for a Brownian motion b_t, starting at x. Let $\tau_\Lambda := \inf\{t > 0 | b_t \notin \Lambda\}$ be the exit time from Λ and $\tau_R := \inf\{t > 0 | b_t \notin \Lambda_{2R}\}$ be the exit time from Λ_{2R}. Then

$$(D_R f)(x) = \mathbb{E}_x\left[\left(e^{-\int_0^1 (V_0+V)(b_s)ds} - e^{-\int_0^1 V_0(b_s)ds}\right)\chi_{[\tau_\Lambda > 1]}(b)\chi_{[\tau_R \le 1]}(b)f(b_1)\right].$$

Using that $\chi_{[\tau_\Lambda>1]}(b) \le 1$, we find

$$|D_R f|(x) \le \mathbb{E}_x\left[e^{-\int_0^1 V_0(b_s)ds}\cdot|e^{-\int_0^1 V(b_s)ds} - 1|\cdot\chi_{[\tau_R\le 1]}(b)\cdot|f(b_1)|\right].$$

Only Brownian paths which both visit $\mathrm{supp}(V)$ and leave B_R within one unit of time contribute to the expectation. Thus, if τ_V is the hitting time for $\mathrm{supp}(V)$ and $\mathfrak{B} = \{\tau_R \le 1, \tau_V \le 1\}$, then

$$|D_R f|(x) \le \mathbb{E}_x\left[e^{-\int_0^1 V_0(b_s)ds}\cdot|e^{-\int_0^1 V(b_s)ds} - 1|\cdot\chi_{\mathfrak{B}}(b)\cdot|f(b_1)|\right].$$

Applying Hölder's inequality

$$|D_n f|(x) \le \left(\mathbb{E}_x\left[e^{-8\int_0^1 V_0(b_s)ds}\right]\right)^{1/8}\left(\mathbb{E}_x\left[|e^{-\int_0^1 V(b_s)ds} - 1|^8\right]\right)^{1/8}$$
$$\times\left(\mathbb{E}_x\left[\chi_{\mathfrak{B}}(b)\right]\right)^{1/4}\left(\mathbb{E}_x\left[|f(b_1)|^2\right]\right)^{1/2}.$$

The first two terms are bounded by 1, uniformly in x.

Letting $b_t = (b_t^{(1)}, ..., b_t^{(n)}) \in \mathbb{R}^d$ and calling $\tau_R^{(j)} := \inf\{t > 0 | b_t^{(j)} \notin [-R, R]\}$ the exit time of the j^{th} coordinate from the interval $[-R, R]$, we have

$$\mathbb{P}_0[\tau_R \leq 1] = \mathbb{P}_0\left[\bigcup_{j=1}^{d}\left\{\tau_R^{(j)} \leq 1\right\}\right] \leq \sum_{j=1}^{d}\mathbb{P}_0\left[\tau_R^{(j)} \leq 1\right] = d \cdot \mathbb{P}_0[\tau_R^{(1)} \leq 1].$$

The projection onto the first coordinate of (b_t) is a one-dimensional Brownian motion and by the reflection principle

$$\mathbb{P}_0[\tau_R^{(1)} \leq 1] = 2\mathbb{P}_0[|b_1^{(1)} \geq R] = \frac{4}{\sqrt{2\pi}}\int_R^\infty e^{-x^2/2}dx \leq \frac{4}{\sqrt{2\pi}}R^{-1}e^{-R^2/2}. \tag{6}$$

Recalling that $R > 2$, and hence $\frac{4}{\sqrt{2\pi}}R^{-1} \leq 1$ we find

$$\mathbb{P}_0[\tau^R \leq 1] \leq d \cdot e^{-R^2/2}.$$

Since every path in $\mathfrak{B}$ must cover the distance $r \geq R - 1/2$ between $\operatorname{supp}(V)$ and the complement of the ball B_R, we find $\mathbb{P}_x[\mathfrak{B}] \leq d \cdot e^{-r^2/2}$. We assumed that $R > 2$, hence $r \geq R - 1/2 \geq R/\sqrt{2}$. Then $\mathbb{P}_x[\mathfrak{B}] \leq de^{-R^2/4}$ and

$$|D_R f|(x) \leq \sqrt[4]{d} \cdot e^{-R^2/16}\left(\mathbb{E}_x|f(b_1)|^2\right)^{1/2} = \sqrt[4]{d} \cdot e^{-R^2/16}\left((e^{\Delta}|f|^2)(x)\right)^{1/2}.$$

Using the fact, that e^{Δ} is an L^1 contraction

$$\|D_n f\|_2 \leq \sqrt[4]{d} \cdot e^{-R^2/16}\|(e^{\Delta}|f|^2)\|_1^{1/2} \leq \sqrt[4]{d} \cdot e^{-R^2/16}\|f^2\|_1^{1/2} = \sqrt[4]{d} \cdot e^{-R^2/16}\|f\|_2.$$

To balance between the two bounds obtained for $\mu_n(A_R)$ and $\|D_R\|$, we choose $R := n^{1/2d}$ and find

$$\mu_n(A_R) \leq \exp(-1/16 \cdot n^{1/d}),$$
$$\|D_n\| \leq \sqrt[4]{d} \cdot \exp(-1/16 \cdot n^{1/d}),$$

that is

$$\mu_n(V_{\text{eff}}^{\Lambda}) \leq (\sqrt[4]{d} + 1) \cdot \exp\left(-1/16 n^{1/d}\right).$$

Since we assumed $R = n^{1/2d} > 2$, this only works for

$$n > N_0 = 4^d$$

□

Let now $g \in C^\infty(\mathbb{R})$ with compactly supported derivative. If $g(H_1) - g(H_0)$ is trace class then there is a unique function $\xi(\lambda, H_1, H_0)$ called the Lifshitz-Krein spectral shift function, such that

$$\operatorname{Tr}\left[g(H_1) - g(H_0)\right] = \int \xi(\lambda, H_1, H_0)\mathrm{d}g(\lambda). \tag{7}$$

This is referred to as Krein's trace identity. $\xi(\cdot, H_1, H_0)$ is independent of the choice of g. In fact, g can be chosen from a substantially larger class of functions:

Proposition 2.3 (Chapter 8.9, Theorem 1 in [7]). *Let H_0, H_1 be positive definite and $H_1^{-\tau} - H_0^{-\tau}$ trace class for some $\tau > 0$. Furthermore let g have two locally bounded derivatives and satisfy*

$$|(\lambda^{\tau+1} g'(\lambda))'| \leq C\lambda^{-1-\varepsilon}, \text{ as } \lambda \to \infty$$

for some $C, \varepsilon > 0$. Then (7) holds for g.

For such admissible functions, it is possible to make a change of variables

$$\xi(\lambda, H_1, H_0) = \operatorname{sgn}(g')\xi(g(\lambda, g(H_1), g(H_0)),$$

see [7], Chapter 8.11. This is referred to as the invariance principle for the spectral shift function. With our choice of H_0 and H_1, $g(\lambda) := \exp(-\lambda)$ is an admissible function, as can be seen via Lemma .

We also define functions $F_t : [0, \infty) \to [0, \infty)$ for $t > 0$ by

$$F_t(x) := \int_0^x (\exp(ty^{1/d}) - 1)dy.$$

Theorem 2.4 (Theorem 2 from [4]). *Let ξ be the spectral shift function for the pair of operators H_0^Λ and H_1^Λ.*

(i) There is a constant K_1, depending on t such that for small enough $t > 0$

$$\int_{-\infty}^T F_t(|\xi(\lambda|)d\lambda \leq K_1 e^T < \infty.$$

t can be chosen to be $t := 1/32$ in which case $K_1 \leq 4 \cdot 32^d \cdot (\sqrt[4]{d} + 1)!$

(ii) There are constants K_1, K_2, only depending on d such that for any bounded function f with compact support within $(-\infty, b]$ we have

$$\int f(\lambda)\xi(\lambda)d\lambda \leq K_1 e^b + K_2 \left(\log(1 + \|f\|_\infty)\right)^d \|f\|_1.$$

We may choose $K_1 := 2 \cdot 32^d \cdot (\sqrt[4]{d} + 1)!$ and $K_2 := 32^d$.

Corollary 2.5. *Let $g \in C^\infty(\mathbb{R})$ with g' of compact support within $(-\infty, b]$ such that $g(H_1^\Lambda) - g(H_0^\Lambda)$ is trace class. Then*

$$\operatorname{Tr}[g(H_1^\Lambda) - g(H_0^\Lambda)] \leq 32^d \left[2(\sqrt[4]{d} + 1)! e^b + \left(\log(1 + \|g'\|_\infty)\right)^d \|g'\|_1\right] \tag{8}$$

Proof of Theorem 2.4. (i) Using the invariance principle for the spectral shift function we have

$$\begin{aligned}\int_{-\infty}^T F_t(|\xi(\lambda, H_1^\Lambda, H_0^\Lambda)|)d\lambda &= \int_{-\infty}^T F_t(|\xi(e^{-\lambda}, e^{-H_1^\Lambda}, e^{-H_0^\Lambda})|)d\lambda \\ &\leq e^T \int_{e^{-T}}^\infty F_t(|\xi(s, e^{-H_1^\Lambda}, e^{-H_0^\Lambda})|)ds.\end{aligned}$$

Since the difference $V_{\mathrm{eff}}^{\Lambda} = e^{-H_1^{\Lambda}} - e^{-H_0^{\Lambda}}$ is trace class, we can apply an estimate of [5] and find

$$\int_{-\infty}^{\infty} F_t(|\xi(s, e^{-H_1}, e^{-H_0})|)ds \leq \sum_{n=1}^{\infty} \mu_n(V_{\mathrm{eff}})(F_t(n) - F_t(x-1))$$
$$= \sum_{n=1}^{N_0} \mu_n(V_{\mathrm{eff}}) \int_{n-1}^{n} (e^{ts^{1/d}} - 1)ds + \sum_{n=N_0+1}^{\infty} \mu_n(V_{\mathrm{eff}}) \int_{n-1}^{n} (e^{ts^{1/d}} - 1)ds$$
$$\leq N_0(e^{tN_0^{1/d}} - 1) + (\sqrt[4]{d} + 1) \sum_{n=1}^{\infty} e^{(t-1/16)n^{1/d}}.$$

If we choose t smaller than $1/16$ this will be finite. We choose $t := 1/32$ and recall that $N_0 = 4^d$ so that we obtain

$$\int_{-\infty}^{\infty} F_t(|\xi(s, e^{-H_1}, e^{-H_0})|)ds \leq 4^d(e^{1/16} - 1) + (\sqrt[4]{d} + 1) \sum_{n=1}^{\infty} e^{-1/32 n^{1/d}}.$$

To estimate the second summand we use

$$\sum_{n=1}^{\infty} e^{-1/32n^{1/d}} \leq \int_0^{\infty} e^{-1/32x^{1/d}} dx = d{\cdot}32^d \int_0^{\infty} e^{-y} y^{d-1} dy = d{\cdot}32^d \Gamma(d) = d!{\cdot}32^d$$

and obtain

$$\int_{-\infty}^{\infty} F_t(|\xi(s, e^{-H_1}, e^{-H_0})|)ds \leq 4^d + (\sqrt[4]{d} + 1) \cdot 32^d d! \leq 2 \cdot 32^d \cdot (\sqrt[4]{d} + 1)!.$$

(ii) We use Young's inequality and dualize the bound from part (i). F_t is non-negative and convex with $F_t'(0) = 0$, hence its Legendre transform G_t is well-defined and satisfies

$$G_t(y) := \sup_{x \geq 0} \{xy - F_t(x)\} \leq y \left(\frac{\log(1+y)}{t} \right)^d \quad \text{for all } y \geq 0.$$

Young's equality yields $yx \leq F_t(x) + G_t(y)$ and with $b := \sup \operatorname{supp}(f)$ we find

$$\int f(\lambda)\xi(\lambda)d\lambda \leq \int_{-\infty}^{b} F_t(|\xi(\lambda|)d\lambda + \int G_t(|f(\lambda)|)d\lambda.$$

Using part (i), we find that the first integral is bounded by $K_1 e^b$. For the second summand we estimate

$$\int G(|f(\lambda)|)d\lambda \leq \int |f(\lambda)| \left(\frac{\log(1 + |f(\lambda)|)}{t} \right)^d d\lambda \leq t^{-d} |\log(1 + \|f\|_\infty)^d \|f\|_1.$$

□

3. Proof of the Wegner estimate

Since the proofs for balls and cubes are identical, we only consider characteristic functions of balls as single-site potentials

$$V_\omega(x) := \sum_{j \in \mathbb{Z}^d} \chi_{B_{\omega_j}}(x - j). \tag{9}$$

Sometimes, we will write $H_\Lambda(\omega) = H_{\omega,\Lambda}$ and $V_\Lambda(\omega) = V_{\omega,\Lambda}$ for notational convenience.

Note that for all $\omega \in [\omega_-, \omega_+]^{\mathbb{Z}^d}$, all cubes $\Lambda \subset \mathbb{R}^d$ with sidelength $L \in 2\mathbb{N}+1$ and all $j \in \mathbb{Z}^d$, $H_{\omega,\Lambda}$ has purely discrete spectrum. Denote the eigenvalues of $H_{\omega,\Lambda}$ by $\{\lambda_i(\omega)\}_{i\in\mathbb{N}}$, enumerated increasingly and counting multiplicities.

Lemma 3.1. *Let* (3) *hold and assume that* $\omega \in [\omega_-, \omega_+]^{\mathbb{Z}^d}$. *Then for all* $i \in \mathbb{N}$ *with* $\lambda_i(\omega) \in (-\infty, b-1]$ *and all* $\delta \leq 1/2 - \omega_+$ *we have*

$$\lambda_i(\omega+\delta) \geq \lambda_i(\omega) + (\delta/2)^M.$$

Proof. Let $H_{\omega+\delta,\Lambda}\phi_i = \lambda_i(\omega+\delta)\phi_i$ for all $i \in \mathbb{N}$. Then

$$\lambda_i(\omega+\delta) = \langle \phi_i, H_{\omega+\delta,\Lambda}\phi_i \rangle .$$

We write $H_{\omega+\delta,\Lambda} = H_{\omega,\Lambda} + (V_{\omega+\delta} - V_\omega)$. Note that $\omega_+ + \delta \leq 1/2$ and

$$V_{\omega+\delta}(x) - V_\omega(x) = \sum_{j\in\tilde{\Lambda}} \chi_{B\omega_j+\delta}(x-j) - \chi_{B\omega_j}(x-j) \tag{10}$$

This is a sum of characteristic functions of mutually disjoint annuli of width δ. Each of these annuli contains a ball of radius $\delta/2$ and we find

$$V_{\omega+\delta}(x) - V_\omega(x) \geq W_{\delta/2}(x)$$

where $W_{\delta/2}$ is characteristic function of a union of balls of radius $\delta/2$ each of which is contained in a different elementary cell of the grid $\mathbb{Z}^d$.
Restrict now to i such that $\lambda_i^L(\omega) \in (-\infty, b-1]$ which implies in particular $\lambda_i^L(\omega+\delta) \in (-\infty, b]$. Then estimate (3) holds and we obtain via the variational characterization of eigenvalues

$$\begin{aligned}
\lambda_i(\omega+\delta) &= \langle \phi_i, H_{\omega+\delta,\Lambda}\phi_i \rangle \\
&= \max_{\phi\in\mathrm{Span}\{\phi_1,\ldots,\phi_i\},\|\phi\|=1} \langle \phi, H_{\omega,\Lambda}\phi\rangle + \langle \phi, (V_{\omega+\delta} - V_\omega)\,\phi\rangle \\
&\geq \max_{\phi\in\mathrm{Span}\{\phi_1,\ldots,\phi_i\},\|\phi\|=1} \langle \phi, H_{\omega,\Lambda}\phi\rangle + \left\langle \phi, W_{\delta/2}\phi\right\rangle \\
&\geq \max_{\phi\in\mathrm{Span}\{\phi_1,\ldots,\phi_i\},\|\phi\|=1} \langle \phi, H_{\omega,\Lambda}\phi\rangle + (\delta/2)^M \\
&\geq \inf_{\dim\mathcal{D}=i}\ \max_{\phi\in\mathcal{D},\|\phi\|=1} \langle \phi, H_{\omega,\Lambda}\phi\rangle + (\delta/2)^M \\
&= \lambda_i(\omega) + (\delta/2)^M.
\end{aligned}$$

□

We choose $\delta := 2\,(4\varepsilon)^{1/M}$, so that the lemma becomes

$$\lambda_i(\omega+\delta) \geq \lambda_i(\omega) + 4\varepsilon.$$

Since we required $\delta \leq 1/2 - \omega_+$, this yields the upper bound on ε from the theorem. Note in particular that this bound is smaller than $1/2$.

Let $\rho \in C^\infty(\mathbb{R}, [-1,0])$ be a smooth, non-decreasing function such that $\rho = -1$ on $(-\infty; -\varepsilon]$ and $\rho = 0$ on $[\varepsilon; \infty)$. We can assume $\|\rho'\|_\infty \leq 1/\varepsilon$. It holds that

$$\chi_{[E-\varepsilon;E+\varepsilon]}(x) \leq \rho(x - E + 2\varepsilon) - \rho(x - E - 2\varepsilon)$$

for all $x \in \mathbb{R}$ which translates into

$$\begin{aligned}&\mathbb{E}\left[\operatorname{Tr}\left[\chi_{[E-\varepsilon;E+\varepsilon]}(H_{\omega,\Lambda})\right]\right]\\ \leq\ &\mathbb{E}\left[\operatorname{Tr}\left[\rho(H_{\omega,\Lambda}-E+2\varepsilon)-\rho(H_{\omega,\Lambda}-E-2\varepsilon)\right]\right]\\ =\ &\mathbb{E}\left[\operatorname{Tr}\left[\rho(H_{\omega,\Lambda}-E-2\varepsilon+4\varepsilon)-\rho(H_{\omega,\Lambda}-E-2\varepsilon)\right]\right].\end{aligned}$$

Lemma 3.2.

$$\operatorname{Tr}\left[\rho\left(H_{\omega,\Lambda}-E-2\varepsilon+4\varepsilon\right)\right]\leq\operatorname{Tr}\left[\rho\left(H_{\omega+\delta,\Lambda}-E-2\varepsilon\right)\right]. \tag{11}$$

Proof. ρ is a monotonous function, hence we have by the previous lemma

$$\rho(\lambda_i(\omega)-E-2\varepsilon+4\varepsilon)\leq\rho(\lambda_i(\omega+\delta)-E-2\varepsilon).$$

We expand the trace in eigenvalues

$$\begin{aligned}\operatorname{Tr}\left[\rho(H_{\omega,\Lambda}-E-2\varepsilon+4\varepsilon)\right]&=\sum_k\rho(\lambda_i(\omega)-E-2\varepsilon+4\varepsilon)\\ \leq\sum_i\rho(\lambda_i(\omega+\delta)-E-2\varepsilon)&=\operatorname{Tr}\left[\rho(H_{\omega+\delta,\Lambda}-E-2\varepsilon)\right].\end{aligned}$$

□

Now let $\tilde{\Lambda} := \Lambda \cap \mathbb{Z}^d$ and $N := |\tilde{\Lambda}|$. The indices which affect the potential in Λ will be enumerated by

$$k : \{1, ...N\} \to \tilde{\Lambda}, \quad n \mapsto k(n).$$

We define functions which describe how the upper bound in (11) varies when we change one random variable $\omega_{k(n)}$ while keeping all the other random variables fixed. In order to do that we need some notation. Given $\omega \in [\omega_-, \omega_+]^{\mathbb{Z}^d}$, $n \in \{1, ..., N\}$, $\delta \in [0, 1/2 - \omega_+]$ and $t \in [\omega_-, \omega_+]$, we define $\tilde{\omega}^{(n,\delta)}(t) \in [\omega_-, 1/2]^{\mathbb{Z}^d}$ inductively via

$$\left(\tilde{\omega}^{(1,\delta)}(t)\right)_j := \begin{cases} t & \text{if } j = k(1)\\ \omega_j & \text{else}\end{cases}$$

$$\left(\tilde{\omega}^{(n,\delta)}(t)\right)_j := \begin{cases} t & \text{if } j = k(n)\\ \left(\tilde{\omega}^{(n-1,\delta)}(\omega_j+\delta)\right)_j & \text{else.}\end{cases}$$

The function $\tilde{\omega}^{(n,\delta)} : [\omega_-, 1/2] \to [\omega_-, 1/2]^{\mathbb{Z}^d}$ is the rank-one perturbation of ω in the $k(n)$-th coordinate with the additional requirement that all sites $k(1), ..., k(n-1)$ have already been blown up by δ.

Now let

$$\Theta_n(t) := \operatorname{Tr}\left[\rho\left(H_\Lambda\left(\tilde{\omega}^{(n,\delta)}(t)\right) - E - 2\varepsilon\right)\right], \text{ for } n = 1, ..., N.$$

Note that

$$\Theta_n(\omega_{k(n)}) = \Theta_{n-1}(\omega_{k(n-1)} + \delta) \text{ for } n = 2, ..., N$$

and

$$\begin{aligned}\Theta_N(\omega_{k(N)}+\delta)&=\operatorname{Tr}\left[\rho\left(H_{\omega+\delta,\Lambda}-E-2\varepsilon\right)\right],\\ \Theta_1(\omega_{k(1)})&=\operatorname{Tr}\left[\rho\left(H_{\omega,\Lambda}-E-2\varepsilon\right)\right].\end{aligned}$$

Hence we can expand the expectation of the upper bound in (11) in a telescopic sum

$$\begin{aligned}&\mathbb{E}\left[\operatorname{Tr}\left[\rho(H_{\omega+\delta,\Lambda}-E-2\varepsilon)\right]-\operatorname{Tr}\left[\rho(H_{\omega,\Lambda}-E-2\varepsilon\right]\right]\\ =&\mathbb{E}\left[\Theta_N(\omega_{k(N)}+\delta)-\Theta_1(\omega_{k(1)})\right]\\ =&\sum_{n=1}^{N}\mathbb{E}\left[\Theta_n(\omega_{k(n)}+\delta)-\Theta_n(\omega_{k(n)})\right].\end{aligned}$$

Since we have a product measure structure, we can apply Fubini's Theorem

$$\mathbb{E}\left[\Theta_n(\omega_{k(n)}+\delta)-\Theta_n(\omega_{k(n)})\right]=\mathbb{E}\left[\int_{\omega_-}^{\omega_+}\Theta_n(\omega_{k(n)}+\delta)-\Theta_n(\omega_{k(n)})\mathrm{d}\mu(\omega_{k(i)})\right].$$

Note that for $t\in[\omega_-,1/2]$, Θ_n is non-decreasing and bounded. In fact, monotonicity follows from the inequality

$$V_\Lambda\left(\tilde{\omega}^{(n,\delta)}(t_1)\right)\leq V_\Lambda\left(\tilde{\omega}^{(n,\delta)}(t_2)\right),$$

whenever $t_1\leq t_2$ and boundedness is due to the fact that 0 and V_∞ provide upper and lower bounds

$$0\leq V_\Lambda\left(\tilde{\omega}^{(n,\delta)}(t)\right)\leq 1.$$

Lemma 3.3. *Let $-\infty<\omega_-<\omega_+\leq+\infty$. Assume that μ is a probability distribution with bounded density ν_μ and support in the interval $[\omega_-,\omega_+]$ or $[\omega_-,\omega_+)$ if $\omega_+=\infty$ and let Θ be a non-decreasing, bounded function. Then for every $\delta>0$*

$$\int_{\mathbb{R}}\left[\Theta(\lambda+\delta)-\Theta(\lambda)\right]\mathrm{d}\mu(\lambda)\leq\|\nu_\mu\|_\infty\cdot\delta\left[\Theta(\omega_++\delta)-\Theta(\omega_-)\right].$$

Proof. We calculate

$$\begin{aligned}&\int_{\mathbb{R}}\left[\Theta(\lambda+\delta)-\Theta(\lambda)\right]\mathrm{d}\mu(\lambda)\\ \leq&\|\nu_\mu\|_\infty\int_{\omega_-}^{\omega_+}\left[\Theta(\lambda+\delta)-\Theta(\lambda)\right]\mathrm{d}\lambda=\|\nu_\mu\|_\infty\left[\int_{\omega_-+\delta}^{\omega_++\delta}\Theta(\lambda)\mathrm{d}\lambda-\int_{\omega_-}^{\omega_+}\Theta(\lambda)\mathrm{d}\lambda\right]\\ =&\|\nu_\mu\|_\infty\left[\int_{\omega_+}^{\omega_++\delta}\Theta(\lambda)\mathrm{d}\lambda-\int_{\omega_-}^{\omega_-+\delta}\Theta(\lambda)\mathrm{d}\lambda\right]\leq\|\nu_\mu\|_\infty\cdot\delta\left[\Theta(\omega_++\delta)-\Theta(\omega_-)\right].\end{aligned}$$

□

Thus, we find

$$\int_{\omega_-}^{\omega_+}\left[\Theta_i(\omega_{k(i)}+\delta)-\Theta_i(\omega_{k(i)})\mathrm{d}\mu(\omega_{k(i)})\right]\leq\|\nu_\mu\|_\infty\cdot\delta\left[\Theta_i(\omega_++\delta)-\Theta_i(\omega_-)\right]$$

We will use the results from the section 2 in the following form:

Proposition 3.4. *Let $H_0:=-\Delta+V_0$ be a Schrödinger operator with a bounded potential $V_0\geq 0$, and let $H_1:=H_0+V$ for some bounded $V\geq 0$ with support in a ball of radius $1/2$. Denote the Dirichlet restrictions to Λ by H_0^Λ and H_1^Λ, respectively. There are constants K_1, K_2 depending only on d such that for any smooth function $g:\mathbb{R}\to\mathbb{R}$ with derivative supported* *in a compact subset of* $(-\infty, b]$ *and the property that* $g(H_1) - g(H_0)$ *is trace class*

$$\mathrm{Tr}\,[g(H_1) - g(H_0)] \leq K_1 e^b + K_2 \left(\ln(1 + \|g'\|_\infty)^d\right) \|g'\|_1$$

A possible choice is $K_1 := 2 \cdot 32^d \cdot (\sqrt[4]{d} + 1)!$ *and* $K_2 := 32^d$.

The expression $\mathrm{Tr}\,[g(H_1) - g(H_0)]$ is well-defined since H_0 and H_1 are both lower semibounded operators with purely discrete spectrum and only the finite set of eigenvalues in $\mathrm{supp}\ g'$ can contribute to the trace.

Proposition 3.4 implies

Lemma 3.5. *Let* $0 < \varepsilon \leq \frac{1}{2}$*. There is a constant* $\tilde{C}$ *depending only on* d *and on* b *such that*

$$\Theta_n(\omega_+ + \delta) - \Theta_n(\omega_-) \leq \tilde{C} |\ln \varepsilon|^d.$$

The constant $\tilde{C}$ *can be chosen equal to* $K_1 e^b + 2^d K_2$ *with* K_1, K_2 *as in Proposition 3.4.*

Proof. Let $g(\cdot) := \rho_{E+2\varepsilon}(\cdot) := \rho(\cdot - (E + 2\varepsilon))$. By our choice of ρ, g has support in $(-\infty, b]$, $\|g'\|_\infty \leq 1/\varepsilon$ and $\|g'\|_1 = 1$. We define the operators

$$H_0 := H\left(\tilde{\omega}^{(n,\delta)}(\omega_-)\right)$$
$$H_1 := H\left(\tilde{\omega}^{(n,\delta)}(\omega_+ + \delta)\right).$$

These are lower semibounded operators with purely discrete spectrum and since g has support in $(-\infty, b]$, the difference $g(H_1) - g(H_0)$ trace class. By the previous Proposition

$$\Theta_n(\omega_+ + \delta) - \Theta_n(\omega_-) = \mathrm{Tr}\,[\rho_{E+2\varepsilon}(H_1) - \rho_{E+2\varepsilon}(H_0)] \leq K_1 e^b + K_2 \left(\ln(1 + 1/\varepsilon)\right)^d.$$

We assumed $0 < \varepsilon \leq \frac{1}{2}$, thus $1 + \varepsilon \leq \varepsilon^{-1}$ and

$$\ln(1 + 1/\varepsilon) = \ln(1 + \varepsilon) - \ln \varepsilon \leq -2 \ln \varepsilon = 2 |\ln \varepsilon|$$

and $1 \leq |\ln \varepsilon| \leq |\ln \varepsilon|^d$ which proves the Lemma. □

Putting everything together yields

$$\mathbb{E}\left[\mathrm{Tr}\left[\chi_{[E-\varepsilon, E+\varepsilon]}(H_{\omega,\Lambda})\right]\right] \leq \left(K_1 e^b + 2^d K_2\right) \|\nu_\mu\|_\infty \cdot \delta\, |\ln \varepsilon|^d\, L^d. \tag{12}$$

and bearing in mind that $\delta = 2 \cdot (4\varepsilon)^{1/M}$ we obtain (4). This proves Theorem 1.2.

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