DISCRETE ALLOY-TYPE MODELS: REGULARITY OF DISTRIBUTIONS AND RECENT RESULTS

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Abstract. We consider discrete random Schrödinger operators on $\ell^2(\mathbb{Z}^d)$ with a potential of discrete alloy-type structure. That is, the potential at lattice site $x \in \mathbb{Z}^d$ is given by a linear combination of independent identically distributed random variables, possibly with sign-changing coefficients. In a first part we show that the discrete alloy-type model is not uniformly τ-Hölder continuous, a frequently used condition in the literature of Anderson-type models with general random potentials. In a second part we review recent results on regularity properties of spectral data and localization properties for the discrete alloy-type model.

1. Introduction

We consider discrete Schrödinger operators on $\ell^2(\mathbb{Z}^d)$, where $\mathbb{Z}^d$ is the d-dimensional integer lattice. The potential of the discrete Schrödinger operator is given by a stochastic field. Thus we are dealing with generalizations of the standard Anderson model. We are interested in properties of distributions of spectral data of such random operators, as well as of their restrictions to finite cubes $\Lambda \subset \mathbb{Z}^d$. An appropriate control of these distributions allows one to conclude almost sure spectral and dynamical localization for the random Schrödinger operator on $\ell^2(\mathbb{Z}^d)$. More precisely, we discuss in the paper the following issues:

- We review results on discrete alloy-type models proven in [ETV10, ETV11] and [PTV11, LPTV13]. They concern localization criteria based on the multiscale analysis and the fractional moment method and can be considered as generalizations of earlier results obtained in [Ves10b, Ves10a, TV10b]. Also, we discuss related results in the recent paper [ESS12] and highlight the role of a reverse Hölder inequality in the argument of [ESS12].
- We present Minami estimates and Poisson statistics of eigenvalues proven in [TV13a] for a class of discrete alloy-type models. This extend results of [Min96] beyond the Anderson model.
- Alloy-type potentials are a specific type of a correlated stochastic field. While there are abstract localization results in the literature concerning correlated random potentials, they rarely cover those of alloy-type. We show this by checking the relevant regularity properties of the conditional distributions of the stochastic field.
- This prompts a careful consideration of conditional distributions. In the literature on random Schrödinger operators these are sometimes not treated correctly. We show how to deal with certain measurability issues and give a (counter)example, which shows how badly conditional distributions may behave, even for innocently looking alloy-type potentials.

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Let us put these statements into context. The Anderson model is a Schrödinger operators on $\ell^2(\mathbb{Z}^d)$ with potential given by an independent identically distributed (i.i.d.) sequence of random variables $V_\cdot(x), x \in \mathbb{Z}^d$. One expects that for energies near the infimum and supremum of the spectrum, as well as at large disorder, this model exhibits localization, i.e. discrete spectrum with exponentially decaying eigenfunctions, almost surely. However, proofs of this claim depend on regularity conditions on the distribution of random variables. For instance, for the Anderson model with Bernoulli distributed variables so far localization was proven only in one space dimension. If we consider more general random potentials, where each random variable $V_\cdot(x), x \in \mathbb{Z}^d$, has the same marginal distribution ν, but they are no longer independent, additional distinctions are necessary. Now one has to impose regularity conditions on the finite-dimensional distributions of the process $V_\cdot(x), x \in \mathbb{Z}^d$, i.e. the joint distribution of a finite subcollection of random variables. Alternatively, one can formulate regularity hypotheses on the conditional distributions. While the ultimate goal is to formulate regularity hypotheses which can be used to derive localization, an intermediate step is to derive regularity of spectral data. More precisely, one wants to show, that if the distribution of the stochastic process $V_\cdot(x), x \in \mathbb{Z}^d$, is sufficiently regular, then the distribution of spectral data is so as well. Abstractly speaking: the pushforward map preserves the regularity of probability measures. Let us give an illustration. If the distribution function of ν is Lipschitz continuous, then the integrated density of states inherits this property. This is called a Wegner estimate.

We review here a number of positive results in this direction for correlated fields $V_\cdot(x), x \in \mathbb{Z}^d$, which arise as an alloy-type potential. In stochastic data analysis such models are known as (multidimensional) moving average processes. The mentioned results include Wegner estimates, uniform bounds on fractional moments and exponential decay of fractional moments. This are results obtained in [ETV11, PTV11]. They have been extended in the recent papers [Krü12] and [ESS12]. The latter one will be discussed in Section 6. In particular, for the case of alloy-type potentials with large disorder we give a short and direct modification of the proof of a subharmonicity inequality crucial for the fractional moment method. Furthermore, we single out a reverse Hölder-inequality as the pivot estimate in the strategy of [ESS12].

While the Wegner estimate concerns a bound on the probability of finding an eigenvalue at all in an energy interval, the Minami estimate bounds the the probability of finding at least two eigenvalue in an energy interval. For a specific class of discrete alloy-type models Minami's result [Min96] has been generalized in [TV13a]. We discuss this in Section 4. The implications for the asymptotic statistics of eigenvalues is presented in Section 5.

The papers [vDK91, AM93, AG98, Hun00, ASFH01, Hun08] give abstract regularity conditions, formulated in terms of conditional distributions, which ensure localization for discrete Schrödinger operators with random potential. We show that these regularity conditions are not satisfied for alloy-type potentials with bounded values, see Section 3 for a precise statement. A very interesting borderline behavior is encountered for alloy-type potentials with Gaussian coupling constants. In this case the above mentioned regularity conditions may be satisfied or not, depending on the specific choice of the single-site potential. In Section 3 we also show how to define carefully the associated concentration function or modulus of continuity. This concerns the measurability of a supremum over an uncountable set. Such measurability issues are encountered in other areas of probability theory, for instance in the context of the Glivenko-Cantelli Theorem or the definition of Markov transition kernels. Using regular versions of the conditional expectation we show in detail how to define the concentration functions rigorously. The reason to devote so much attention to this topic is, that in the literature on random Schrödinger operators conditional distributions are not always treated correctly. There is the misconception that a moving average process inherits the regularity properties of the i.i.d random variables on which it is based. Section 3 shows that if one starts with regularly distributed i.i.d. random variables and uses them to define a discrete alloy-type potential (or moving average process) the resulting conditional distributions are quite singular. These results have been formulated before in the technical reports [TV10a, TV13b].

To summarize, multidimensional Anderson models without independence condition are still not very well understood. Exceptions are Gaussian processes treated rigorously in [vDK91, ASFH01] and discrete alloy-type potentials treated in the above mentioned papers.

2. Notation and model

2.1. **General random Schrödinger operators on $\mathbb{Z}^d$.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\eta_k : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $k \in \mathbb{Z}^d$, be real-valued random variables. We define the product space $Z = \times_{k \in \mathbb{Z}^d} \mathbb{R}$ equipped with the product σ-algebra $\mathcal{Z} = \otimes_{k \in \mathbb{Z}^d} \mathcal{B}(\mathbb{R})$. The collection $(\eta_k)_{k \in \mathbb{Z}^d}$ will be denoted by

$$\eta := (\eta_k)_{k \in \mathbb{Z}^d} : (\Omega, \mathcal{A}) \to (Z, \mathcal{Z}).$$

The expectation with respect to the probability measure $\mathbb{P}$ will be denoted by $\mathbb{E}$. A discrete random Schrödinger operator is given by a family of self-adjoint operators

$$H_\omega = -\Delta + \lambda V_\omega, \quad \omega \in \Omega, \tag{1}$$

on $\ell^2(\mathbb{Z}^d)$. Here $\lambda > 0$ measures the strength of the disorder present in the model, Δ denotes the discrete Laplace operator and V_ω is a multiplication operator. They are defined by

$$(\Delta\psi)(x) = \sum_{|y-x|_1} \psi(y), \quad \text{and} \quad (V_\omega\psi)(x) = \eta_x(\omega)\psi(x).$$

We assume that H_ω is for each $\omega \in \Omega$ a self-adjoint operator (on some dense domain $D_\omega \subset \ell^2(\mathbb{Z}^d)$). This is for example satisfied if the random potentials η_k, $k \in \mathbb{Z}^d$ are uniformly bounded random variables. If the potential values are not uniformly bounded, we recall that H_ω is essentially self-adjoint on the set of compactly supported functions, see e.g. [Kir08].

For the operator H_ω in (1) and $z \in \mathbb{C} \setminus \sigma(H_\omega)$ we define the corresponding *resolvent* by $G_\omega(z) = (H_\omega - z)^{-1}$. For the *Green function*, which assigns to each $(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$ the corresponding matrix element of the resolvent, we use the notation

$$G_\omega(z; x, y) := \langle \delta_x, (H_\omega - z)^{-1} \delta_y \rangle.$$

For $\Gamma \subset \mathbb{Z}^d$, $\delta_k \in \ell^2(\Gamma)$ denotes the Dirac function given by $\delta_k(k) = 1$ for $k \in \Gamma$ and $\delta_k(j) = 0$ for $j \in \Gamma \setminus \{k\}$. Let $\Gamma \subset \mathbb{Z}^d$. We define the canonical restriction $p_\Gamma : \ell^2(\mathbb{Z}^d) \to \ell^2(\Gamma)$ by

$$p_\Gamma \psi := \sum_{k \in \Gamma} \psi(k) \delta_k,$$

where the Dirac function has to be understood as an element of $\ell^2(\Gamma)$. Note that the corresponding embedding $\iota_\Gamma := (p_\Gamma)^* : \ell^2(\Gamma) \to \ell^2(\mathbb{Z}^d)$ is given by

$$\iota_\Gamma \phi := \sum_{k \in \Gamma} \phi(k) \delta_k,$$

where here the Dirac function has to be understood as an element of $\ell^2(\mathbb{Z}^d)$. For an arbitrary set $\Gamma \subset \mathbb{Z}^d$ we define the restricted operators $\Delta_\Gamma, V_{\omega,\Gamma}, H_{\omega,\Gamma} : \ell^2(\Gamma) \to \ell^2(\Gamma)$ by $\Delta_\Gamma := p_\Gamma \Delta \iota_\Gamma$, $V_{\omega,\Gamma} := p_\Gamma V_\omega \iota_\Gamma$ and

$$H_{\omega,\Gamma} := p_\Gamma H_\omega \iota_\Gamma = -\Delta_\Gamma + \lambda V_{\omega,\Gamma}.$$

Furthermore, we define $G_\Gamma(z) := (H_\Gamma - z)^{-1}$ and $G_\Gamma(z; x, y) := \langle \delta_x, G_\Gamma(z) \delta_y \rangle$ for $z \in \mathbb{C} \setminus \sigma(H_\Gamma)$ and $x, y \in \Gamma$.

2.2. **Discrete alloy-type model.** Of particular interest will be the case where the random variables η_k are given by a linear combination of i.i.d. random variables, giving rise to a discrete alloy-type potential. While some abstract definitions in Section 3 hold for arbitrary random fields η, our main results concern the discrete alloy-type potential.

Assumption (A). The probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is given by the product space $\Omega = \times_{k \in \mathbb{Z}^d} \mathbb{R}$, $\mathcal{A} = \otimes_{k \in \mathbb{Z}^d} \mathcal{B}(\mathbb{R})$ and $\mathbb{P} = \otimes_{k \in \mathbb{Z}^d} \mu$, where μ is some probability measure on $\mathbb{R}$. The random variables $\eta_k : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $k \in \mathbb{Z}^d$, are given by

$$\eta_k(\omega) = \sum_{i \in \mathbb{Z}^d} \omega_i u(x - i) \tag{2}$$

for some summable function $u : \mathbb{Z}^d \to \mathbb{R}$.

If Assumption (A) is satisfied, we call the collection of random variables η_k, $k \in \mathbb{Z}^d$, given by Eq. (2) a *discrete alloy-type potential*, the corresponding family of operators

$$H_\omega = -\Delta + \lambda V_\omega, \quad (V_\omega \psi)(x) = \eta_x(\omega) \psi(x), \quad \omega \in \Omega,$$

on $\ell^2(\mathbb{Z}^d)$ a *discrete alloy-type model*, and the function u a *single-site potential*. Moreover, we set $\Theta = \operatorname{supp} u$.

In the case where the single-site potential $u = \delta_0$, the random Hamiltonian (1) is exactly the standard Anderson model.

3. Conditional distributions and modulus of continuity

3.1. **Definition and main result.** Let $m \in \mathbb{Z}^d$, $Z_m^\perp = \times_{k \in \mathbb{Z}^d \setminus \{m\}} \mathbb{R}$ and $\mathcal{Z}_m^\perp = \otimes_{k \in \mathbb{Z}^d \setminus \{m\}} \mathcal{B}(\mathbb{R})$. We introduce the random variable

$$\eta_m^\perp : (\Omega, \mathcal{A}) \to (Z_m^\perp, \mathcal{Z}_m^\perp), \quad \eta_m^\perp(\omega) = (\eta_k(\omega))_{k \in \mathbb{Z}^d \setminus \{m\}}.$$

We denote by $\mathbb{P}_{\eta_m^\perp} : Z_m^\perp \to [0,1]$ the distribution of $\eta_m^\perp$ with respect to $\mathbb{P}$, i.e. $\mathbb{P}_{\eta_m^\perp}(B) := \mathbb{P}(\{\omega \in \Omega \colon \eta_m^\perp(\omega) \in B\})$. For $m \in \mathbb{Z}^d$, $a \in \mathbb{R}$ and $\varepsilon > 0$ we define the conditional expectation

$$Y_m^{\varepsilon,a} := \mathbb{P}\bigl(\eta_m \in [a, a+\varepsilon] \mid \eta_m^\perp\bigr) := \mathbb{E}\bigl(\mathbf{1}_{\{\eta_m \in [a,a+\varepsilon]\}} \mid \eta_m^\perp\bigr).$$

A conditional expectation $Y_m^{\varepsilon,a} = \mathbb{E}(\mathbf{1}_{\{\eta_m \in [a,a+\varepsilon]\}} \mid \eta_m^\perp)$ is a random variable $Y_m^{\varepsilon,a} : \Omega \to [0,1]$ with the property that

(i) $Y_m^{\varepsilon,a}$ is $\mathcal{F}$-measurable, where $\mathcal{F} = \sigma(\eta_m^\perp)$, and that

(ii) for all $A \in \mathcal{F}$ we have $\mathbb{E}(\mathbf{1}_{\{\eta_m \in [a,a+\varepsilon]\}} \mathbf{1}_A) = \mathbb{E}(Y_m^{\varepsilon,a} \mathbf{1}_A)$.

Since $\mathbf{1}_{\{\eta_m \in [a,a+\varepsilon]\}} \in \mathcal{L}^1(\Omega, \mathcal{A}, \mathbb{P})$, $Y_m^{\varepsilon,a}$ exists. There may exist several functions $Y_m^{\varepsilon,a}$ which satisfy conditions (i) and (ii). They are called *versions* of $\mathbb{E}(\mathbf{1}_{\{\eta_m \in [a,a+\varepsilon]\}} \mid \eta_m^\perp)$. Two such versions coincide $\mathbb{P}$-almost everywhere. For convenience, for each $a \in \mathbb{R}$ and $\varepsilon > 0$ we fix one version $Y_m^{\varepsilon,a}$ of the conditional expectation. Since $Y_m^{\varepsilon,a}$ is $\mathcal{F}$-measurable, the factorization lemma tells us that (for each a and ε) there is a measurable function $g_m^{\varepsilon,a} : (Z_m^\perp, \mathcal{Z}_m^\perp) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $Y_m^{\varepsilon,a} = g_m^{\varepsilon,a} \circ \eta_m^\perp$, i.e. for all $\omega \in \Omega$ we have

$$Y_m^{\varepsilon,a}(\omega) = g_m^{\varepsilon,a}(\eta_m^\perp(\omega)). \tag{3}$$

We introduce several quantities used in the literature to describe the regularity of (the conditional distribution) of the random field η_k, $k \in \mathbb{Z}^d$. For $m \in \mathbb{Z}^d$ we denote by $S_m \colon [0,\infty) \to [0,1]$,

$$S_m(\varepsilon) := \sup_{a \in \mathbb{R}} \mathbb{P}(\{\omega \in \Omega \colon \eta_m \in [a, a+\varepsilon]\}),$$

the *global modulus of continuity* or *the concentration function* of the distribution of η_m. For $\Lambda \subset \mathbb{Z}^d$ and $\varepsilon > 0$ we define

$$\hat{S}_\Lambda(\varepsilon) := \sup_{m \in \Lambda} \sup_{a \in \mathbb{R}} \operatorname*{ess\,sup}_{\eta_m^\perp \in Z_m^\perp} g_m^{\varepsilon,a}(\eta_m^\perp).$$

Here, the essential supremum refers to the measure $\mathbb{P}_{\eta_m^\perp}$, that is,

$$\operatorname*{ess\,sup}_{\eta_m^\perp \in Z_m^\perp} g_m^{\varepsilon,a}(\eta_m^\perp) = \inf\Big\{ b \in \mathbb{R} \colon \mathbb{P}_{\eta_m^\perp}\big(\{\eta_m^\perp \in Z_m^\perp \colon g_m^{\varepsilon,a}(\eta_m^\perp) > b\}\big) = 0 \Big\}.$$

Denote by $\tilde{S}_m^\varepsilon$ the *conditional global modulus of continuity* or the *conditional concentration function* of the distribution of η_m, i.e.

$$\tilde{S}_m^\varepsilon : \Omega \to [0,1], \quad \tilde{S}_m^\varepsilon = \sup_{a \in \mathbb{R}} Y_m^{\varepsilon,a}.$$

Since we are taking here a supremum over an uncountable set, it is not clear whether the resulting function is still measurable. In fact, this depends on how we chose the version of the conditional expectation (for each of the uncountable many $a \in \mathbb{R}$). We show in Lemma 3.1 that if we choose a regular version of $Y_m^{\varepsilon,a}$ (which always exists since η_m is real-valued), then $\tilde{S}_m^\varepsilon$ is $\mathcal{F}$-measurable. In what follows we always assume that $\tilde{S}_m^\varepsilon$ is $\mathcal{F}$-measurable and we denote by $g_m^\varepsilon : (Z_m^\perp, \mathcal{Z}_m^\perp) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ the measurable function which comes up with the factorization lemma and satisfies $\tilde{S}_m^\varepsilon = g_m^\varepsilon \circ \eta_m^\perp$. Finally, we define

$$\tilde{S}_\Lambda(\varepsilon) := \sup_{m \in \Lambda} \operatorname*{ess\,sup}_{\eta_m^\perp \in Z_m^\perp} g_m^\varepsilon(\eta_m^\perp),$$

where the essential supremum again refers to the measure $\mathbb{P}_{\eta_m^\perp}$.

Lemma 3.1. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\mathcal{C} \subset \mathcal{A}$ a σ-algebra and $X \colon \Omega \to \mathbb{R}$ a random variable. Let further $Q \colon \Omega \times \mathcal{B}(\mathbb{R}) \to [0,1]$ be a regular version of the conditional distribution of X with respect to $\mathcal{C}$. Then for all $\varepsilon > 0$ the function*

$$\sup_{a \in \mathbb{R}} Q(\cdot, [a, a+\varepsilon]) : \Omega \to [0,1]$$

is $\mathcal{C}$-measurable.

For the proof we will use results on the regular version of the condition a distribution of a random variable with respect to a sub-σ-algebra. These can be found, e.g., in §44 of [Bau91].

Proof of Lemma 3.1. For each $\varepsilon > 0$ and $a \in \mathbb{R}$

$$\Omega \ni \omega \mapsto Q(\omega, [a, a+\varepsilon])$$

is $\mathcal{C}$-measurable. Consequently, for each $\varepsilon > 0$

$$\sup_{b,\delta \in \mathbb{Q}, \delta \in [0,\varepsilon]} Q(\omega, [b, b+\delta])$$

is $\mathcal{C}$-measurable as well. It remains to show

$$\sup_{a \in \mathbb{R}} Q(\omega, [a, a+\varepsilon]) = \sup_{b,\delta \in \mathbb{Q}, \delta \in [0,\varepsilon]} Q(\omega, [b, b+\delta]).$$

Fix $c \in \mathbb{R}$. Since Q is a regular version of the conditional distribution we have for all $\omega \in \Omega$

$$Q(\omega, [c, c+\varepsilon]) = \sup_{b,\delta \in \mathbb{Q}, b \geq c, \delta \geq 0, b+\delta \leq c+\varepsilon} Q(\omega, [b, b+\delta]).$$

(For an arbitrary version of the conditional distribution we would have this statement only for almost all ω, with the exceptional set depending on c.) The last quantity equals

$$\sup_{b,\delta \in \mathbb{Q}, b \geq c, \delta \geq 0, b+\delta \leq c+\varepsilon, \delta \leq \varepsilon} Q(\omega, [b, b+\delta])$$

and is bounded from above by

$$\begin{aligned} \sup_{b,\delta \in \mathbb{Q}, b \geq c, \delta \geq 0, \delta \leq \varepsilon} Q(\omega, [b, b+\delta]) &\leq \sup_{b,\delta \in \mathbb{Q}, \delta \geq 0, \delta \leq \varepsilon} Q(\omega, [b, b+\delta]) \\ &\leq \sup_{b \in \mathbb{Q}} Q(\omega, [b, b+\varepsilon]) \leq \sup_{b \in \mathbb{R}} Q(\omega, [b, b+\varepsilon]). \end{aligned}$$

This completes the proof. □

The papers [vDK91, AM93, AG98, Hun00, ASFH01, Hun08] make use of certain regularity conditions, formulated in terms of the conditional modulus of continuity, which are used to derive localization for Anderson-type models as in (1) with correlated potentials. Since the regularity conditions of the mentioned papers are all in the same flavor, we formulate exemplary the condition from [ASFH01].

Condition (B). The collection η_k, $k \in \mathbb{Z}^d$, is said to be (uniformly) *τ-Hölder continuous* for $\tau \in (0,1]$ if there is a constant C such that for all $\varepsilon > 0$

$$\hat{S}_{\mathbb{Z}^d}(\varepsilon) := \sup_{m \in \mathbb{Z}^d} \sup_{a \in \mathbb{R}} \operatorname*{ess\,sup}_{\eta_m^\perp \in Z_m^\perp} g_m^{\varepsilon,a}(\eta_m^\perp) \leq C \varepsilon^\tau.$$

Our main results on the modulus of continuity is the following theorem. It applies to a class of discrete alloy-type potentials, including a case where the measure μ has unbounded support.

Theorem 3.2. *Let Assumption (A) be satisfied, $d = 1$ and either*

(a) $\Theta = \{0, \ldots, n-1\}$ *for some* $n \in \mathbb{N}$, $\sup \operatorname{supp} \mu = 1$ *and* $\inf \operatorname{supp} \mu = 0$, ***or***

(b) $\Theta = \{-1, 0\}$, $u(0) = 1$, $|u(1)|^2 = 1$ *and* μ *be the normal distribution with mean zero and variance* σ^2.

Then for any $\Lambda \subset \mathbb{Z}$ *and any* $\varepsilon > 0$ *we have*

$$\hat{S}_\Lambda(\varepsilon) := \sup_{m\in\Lambda} \sup_{a\in\mathbb{R}} \operatorname*{ess\,sup}_{\eta_m^\perp \in Z_m^\perp} g_m^{\varepsilon,a}(\eta_m^\perp) = 1 \tag{4}$$

and

$$\tilde{S}_\Lambda(\varepsilon) := \sup_{m\in\Lambda} \operatorname*{ess\,sup}_{\eta_m^\perp \in Z_m^\perp} g_m^{\varepsilon}(\eta_m^\perp) = 1. \tag{5}$$

The above Theorem shows that Condition (B) is not satisfied for the discrete alloy-type potential, if any of the two cases (a) or (b) holds. This is in sharp contrast to the fact that the concentration function S_m of the distribution of such η_m may be very well τ-Hölder continuous, as the following example shows.

Example 3.3. Let Assumption (A) be satisfied, $d = 1$, $\Theta = \{0, 1\}$, $u(0) = u(1) = 1$ and μ be the uniform distribution on $[0, 1]$, which is a special case of case (a) in Theorem 3.2. Then we have for $m \in \mathbb{Z}$ and $\varepsilon > 0$

$$S_m(\varepsilon) = \sup_{a\in\mathbb{R}} \mathbb{P}(\{\omega_m + \omega_{m-1} \in [a, a+\varepsilon]\}) = \begin{cases} \varepsilon - \frac{\varepsilon^2}{4} & \text{if } \varepsilon \in (0, 2], \\ 1 & \text{if } \varepsilon > 2. \end{cases}$$

If one considers finite a volume restriction H_{ω,Λ_L}, $\Lambda_L = \{y \in \mathbb{Z}^d : |y|_\infty \leq L\}$, an analogue to Condition (B) which is sufficient for localization would be the following: There is some $\tau \in (0, 1]$ and a constant C such that

$$\sup_{L\in\mathbb{N}} \sup_{m\in\Lambda_L} \sup_{a\in\mathbb{R}} \operatorname*{ess\,sup}_{(\eta_k)_{k\in\Lambda_L\setminus\{m\}}} \mathbb{P}\left(\eta_m \in [a, a+\varepsilon] \mid (\eta_k)_{k\in\Lambda_L\setminus\{m\}}\right) \leq C\varepsilon^\tau. \tag{6}$$

As can be seen from the proof of Theorem 3.2, this condition is also not satisfied for the discrete alloy-type potential if any of the two cases (a) or (b) holds. On the contrary, the proof of Theorem 3.2 suggests that Condition (B) is satisfied for the discrete alloy-type potential, if $\Theta = \{-1, 0\}$, $u(0) = 1$, $|u(1)|^2 \neq 1$ and μ is the normal distribution, see Proposition 3.6 and Remark 3.7 below.

The result of Theorem 3.2 also shows that the key Lemma 3 in [Klo12] is not correct. Lemma 3 in [Klo12] states (in our notation) that the conditional distributions of the random variables η_m exhibits qualitatively the same regularity as the distributions of the random variables ω_m.

The proof of Theorem 3.2 is split into two parts. First we consider in Subsection 3.2 elementary conditional probabilities (conditioned on an event, not on a σ-algebra) and derive appropriate bounds. Thereafter we show how to transfer these bounds to probabilities conditioned on a σ-algebra in Subsection 3.3.

3.2. **Elementary conditional probabilities.**

Proposition 3.4. *Let Assumption (A) be satisfied,* $d = 1$*,* $\Theta = \{0, 1, \ldots, n-1\}$ *for some* $n \in \mathbb{N}$*,* $\inf \operatorname{supp} \mu = 0$ *and* $\sup \operatorname{supp} \mu = 1$*. There are constants* $c, m, s^+ \in (-\infty, \infty)$*, depending only on u, such that for all* $\delta > 0$ *and* $\delta \geq \delta' > 0$

$$\mathbb{P}\big(\eta_0 \in [m - c\delta, m + c\delta] \mid \eta_{-1}, \eta_{n-1} \in [s^+ - \delta', s^+]\big) = 1.$$

The values of the constants c*,* m *and* s^+ *can be inferred from the proof.*

Notice that, under the assumptions of Proposition 3.4, η_{-1} and η_{n-1} are stochastically independent and $\mathbb{P}(\eta_{-1}, \eta_{n-1} \in [s^+ - \delta', s^+]) > 0$, where s^+ is defined in the proof of Proposition 3.4.

Proof of Proposition 3.4. Let $\Theta^+ := \{k \in \mathbb{Z} : u(k) > 0\}$, $\Theta^- := \{k \in \mathbb{Z}^1 : u(k) < 0\}$, $u_{\max} = \max_{k\in\Theta}|u(k)|$, $u_{\min} = \min_{k\in\Theta}|u(k)|$ and $s^+ = \sum_{k\in\Theta^+} u(k)$. Let us further introduce two subsets of Θ which are important in our study. The first one is

$$\Theta_1 = \begin{cases} \Theta^+ + 1 & \text{if } n-1 \notin \Theta^+, \\ ((\Theta^+ + 1) \cap \Theta) \cup \{0\} & \text{if } n-1 \in \Theta^+, \end{cases}$$

with $\Theta^+ + 1 = \{k \in \mathbb{N} : (k-1) \in \Theta^+\}$. The second subset is the complement $\Theta_0 = \Theta \setminus \Theta_1$. To end the proof we show the following interval arithmetic result:

Let $\delta \geq \delta' > 0$ and $\eta_{-1}, \eta_{n-1} \in [s^+ - \delta', s^+]$. Then

$$\eta_0 \in [m - c\delta', m + c\delta'] \subset [m - c\delta, m + c\delta] \tag{7}$$

with $c = n u_{\max}/u_{\min}$ and $m = \sum_{k\in\Theta_1} u(k)$.

We divide the proof of (7) into three parts. The first step is to argue that

$$\omega_{-1-k} \in \begin{cases} \left[1 - \frac{\delta'}{u_{\min}}, 1\right] & \text{for } k \in \Theta^+, \\ \left[0, \frac{\delta'}{u_{\min}}\right] & \text{for } k \in \Theta^-. \end{cases} \tag{8}$$

For the proof of the first part of (8) we use the assumption $\eta_{-1} \geq s^+ - \delta'$ and obtain

$$s^+ - \delta' \leq \eta_{-1} = \sum_{k\in\Theta} u(k)\omega_{-1-k} \leq \sum_{k\in\Theta^+} u(k)\omega_{-1-k},$$

and hence $\sum_{k\in\Theta^+} u(k)(1 - \omega_{-1-k}) \leq \delta'$. We conclude that for all $k \in \Theta^+$ we have $u(k)(1 - \omega_{-1-k}) \leq \delta'$ which gives the first part of (8). For the proof of the second part of (8) we use again the assumption $\eta_{-1} \geq s^+ - \delta'$ and obtain

$$\sum_{k\in\Theta^+} u(k)\omega_{-1-k} - \delta' \leq s^+ - \delta' \leq \eta_{-1} = \sum_{k\in\Theta^+} u(k)\omega_{-1-k} + \sum_{k\in\Theta^-} u(k)\omega_{-1-k}$$

which gives $-\delta' \leq \sum_{k\in\Theta^-} u(k)\omega_{-1-k}$. Thus, for all $k \in \Theta^-$ we have $\omega_{-1-k} \leq -\delta'/u(k) = \delta'/|u(k)|$ which gives the second part of (8). In a second step we argue that

$$\omega_{-k+n-1} \in \begin{cases} \left[1 - \frac{\delta'}{u_{\min}}, 1\right] & \text{for } k \in \Theta^+, \\ \left[0, \frac{\delta'}{u_{\min}}\right] & \text{for } k \in \Theta^-. \end{cases} \tag{9}$$

The proof of (9) can be done in analogy to the proof of (8), but using the assumption $\eta_{n-1} \geq s^+ - \delta'$. In a third step we ask the question for which $k \in \Theta$ we have $\omega_{-k} \in [1 - \delta'/u_{\min}, 1]$. Using the definition of the set Θ_1 we find with (8) and (9) that

$$\omega_{-k} \in \begin{cases} \left[1 - \frac{\delta'}{u_{\min}}, 1\right] & \text{for } k \in \Theta_1, \\ \left[0, \frac{\delta'}{u_{\min}}\right] & \text{for } k \in \Theta_0. \end{cases} \tag{10}$$

Now, the desired result (7) follows from (10) and the decomposition

$$\eta_0 = \sum_{k\in\Theta} u(k)\omega_{-k} = \sum_{k\in\Theta_1} u(k)\omega_{-k} + \sum_{k\in\Theta_0} u(k)\omega_{-k}.$$

Hence, the proof is complete. □

Remark 3.5. The assumption $\inf \operatorname{supp} \mu = 1$ and $\sup \operatorname{supp} \mu = 1$ in Proposition 3.4 is not crucial. What matters is that $\operatorname{supp} \mu$ is a bounded set.

In the case where $\operatorname{supp}\mu$ is an unbounded set the situation is somehow different. We illustrate the effects in the case where μ is Gaussian. For $l \in \mathbb{N}$ let $A_l \in \mathbb{R}^{l \times l+1}$ be the matrix with coefficients in the canonical basis given by $A_l(i,i) = 1$, $A_l(i,i+1) = u(-1)$ for $i \in \{1,\dots,l\}$, and zero otherwise, namely

$$A_l = \begin{pmatrix} 1 & u(-1) & & & \\ & \ddots & \ddots & & \\ & & \ddots & u(-1) & \\ & & & 1 & u(-1) \end{pmatrix} \in \mathbb{R}^{l \times l+1}.$$

Proposition 3.6. *Let Assumption (A) be satisfied, $d = 1$, $l, m \geq 1$, $\Theta = \{-1, 0\}$, $u(0) = 1$ and μ be the normal distribution with mean zero and variance σ^2. Let further $v^+ \in \mathbb{R}^l$ and $v^- \in \mathbb{R}^m$. Then the distribution of η_0 conditioned on $(\eta_k)_{k=1}^l = v^+$ and $(\eta_{-m+k-1})_{k=1}^m = v^-$ is Gaussian with variance*

$$\gamma = \sigma^2 \Big(u(-1)^2 - 1 + \frac{1}{s_m} + \frac{1}{s_l} \Big), \quad \textit{where} \quad s_l := \sum_{i=1}^{l} \big(u(-1)\big)^{2i},$$

and mean

$$m = u(-1) \left(\sum_{i=1}^{m} (A_m A_m^{\mathrm{T}})^{-1}(m,i)\, v_i^- + \sum_{i=1}^{l} (A_l A_l^{\mathrm{T}})^{-1}(1,i)\, v_i^+ \right).$$

Remark 3.7. Let $l, m \geq 1$. If $|u(-1)| \neq 1$, Proposition 3.6 gives that the distribution of η_0 conditioned on fixed potential values η_k, $k \in \{-m,\dots,l\} \setminus \{0\}$, is again Gaussian with variance bounded from below by $\sigma^2 |u^2(-1) - 1|$. This shows that the random field η_k, $k \in \mathbb{Z}^d$, satisfies the regularity condition formulated in Ineq. (6) if $\Theta = \{-1, 0\}$, $u(0) = 1$, $|u(-1)| \neq 1$ and μ is Gaussian. Moreover, the regularity condition from Ineq. (6) is not satisfied if $\Theta = \{-1, 0\}$, $u(0) = 1$, $|u(-1)| = 1$ and μ is Gaussian.

The proof of Proposition 3.6 is based on following classical result which may be found in [Por94].

Proposition 3.8. *Let X be normally distributed on $\mathbb{R}^d$, $Y = a \cdot X$ where $a \in \mathbb{R}^d$, and $W = BX$ where $B \in \mathbb{R}^{m \times d}$. Assume W has a non-singular distribution. Then the distribution of Y conditioned on $W = v \in \mathbb{R}^m$ is the Gaussian distribution having mean*

$$\mathbf{E}(Y) + \operatorname{cov}(Y, W) \operatorname{cov}(W, W)^{-1} [v - \mathbf{E}(W)]$$

and variance

$$\operatorname{cov}(Y, Y) - \operatorname{cov}(Y, W) \operatorname{cov}(W, W)^{-1} \operatorname{cov}(W, Y).$$

Notice, if we apply A_l on the vector $\omega_{[x,x+l]} = (\omega_{x+k-1})_{k=1}^{l+1}$, we obtain a vector containing the potential values η_k, $k \in \{x, x+1, \dots, x+l\}$. Moreover, the vector $(\eta_{x+k-1})_{k=1}^l = A_l \omega_{[x,x+l]}$ is normally distributed with mean zero and covariance $\sigma^2 A_l A_l^{\mathrm{T}}$. The matrix $A_l A_l^T$ has the form

$$A_l A_l^T = \begin{pmatrix} 1 + u^2(-1) & u(-1) & & \\ u(-1) & 1 + u^2(-1) & \ddots & \\ & \ddots & \ddots & u(-1) \\ & & u(-1) & 1 + u^2(-1) \end{pmatrix} \in \mathbb{R}^{l \times l}.$$

By induction we find that the determinant of $A_l A_l^{\mathrm{T}}$ is given by

$$\det(A_l A_l^{\mathrm{T}}) = s_l > 0 \quad \text{where} \quad s_l = \sum_{i=1}^{l} \bigl(u(-1)\bigr)^{2i}.$$

Since the minor M_{11} and M_{ll} of $A_l A_l^{\mathrm{T}}$ equals $A_{l-1} A_{l-1}^{\mathrm{T}}$ we obtain by Cramers rule for the elements $(1,1)$ and (l,l) of the inverse of $A_{l-1} A_{l-1}^{\mathrm{T}}$

$$(A_l A_l^{\mathrm{T}})^{-1}(1,1) = (A_l A_l^{\mathrm{T}})^{-1}(l,l) = \frac{s_{l-1}}{s_l}. \tag{11}$$

Proof of Proposition 3.6. Let $X := (\omega_{-m-1+k})_{k=1}^{l+m+2} \in \mathbb{R}^{m+n+2}$, $a = (a_i)_{i=1}^{l+m+2} \in \mathbb{R}^{l+m+2}$ the vector with coefficients $a_{m+1} = 1$, $a_{m+2} = u(-1)$ and zero otherwise. Let us further define the block-matrix

$$B = \begin{pmatrix} A_m & 0 \\ 0 & A_l \end{pmatrix} \in \mathbb{R}^{(m+l)\times(m+l+2)}.$$

Notice that $Y := a \cdot X = \eta_0$,

$$A_m \omega_{[-m,0]} = (\eta_{-m+k-1})_{k=1}^{m}, \quad \text{and} \quad A_l \omega_{[1,l+1]} = (\eta_k)_{k=1}^{l},$$

where $\omega_{[-m,0]} = (\omega_{-m+k-1})_{k=1}^{m+1}$ and $\omega_{[1,l+1]} = (\omega_k)_{k=1}^{l+1}$. Hence $W := BX$ is the $m+l$-dimensional vector containing the potentials η_k, $k \in \{-m,\dots,l\} \setminus \{0\}$. Notice that Y and W have mean zero, since X has mean zero. We apply Proposition 3.8 with these choices of X, Y and W, and obtain that the distribution of η_0 conditioned on $(\eta_{-m+k-1})_{k=1}^{m} = v^-$ and $(\eta_k)_{k=1}^{l} = v^+$ is Gaussian with mean

$$m = \operatorname{cov}(Y,W)\operatorname{cov}(W,W)^{-1} v$$

and variance

$$\gamma = \operatorname{cov}(Y,Y) - \operatorname{cov}(Y,W)\operatorname{cov}(w,w)^{-1}\operatorname{cov}(W,Y),$$

where $v = (v^-, v^+)^{\mathrm{T}}$. It is straightforward to calculate $\operatorname{cov}(Y,Y) = \sigma^2(1+u(-1)^2)$ and $\operatorname{cov}(W,Y) = z = (z^-, z^+)^{\mathrm{T}}$, where $z^- = (0,\dots,0,\sigma^2 u(-1))^{\mathrm{T}} \in \mathbb{R}^m$ and $z^+ = (\sigma^2 u(-1), 0, \dots, 0)^{\mathrm{T}} \in \mathbb{R}^l$. We also have

$$\operatorname{cov}(W,W) = \sigma^2 \begin{pmatrix} A_m A_m^{\mathrm{T}} & 0 \\ 0 & A_l A_l^{\mathrm{T}} \end{pmatrix}.$$

Hence by Eq. (11)

$$\begin{aligned}
\gamma &= \sigma^2(1+u(-1)^2) - \sigma^{-2} z^{\mathrm{T}} \begin{pmatrix} (A_m A_m^{\mathrm{T}})^{-1} & 0 \\ 0 & (A_l A_l^{\mathrm{T}})^{-1} \end{pmatrix}^{-1} z \\
&= \sigma^2(1+u(-1)^2) - \sigma^{-2}\left[\sigma^4 u^2(-1)\frac{s_{m-1}}{s_m} + \sigma^4 u^2(-1)\frac{s_{l-1}}{s_l}\right] \\
&= \sigma^2(1+u(-1)^2) - \sigma^2\left(1 - \frac{1}{s_m}\right) - \sigma^2\left(1 - \frac{1}{s_m}\right),
\end{aligned}$$

and

$$m = \bigl[{z^-}^{\mathrm{T}}(\sigma^2 A_m A_m^{\mathrm{T}})^{-1} v^- + {z^+}^{\mathrm{T}}(\sigma^2 A_l A_l^{\mathrm{T}})^{-1} v^+\bigr].$$

This proves the statement of the proposition. □

The case of Proposition 3.6 where either m or l equals zero can be proven analogously and is indeed contained in the statement of Proposition 3.6 in the sense that $s_0 = 1$. However, to avoid confusion let us reformulate the case $m = 0$.

Proposition 3.9. *Let Assumption (A) be satisfied, $d = 1$, $l \geq 1$, $\Theta = \{-1, 0\}$, $u(0) = 1$ and ρ be the Gaussian density with mean zero and variance σ^2. Let further $v \in \mathbb{R}^l$. Then the distribution of η_0 conditioned on $(\eta_k)_{k=1}^l = v$ is Gaussian with variance*

$$\gamma = \sigma^2\Big(u(-1)^2 + \frac{1}{s_l}\Big) \quad \text{and mean} \quad m = u(-1)\sum_{i=1}^{l}(A_l A_l^{\mathrm{T}})^{-1}(1,i)\, v_i.$$

3.3. **Proof of Theorem 3.2.**

Proposition 3.10. *Let Assumption (A) be satisfied, $d = 1$ and either*

(a) $\Theta = \{0, \dots, n-1\}$ *for some* $n \in \mathbb{N}$, $\sup\operatorname{supp}\mu = 1$, $\inf\operatorname{supp}\mu = 0$, *$m$ be as in Lemma 3.4, $\varepsilon > 0$ and $a = m - \varepsilon/2$,* ***or***

(b) $\Theta = \{-1, 0\}$, $u(0) = 1$, $|u(1)|^2 = 1$, *μ be the normal distribution with mean zero and variance σ^2, $\varepsilon > 0$ and $a = -\varepsilon/2$.*

Then,

$$\operatorname*{ess\,sup}_{\eta_0^\perp \in Z_0^\perp} g_0^{\varepsilon,a}(\eta_0^\perp) = 1.$$

Proof. Assume the converse, i.e. $b := \operatorname{ess\,sup}_{\eta_0^\perp} g_0^{\varepsilon,a}(\eta_0^\perp) < 1$. By definition of the conditional expectation we have for all $B \in \sigma(\eta_0^\perp)$ that

$$\mathbb{E}\big(\mathbf{1}_B \mathbf{1}_{\{\eta_0 \in [a, a+\varepsilon]\}}\big) = \mathbb{E}\big(\mathbf{1}_B Y_0^{\varepsilon,a}\big). \tag{12}$$

Let $l \in \mathbb{N}$, s^+ and c be as in Lemma 3.4, and choose

$$B = \begin{cases} \{\omega \in \Omega \colon \eta_{-1}, \eta_{n-1} \in [s^+ - \varepsilon/(2c), s^+]\} & \text{if (a) is satisfied} \\ \{\omega \in \Omega \colon \eta_k = 0, k \in \{-l, \dots, l\} \setminus \{0\}\} & \text{if (b) is satisfied} \end{cases}$$

which is $\sigma(\eta_0^\perp)$-measurable. Lemma 3.4 and Proposition 3.6 tells us that the left hand side of Eq. (12) equals

$$\mathbb{P}(B \cap \{\eta_0 \in [a, a+\varepsilon]\}) = \begin{cases} 1 \cdot \mathbb{P}(B) & \text{if (a) is satisfied,} \\ \mathcal{N}_{0,\gamma}([a, a+\varepsilon]) \cdot \mathbb{P}(B) & \text{if (b) is satisfied,} \end{cases}$$

where $\gamma = \sigma^2(2/l)$. Here, $\mathcal{N}_{0,\gamma}$ denotes the normal distribution with mean zero and variance γ. Now we choose l large enough, such that $\mathcal{N}_{0,\gamma}([a, a+\varepsilon]) > b$. For the right hand side of Eq. (12) we use the factorized version (3) of $Y_0^{\varepsilon,a}$ and obtain by substitution

$$\mathbb{E}\big(\mathbf{1}_B Y_0^{\varepsilon,a}\big) = \int_{Z_0^\perp} \mathbf{1}_{B'}(\eta_0^\perp) g_0^{\varepsilon,a}(\eta_0^\perp) \mathrm{d}\mathbb{P}_{\eta_0^\perp}(\eta_0^\perp),$$

where

$$B' = \{\eta_0^\perp \in Z_0^\perp \colon \eta_{-1}, \eta_{n-1} \in [s^+ - \varepsilon/(2c), s^+]\}.$$

Since $b < 1$ by our assumption we obtain

$$\mathbb{E}(\mathbf{1}_B Y_0^{\varepsilon,a}) \leq b\mathbb{P}_{\eta_0^\perp}(B') = b\mathbb{P}(B) < \begin{cases} 1 \cdot \mathbb{P}(B) & \text{if (a) is satisfied,} \\ \mathcal{N}_{0,\gamma}([a, a+\varepsilon]) \cdot \mathbb{P}(B) & \text{if (b) is satisfied.} \end{cases}$$

This is a contradiction to Eq. (12). □

Proof of Theorem 3.2. The first equality (4) follows from translation invariance and Proposition 3.10. For the second statement (5) we use the pointwise inequality $g_0^\varepsilon(\eta_0^\perp) \geq g_0^{\varepsilon,a}(\eta_0^\perp)$. If we take first the essential supremum with respect to $\eta_0^\perp$ and then supremum with respect to a on both sides, we obtain using Proposition 3.10

$$\operatorname*{ess\,sup}_{\eta_0^\perp \in Z_0^\perp} g_0^\varepsilon(\eta_0^\perp) \geq 1.$$

The result now follows by translation invariance. □

4. How regularity properties turn into regularity of spectral data

Throughout Section 4, 5 and 6 we assume that Assumption (A) is satisfied, i.e. the random field $\eta_k : (\Omega, \mathcal{A}) \to (R, \mathcal{B}(\mathbb{R}))$, $k \in \mathbb{Z}^d$, is the discrete alloy-type potential given in Eq. (2). Next we list several additional regularity assumptions which may hold or not hold. All of them can be interpreted as assumptions of the distribution of the stochastic process $\eta_m, m \in \mathbb{Z}^d$.

Assumption (C). The measure μ has compact support, a probability density $\rho \in W^{1,1}(\mathbb{R})$, Θ is finite and the single-site potential satisfies $\overline{u} = \sum_{k \in \mathbb{Z}^d} u(k) > 0$.

Assumption (D). Θ is a finite set, the measure μ has bounded support and a probability density $\rho \in L^\infty(\mathbb{R})$, and the function u satisfies $u(k) > 0$ for all $k \in \partial^{\mathrm{i}} \Theta := \{k \in \Theta \mid k \text{ has less than } 2d \text{ neighbors in } \Theta\}$.

Assumption (E). The measure μ has bounded support and a probability density $\rho \in \mathrm{BV}(\mathbb{R})$ and there are constants $C, \alpha > 0$ such that for all $k \in \mathbb{Z}^d$ we have $|u(k)| \leq C \mathrm{e}^{-\alpha \|k\|_1}$.

If Assumption (E) is satisfied, we define a constant N as follows. For $\delta \in (0, 1 - \mathrm{e}^{-\alpha})$ we consider the to u associated generating function $F : D_\delta \subset \mathbb{C}^d \to \mathbb{C}$,

$$D_\delta = \{z \in \mathbb{C}^d : |z_1 - 1| < \delta, \dots, |z_d - 1| < \delta\}, \quad F(z) = \sum_{k \in \mathbb{Z}^d} u(-k) z^k.$$

Notice that the sum $\sum_{k \in \mathbb{Z}^d} u(-k) z^k$ is normally convergent in D_δ by our choice of δ and the exponential decay condition of Assumption (E). By Weierstrass' theorem, F is a holomorphic function. Since F is holomorphic and not identically zero, we have $(D_z^I F)(\mathbf{1}) \neq 0$ for at least one $I \in \mathbb{N}_0^d$. Therefore, there exists a multi-index $I_0 \in \mathbb{N}_0^d$ (not necessarily unique), such that we have

$$(D_z^I F)(\mathbf{1}) = \begin{cases} c_u \neq 0, & \text{if } I = I_0, \\ 0, & \text{if } I < I_0. \end{cases} \tag{13}$$

Such a I_0 can be found by diagonal inspection: Let $n \geq 0$ be the largest integer such that $D_z^I F(\mathbf{1}) = 0$ for all $\|I\|_1 < n$. Then choose a multi-index $I_0 \in \mathbb{N}_0^d$, $|I_0|_1 = n$ with $(D_z^{I_0} F)(\mathbf{1}) \neq 0$. We finally set $N = |I_0|_1$.

Assumption (F). Assume that Θ is a finite set, the Fourier transform $\hat{u} \colon [0, 2\pi)^d \to \mathbb{C}$ of u, i.e.

$$\hat{u}(\theta) = \sum_{k \in \mathbb{Z}^d} u(k) \mathrm{e}^{\mathrm{i} k \cdot \theta},$$

does not vanish, and that the measure μ has bounded support and a density $\rho \in W^{2,1}(\mathbb{R})$.

If Assumption (F) is satisfied, we define the constant C_u as follows. Let $A : \ell^1(\mathbb{Z}^d) \to \ell^1(\mathbb{Z}^d)$ be the linear operator whose coefficients in the canonical orthonormal basis are given by $A(j,k) = u(j-k)$ for $j,k \in \mathbb{Z}^d$. Since u has compact support, the operator A is bounded. If $\hat{u}$ does not vanish (as required by Assumption (F)), the operator A has a bounded inverse by the so-called $1/f$-Theorem of Wiener and we have

$$C_u := \|A^{-1}\|_1 < \infty, \tag{14}$$

see [Ves10a] for details.

We list several results on the regularity of spectral data under (some of) the above conditions on the stochastic process defining the random potential. While these results by themselves are probabilistic statements, describing the regularity of push-forward (or image) measures, they are of crucial importance for the study of spectral properties of random Schrödinger operators. This is described explicitly in the subsequent Section 5.

The first result concerns the uniform boundedness of the average of a fractional power of the Green function. It is sometimes called a-priori bound of the fractional moment method.

Theorem 4.1 ([ETV11])**.** *Let $\Lambda \subset \mathbb{Z}^d$ finite, $s \in (0,1)$ and Assumption (C) be satisfied. Then we have for all $x, y \in \Lambda$ and $z \in \mathbb{C} \setminus \mathbb{R}$*

$$\mathbb{E}\left(|G_{\omega,\Lambda}(z;x,y)|^s\right) \leq \frac{8}{(\overline{u})^s} \frac{s^{-s}}{1-s} \|\rho'\|_{L^1}^s C^s \frac{1}{\lambda^s},$$

where

$$C = \left(\frac{\mathrm{e}^c + 1}{\mathrm{e}^c - 1}\right)^d \quad \text{and} \quad c = \frac{1}{\operatorname{diam}\Theta} \ln\left(1 + \frac{\overline{u}}{2\|u\|_{\ell^1}}\right).$$

While the last statement is uniform in the lattice points x, y it does not exploit the intuition that the Green function should decay, as the $|x-y|$ grows. Such a statement is given, for discrete alloy-type models, in the following theorem. It is the core of the fractional moment method.

Theorem 4.2 ([ETV11])**.** *Let $\Gamma \subset \mathbb{Z}^d$, $s \in (0, 1/3)$ and suppose that Assumption (D) is satisfied. Then for a sufficiently large λ there are constants $C, m \in (0,\infty)$, depending only on d, ρ, u, s and λ, such that for all $z \in \mathbb{C} \setminus \mathbb{R}$ and all $x, y \in \Gamma$*

$$\mathbb{E}\left(|G_{\omega,\Gamma}(z;x,y)|^{s/2|\Theta|}\right) \leq C\mathrm{e}^{-m|x-y|}. \tag{15}$$

As we will comment below, exponential bounds of the type (15) allow to conclude spectral localization.

For $L > 0$ we denote by $\Lambda_L = \{y \in \mathbb{Z}^d \colon |y|_\infty \leq L\}$ the cube centered at the origin. The following result is a bound on the expected number of eigenvalues in a given energy region, for a finite cube random Hamiltonian.

Theorem 4.3 ([PTV11])**.** *Let Assumption (E) be satisfied and $\lambda > 0$. Then there exists $C_{\mathrm{W}} > 0$ depending only on u, such that for any $L > 0$ and any bounded interval $I \subset \mathbb{R}$*

$$\mathbb{E}\left(\operatorname{Tr} \chi_I(H_{\omega,\Lambda_L})\right) \leq \lambda^{-1} C_{\mathrm{W}} \|\rho\|_{\mathrm{Var}} |I| (2l+1)^{2d+N},$$

where N is defined by Eq. (13)*.*

The above inequality shows the Lipschitz-continuity of the distribution function $E \mapsto \mathbb{E}\left(\operatorname{Tr} \chi_{(-\infty,E]}(H_{\omega,\Lambda_L})\right)$. Hence, we have result about regularity of spectral data.

The following theorem is a so-called Minami-estimate. It generalizes the key estimate in [Min96]. Minami's result applies to the standard Anderson model, while our theorem concerns also certain correlated random potentials.

Theorem 4.4 ([TV13a])**.** *Let $\Lambda \subset \mathbb{Z}^d$ be finite and Assumption (F) satisfied. Then we have for all $x, y \in \Lambda$ with $x \neq y$, all $z \in \mathbb{C}$ with $\Im z > 0$ and all $\lambda > 0$*

$$\mathbb{E}\left(\det\left\{\Im\begin{pmatrix} G_{\omega,\Lambda}(z;x,x) & G_{\omega,\Lambda}(z;x,y) \\ G_{\omega,\Lambda}(z;y,x) & G_{\omega,\Lambda}(z;y,y)\end{pmatrix}\right\}\right) \leq \left(\frac{\pi}{\lambda}\right)^2 C_{\mathrm{Min}},$$

where

$$C_{\mathrm{Min}} = \frac{C_u^2}{4}\max\{\|\rho'\|_1^2, \|\rho''\|_1\}$$

and C_u is the constant from Eq. (14)*.*

Minami's estimate has an important corollary, a bound on the probability of finding at least two eigenvalues of $H_{\omega,\Lambda}$ in a given energy interval.

Corollary 4.5. *Let Assumption (F) be satisfied, $\Lambda \subset \mathbb{Z}^d$ finite and $I \subset \mathbb{R}$ be a bounded interval. Then we have for all $\lambda > 0$*

$$\begin{aligned} \mathbb{P}\big\{\operatorname{Tr}\chi_I(H_{\omega,\Lambda}) \geq 2\big\} &\leq \frac{1}{2}\mathbb{E}\big((\operatorname{Tr}\chi_I(H_{\omega,\Lambda}))^2 - \operatorname{Tr}\chi_I(H_{\omega,\Lambda})\big) \\ &\leq \frac{1}{2}\left(\frac{\pi}{\lambda}\right)^2 C_{\mathrm{Min}}|I|^2|\Lambda|^2. \end{aligned} \tag{16}$$

Thus we can control the probability that two eigenvalues fall close to each other: Again a regularity statement for spectral data.

5. Physical implications of the regularity of spectral data

One motivation for proving the regularity results of spectral data is that they are the main ingredient for localization proofs. There are different signatures of localization. We discuss two of them: spectral localization and Poisson statistics. Spectral localization or Anderson localization is the phenomenon that there are energy intervals I such that for almost all configurations of the randomness, the spectrum of H_ω consists only of eigenvalues.

Definition 5.1. Let $I \subset \mathbb{R}$. We say that H_ω exhibits exponential localization in I if, for almost all $\omega \in \Omega$, $\sigma_{\mathrm{c}}(H_\omega) \cap I = \emptyset$ and the eigenfunctions corresponding to the eigenvalues of H_ω in I decay exponentially. If $I = \mathbb{R}$ we simply say that H_ω exhibits exponential localization.

Beside this spectral interpretation of localization there are also interpretations from the dynamical point of view. Since we put our focus here on spectral localization we do not give a definition here. However, for the discussion of various notions of dynamical localization we refer to [Kle08].

In space dimension $d > 1$ there are exactly two methods to prove localization: the multiscale analysis [FS83, FMSS85] and the fractional moment method [AM93]. The output of the fractional moment method is the exponential decay of an averaged fractional power of the Green function, i.e. an inequality of the form (15). There is a variety of methods for concluding localization from this so-called fractional moment bound (15), for example using the Simon-Wolff criterion [AM93], via the RAGE-theorem [Gra94, Hun00], using a method called eigenfunction correlators [AEN+06], or going the way via the output of multiscale analysis [ETV10, ETV11]. In this sense, it is not surprising that Theorem 4.2 yields the following localization result.

Theorem 5.2 ([ETV11])**.** *Let Assumption (D) be satisfied and λ sufficiently large. Then, for almost all $\omega \in \Omega$, H_ω exhibits exponential localization.*

The multiscale ananysis is an induction argument which shows the exponential decay of the Green function with high probability on larger and larger scales. The induction anchor is the so-called initial length scale estimate. The main ingredient for the induction step is a Wegner estimate, which is formulated for our specific model in Theorem 4.3. Since the initial length scale estimate follows from a Wegner estimate in the case of large disorder, we obtain the following improvement of Theorem 5.2.

Theorem 5.3 ([LPTV13])**.** *Let Assumption (E) be satisfied and λ be sufficiently large. Then, for almost all $\omega \in \Omega$, H_ω exhibits exponential localization.*

Due to the lack of monotonicity it is not possible by standard methods to obtain an initial length scale estimate in the case of small disorder $\lambda > 0$ under the general Assumption (E). However, if the single-site potential has only a small negative part, it is possible to deduce an initial length scale estimate at the bottom of the spectrum by using perturbative arguments. This has been implemented for compactly supported single-site potentials in the continuous setting in [Ves02] and adapted to exponentially decaying (not compactly supported) single-site potentials in the discrete setting in [LPTV13]. Together with the Wegner estimate from Theorem 4.3 one obtains localization via multiscale analysis in the weak disorder regime.

Assumption (G). We say that Assumption (G) is satisfied for $\delta > 0$, if there exists a decomposition $u = u_+ - \delta u_-$ with $u_+, u_- \in \ell^1(\mathbb{Z}^d; \mathbb{R}_0^+)$, and $\|u_-\|_1 \leq 1$. For the measure μ we assume $\operatorname{supp} \mu = [0, \omega_+]$ for some $\omega_+ > 0$.

Theorem 5.4 ([LPTV13])**.** *Let Assumption (E) and (C) be satisfied and $\lambda > 0$. Then there exists $\delta > 0$ and $\varepsilon > 0$, such that if Assumption (G) is satisfied for δ, then, for almost all $\omega \in \Omega$, H_ω exhibits exponential localization in $[-\varepsilon, \varepsilon]$.*

See also [CE12] for a more general result in three space dimensions.

Another signature of localization is Poisson statistics. Physicists expect that there is no level repulsion of energy levels in the localized regime. This manifests itself in the sense that the point process associated to the rescaled eigenvalues of H_{ω,Λ_L} converges to a Poisson process.

To be more precise, we introduce all the basic definitions. Let $L \in \mathbb{N}$ and and $E_1^\omega(\Lambda_L) \leq E_2^\omega(\Lambda_L) \leq \ldots \leq E_{|\Lambda_L|}^\omega(\Lambda_L)$ be the eigenvalues of H_{ω,Λ_L} repeated according to multiplicity. Since $(H_\omega)_\omega$ is an ergodic family of random operators, the IDS exists as a (non-random) distribution function $N : \mathbb{R} \to [0, 1]$, satisfying for almost all $\omega \in \Omega$

$$N(E) = \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \#\{j \in \mathbb{N} \colon E_j^\omega(\Lambda_L) \leq E\},$$

at all continuity points of N. In particular, if Assumption (F) is satisfied, the IDS is known to be Lipschitz continuous [Ves10a]. Let us now introduce a second hypothesis which may be interpreted as a quantitative growth condition on the IDS or a positivity assumption on the density of states measure.

Assumption (H). Let $E_0 \in \mathbb{R}$ and $\kappa \geq 0$. We say that Assumption (Pos) is satisfied for E_0 and κ if for all $a < b$ there exists $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there holds

$$|N(E_0 + a\varepsilon) - N(E_0 + b\varepsilon)| \geq C\varepsilon^{1+\kappa}.$$

For $E_0 \in \mathbb{R}$ we consider the rescaled spectrum $\xi^\omega = (\xi_j^\omega)_{j=1}^{|\Lambda_L|}$, defined by

$$\xi_j^\omega = \xi_j^\omega(L, E_0) = |\Lambda_L| \left(N(E_j^\omega(\Lambda_L)) - N(E_0) \right), \quad j = 1, \dots, |\Lambda_L|, \tag{17}$$

and the associated point process $\Xi : \Omega \to \mathcal{M}_\mathrm{p}$ given by

$$\Xi^\omega = \Xi^\omega_{L,E_0} = \sum_{j=1}^{|\Lambda_L|} \delta_{\xi_j^\omega}, \tag{18}$$

where δ_x is the Dirac measure concentrated at x and $\mathcal{M}_\mathrm{p}$ is the set of all integer valued Radon measures on $\mathbb{R}$. A point process Υ is called Poisson point process with intensity measure μ if

$$\mathbb{P}\bigl\{\omega \in \Omega \colon \Upsilon^\omega(A) = k\bigr\} = \mathrm{e}^{-\mu(A)} \frac{\mu(A)^k}{k!}, \quad k = 1, 2, \dots$$

holds for each bounded Borel set $A \in \mathcal{B}(\mathbb{R})$ and for disjoint sets $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$, the random variables $\Upsilon(A_1), \dots, \Upsilon(A_n)$ are independent. Let $\Upsilon_n : \Omega \to \mathcal{M}_\mathrm{p}$, $n \in \mathbb{N}$, be a sequence of point processes defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. This sequence is said to converge weakly to a point process $\Upsilon : \tilde{\Omega} \to \mathcal{M}_\mathrm{p}$ defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$, if and only if for any bounded continuous function $\phi : \mathcal{M}_\mathrm{p} \to \mathbb{R}$ there holds

$$\lim_{n\to\infty} \int_\Omega \phi(\Upsilon_n^\omega)\mathbb{P}(\mathrm{d}\omega) = \int_{\tilde{\Omega}} \phi(\Upsilon^{\tilde{\omega}})\tilde{\mathbb{P}}(\mathrm{d}\omega).$$

Finally, we introduce a characterization for a region of localization. We refer to [TV13a] for a discussion of the validity of Assumption (I). Roughly speaking, it is satisfied whenever one of the Theorems 5.2, 5.3, or 5.4 holds.

Assumption (I). Let $I \subset \mathbb{R}$. We assume that for all $E \in I$ there exists $\Theta > 3d - 1$ such that

$$\limsup_{L\to\infty} \mathbb{P}\left\{\forall x, y \in \Lambda_L,\ |x - y|_\infty \geq \frac{L}{2} : |G_{\omega,\Lambda_L}(E; x, y)| \leq L^{-\Theta}\right\} = 1.$$

Let Σ denote the almost sure spectrum of the (ergodic) family of operators H_ω, $\omega \in \Omega$.

Theorem 5.5 ([TV13a])**.** *Let Assumption (F) be satisfied, $I \subset \Sigma$ be a bounded interval and $E_0 \in I$. Assume that Assumption (I) is satisfied in I and Assumption (H) is satisfied for E_0 and some $\kappa \in [0, 1/(1 + d))$.*

Then the point process Ξ, defined in Eq. (18)*, converges for $L \to \infty$ weakly to a Poisson process on $\mathbb{R}$ with Lebesgue measure as the intensity measure.*

The result of Theorem 5.5 follows from Minami's estimate (formulated in Theorem 4.4) by using an abstract result from [GK12]. Roughly speaking, the criterion of [GK12] states, that for a large class of discrete random Schrödinger operators Minami's estimate and a Wegner estimate implies Poisson statistics in any region of localization.

6. Reverse Hölder inequality and fractional moments

In this section we review a result of Elgart Shamis and Sodin [ESS12]. They apply the fractional moment method for a large class of discrete alloy-type models. The main new ingredient in comparison to other proofs via the fractional moment method is an estimate on the integral of a fractional power of a rational function, respectively, an iterated version thereof.

We would like to point out that the estimate which is effectively used in [ESS12], is a reverse Hölder inequality. Such inequalities play an important role in harmonic analysis, e.g. in the theory of Muckenhoupt weights. In this section we modify the method of [ESS12] for the discrete alloy-type model at large disorder without the use of the iterated version of the reverse Hölder inequality.

As mentioned before, we assume throughout this section that Assumption (A) is satisfied. First we state additional regularity assumptions for the model, see [ESS12].

Assumption (J). There exists $\alpha \in (0,1]$ and $C_1 > 0$, such that $\mu([t-\varepsilon, t+\varepsilon]) \leq C_1 \varepsilon^\alpha$ for all $\varepsilon > 0$ and $t \in \mathbb{R}$.

Assumption (K). μ has a finite q-moment, i.e. there exists $q > 0$ and a constant $C_2 > 0$ such that $\int |x^q| \mu(\mathrm{d}x) \leq C_2$.

Assumption (L). We assume that Θ is a finite set and that $0 \in \Theta$.

The next lemma provides the usual boundedness of fractional moments.

Lemma 6.1. *Let Assumption (J) be satisfied and $s \in (0, \alpha)$. Then we have for all $b \in \mathbb{C}$*

$$\int_{\mathbb{R}} \frac{1}{|x-b|^s} \mu(\mathrm{d}x) \leq C_1^{s/\alpha} \frac{\alpha}{\alpha - s}$$

Proof. We assume $b \in \mathbb{R}$, if $b \notin \mathbb{R}$ we estimate the integrand by replacing b by its real part. Layer Cake gives us

$$I := \int_{\mathbb{R}} \frac{1}{|x-b|^s} \mu(\mathrm{d}x) = \int_0^\infty \mu(\{x \in \mathbb{R} \colon |x-b|^{-s} > t\}) \mathrm{d}t.$$

We split the domain of integration for some $\kappa > 0$ according to $[0,\infty) = [0,\kappa) \cup [\kappa, \infty)$ and obtain

$$I = \int_0^\kappa \mu(\{x \in \mathbb{R} \colon |x-b|^{-s} > t\}) \mathrm{d}t + \int_\kappa^\infty \mu(\{x \in \mathbb{R} \colon |x-b|^{-s} > t\}) \mathrm{d}t.$$

Since μ is a probability measure, we can estimate the first integral by κ. For the second integral we get due to Assumption (J)

$$\begin{aligned} \int_\kappa^\infty \mu(\{x \in \mathbb{R} \colon |x-b|^{-s} > t\}) \mathrm{d}t &= \int_\kappa^\infty \mu([b - t^{-1/s}, b + t^{-1/s}]) \mathrm{d}t \\ &\leq \int_\kappa^\infty C_1 t^{-\alpha/s} \mathrm{d}t \\ &= C_1 \frac{s}{\alpha - s} \kappa^{-\alpha/s + 1}. \end{aligned}$$

If we choose $\kappa = C_1^{s/\alpha}$ we obtain the statement of the lemma. □

The main new idea of [ESS12], formulated there in Proposition 3.1, implies the following reverse Hölder inequality.

Proposition 6.2 ([ESS12]). *Let Assumptions (J) and (K) be satisfied, Q_1 and Q_2 be two polynomials of degree smaller or equal k, and $s \in (0, q\alpha/\min\{k(4\alpha+q), \alpha/(2k)\})$. Then there is a constant $\tilde C = \tilde C(\alpha, q, k, s, C_1, C_2)$ such that*

$$\Bigl(\int_{\mathbb{R}} \frac{|Q_1(x)|^{2s}}{|Q_2(x)|^{2s}} \mathrm{d}\mu(x)\Bigr)^{1/2} \leq \tilde C \int_{\mathbb{R}} \frac{|Q_1(x)|^{s}}{|Q_2(x)|^{s}} \mathrm{d}\mu(x)$$

The next statement is contained in [ESS12] as well. We give a short, direct proof, for discrete alloy-type models, which makes use of Lemma 6.1 and Proposition 6.2 only.

Theorem 6.3. *Let Assumption (J), (K) and (L) be satisfied, $s \in (0, q\alpha/\min\{|\Theta|(4\alpha+q), \alpha/(2k)\})$ and $\Lambda \subset \mathbb{Z}^d$. Then there is a constant $C = C(\alpha, q, |\Theta|, s, C_1, C_2, u(0))$, such that for all $\lambda > 0$, $E \in \mathbb{R}$ and $x, y \in \Lambda$ with $x \neq y$*

$$\mathbb{E}(|G_{\omega,\Lambda}(E;x,y)|^s) \leq \frac{C}{\lambda^s} \sum_{|e|=1} \mathbb{E}(|G_{\omega,\Lambda}(E;x,y+e)|^s). \tag{19}$$

Here we use the convention that $G_{\omega,\Lambda}(E;x,y) = 0$ if $x \notin \Lambda$ or $y \notin \Lambda$. Moreover, we note that the set of $\omega \in \Omega$ such that $E \in \mathbb{R}$ is in the spectrum of $H_{\omega,\Lambda}$ has $\mathbb{P}$-measure zero. This justifies to deal with real energies.

Proof of Theorem 6.3. By definition of $G_{\omega,\Lambda}(E)$ we have for $x \neq y$

$$\begin{aligned} 0 = \langle \delta_x, G_{\omega,\Lambda}(E)(H_{\omega,\Lambda} - E)\delta_y\rangle &= \sum_{i\in\Lambda} G_{\omega,\Lambda}(E;x,i)\langle \delta_i, (H_{\omega,\Lambda} - E)\delta_y\rangle \\ &= -\sum_{|e|=1} G_{\omega,\Lambda}(E;x,y+e) + (\lambda V_\omega(y) - E) G_{\omega,\Lambda}(E;x,y) \end{aligned}$$

Hence,

$$|V_\omega(y) - E/\lambda|^s |G_{\omega,\Lambda}(E;x,y)|^s \leq \frac{1}{\lambda^s} \sum_{|e|=1} |G_{\omega,\Lambda}(E;x,y)|^s.$$

Next we provide a lover bound on the expectation of the left hand side. By Cauchy Schwarz we have for all $k \in \mathbb{Z}^d$

$$\begin{aligned} &\mathbb{E}_{\{k\}}\bigl(|G_{\omega,\Lambda}(E;x,y)|^{s/2}\bigr)^2 \\ &\qquad \leq \mathbb{E}_{\{k\}}\bigl(|G_{\omega,\Lambda}(E;x,y)|^s |V_\omega(y) - E/\lambda|^s\bigr) \mathbb{E}_{\{k\}}\bigl(|V_\omega(y) - E/\lambda|^{-s}\bigr). \end{aligned}$$

Here $\mathbb{E}_{\{k\}}$ denotes the expectation with respect to the random variable ω_k, i.e. $\mathbb{E}_{\{k\}}(\cdot) = \int_{\mathbb{R}} (\cdot) \mu(\mathrm{d}\omega_k)$ By Lemma 6.1 (and since $0 \in \Theta$) we have

$$\mathbb{E}_{\{y\}}\bigl(|V_\omega(y) - E/\lambda|^{-s}\bigr) = \int_{\mathbb{R}} \frac{1}{|V_\omega(y) - E/\lambda|^s} \mu(\mathrm{d}\omega_y) \leq \frac{1}{|u(0)|^s} C_{\mathrm{A1}}^{s/\alpha} \frac{\alpha}{\alpha - s}.$$

Hence,

$$\mathbb{E}_{\{y\}}\bigl(|G_{\omega,\Lambda}(E;x,y)|^{s/2}\bigr)^2 \leq \frac{C_{\mathrm{A1}}^{s/\alpha}}{|u(0)|^s} \frac{\alpha}{\alpha - s} \mathbb{E}_{\{y\}}\bigl(|G_{\omega,\Lambda}(E;x,y)|^s |V_\omega(y) - E/\lambda|^s\bigr).$$

By Cramers rule, the Green function is a ratio of two polynomials. More precisely, we have

$$G_{\omega,\Lambda}(E;x,y) = \frac{\det C_{y,x}}{\det(H_{\omega,\Lambda} - E)},$$

where $C_{i,j} = (-1)^{i+j} M_{i,j}$, and where $M_{i,j}$ is obtained from the matrix $H_{\omega,\Lambda} - E$ by deleting row i and column j. Now we observe that both, the numerator and the denominator, are polynomials in ω_y of order $k \leq |\Theta|$. By Lemma 6.2 there is a constant $\tilde{C} = \tilde{C}(\alpha, q, |\Theta|, s, C_1, C_2)$ such that

$$\mathbb{E}_{\{y\}}\bigl(|G_{\omega,\Lambda}(E;x,y)|^s\bigr)^2 \geq \tilde{C}^{-1}\mathbb{E}_{\{y\}}\bigl(|G_{\omega,\Lambda}(E;x,y)|^{2s}\bigr)$$

Hence,

$$\mathbb{E}_{\{y\}}\bigl(|G_{\omega,\Lambda}(E;x,y)|^s\bigr) \leq \tilde{C}\frac{C_{\mathrm{A1}}^{s/\alpha}}{|u(0)|^s}\frac{\alpha}{\alpha - s}\mathbb{E}_{\{y\}}\bigl(|G_{\omega,\Lambda}(E;x,y)|^s|V_\omega(y) - E/\lambda|^s\bigr).$$

Putting everything together we obtain the statement of the theorem. □

Remark 6.4. The conclusion of Theorem 6.3 implies the fractional moment bound as in Ineq. (15), if $\mathbb{E}(|G_{\omega,\Lambda}(E;x,y)|^s)$ is uniformly bounded and λ is sufficiently large. This is elaborated e.g. in [Gra94]. So the question remains, whether

$$\sup_{\Lambda\subset\mathbb{Z}^d, x,y\in\Lambda} \mathbb{E}(|G_{\omega,\Lambda}(E;x,y)|^s) < \infty.$$

An elementary argument how to deduce this uniform bound from (19) in the large disorder regime, is given in Corollaries 2.2.1 and 2.2.2 of [ESS12]. Hence, exponential decay and localization follows.

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