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J. Harant, S. Richter, H. Sachs

Preprint 2013-14

Preprintreihe der Fakultät für Mathematik

ISSN 1614-8835

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Jochen Harant ², Sebastian Richter ¹, Horst Sachs ²

¹ Chemnitz University of Technology, Department of Mathematics, Germany

² Ilmenau University of Technology, Department of Mathematics, Germany

email: jochen.harant@tu-ilmenau.de, sebastian.richter@mathematik.tu-chemnitz.de,
horst.sachs@tu-ilmenau.de

Abstract. Let G be a simple, undirected, and connected graph on $n \geq 2$ vertices with eigenvalues $\lambda_1 \leq \dots \leq \lambda_n$. Motivated by research to upper bounds on independence, the following concept of graph packing is considered. Two vertex disjoint induced subgraphs H and H' of G are *independent* in G if G does not contain an edge between H and H' . Let $\alpha(G, H)$ be the maximum number of vertex disjoint copies of H contained in G as induced and pairwise independent subgraphs. Note that $\alpha(G, K_1)$ is the independence number of G . We present upper bounds on $\alpha(G, H)$ generalizing the result $\alpha(G, K_1) \leq \frac{-\lambda_1}{r-\lambda_1}n$ of A.J. Hoffman for an r -regular graph G . If G is an arbitrary graph with minimum degree δ , then W.H. Haemers extended Hoffman's inequality to $\alpha(G, K_1) \leq \frac{-\lambda_1 \lambda_n}{\delta^2 - \lambda_1 \lambda_n}n$. In case $H = K_1$, our bounds are not comparable with that one of W.H. Haemers. Furthermore, we generalize the result $\alpha(G, K_1) \leq \min\{|\{i|\lambda_i \leq 0\}|, |\{i|\lambda_i \geq 0\}|\}$ of D. M. Cvetković. The possibility to derive lower bounds on $\alpha(G, H)$ is discussed. The results are used to derive upper bounds also for the ordinary packing number, where the condition of independence of the copies of H is dropped.

Keywords. independence number, packing of graphs

1 Introduction

We consider two fixed finite, undirected, and simple* graphs: Let $G = (V, E)$ be a connected graph on at least two vertices, where $V = \{1, \dots, n\}$ and E (with $|E| = m$) denote the vertex set and the edge set of G , respectively. Furthermore, let H be a graph on $V(H)$ with edge set $E(H)$, where $|V(H)| = h$ and $|E(H)| = e$. For further notation and terminology we refer to [7].

Packing of graphs is a very popular topic in graph theory and several versions of the packing concept are investigated in the literature. An H -packing $\{H_1, \dots, H_p\}$ of G is a set of vertex disjoint (not necessarily induced) subgraphs H_1, \dots, H_p of G , each isomorphic to H ([13, 14, 15]). Thus, a K_1 -packing of G is just a subset of V and a K_2 -packing of G is a matching of G . An H -packing $\{H_1, \dots, H_p\}$ of G is *maximum* if it covers the greatest possible number of vertices of G , that is if p is maximum, and is *perfect* (or an H -factor), if it

*We remark that most of the results of the present paper can be generalized to arbitrary undirected graphs with weighted edges provided that the weighted degree of each vertex is positive and that the sum of the weights of all loops equals zero.

covers all vertices of G . In [15], also a stronger (and more natural) version of H -packing is considered: The H -packing $\{H_1, \dots, H_p\}$ is called *strict* if H_1, \dots, H_p are induced subgraphs of G . If H has a component on at least three vertices, then the *H -packing problem* (decide whether G admits an H -factor) is *NP*-complete for both packing versions ([14, 15]). On the other hand, when H has only components isomorphic to K_1 or K_2 , then a maximum H -packing can be found in polynomial time using the matching algorithm.

In the present paper, we are particularly interested in the case that the copies H_1, \dots, H_p of H are pairwise *independent*, that is G does not contain edges between them. This is motivated by the observation that if $H = K_1$, then the union of H_1, \dots, H_p forms an independent vertex set of G , hence, this *independent packing* is a generalization of the concept of independence in graphs. For our intention to derive upper bounds on the number p of packed copies H_1, \dots, H_p of H in terms of eigenvalues (or generalized eigenvalues) of G and of H , it is important that H_1, \dots, H_p are both induced and independent subgraphs of G . Thus, let $\alpha(G, H)$ denote the maximum number of vertex disjoint copies of H contained in G as induced and pairwise independent subgraphs. Since $\alpha(G, K_1)$ is the independence number of G , it is already *NP*-hard to find such a maximum *independent induced H -packing* if $H = K_1$.

Although H -packing and strict H -packing are more general concepts, there are options to reduce them to independent induced H -packing. The *subdivision graph* $S(G)$ of G is the graph obtained by inserting a new vertex into every edge of G . Let $L(G)$ denote the *line graph* of G . This is the graph whose vertex set is in one-to-one correspondence with the set of edges of G , with two vertices of $L(G)$ being adjacent if and only if the corresponding edges in G have an end vertex in common.

If $\{H_1, \dots, H_p\}$ is an H -packing or a strict H -packing of G , then it is easy to see that the graphs $S(H_1), \dots, S(H_p)$ and $L(H_1), \dots, L(H_p)$ are induced and pairwise independent subgraphs of $S(G)$ and $L(G)$, respectively. Hence, $\alpha(S(G), S(H)) \geq p$ and $\alpha(L(G), L(H)) \geq p$. Since we will derive upper bounds on $\alpha(G, H)$ in terms of eigenvalues of G and H , let us remark that there are results about connections between the sets of eigenvalues of G , $L(G)$, and $S(G)$ ([5]). Let A and E_n denote the adjacency matrix of G ([5]) and the $(n \times n)$ -identity-matrix, respectively. For example, if $c_G(\lambda) = |\lambda E_n - A|$ (c_G is the *characteristic polynomial* of G), then, for an r -regular graph G , we have $c_{L(G)}(\lambda) = (\lambda+2)^{\frac{(r-2)n}{2}} c_G(\lambda-r+2)$ and $c_{S(G)}(\lambda) = \lambda^{\frac{(r-2)n}{2}} c_G(\lambda^2 - r)$ (see Theorems 2.15 and 2.17 in [5]). Examples for this approach are given in Chapter 2.

A reduction of independent induced H -packing to the ordinary independence concept is also possible. Obviously, G contains at least $\alpha(G, H)\alpha(H, K_1)$ independent vertices, hence, the inequality $\alpha(G, H) \leq \frac{\alpha(G, K_1)}{\alpha(H, K_1)}$ leads to upper bounds on $\alpha(G, H)$. In Chapter 2, we give an example for this approach.

Consider the simple and undirected graph $G(H)$, where $V(G(H))$ is the set of all induced subgraphs U of G isomorphic to H . Furthermore, for $U, U' \in V(G(H))$, the edge (U, U')

belongs to the edge set $E(G(H))$ of $G(H)$ if and only if U and U' are not independent in G . Obviously, $\alpha(G, H) = \alpha(G(H), K_1)$.

It is clear that it is not easy to establish $G(H)$ for given G and H in general. If at least the degrees $d(U)$ (or bounds on them) for $U \in V(G(H))$ in $G(H)$ can be determined, then many results about lower bounds on $\alpha(G(H), K_1)$ in terms of the $d(U)$ can be found in the literature.

If the definition of $G(H)$ is slightly changed, then this approach also can be used for reducing H -packing and strict H -packing to their independence version.

Upper bounds on $\alpha(G, K_1)$ have barely been touched in the literature. To our knowledge, for an arbitrary graph H , except for relations such as $\alpha(G, H) \leq \frac{\alpha(G, K_1)}{\alpha(H, K_1)}$ the only known upper bound on $\alpha(G, H)$ is given in the following inequality (1). Surprisingly, a nontrivial lower bound on $\alpha(G, H)$ for general H is not known at all. For the rest of the paper, we deal with upper bounds on $\alpha(G, H)$ in terms of eigenvalues or generalized eigenvalues of G and H .

Counting the edges of \overline{G} , the complement of G , we arrive at $|E(\overline{G})| \geq \alpha(G, H)|E(\overline{H})| + \binom{\alpha(G, H)}{2}h^2$ or, equivalently,

$$\alpha(G, H) \leq \frac{2e + h + \sqrt{4h^2(n^2 - n - 2m) + (2e + h)^2}}{2h^2}. \quad (1)$$

In case $H = K_1$, these inequalities reduce to $|E(\overline{G})| \geq \binom{\alpha(G, K_1)}{2}$ and

$$\alpha(G, K_1) \leq \frac{1 + \sqrt{4(n^2 - n - 2m) + 1}}{2}. \quad (2)$$

Note that the adjacency matrix A of G is symmetric and let $\lambda_1 \leq \dots \leq \lambda_n$ denote the eigenvalues of A , also called the *eigenvalues of G* .

First of all, we will focus on the following three upper bounds on $\alpha(G, K_1)$, all involving eigenvalues of G . Starting point of our investigations is the following result of A. J. Hoffman ([2, 11, 16]).

If G is an r -regular graph, then

$$\alpha(G, K_1) \leq \frac{-\lambda_1}{r - \lambda_1}n. \quad (3)$$

Note that $\lambda_n = r$ if G is r -regular [2]. Let δ denote the minimum degree of G . In [10, 11], W.H. Haemers proved the following extension of Hoffman's result for arbitrary graphs:

$$\alpha(G, K_1) \leq \frac{-\lambda_1 \lambda_n}{\delta^2 - \lambda_1 \lambda_n}n. \quad (4)$$

If all eigenvalues of G are taken into consideration, then D. M. Cvetković [2, 4, 5] proved

$$\alpha(G, K_1) \leq \min\{|\{i \in \{1, \dots, n\} | \lambda_i \leq 0\}|, |\{i \in \{1, \dots, n\} | \lambda_i \geq 0\}|\}. \quad (5)$$

Section 2 starts with Theorem 1 presenting a necessary condition for a graph U to be an induced subgraph of G .

New upper bounds on $\alpha(G, H)$ depending on only very few parameters of G are established in Corollary 1, Theorem 2, and Theorem 3. In case $H = K_1$, these bounds are compared with that one in inequality (1) and with Haemers' bound (4).

Moreover, an extension of the result (5) of Cvetković is presented in Theorem 4.

Proofs are postponed to Section 3.

2 Upper Bounds on $\alpha(G, H)$

We start this section with some notation and definitions from linear algebra used for the rest of the paper.

In the sequel, $\underline{x} = (x_1, \dots, x_n)^T$ denotes a vector of R^n , where $\underline{1} = (1, \dots, 1)^T$ and $\underline{0} = (0, \dots, 0)^T$. Let B and C denote real symmetric square matrices with n rows, where C is positive definite. Define the inner product $\langle \underline{x}, \underline{y} \rangle_C = \underline{x}^T C \underline{y}$ for vectors $\underline{x}, \underline{y} \in R^n$ and let \underline{x} and \underline{y} be called *C-orthogonal* if $\langle \underline{x}, \underline{y} \rangle_C = 0$. If $\langle \underline{x}, \underline{x} \rangle_C = 1$ then \underline{x} is called *C-normal*. A set of *C-normal* vectors being pairwise *C-orthogonal* is a *C-orthonormal* set. We consider the generalized eigenvalue problem $B\underline{x} = \mu C\underline{x}$ for $\mu \in R$ and $\underline{x} \in R^n$ with $\underline{x} \neq \underline{0}$ (see [20]). If the pair (μ, \underline{x}) is a solution of this equation, then \underline{x} is a *C-eigenvector* of B and μ is the corresponding *C-eigenvalue* of B .

If A is the adjacency matrix of G , \underline{x} is a *C-eigenvector* of A , and μ is a *C-eigenvalue* of A , then we say that \underline{x} is a *C-eigenvector* of G and μ is a *C-eigenvalue* of G .

Let d_i denote the degree of vertex $i \in V$ in G and D be the *degree matrix* of G , that is a diagonal matrix, where d_i is the i -th element of the main diagonal. Recall that $d_i > 0$ for $i = 1, \dots, n$, since G is a connected graph on at least two vertices, hence, D is positive definite.

Exclusively, we will consider the cases $C = E_n$ and $C = D$. We write *eigenvalue* and *eigenvector* instead of E_n -*eigenvalue* and E_n -*eigenvector*, respectively. We remark that D -eigenvalues and D -eigenvectors of G are also called normalized eigenvalues and normalized eigenvectors ([18]).

Let $\mu_1 \leq \dots \leq \mu_n$ be the D -eigenvalues of G and H have the eigenvalues $\eta_1 \leq \dots \leq \eta_h$. Note that $\mu_n = 1$ and that $\eta_1 < 0$ and $\eta_h > 0$ if and only if H contains at least one edge ([2, 5, 8, 9, 17, 18]). Eventually, let Δ denote the maximum degree of G .

In [11], W.H. Haemers proved a necessary condition for a regular graph G to contain a copy of a given graph U as an induced subgraph. Our first result is Theorem 1 generalizing this result of Haemers to arbitrary graphs.

Theorem 1

If G contains an induced subgraph U of order n' and size m' , then

$$(1 - \mu_1)\delta^2 n'^2 + 2\mu_1 m \delta n' \leq 4mm'.$$

If U is the union of $\alpha(G, H)$ copies of H as pairwise independent subgraphs of G , then $n' = \alpha(G, H)h$ and $m' = \alpha(G, H)e$. From Theorem 1, we obtain

Corollary 1

$$\alpha(G, H) \leq \frac{4e - 2\mu_1 h \delta}{(1 - \mu_1)\delta^2 h^2} m. \quad (6)$$

Using specific inequalities (e.g. the Cauchy-Schwarz inequality), other upper bounds on $\alpha(G, H)$ can be derived. An example is the following

Theorem 2

$$\alpha(G, H) \leq \sqrt[3]{\frac{4e - 2\mu_1 \Delta h}{h^4} \cdot \frac{(\frac{n^2}{2m} + (\frac{1}{\delta} - \frac{1}{\Delta})(n - 1 - \frac{2m}{n}))^2}{1 - \mu_1}} \cdot m \quad (7)$$

and, if H is connected, then

$$\alpha(G, H) \leq \sqrt[3]{2(\eta_h - \mu_1 \Delta) \frac{(\frac{n^2}{2m} + (\frac{1}{\delta} - \frac{1}{\Delta})(n - 1 - \frac{2m}{n}))^2}{1 - \mu_1}} \cdot m \quad (8)$$

Note that $-r \leq \lambda \leq \lambda_n = r$ for all eigenvalues λ of an r -regular graph G [2, 5]. Moreover, if G is r -regular, then $2m = rn$, $\Delta = \delta = r$, and $D = rE_n$. Hence, in this case, μ is a D -eigenvalue of G if and only if $r\mu$ is an eigenvalue of G . Furthermore, $\mu_1 = -1$ if and only if G is bipartite [2].

If G is r -regular, then $\alpha(G, H) \leq \frac{(2e - \lambda_1 h)n}{(r - \lambda_1)h^2}$ by Corollary 1. This inequality can be improved as shown in Theorem 3. We remark that $\lambda_1 \leq \eta_1$ if H is an induced subgraph of G , i.e. if $\alpha(G, H) \geq 1$ (see Lemma 4 in Section 3).

Theorem 3

Let G be r -regular and A_H be the adjacency matrix of H .

If $\alpha(G, H) \geq 1$, then $(A_H - \lambda_1 E_h)\underline{x} = \underline{1}$ is solvable, and, for any solution \underline{x} of this equation,

$$\alpha(G, H) \leq \frac{n}{(r - \lambda_1)\underline{1}^T \underline{x}} \leq \frac{(2e - \lambda_1 h)n}{(r - \lambda_1)h^2}. \quad (9)$$

Moreover, if $\lambda_1 < \eta_1$, then $A_H - \lambda_1 E_h$ is regular and $\alpha(G, H) \leq \frac{n}{(r - \lambda_1)s}$, where s denotes the sum of all entries of $(A_H - \lambda_1 E_h)^{-1}$.

Recall that $A_H - \lambda_1 E_h$ is symmetric. If $(A_H - \lambda_1 E_h)x_1 = \underline{1}$ and $(A_H - \lambda_1 E_h)x_2 = \underline{1}$, then $\underline{1}^T x_1 = x_2^T (A_H - \lambda_1 E_h)x_1 = x_2^T \underline{1} = \underline{1}^T x_2$. Thus, in Theorem 3, the value $\underline{1}^T x$ is independent of the choice of the solution x .

If in Theorem 3, additionally, H is assumed to be ρ -regular, then $x = (\frac{1}{\rho - \lambda_1}, \dots, \frac{1}{\rho - \lambda_1})^T$ is a solution of $(A_H - \lambda_1 E_h)x = \underline{1}$, hence, $\frac{n}{(r - \lambda_1)\underline{1}^T x} = \frac{(\rho - \lambda_1)n}{(r - \lambda_1)h} = \frac{(2e - \lambda_1 h)n}{(r - \lambda_1)h^2} = \frac{4e - 2\mu_1 h \delta}{(1 - \mu_1)\delta^2 h^2} m$, hence, the upper bound $\frac{n}{(r - \lambda_1)\underline{1}^T x}$ on $\alpha(G, H)$ in Theorem 3 equals that one in Corollary 1.

For example, if we consider the nonregular graph $H = P_3$, then $\alpha(G, P_3) \geq 1$ implies $\lambda_1 \leq \eta_1 = -\sqrt{2}$. Moreover, if G is regular, then, by Theorem 3, $(A_{P_3} - \lambda_1 E_3)x = \underline{1}$ is solvable and it is easy to see that $\underline{1}^T x = \frac{3\lambda_1 + 4}{-\lambda_1^2 + 2}$ for a solution x . Thus, if G is r -regular and $\alpha(G, P_3) \geq 1$, then $\lambda_1 < -\sqrt{2}$ and $\alpha(G, P_3) \leq \frac{(-\lambda_1^2 + 2)n}{(r - \lambda_1)(3\lambda_1 + 4)}$ by Theorem 3. Each of the upper bounds on $\alpha(G, P_3)$ resulting from inequalities (6), (7), and (8) is weaker in this case.

Recall the inequality $\alpha(G, H) \leq \frac{\alpha(G, K_1)}{\alpha(H, K_1)}$. Let G be r -regular and non complete. Since G contains a subgraph isomorphic to P_3 , it follows $\lambda_1 \leq -\sqrt{2}$. Moreover, let H have at least one edge. It is well-known ([12]) that $\alpha(H, K_1) \geq \frac{h^2}{2e + h}$. Using inequality (3), we obtain $\alpha(G, H) \leq \frac{-\lambda_1 n(2e + h)}{(r - \lambda_1)h^2} = b_1$. However, the upper bound $b_2 = \frac{(2e - \lambda_1 h)n}{(r - \lambda_1)h^2}$ on $\alpha(G, H)$ by inequality (9) is stronger than b_1 since $\frac{b_1}{b_2} = \frac{-\lambda_1(2e + h)}{2e - \lambda_1 h} > 1$.

If $H = K_1$, then $h = 1$, $e = 0$, and $\eta_h = 0$, hence, (7) and (8) are equivalent in this case. Corollary 2 follows from (6) and (7). Inequality (9) is Hoffman's result (3) if $H = K_1$.

Corollary 2

$$\alpha(G, K_1) \leq \frac{-2\mu_1}{(1 - \mu_1)\delta} m \quad (10)$$

and

$$\alpha(G, K_1) \leq \sqrt[3]{\frac{(\frac{n^2}{2m} + (\frac{1}{\delta} - \frac{1}{\Delta})(n - 1 - \frac{2m}{n}))^2 (-2\mu_1 \Delta m)}{1 - \mu_1}}. \quad (11)$$

If G is r -regular, then Hoffman's result (3) is equivalent to inequality (10). Furthermore, (11) is weaker than (3) if G is r -regular, because $\frac{-\lambda_1}{r - \lambda_1} \leq \frac{1}{2}$.

If all eigenvalues of G are taken into consideration, then, by interlacing (see Lemma 4), the following interesting extension of Cvetkovic's result (inequality (5)) can be proved (note that $\eta_1 = 0$ if $H = K_1$).

Theorem 4

$\alpha(G, H) \leq \min\{\frac{1}{q}|\{i \in \{1, \dots, n\} | \lambda_i \leq \eta_q\}|, \frac{1}{h - q + 1}|\{i \in \{1, \dots, n\} | \lambda_i \geq \eta_q\}|\}$
for all $q \in \{1, \dots, h\}$.

To illustrate our results, we consider the Petersen graph Π of order 10 and size 15. Its eigenvalues are $(-2)^{(4)}$, $1^{(5)}$, $3^{(1)}$, where $\lambda^{(t)}$ means that the eigenvalue λ has multiplicity t , and $\mu_1 = -\frac{2}{3}$, because Π is cubic.

It follows $c_{L(\Pi)}(\lambda) = (\lambda + 2)^5 c_{\Pi}(\lambda - 1)$ and $c_{S(\Pi)}(\lambda) = \lambda^5 c_{\Pi}(\lambda^2 - 3)$ by the mentioned Theorems 2.15 and 2.17 in [5]. Thus, $L(\Pi)$ and $S(\Pi)$ have the eigenvalues $(-2)^{(5)}$, $(-1)^{(4)}$, $2^{(5)}$, $4^{(1)}$ and $(-\sqrt{6})^{(1)}$, $(-2)^{(5)}$, $(-1)^{(4)}$, $0^{(5)}$, $1^{(4)}$, $2^{(5)}$, $\sqrt{6}^{(1)}$, respectively. Since $L(\Pi)$ is 4-regular and $S(\Pi)$ is bipartite, $\mu_1 = \frac{-1}{2}$ and $\mu_1 = -1$ for $L(\Pi)$ and $S(\Pi)$, respectively.

Moreover, let $H = P_3$ be the path on three vertices with eigenvalues $(-\sqrt{2})^{(1)}$, $0^{(1)}$, $\sqrt{2}^{(1)}$. It follows that $S(P_3)$ is the path P_5 on 5 vertices with eigenvalues $(-\sqrt{3})^{(1)}$, $(-1)^{(1)}$, $0^{(1)}$, $1^{(1)}$, $\sqrt{3}^{(1)}$ and $L(P_3) = K_2$ with eigenvalues $(-1)^{(1)}$, $1^{(1)}$.

Since Π contains no triangles, a subgraph of Π isomorphic to P_3 is also an induced subgraph of Π , thus, a P_3 -packing of Π is also strict. Π contains $p = 3$ (and not more) vertex disjoint copies of P_3 as subgraphs.

Obviously, $\alpha(\Pi, P_3) = 1$ and $p = \alpha(S(\Pi), P_5) = \alpha(L(\Pi), K_2)$.

We remark that $p \leq \lfloor \frac{10}{3} \rfloor = 3$, because Π and P_3 have 10 and 3 vertices, respectively, thus, such a trivial upper bound may be sharp.

The following table presents upper bounds on $\alpha(\Pi, P_3)$, $\alpha(S(\Pi), P_5)$, and $\alpha(L(\Pi), K_2)$ obtained by inequalities (6), (7), (8), and (9) (note that $S(\Pi)$ is not regular), and Theorem 4.

| | (6) | (7) | (8) | (9) | Theorem 4 |
|-------------------------------|-----|-----|-----|-----|-----------|
| $\alpha(\Pi, P_3) = 1$ | 2 | 2 | 8 | 2 | 1 |
| $p = \alpha(S(\Pi), P_5) = 3$ | 5 | 6 | 30 | | 3 |
| $p = \alpha(L(\Pi), K_2) = 3$ | 3 | 5 | 11 | 3 | 4 |

For another example, let G be the graph of Figure 1, which can be found in [5], page 303 and in [6], page 327, with eigenvalues $(-2.21\dots)^{(1)}$, $(-2)^{(1)}$, $(-1.86\dots)^{(1)}$, $(-1)^{(3)}$, $0^{(1)}$, $0.53\dots^{(1)}$, $1.21\dots^{(1)}$, $1.67\dots^{(1)}$, $2.65\dots^{(1)}$, $3^{(1)}$.

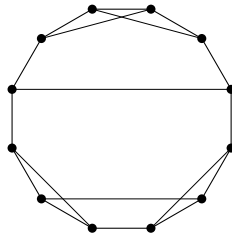


Figure 1

Moreover, let $H = K_3$ with eigenvalues $(-1)^{(2)}$, $2^{(1)}$.

The upper bound on $\alpha(G, K_3) = 2$ by Theorem 4 is sharp.

Obviously, G contains $p = 3$ vertex disjoint copies of K_3 as subgraphs.

Note that $L(K_3) = K_3$. Using $c_{L(G)}(\lambda) = (\lambda + 2)^6 c_G(\lambda - 1)$, $L(G)$ has the eigenvalues $(-2)^{(6)}$, $(-1.21\dots)^{(1)}$, $(-1)^{(1)}$, $(-0.86\dots)^{(1)}$, $0^{(3)}$, $1^{(1)}$, $1.53\dots^{(1)}$, $2.21\dots^{(1)}$, $2.67\dots^{(1)}$, $3.65\dots^{(1)}$, $4^{(1)}$.

In this case, the trivial upper bounds $\lfloor \frac{n(G)}{n(K_3)} \rfloor = 4$ and $\lfloor \frac{m(G)}{m(K_3)} \rfloor = 6$ on p both are not sharp,

however, by Theorem 4, we obtain $p \leq \alpha(L(G), K_3) \leq 3$.

Unlike the upper bound on $\alpha(G, H)$ in Theorem 4, the upper bounds of Corollary 1, Theorem 2, and Theorem 3 only involve some of the values $n, m, \lambda_1, \lambda_n, \mu_1, \delta$, and Δ as information about G , similar as the bounds in (1), (3), and (4). One should expect that an upper bound on $\alpha(G, H)$ can be arbitrarily bad in this case. To confirm this, we consider the following example.

Denote by C_p the cycle on p vertices and, for a positive even integer $q \geq 6$, let $n = 2q^2$. Furthermore, let G_1 be the cartesian product of K_2 and $C_{\frac{n}{2}}$ and G_2 be obtained by "blowing up" each vertex of the complete bipartite graph $K_{q,q}$ to C_q (note that G_2 is not unique). It is easy to see that G_1 and G_2 are 3-regular and bipartite graphs on n vertices, hence, both graphs coincide in the values $n, m, \lambda_1, \lambda_n, \mu_1, \delta$, and Δ (both are cubic), however, $\alpha(G_1, C_4) = \lfloor \frac{n}{6} \rfloor$ and $\alpha(G_2, C_4) = 0$.

This example also shows the impossibility to derive nontrivial lower bounds on $\alpha(G, H)$ without using additional properties of G .

For two upper bounds b_1 and b_2 on the independence number of a graph, we write $b_1 \prec_{\mathcal{F}} b_2$ if there is an infinite sequence \mathcal{F} of graphs such that $b_1(G) < b_2(G)$ for all graphs $G \in \mathcal{F}$. We say that b_1 and b_2 are *incomparable* if there are infinite sequences \mathcal{F}_1 and \mathcal{F}_2 such that $b_1 \prec_{\mathcal{F}_1} b_2$ and $b_2 \prec_{\mathcal{F}_2} b_1$.

Proposition 1

The upper bounds on $\alpha(G, K_1)$ in inequalities (2), (4), (10), and (11) are pairwise incomparable.

3 Proofs

Lemma 1 ([20])

If B and C are real symmetric square matrices with n rows, where C is positive definite, then there is a C -orthonormal basis of R^n consisting of C -eigenvectors of B .

The assertion of Lemma 2 is a simple observation. For completeness, we give a short proof here.

Lemma 2

Let B and C be real symmetric square matrices with n rows, where C is positive definite. If $\{\underline{u}_1, \dots, \underline{u}_n\}$ is a C -orthonormal base of R^n such that \underline{u}_i is a C -eigenvector with corresponding C -eigenvalue μ_i for $i = 1, \dots, n$, then, for any vector $\underline{x} \in R^n$,

$$(\mu_2 - \mu_1)(\underline{x}^T C \underline{u}_2)^2 + \dots + (\mu_n - \mu_1)(\underline{x}^T C \underline{u}_n)^2 + \mu_1 \underline{x}^T C \underline{x} = \underline{x}^T B \underline{x}.$$

As a consequence,

$$(\mu_n - \mu_1)(\underline{x}^T C \underline{u}_n)^2 + \mu_1 \underline{x}^T C \underline{x} \leq \underline{x}^T B \underline{x}. \quad (12)$$

Proof of Lemma 2.

Given \underline{x} , there are real numbers a_1, \dots, a_n such that $\underline{x} = a_1 \underline{u}_1 + \dots + a_n \underline{u}_n$. Then $\underline{x}^T B \underline{x} = \mu_1 a_1^2 + \dots + \mu_n a_n^2$, $\underline{x}^T C \underline{x} = a_1^2 + \dots + a_n^2$, and $\underline{x}^T C \underline{u}_i = a_i$ for $i = 1, \dots, n$. The desired equality is equivalent to $(\mu_2 - \mu_1) a_2^2 + \dots + (\mu_n - \mu_1) a_n^2 + \mu_1 (a_1^2 + \dots + a_n^2) = \mu_1 a_1^2 + \dots + \mu_n a_n^2$. \square

In the sequel, put $B = A$ and $C = D$. Especially, it follows $-1 \leq \mu_1 < 0$ ([8, 18, 20]).

The vector $\frac{1}{\sqrt{2m}} \underline{1}$ is a D -normal D -eigenvector of G with corresponding D -eigenvalue $\mu_n = 1$, thus, inequality (12) implies

Lemma 3 *If x_1, \dots, x_n are real numbers, then*

$$(1 - \mu_1) \left(\sum_{i=1}^n d_i x_i \right)^2 + 2\mu_1 m \sum_{i=1}^n d_i x_i^2 \leq 4m \sum_{ij \in E} x_i x_j. \quad (13)$$

Obviously, $\sum_{i=1}^n d_i x_i = \sum_{ij \in E} (x_i + x_j)$ and $\sum_{i=1}^n d_i x_i^2 = \sum_{ij \in E} (x_i^2 + x_j^2)$. If $\mu_1 = -1$, i.e. if G is bipartite, then (13) is equivalent to

$$\left(\sum_{ij \in E} (x_i + x_j) \right)^2 \leq m \sum_{ij \in E} (x_i + x_j)^2. \quad (14)$$

Note that inequality (14) is a consequence of the Cauchy-Schwarz inequality and, therefore, (14) is valid also for an arbitrary (not necessarily bipartite) graph G .

Using (14),

$$\begin{aligned} c(\mu_1) &= 2m \sum_{i=1}^n d_i x_i^2 - \left(\sum_{i=1}^n d_i x_i \right)^2 = 2m \sum_{ij \in E} (x_i^2 + x_j^2) - \left(\sum_{ij \in E} (x_i + x_j) \right)^2 \\ &\geq m \left(2 \sum_{ij \in E} (x_i^2 + x_j^2) - \sum_{ij \in E} (x_i + x_j)^2 \right) = m \sum_{ij \in E} (x_i - x_j)^2 \geq 0 \end{aligned}$$

for the coefficient $c(\mu_1)$ of μ_1 in inequality (13). Hence, the left side of (13) is a non-decreasing function in μ_1 and, if $\mu_1 > -1$ and $c(\mu_1) > 0$, then (13) is stronger than (14).

Corollary 3 *If $x_1, \dots, x_n \geq 0$ are real numbers, then*

$$(1 - \mu_1) \left(\sum_{i=1}^n \sqrt{x_i} \right)^4 \leq \left(\sum_{i=1}^n \frac{1}{d_i} \right)^2 \left(4m \sum_{ij \in E} x_i x_j - 2\mu_1 m \sum_{i=1}^n d_i x_i^2 \right). \quad (15)$$

Proof of Corollary 3.

Recall that $\mu_1 < 0$. We use inequality (13) and the Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right) \text{ for real numbers } a_i, b_i, i = 1, \dots, n.$$

With $a_i^2 = d_i x_i$ and $b_i^2 = \frac{1}{d_i}$, we have $\left(\sum_{i=1}^n \frac{1}{d_i} \right) \left(\sum_{i=1}^n d_i x_i \right) \geq \left(\sum_{i=1}^n \sqrt{x_i} \right)^2$ implying (15). \square

Inequality (13) and $\lambda_1 = r\mu_1$, if G is r -regular, imply

Corollary 4 *If G is r -regular and x_1, \dots, x_n are real numbers then*

$$(r - \lambda_1) \left(\sum_{i=1}^n x_i \right)^2 + \lambda_1 n \sum_{i=1}^n x_i^2 \leq 2n \sum_{ij \in E} x_i x_j. \quad (16)$$

Proof of Theorem 1.

Let $\underline{x} = (x_1, \dots, x_n)$ with $x_i = 1$ if $i \in V(U)$ and $x_i = 0$, otherwise.

By Lemma 3,

$$(1 - \mu_1) \left(\sum_{i \in V(U)} d_i \right)^2 + 2\mu_1 m \sum_{i \in V(U)} d_i = (1 - \mu_1) \left(\sum_{i=1}^n d_i x_i \right)^2 + 2\mu_1 m \sum_{i=1}^n d_i x_i^2 \leq 4m \sum_{ij \in E} x_i x_j,$$

hence, with $\sum_{ij \in E} x_i x_j = m'$ and $\sum_{i \in V(U)} d_i \geq \delta n'$,

$$(1 - \mu_1) \delta n' + 2\mu_1 m \leq (1 - \mu_1) \sum_{i \in V(U)} d_i + 2\mu_1 m \leq \frac{4mm'}{\sum_{i \in V(U)} d_i} \leq \frac{4mm'}{\delta n'}$$

and Theorem 1 is proved. \square

Proof of Theorem 2.

To prove (7), we use inequality (15) and obtain

$$(1 - \mu_1) \alpha(G, H)^3 h^4 \leq \left(\sum_{i=1}^n \frac{1}{d_i} \right)^2 (4me - 2\mu_1 m \Delta h).$$

In [3], it is proved that

$$\sum_{i=1}^n \frac{1}{d_i} \leq \frac{n^2}{2m} + \left(\frac{1}{\delta} - \frac{1}{\Delta} \right) \left(n - 1 - \frac{2m}{n} \right), \quad (17)$$

hence, (7) follows.

Now we prove inequality (8).

Let $V(H) = \{v_1, \dots, v_h\}$, $\alpha = \alpha(G, H)$, and H_1, \dots, H_α be α vertex disjoint mutually independent copies of H in G .

Given $t \in \{1, \dots, \alpha\}$, let $\phi_t : V(H) \rightarrow V(H_t)$ be a graph isomorphism from H to H_t for $t = 1, \dots, \alpha$.

Furthermore, let $\underline{z} = (z_1, \dots, z_h)^T$ with $z_q > 0$ for $q = 1, \dots, h$ and $\sum_{q=1}^h z_q^2 = 1$ be an eigenvector to the largest eigenvalue η_h of H . In [9, 17], it is proved that this choice of \underline{z} is possible, because H is connected, η_h has multiplicity one in this case, and \underline{z} is unique.

Next, let x_1, \dots, x_n be defined as follows:

If $t \in \{1, \dots, \alpha\}$ and $i \in V(H_t)$, then there is a suitable $q \in \{1, \dots, h\}$ such that $i = \phi_t(v_q)$.

Set $x_i = z_q$ in this case. If $i \in V \setminus (V(H_1) \cup \dots \cup V(H_\alpha))$, then let $x_i = 0$.

We obtain $\sum_{i=1}^n x_i = \alpha \sum_{q=1}^h z_q$, $\sum_{i=1}^n x_i^2 = \alpha \sum_{q=1}^h z_q^2 = \alpha$, $\sum_{i=1}^n \sqrt{x_i} = \alpha \sum_{q=1}^h \sqrt{z_q} \geq \alpha \sum_{q=1}^h z_q^2 = \alpha$, and

$$2 \sum_{ij \in E} x_i x_j = 2\alpha \left(\sum_{v_q v_{q'} \in E(H)} z_q z_{q'} \right) = \alpha \underline{z}^T A_H \underline{z} = \alpha \eta_h \sum_{q=1}^h z_q^2 = \alpha \eta_h.$$

With Corollary 3 and $d_i \leq \Delta$ for $i = 1, \dots, n$, it follows $(1 - \mu_1)\alpha^3 \leq 2 \left(\sum_{i=1}^n \frac{1}{d_i} \right)^2 (\eta_h - \mu_1 \Delta) m$, thus, (8) follows with inequality (17) and Theorem 2 is proved. \square

Proof of Theorem 3.

Again, given $t \in \{1, \dots, \alpha\}$, let $\phi_t : V(H) \rightarrow V(H_t)$ be a graph isomorphism from H to H_t for $t = 1, \dots, \alpha$.

For real numbers z_1, \dots, z_h with $\sum_{q=1}^h z_q = 1$, let x_1, \dots, x_n be defined as follows:

If $t \in \{1, \dots, \alpha\}$ and $i \in V(H_t)$, then there is a suitable $q \in \{1, \dots, h\}$ such that $i = \phi_t(v_q)$. Set $x_i = z_q$ in this case. If $i \in V \setminus (V(H_1) \cup \dots \cup V(H_\alpha))$, then let $x_i = 0$.

With $\underline{z} = (z_1, \dots, z_h)^T$, we obtain

$$\sum_{i \in V} x_i = \alpha \left(\sum_{q=1}^h z_q \right) = \alpha, \quad \sum_{i \in V} x_i^2 = \alpha \left(\sum_{q=1}^h z_q^2 \right), \quad \text{and}$$

$$2 \sum_{ij \in E} x_i x_j = 2\alpha \left(\sum_{v_q v_{q'} \in E(H)} z_q z_{q'} \right) = \alpha \underline{z}^T A_H \underline{z}.$$

Corollary 4 implies $(r - \lambda_1)\alpha + \lambda_1 \left(\sum_{q=1}^h z_q^2 \right) n \leq \underline{z}^T A_H \underline{z} n$, hence, with

$$M = (A_H - \lambda_1 E_h), \quad \alpha \leq \frac{n}{(r - \lambda_1)} \min \underline{z}^T M \underline{z} = \frac{n}{(r - \lambda_1)} MIN,$$

where the minimum is taken over all vectors $\underline{z} = (z_1, \dots, z_h)^T$ with $\sum_{q=1}^h z_q = 1$.

Note that this minimum exists, because, $\alpha(G, H) \geq 1$ implies $\lambda_1 \leq \eta_1$, hence, all eigenvalues $\eta_1 - \lambda_1, \eta_2 - \lambda_1, \dots, \eta_h - \lambda_1$ of M are non-negative. It follows that $A_H - \lambda_1 E_h$ is positive semidefinite.

To investigate this value MIN , we consider the Lagrange function

$$f(\underline{z}, \kappa) = \underline{z}^T M \underline{z} - 2\kappa \left(\sum_{q=1}^h z_q - 1 \right)$$

with Lagrange multiplier 2κ and the necessary optimality conditions $f_{z_q} = 0$ for $q = 1, \dots, h$ (for more details to Lagrange Theory see [1]).

We obtain that the equations $M \underline{z} = \kappa \underline{1}$ and $\underline{1}^T \underline{z} = 1$ are simultaneously solvable.

Next we will show that κ is unique. If $M \underline{z}_1 = \kappa_1 \underline{1}$, $\underline{1}^T \underline{z}_1 = 1$, $M \underline{z}_2 = \kappa_2 \underline{1}$, and $\underline{1}^T \underline{z}_2 = 1$, then $\kappa_1 = \kappa_1 \underline{1}^T \underline{z}_2 = \underline{z}_1^T M \underline{z}_2 = \kappa_2 \underline{z}_1^T \underline{1} = \kappa_2$.

With $1 \leq \alpha \leq \frac{n}{(r-\lambda_1)}MIN$, it follows $MIN = \underline{z}^T M \underline{z} = \kappa > 0$.

If $\underline{x} = \frac{1}{\kappa} \underline{z}$, then $M \underline{x} = \underline{1}$ and $\underline{1}^T \underline{x} = \frac{1}{\kappa}$.

Moreover, if $\underline{u} = (\frac{1}{h}, \dots, \frac{1}{h})^T \in R^h$ then $MIN \leq \underline{u}^T M \underline{u} = \frac{2e-\lambda_1 h}{h^2}$ and inequality (9) is proved.

If $\lambda_1 < \eta_1$, then M is regular and $1 = \underline{1}^T \underline{z} = \kappa \underline{1}^T M^{-1} \underline{1} = \kappa s$, hence, $\alpha \leq \frac{n}{(r-\lambda_1)s}$. \square

Proof of Theorem 4.

We will use the following Lemma 4 ([2, 5]).

Lemma 4 (Cauchy's inequalities, Interlacing theorem)

If U is an induced subgraph of G with eigenvalues $\phi_1 \leq \dots \leq \phi_t$, then $\lambda_i \leq \phi_i \leq \lambda_{n-t+i}$ for $i = 1, \dots, t$.

Let $\alpha = \alpha(G, H)$ and U be the subgraph of G consisting of α mutually independent induced copies of H . Then U has αh not necessarily distinct eigenvalues and, for $q = 1, \dots, h$, each eigenvalue μ_q of H is α times an eigenvalue of U .

Hence, U has $q\alpha$ not necessarily distinct eigenvalues at most η_q and $(h - q + 1)\alpha$ not necessarily distinct eigenvalues at least η_q .

By Lemma 4,

$$|\{i \in \{1, \dots, n\} | \lambda_i \leq \eta_q\}| \geq q\alpha \text{ and } |\{i \in \{1, \dots, n\} | \lambda_i \geq \eta_q\}| \geq (h - q + 1)\alpha. \quad \square$$

Proof of Proposition 1.

For $a, b \in \{2, 4, 10, 11\}$, we write $(a) \prec_{\mathcal{F}} (b)$ if there is an infinite sequence \mathcal{F} of graphs such that this relation is true for the two bounds in inequality (a) and inequality (b).

Using that G is bipartite if and only if $\lambda_1 = -\lambda_n$ or, equivalently, $\mu_1 = -1$ and that $\lambda_n = r$ if G is r -regular, it is easy to check that the following holds:

If \mathcal{F} is a sequence of

cycles on $n \geq 4$ vertices, then (4) $\prec_{\mathcal{F}}$ (2),

bipartite graphs with $m \geq n$ and $\delta = 1$, then (2) $\prec_{\mathcal{F}}$ (10),

cycles on $n \geq 4$ (n even) vertices, then (10) $\prec_{\mathcal{F}}$ (2),

complete bipartite graphs $K_{r,r}$ with $r \geq 2$, then (2) $\prec_{\mathcal{F}}$ (11),

cycles on $n \geq 6$ (n even) vertices, then (11) $\prec_{\mathcal{F}}$ (2),

bipartite graphs with $m \geq n$ and $\delta = 1$, then (4) $\prec_{\mathcal{F}}$ (10),

regular graphs, then (4) $\prec_{\mathcal{F}}$ (11),

bipartite and regular graphs, then (10) $\prec_{\mathcal{F}}$ (11), and

bipartite graphs with $2\sqrt{\delta\Delta}n < m$, then (11) $\prec_{\mathcal{F}}$ (10).

We only prove (2) $\prec_{\mathcal{F}}$ (4), (10) $\prec_{\mathcal{F}}$ (4), and (11) $\prec_{\mathcal{F}}$ (4) as follows.

Consider the sequence \mathcal{F} of graphs G_n obtained from the cycle C_n ($n \geq 6$, even) with $V(C_n) = \{1, \dots, n\}$ and $E(C_n) = \{i, i+1 \mid i = 1, \dots, n\}$ (index i modulo n) by adding the edge between the vertices 1 and 4. Recall that $\mu_n = 1$, $\mu_1 = -1$, and $\lambda_1 = -\lambda_n$ since G_n is bipartite. Moreover, $m = n+1$ and $\delta = 2$. The inequality $\frac{-\lambda_1\lambda_n}{\delta^2-\lambda_1\lambda_n}n > \frac{-2\mu_1}{(1-\mu_1)\delta}m$ leads to

$$\frac{(\lambda_n)^2}{4+(\lambda_n)^2}n > \frac{n+1}{2} \text{ being equivalent to } \lambda_n > 2\sqrt{\frac{n+1}{n-1}}.$$

The last inequality is true because $\underline{x}^T \underline{x} = \frac{n+2}{2}$ and $\underline{x}^T A \underline{x} = 2 \sum_{ij \in E} x_i x_j = n + 4\sqrt{2} - 2$ for the vector $\underline{x} = (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, \dots, \frac{1}{\sqrt{2}})^T$, hence, $\lambda_n \geq \frac{\underline{x}^T A \underline{x}}{\underline{x}^T \underline{x}} = \frac{2n+8\sqrt{2}-4}{n+2}$, and it is easy to check that the last expression is larger than $2\sqrt{\frac{n+1}{n-1}}$ for $n \geq 6$, and (10) $\prec_{\mathcal{F}}$ (4) is proved.

It remains to show that (2) $\prec_{\mathcal{F}}$ (4) and (11) $\prec_{\mathcal{F}}$ (4).

The complete graph K_n on n vertices has the eigenvalues $\lambda_1 = \dots = \lambda_{n-1} = -1$ and $\lambda_n = n-1$.

It follows that $c_{K_n}(\lambda) = (\lambda - (n-1))(\lambda + 1)^{n-1}$.

For $n \geq n_0$, consider the sequence \mathcal{F} of graphs G_n obtained from K_{n-1} with $V(K_{n-1}) = \{1, \dots, n-1\}$ by adding a vertex n and the edge $(n-1, n)$. We will show that (2) $\prec_{\mathcal{F}}$ (4) and (11) $\prec_{\mathcal{F}}$ (4) follow if n_0 is large enough.

By Laplace expansion,

$$\begin{aligned} c_{G_n}(\lambda) &= |\lambda E_n - A| = \lambda c_{K_{n-1}}(\lambda) - c_{K_{n-2}}(\lambda) \\ &= \lambda(\lambda - (n-2))(\lambda + 1)^{n-2} - (\lambda - (n-3))(\lambda + 1)^{n-3} \\ &= (\lambda + 1)^{n-3}(\lambda^3 - (n-3)\lambda^2 - (n-1)\lambda + (n-3)). \end{aligned}$$

Since $f(-2) < 0$, $f(-1) > 0$, $f(2) < 0$, and $f(n) > 0$ for

$f(\lambda) = \lambda^3 - (n-3)\lambda^2 - (n-1)\lambda + (n-3)$, it follows $-2 < \lambda_1 < -1$ and $2 < \lambda_n < n$, hence, $-\lambda_1\lambda_n > 2$.

Using $\delta = 1$, we obtain $\frac{-\lambda_1\lambda_n}{\delta^2-\lambda_1\lambda_n}n > \frac{2n}{3}$.

Moreover, $\frac{1+\sqrt{4(n^2-n-2m)+1}}{2} = \frac{1+\sqrt{8n-15}}{2}$ and (2) $\prec_{\mathcal{F}}$ (4) is proved for large n_0 .

Let I_n and $D_n(a_1, \dots, a_n)$ be the $(n \times n)$ -matrix with entries all equal to 1 and the $(n \times n)$ -diagonal-matrix with a_i as the i -th element at the main diagonal, respectively.

Furthermore, let

$$S_{n-1} = D_{n-1}(\mu(n-2) + 1, \dots, \mu(n-2) + 1, \mu(n-1) + 1) - I_{n-1} \text{ and} \\ T_{n-2} = D_{n-2}(\mu(n-2) + 1, \dots, \mu(n-2) + 1) - I_{n-2}.$$

$$\text{Then } |\mu D - A| = \mu |S_{n-1}| - |T_{n-2}| = \mu |S_{n-1}| - c_{K_{n-2}}(\mu(n-2)).$$

Let S'_{n-1} and S''_{n-1} be obtained from S_{n-1} by replacing the last row by $(-1, \dots, -1, \mu(n-2))$ and $(0, \dots, 0, \mu)$, respectively.

$$\text{Then } |S_{n-1}| = |S'_{n-1}| + |S''_{n-1}| = c_{K_{n-1}}(\mu(n-2)) + \mu c_{K_{n-2}}(\mu(n-2)), \text{ hence,} \\ |\mu D - A| = \mu(c_{K_{n-1}}(\mu(n-2)) + (\mu^2 - 1)c_{K_{n-2}}(\mu(n-2))) \\ = \mu((\mu(n-2) + 1)^{n-2}(\mu(n-2) - (n-2))) + (\mu^2 - 1)((\mu(n-2) + 1)^{n-3}(\mu(n-2) - (n-3))) \\ = (\mu(n-2) + 1)^{n-3}(\mu - 1)(\mu^2(n-2)(n-1) + \mu(n-1) - (n-3)).$$

$$\text{It follows } \mu_1 = \frac{1-n-\sqrt{4n^3-23n^2+42n-23}}{2(n^2-3n+2)}.$$

Since $2m = n^2 - 3n + 4$, $\Delta = n - 1$, and $\delta = 1$, we have

$$\sqrt[3]{\frac{(\frac{n^2}{2m} + (\frac{1}{\delta} - \frac{1}{\Delta})(n-1 - \frac{2m}{n}))^2(-2\mu_1\Delta m)}{(1-\mu_1)}} < \sqrt[3]{\frac{9(-2\mu_1\Delta m)}{(1-\mu_1)}} < \sqrt[3]{\frac{9(-\mu_1)}{(1-\mu_1)}} n.$$

With $\lim_{n \rightarrow \infty} \mu_1 = 0$, also (11) $\prec_{\mathcal{F}}$ (4) is proved if n_0 is large enough. \square

Acknowledgement. The authors would like to express their gratitude to P. E. John and J. Thierfelder, Ilmenau University of Technology, for their valuable advice.

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