Reconstruction of sparse Legendre and Gegenbauer expansions

Daniel Potts[*] Manfred Tasche[‡]

We present a new deterministic algorithm for the reconstruction of sparse Legendre expansions from a small number of given samples. Using asymptotic properties of Legendre polynomials, this reconstruction is based on Prony–like methods. Furthermore we show that the suggested method can be extended to the reconstruction of sparse Gegenbauer expansions of low positive order.

Key words and phrases: Legendre polynomials, sparse Legendre expansions, Gegenbauer polynomials, ultraspherical polynomials, sparse Gegenbauer expansions, sparse recovering, sparse Legendre interpolation, sparse Gegenbauer interpolation, asymptotic formula, Prony–like method.

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1 Introduction

In this paper, we present a new deterministic approach to the reconstruction of sparse Legendre and Gegenbauer expansions, if relatively few samples on a special grid are given.

Recently the reconstruction of sparse trigonometric polynomials has attained much attention. There exist recovery methods based on random sampling related to compressed sensing (see e.g. [17, 10, 5, 4] and the references therein) and methods based on deterministic sampling related to Prony–like methods (see e.g. [15] and the references therein).

Both methods are already generalized to other polynomial systems. Rauhut and Ward [18] presented a recovery method of a polynomial of degree at most $N-1$ given in Legendre expansion with M nonzero terms, where $\mathcal{O}(M\,(\log N)^4)$ random samples are

[*]potts@mathematik.tu-chemnitz.de, Chemnitz University of Technology, Department of Mathematics, D–09107 Chemnitz, Germany

[‡]manfred.tasche@uni-rostock.de, University of Rostock, Institute of Mathematics, D–18051 Rostock, Germany

taken independently according to the Chebyshev probability measure of $[-1, 1]$. The recovery algorithms in compressive sensing are often based on ℓ_1–minimization. Exact recovery of sparse functions can be ensured only with a certain probability. Peter, Plonka, and Roşca [14] have presented a Prony–like method for the reconstruction of sparse Legendre expansions, where only $2M + 1$ function resp. derivative values at one point are given.

Recently, the authors have described a unified approach to Prony–like methods in [15] and applied to the recovery of sparse expansions of Chebyshev polynomials of first and second kind in [16]. Similar sparse interpolation problems for special polynomial systems are formerly explored in [11, 7, 2, 3] and also solved by Prony–like methods. A very general approach for the reconstruction of sparse expansions of eigenfunctions of suitable linear operators were suggested by Plonka and Peter in [13]. New reconstruction formulas for M-sparse expansions of orthogonal polynomials using the Sturm–Liouville operator, where presented. However one hast to use sampling points and derivative values.

In this paper we present a new method for the reconstruction of sparse Legendre expansions which is based on local approximation of Legendre polynomials by cosine functions. Therefore this algorithm is closely related to [16]. Note that fast algorithms for the computation of Fourier–Legendre coefficients in a Legendre expansion (see [6, 1]) are based on similar asymptotic formulas of the Legendre polynomials. However the key idea is that the convenient scaled Legendre polynomials behave very similar as the cosine functions near by zero. Therefore we use a sampling grid located near by zero. Finally we generalize this method to sparse Gegenbauer expansions of low positive order.

The outline of this paper is as follows. In Section 2, we collect some useful properties of Legendre polynomials. In Section 3, we present the new reconstruction method for sparse Legendre expansions. We extend our recovery method in Section 4 to the case of sparse Gegenbauer expansions of low positive order. Finally we show some results of numerical experiments in Section 5.

2 Properties of Legendre polynomials

For each $n \in \mathbb{N}_0$, the Legendre polynomials P_n can be recursively defined by

$$P_{n+2}(x) := \frac{2n+3}{n+2}\, x\, P_{n+1}(x) - \frac{n+1}{n+2}\, P_n(x) \quad (x \in \mathbb{R})$$

with $P_0(x) := 1$ and $P_1(x) := x$. The Legendre polynomial P_n can be represented in the explicit form

$$P_n(x) = \frac{1}{2^n} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j\,(2n-2j)!}{j!\,(n-j)!\,(n-2j)!}\, x^{n-2j}\,.$$

Hence it follows that for $m \in \mathbb{N}_0$

$$P_{2m}(0) = \frac{(-1)^m\,(2m)!}{2^{2m}\,(m!)^2}\,, \quad P_{2m+1}(0) = 0\,, \tag{2.1}$$

$$P'_{2m+1}(0) = \frac{(-1)^m\,(2m+1)!}{2^{2m}\,(m!)^2}\,, \quad P'_{2m}(0) = 0\,. \tag{2.2}$$

Further, Legendre polynomials of even degree are even and Legendre polynomials of odd degree are odd, i.e.

$$P_n(-x) = (-1)^n\,P_n(x)\,. \tag{2.3}$$

Moreover, the Legendre polynomial P_n satisfies the following homogeneous linear differential equation of second order

$$(1-x^2)\,P''_n(x) - 2x\,P'_n(x) + n^2\,P_n(x) = 0\,. \tag{2.4}$$

In the Hilbert space $L^2_{1/2}([-1,\,1])$ with the constant weight $\frac{1}{2}$, the *normed Legendre polynomials*

$$L_n(x) := \sqrt{2n+1}\,P_n(x) \quad (n \in \mathbb{N}_0) \tag{2.5}$$

form an orthonormal basis, since

$$\frac{1}{2}\int_{-1}^{1} L_n(x)\,L_m(x)\,\mathrm{d}x = \delta_{n-m} \quad (m,\,n \in \mathbb{N}_0)\,.$$

Note that the uniform norm

$$\max_{x\in[-1,1]} |L_n(x)| = |L_n(-1)| = |L_n(1)| = (2n+1)^{1/2}$$

is increasing with respect to n.

Let M be a positive integer. A polynomial

$$H(x) := \sum_{k=0}^{d} b_k\,L_k(x) \tag{2.6}$$

of degree d with $d \gg M$ is called *M-sparse in the Legendre basis* or simply a *sparse Legendre expansion*, if M coefficients b_k are nonzero and if the other $d-M+1$ coefficients vanish. Then such an M-sparse polynomial H can be represented in the form

$$H(x) = \sum_{j=1}^{M_0} c_{0,j}\,L_{n_{0,j}}(x) + \sum_{k=1}^{M_1} c_{1,k}\,L_{n_{1,k}}(x) \tag{2.7}$$

with $c_{0,j} := b_{n_{0,j}} \neq 0$ for all even $n_{0,j}$ with $0 \le n_{0,1} < n_{0,2} < \ldots < n_{0,M_0}$ and with $c_{1,k} := b_{n_{1,k}} \neq 0$ for all odd $n_{1,k}$ with $1 \le n_{1,1} < n_{1,2} < \ldots < n_{1,M_1}$. The positive integer $M = M_0 + M_1$ is called the *Legendre sparsity* of the polynomial H. The numbers M_0, $M_1 \in \mathbb{N}_0$ are the *even* and *odd Legendre sparsities*, respectively.

Remark 2.1 The sparsity of a polynomial depends essentially on the chosen polynomial basis. If

$$T_n(x) := \cos(n \arccos x) \quad (x \in [-1,\, 1])$$

denotes the nth Chebyshev polynomial of first kind, then the nth Legendre polynomial P_n can be represented in the Chebyshev basis by

$$P_n(x) = \frac{1}{2^{2n}} \sum_{j=0}^{\lfloor n/2 \rfloor} (2 - \delta_{n-2j})\, \frac{(2j)!\,(2n-2j)!}{(j!)^2\,((n-j)!)^2}\, T_{n-2j}(x)\,.$$

Thus a sparse polynomial in the Legendre basis is in general not a sparse polynomial in the Chebyshev basis. In other words, one has to solve the reconstruction problem of a sparse Legendre expansion without change of the Legendre basis. □

Example 2.2 Let (r, φ, θ) be the spherical coordinates. We consider an axially symmetric solution $\Phi(r, \theta)$ of the Laplace equation inside the unit ball which fulfills the Dirichlet boundary condition

$$\Phi(1, \theta) = F(\cos\theta) \quad (\theta \in [0,\, \pi])\,,$$

where $F:\, [-1,\, 1] \to \mathbb{R}$ is continuously differentiable. Then the solution $\Phi(r, \theta)$ reads as follows

$$\Phi(r, \theta) = \sum_{n=0}^{\infty} a_n\, r^n\, L_n(\cos\theta) \quad (r \in [0,\, 1],\ \theta \in [0,\, \pi])$$

with the Fourier–Legendre coefficients

$$a_n = \frac{1}{2} \int_{-1}^{1} F(x)\, L_n(x)\, \mathrm{d}x = \frac{1}{2} \int_{0}^{\pi} F(\cos\theta)\, L_n(\cos\theta)\, \sin\theta\, \mathrm{d}\theta\,.$$

If only a finite number of the Fourier–Legendre coefficients a_n does not vanish, then $\Phi(r, \theta)$ is a sparse Legendre expansion. □

As in [18], we transform the Legendre polynomial system $\{L_n;\, n \in \mathbb{N}_0\}$ into a uniformly bounded orthonormal system. We introduce the functions $Q_n:\, [-1,\, 1] \to \mathbb{R}$ for each $n \in \mathbb{N}_0$ by

$$Q_n(x) := \sqrt{\frac{\pi}{2}}\, \sqrt[4]{1-x^2}\, L_n(x) = \sqrt{\frac{(2n+1)\pi}{2}}\, \sqrt[4]{1-x^2}\, P_n(x)\,. \tag{2.8}$$

Note that these functions Q_n have the same symmetry properties (2.3) as the Legendre polynomials, namely

$$Q_n(-x) = (-1)^n\, Q_n(x) \quad (x \in [-1,\, 1])\,. \tag{2.9}$$

Further the functions Q_n are orthonormal in the Hilbert space $L^2_w([-1,\,1])$ with the Chebyshev weight $w(x) := \frac{1}{\pi}\,(1-x^2)^{-1/2}$, since for all $m,\,n \in \mathbb{N}_0$

$$\int_{-1}^{1} Q_n(x)\,Q_m(x)\,w(x)\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} L_n(x)\,L_m(x)\,\mathrm{d}x = \delta_{n-m}\,.$$

Note that

$$\int_{-1}^{1} w(x)\,\mathrm{d}x = 1\,.$$

In the following, we use the standard substitution $x = \cos\theta$ ($\theta \in [0,\,\pi]$) and obtain

$$Q_n(\cos\theta) = \sqrt{\frac{\pi}{2}}\,\sqrt{\sin\theta}\,L_n(\cos\theta) = \sqrt{\frac{(2n+1)\pi}{2}}\,\sqrt{\sin\theta}\,P_n(\cos\theta) \quad (\theta \in [0,\,\pi])\,.$$

Lemma 2.3 *For all $n \in \mathbb{N}_0$, the functions $Q_n(\cos\theta)$ are uniformly bounded on the interval $[0,\,\pi]$, i.e.*

$$|Q_n(\cos\theta)| < 2 \quad (\theta \in [0,\,\pi])\,. \tag{2.10}$$

Proof. For $n \geq 2$, we know by [19, p. 165] that for all $\theta \in [0,\,\pi]$

$$\sqrt{\frac{\pi}{2}}\,\sqrt{\sin\theta}\,|P_n(\cos\theta)| < \frac{1}{\sqrt{n}}\,.$$

Then for the normed Legendre polynomials L_n, we obtain the estimate

$$\sqrt{\frac{\pi}{2}}\,\sqrt{\sin\theta}\,|L_n(\cos\theta)| < \sqrt{\frac{2n+1}{n}}$$

and hence

$$|Q_n(\cos\theta)| < \sqrt{2 + \frac{1}{n}} < 2\,.$$

By

$$Q_0(\cos\theta) = \sqrt{\frac{\pi}{2}}\,\sqrt{\sin\theta}\,, \quad Q_1(\cos\theta) = \sqrt{\frac{3\pi}{2}}\,\sqrt{\sin\theta}\,\cos\theta\,,$$

we see immediately that the estimate (2.10) is also true for $n = 0$ and $n = 1$. ∎

By (2.4), the function $Q_n(\cos\theta)$ satisfies the following linear differential equation of second order (see [19, p. 165])

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\,Q_n(\cos\theta) + \Big((n+\frac{1}{2})^2 + \frac{1}{4\,(\sin\theta)^2}\Big)\,Q_n(\cos\theta) = 0 \quad (\theta \in (0,\,\pi))\,. \tag{2.11}$$

Now we use the known asymptotic properties of the Legendre polynomials. By the asymptotic formula of Laplace, see [19, p. 194], one knows that for $\theta \in (0,\,\pi)$

$$Q_n(\cos\theta) = \sqrt{2}\,\cos[(n+\frac{1}{2})\,\theta - \frac{\pi}{4}] + \mathcal{O}(n^{-1}) \quad (n \to \infty)\,. \tag{2.12}$$

The error bound holds uniformly in $[\varepsilon, \pi - \varepsilon]$ with $\varepsilon \in (0, \frac{\pi}{2})$. Instead of (2.12), we will use another asymptotic formula of $Q_n(\cos\theta)$, which has a better approximation property in a small neighborhood of $\theta = \frac{\pi}{2}$ for arbitrary $n \in \mathbb{N}_0$.

By the method of Liouville–Stekloff, see [19, p. 210], we show that for *arbitrary* $n \in \mathbb{N}_0$, the function $Q_n(\cos\theta)$ is approximately equal to some multiple of $\cos[(n + \frac{1}{2})\,\theta - \frac{\pi}{4}]$ in the small neighborhood of $\theta = \frac{\pi}{2}$.

Theorem 2.4 *For each $n \in \mathbb{N}_0$, the function $Q_n(\cos\theta)$ can be represented by the asymptotic formula of Laplace–type*

$$Q_n(\cos\theta) = \lambda_n \, \cos[(n + \frac{1}{2})\,\theta - \frac{\pi}{4}] + R_n(\theta) \quad (\theta \in [0, \pi]) \tag{2.13}$$

with the scaling factor

$$\lambda_n := \begin{cases} \sqrt{\frac{(4m+1)\pi}{2}} \, \frac{(2m)!}{2^{2m}\,(m!)^2} & n = 2m\,, \\ \sqrt{\frac{\pi}{4m+3}} \, \frac{(2m+1)!}{2^{2m}\,(m!)^2} & n = 2m+1 \end{cases}$$

and the error term

$$R_n(\theta) := -\frac{1}{4n+2} \int_{\pi/2}^{\theta} \frac{\sin[(n+\frac{1}{2})(\theta - \tau)]}{(\sin\tau)^2} \, Q_n(\cos\tau)\, \mathrm{d}\tau \quad (\theta \in (0, \pi))\,. \tag{2.14}$$

The error term $R_n(\theta)$ fulfills the conditions $R_n(\frac{\pi}{2}) = R_n'(\frac{\pi}{2}) = 0$ and has the symmetry property

$$R_n(\pi - \theta) = (-1)^n \, R_n(\theta)\,. \tag{2.15}$$

Further, the error term can be estimated by

$$|R_n(\theta)| \le \frac{1}{2n+1}\, |\cot\theta|\,. \tag{2.16}$$

Proof. 1. Using the method of Liouville–Stekloff (see [19, p. 210]), we derive the asymptotic formula (2.13) from the differential equation (2.11), which can be written in the form

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\, Q_n(\cos\theta) + (n + \frac{1}{2})^2\, Q_n(\cos\theta) = -\frac{1}{4\,(\sin\theta)^2}\, Q_n(\cos\theta) \quad (\theta \in (0, \pi))\,. \tag{2.17}$$

Since the homogeneous linear differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\, Q_n(\cos\theta) + (n + \frac{1}{2})^2\, Q_n(\cos\theta) = 0$$

has the fundamental system

$$\cos[(n + \frac{1}{2})\,\theta - \frac{\pi}{4}]\,, \quad \sin[(n + \frac{1}{2})\,\theta - \frac{\pi}{4}]\,,$$

the differential equation (2.17) can be transformed into the Volterra integral equation

$$\begin{aligned} Q_n(\cos\theta) &= \lambda_n \cos[(n+\frac{1}{2})\,\theta - \frac{\pi}{4}] + \mu_n \sin[(n+\frac{1}{2})\,\theta - \frac{\pi}{4}] \\ & \quad - \frac{1}{4n+2} \int_{\pi/2}^{\theta} \frac{\sin[(n+\frac{1}{2})(\theta-\tau)]}{(\sin\tau)^2}\, Q_n(\cos\tau)\,\mathrm{d}\tau \quad (\theta \in (0,\,\pi)) \end{aligned}$$

with certain real constants λ_n and μ_n. The integral (2.14) and its derivative vanish for $\theta = \frac{\pi}{2}$.

2. Now we determine the constants λ_n and μ_n. For arbitrary even $n = 2m$ ($m \in \mathbb{N}_0$), the function $Q_{2m}(\cos\theta)$ can be represented in the form

$$Q_{2m}(\cos\theta) = \lambda_{2m} \cos[(2m+\frac{1}{2})\,\theta - \frac{\pi}{4}] + \mu_{2m} \sin[(2m+\frac{1}{2})\,\theta - \frac{\pi}{4}] + R_{2m}(\theta)\,.$$

Hence the condition $R_{2m}(\frac{\pi}{2}) = 0$ means that $Q_{2m}(0) = (-1)^m \lambda_{2m}$. Using (2.8), (2.5) and (2.1), we obtain that

$$\lambda_{2m} = \sqrt{\frac{(4m+1)\pi}{2}}\, \frac{(2m)!}{2^{2m}\,(m!)^2}\,.$$

From $P'_{2m}(0) = 0$ by (2.2) it follows that the derivative of $Q_{2m}(\cos\theta)$ vanishes for $\theta = \frac{\pi}{2}$. Thus the second condition $R'_{2m}(\frac{\pi}{2}) = 0$ implies that $0 = \mu_{2m}\,(2m+\frac{1}{2})\,(-1)^m$, i.e. $\mu_{2m} = 0$. Note that by the formula of Stirling $\lim\limits_{m\to\infty} \lambda_{2m} = \sqrt{2}$.

3. If $n = 2m+1$ ($m \in \mathbb{N}_0$) is odd, then

$$Q_{2m+1}(\cos\theta) = \lambda_{2m+1} \cos[(2m+\frac{3}{2})\theta - \frac{\pi}{4}] + \mu_{2m+1} \sin[(2m+\frac{3}{2})\,\theta - \frac{\pi}{4}] + R_{2m+1}(\theta)\,.$$

Hence the condition $R_{2m+1}(\frac{\pi}{2}) = 0$ implies by $P_{2m+1}(0) = 0$ (see (2.1)) that $0 = \mu_{2m+1}\,(-1)^m$, i.e. $\mu_{2m+1} = 0$. The second condition $R'_{2m+1}(\frac{\pi}{2}) = 0$ reads as follows

$$\begin{aligned} -\sqrt{\frac{(4m+3)\pi}{2}}\, P'_{2m+1}(0) &= -\lambda_{2m+1}\,(2m+\frac{3}{2})\,\sin(m\pi + \frac{\pi}{2}) \\ &= -\lambda_{2m+1}\,(2m+\frac{3}{2})\,(-1)^m\,. \end{aligned}$$

Thus we obtain by (2.2) that

$$\lambda_{2m+1} = \sqrt{\frac{\pi}{4m+3}}\, \frac{(2m+1)!}{2^{2m}\,(m!)^2}\,.$$

Note that by the formula of Stirling $\lim\limits_{m\to\infty} \lambda_{2m+1} = \sqrt{2}$.

4. As shown, the error term $R_n(\theta)$ has the explicit representation (2.14). Using (2.10), we estimate this integral and obtain

$$|R_n(\theta)| \le \frac{2}{4n+2} \Big| \int_{\pi/2}^{\theta} \frac{1}{(\sin\tau)^2}\,\mathrm{d}\tau \Big| = \frac{1}{2n+1}\,|\cot\theta|\,.$$

The symmetry property (2.15) of the error term

$$R_n(\theta) = Q_n(\cos\theta) - \lambda_n \cos[(n+\frac{1}{2})\theta - \frac{\pi}{4}]$$

follows from the fact that $Q_n(\cos\theta)$ and $\cos[(n+\frac{1}{2})\theta - \frac{\pi}{4}]$ possess the same symmetry properties as (2.9). This completes the proof. ■

Remark 2.5 For arbitrary $m \in \mathbb{N}_0$, the scaling factors λ_n in (2.13) can be expressed in the following form

$$\lambda_n = \begin{cases} \sqrt{\frac{(4m+1)\pi}{2}}\,\alpha_m & n = 2m\,, \\ \sqrt{\frac{\pi}{4m+3}}\,(2m+2)\,\alpha_{m+1} & n = 2m+1 \end{cases}$$

with

$$\alpha_0 := 1\,, \quad \alpha_n := \frac{1\cdot 3\cdots(2n+1)}{2\cdot 4\cdots(2n)} \quad (n \in \mathbb{N})\,.$$

□

In Figure 2.1, we plot the expression $|\tan(\theta)\,R_n(\theta)|$ for some polynomial degrees n.

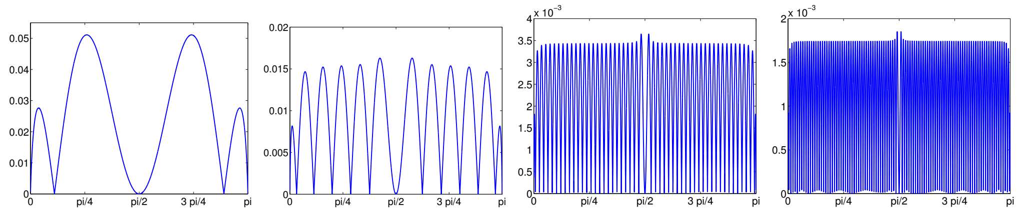

Figure 2.1: Expression $|\tan(\theta)\,R_n(\theta)|$ for $n \in \{3,\,11,\,51,\,101\}$.

We have proved the estimate (2.16) in Theorem 2.4. In the Table 2.1, one can see that the maximum value of $|\tan(\theta)\,R_n(\theta)|$ on the interval $[0,\,\pi]$ is much smaller than $\frac{1}{2n+1}$.

n	3	5	7	9	11	13
$\max\limits_{\theta\in[0,\pi]} \lvert\tan(\theta)R_n(\theta)\rvert$	0.0511	0.0336	0.0249	0.0197	0.0163	0.0139

Table 2.1: Maximum value of $|\tan(\theta)\,R_n(\theta)|$ for some polynomial degrees n.

We observe that the approximation of $R_n(\theta)$ is very accurate in a small neighborhood of $\theta = \frac{\pi}{2}$. By the substitution $t = \theta - \frac{\pi}{2} \in [-\frac{\pi}{2},\,\frac{\pi}{2}]$ and (2.9), we obtain

$$\begin{aligned} Q_n(\sin t) &= (-1)^n\,\lambda_n\,\cos[(n+\frac{1}{2})\,t + \frac{n\pi}{2}] + (-1)^n\,R_n(t+\frac{\pi}{2}) \\ &= (-1)^n\,\lambda_n\,\cos(\frac{n\pi}{2})\cos[(n+\frac{1}{2})\,t] - (-1)^n\,\lambda_n\,\sin(\frac{n\pi}{2})\sin[(n+\frac{1}{2})\,t] \\ &\quad + (-1)^n\,R_n(t+\frac{\pi}{2})\,. \end{aligned} \tag{2.18}$$

3 Prony–like method

In a first step we determine the even and odd polynomial degrees $n_{0,j}$, $n_{1,k}$ in (2.7), similar as in [16]. We use (2.8) and consider the function

$$\sqrt{\frac{\pi}{2}}\,\sqrt[4]{1-x^2}\,H(x)=\sum_{j=1}^{M_0}c_{0,j}\,Q_{n_{0,j}}(x)+\sum_{k=1}^{M_1}c_{1,k}\,Q_{n_{1,k}}(x)\,. \tag{3.1}$$

Now we use the approximation from Theorem 2.4. This means that we have to determine the even and odd polynomial degrees $n_{0,j}$, $n_{1,k}$ from sampling values of the function

$$\sqrt{\frac{\pi}{2}}\,\sqrt{\cos t}\,H(\sin t)\approx F(t)$$

and by using (2.18) we infer

$$F(t):=\sum_{j=1}^{M_0}\tilde{c}_{0,j}\,\cos[(n_{0,j}+\frac{1}{2})t]+\sum_{k=1}^{M_1}\tilde{c}_{1,k}\sin[(n_{1,k}+\frac{1}{2})t]\,.$$

To this end we consider

$$\frac{F(t)+F(-t)}{2}=\sum_{j=1}^{M_0}\tilde{c}_{0,j}\,\cos[(n_{0,j}+\frac{1}{2})t] \tag{3.2}$$

and

$$\frac{F(t)-F(-t)}{2}=\sum_{k=1}^{M_1}\tilde{c}_{1,k}\,\sin[(n_{1,k}+\frac{1}{2})t]\,. \tag{3.3}$$

Now we proceed similar as in [16], but we use only sampling points near by 0, due to the small values of the error term $R_n(t+\frac{\pi}{2})$ in a small neighborhood of $t=0$ (see (2.16)). Let $N\in\mathbb{N}$ be sufficiently large such that $N>M$ and $2N-1$ is an upper bound of the degree of the polynomial (2.6). For $u_N:=\sin\frac{\pi}{2N-1}$ we form the nonequidistant sine–grid $\{u_{N,k}:=\sin\frac{k\pi}{2N-1};\ k=1-2M,\ldots,2M-1\}$ in the interval $[-1,\,1]$.
We consider the following problem of *sparse Legendre interpolation*: For given sampled data

$$h_k:=\sqrt{\frac{\pi}{2}}\,\sqrt{\cos\frac{k\pi}{2N-1}}\,H(\sin\frac{k\pi}{2N-1})\quad(k=1-2M,\ldots,2M-1)$$

determine all parameters $n_{0,j}$ $(j=1,\ldots,M_0)$ of the sparse cosine sum (3.2), determine all parameters $n_{1,k}$ $(k=1,\ldots,M_1)$ of the sparse sine sum (3.3) and finally determine all coefficients $c_{0,j}$ $(j=1,\ldots,M_0)$ and $c_{1,k}$ $(k=1,\ldots,M_1)$ of the sparse Legendre expansion (2.7).

3.1 Sparse even Legendre interpolation

For a moment, we assume that the even Legendre sparsity M_0 of the polynomial (2.7) is known. Then we see that the above interpolation problem is closely related to the *interpolation problem of the sparse, even trigonometric polynomial*

$$\frac{h_k + h_{-k}}{2} \approx f_k := \sum_{j=1}^{M_0} \tilde{c}_{0,j} \, \cos \frac{(n_{0,j} + 1/2)k\pi}{2N - 1} \quad (k = 0, \ldots, 2M_0 - 1)\,, \tag{3.4}$$

where the sampled values f_k $(k = 0, \ldots, 2M_0 - 1)$ are approximatively given.
We introduce the *Prony polynomial* Π_0 of degree M_0 with the leading coefficient 2^{M_0-1}, whose roots are $\cos \frac{(n_{0,j}+1/2)\pi}{2N-1}$ $(j = 1, \ldots, M_0)$, i.e.

$$\Pi_0(x) = 2^{M_0-1} \prod_{j=1}^{M_0} \big(x - \cos \frac{(n_{0,j} + 1/2)\pi}{2N - 1}\big)\,.$$

Then the Prony polynomial Π_0 can be represented in the Chebyshev basis by

$$\Pi_0(x) = \sum_{l=0}^{M_0} p_{0,l} \, T_l(x) \quad (p_{0,M_0} := 1)\,. \tag{3.5}$$

The coefficients $p_{0,l}$ of the Prony polynomial (3.5) can be characterized as follows:

Lemma 3.1 *(see [16]) For all $k = 0, \ldots, M_0 - 1$, the sampled data f_k and the coefficients $p_{0,l}$ of the Prony polynomial* (3.5) *satisfy the equations*

$$\sum_{j=0}^{M_0-1} (f_{j+k} + f_{|j-k|}) \, p_{0,j} = -(f_{k+M_0} + f_{|M_0-k|})\,.$$

Using Lemma 3.1, one obtains immediately a Prony method for sparse even Legendre interpolation in the case of known even Legendre sparsity. This algorithm is similar to Algorithm 2.7 in [16] and omitted here.
In practice, the even/odd Legendre sparsities M_0, M_1 of the polynomial (2.7) of degree at most $2N - 1$ are unknown. Then we can apply the same technique as in Section 3 of [16]. We assume that an upper bound $L \in \mathbb{N}$ of $\max\{M_0, M_1\}$ is known, where $N \in \mathbb{N}$ is sufficiently large with $\max\{M_0, M_1\} \le L \le N$. In order to improve the numerical stability, we allow to choose more sampling points. Therefore we introduce an additional parameter K with $L \le K \le N$ such that we use $K + L$ sampling points of (2.7), more precisely we assume that sampled data f_k $(k = 0, \ldots, L + K - 1)$ from (3.4) are given. With the $L + K$ sampled data $f_k \in \mathbb{R}$ $(k = 0, \ldots, L + K - 1)$, we form the *rectangular Toeplitz–plus–Hankel matrix*

$$\boldsymbol{H}^{(0)}_{K,L+1} := \big(f_{l+m} + f_{|l-m|}\big)_{l,m=0}^{K-1,L}\,. \tag{3.6}$$

Note that $\boldsymbol{H}^{(0)}_{K,L+1}$ is rank deficient with rank M_0 (see Lemma 3.1 in [16]).

3.2 Sparse odd Legendre interpolation

For a moment, we assume that the odd Legendre sparsity M_1 of the polynomial (2.7) is known. Then we see that the above interpolation problem is closely related to the *interpolation problem of the sparse, odd trigonometric polynomial*

$$\frac{h_k - h_{-k}}{2} \approx g_k := \sum_{j=1}^{M_1} \tilde{c}_{1,j} \, \sin \frac{(n_{1,j} + 1/2)k\pi}{2N-1} \quad (k = 0, \ldots, 2M_1 - 1)\,, \tag{3.7}$$

where the sampled values g_k $(k = 0, \ldots, 2M_1 - 1)$ are approximatively given.
We introduce the *Prony polynomial* Π_1 of degree M_1 with the leading coefficient 2^{M_1-1}, whose roots are $\cos \frac{(n_{1,j}+1/2)\pi}{2N-1}$ $(j = 1, \ldots, M_1)$, i.e.

$$\Pi_1(x) = 2^{M_1-1} \prod_{j=1}^{M_1} \big(x - \cos \frac{(n_{1,j} + 1/2)\pi}{2N-1}\big)\,.$$

Then the Prony polynomial Π_1 can be represented in the Chebyshev basis by

$$\Pi_1(x) = \sum_{l=0}^{M_1} p_{1,l}\, T_l(x) \quad (p_{1,M_1} := 1)\,. \tag{3.8}$$

The coefficients $p_{1,j}$ of the Prony polynomial (3.8) can be characterized as follows:

Lemma 3.2 *For all $k = 0, \ldots, M_1 - 1$, the sampled data g_k and the coefficients $p_{1,l}$ of the Prony polynomial* (3.8) *satisfy the equations*

$$\sum_{j=0}^{M_1-1} (g_{j+k} + g_{j-k})\, p_{1,j} = -(g_{k+M_1} + g_{M_1-k})\,. \tag{3.9}$$

Proof. Using $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \, \cos \beta$, we obtain by (3.7) that

$$\begin{aligned} g_{j+k} + g_{j-k} &= 2 \sum_{l=1}^{M_1} \tilde{c}_{1,l} \Big(\sin \frac{(n_{1,l} + 1/2)(j+k)\pi}{2N-1} + \sin \frac{(n_{1,l} + 1/2)(j-k)\pi}{2N-1} \Big) \\ &= 2 \sum_{l=1}^{M_1} \tilde{c}_{1,l} \, \sin \frac{(n_{1,l} + 1/2)j\pi}{2N-1} \, \cos \frac{(n_{1,l} + 1/2)k\pi}{2N-1}\,. \end{aligned}$$

Thus we conclude that

$$\begin{aligned} \sum_{j=0}^{M_1} \big(g_{j+k} + g_{j-k}\big)\, p_{1,j} &= 2 \sum_{l=1}^{M_1} \tilde{c}_{1,l} \, \sin \frac{(n_{1,l} + 1/2)k\pi}{2N-1} \sum_{j=0}^{M_1} p_{1,j} \, \cos \frac{(n_{1,l} + 1/2)j\pi}{2N-1} \\ &= 2 \sum_{l=1}^{M_1} \tilde{c}_{1,l} \, \sin \frac{(n_{1,l} + 1/2)k\pi}{2N-1} \, \Pi_1\big(\cos \frac{(n_{1,l} + 1/2)\pi}{2N-1}\big) = 0\,. \end{aligned}$$

By $p_{1,M_1} = 1$, this implies the assertion (3.9). ■

Using Lemma 3.2, one can formulate a Prony method for sparse odd Legendre interpolation in the case of known odd Legendre sparsity. This algorithm is similar to Algorithm 2.7 in [16] and omitted here.

In general, the even/odd Legendre sparsities M_0 and M_1 of the polynomial (2.7) of degree at most $2N-1$ are unknown. Similarly as in Subsection 3.1, let $L \in \mathbb{N}$ be a convenient upper bound of $\max\{M_0,\, M_1\}$, where $N \in \mathbb{N}$ is sufficiently large with $\max\{M_0,\, M_1\} \le L \le N$. In order to improve the numerical stability, we allow to choose more sampling points. Therefore we introduce an additional parameter K with $L \le K \le N$ such that we use $K+L$ sampling points of (2.7), more precisely we assume that sampled data g_k $(k = 0, \ldots, L+K-1)$ from (3.7) are given. With the $L+K$ sampled data $g_k \in \mathbb{R}$ $(k = 0, \ldots, L+K-1)$ we form the *rectangular Toeplitz–plus–Hankel matrix*

$$\boldsymbol{H}^{(1)}_{K,L+1} := \big(g_{l+m} + g_{l-m}\big)_{l,m=0}^{K-1,L} \,. \tag{3.10}$$

Note that $\boldsymbol{H}^{(1)}_{K,L+1}$ is rank deficient with rank M_1. This is an analogous result to Lemma 3.1 in [16].

3.3 Sparse Legendre interpolation

In this subsection, we sketch a Prony–like method for the computation of the polynomial degrees $n_{0,j}$ and $n_{1,k}$ of the sparse Legendre expansion (2.7). Mainly we use singular value decompositions (SVD) of the Toeplitz–plus–Hankel matrices (3.6) and (3.10). For details of this method see Section 3 in [16]. We start with the singular value factorizations

$$\begin{aligned} \boldsymbol{H}^{(0)}_{K,L+1} &= \boldsymbol{U}^{(0)}_{K}\, \boldsymbol{D}^{(0)}_{K,L+1}\, \boldsymbol{W}^{(0)}_{L+1}\,, \\ \boldsymbol{H}^{(1)}_{K,L+1} &= \boldsymbol{U}^{(1)}_{K}\, \boldsymbol{D}^{(1)}_{K,L+1}\, \boldsymbol{W}^{(1)}_{L+1}\,, \end{aligned}$$

where $\boldsymbol{U}^{(0)}_{K}$, $\boldsymbol{U}^{(1)}_{K}$, $\boldsymbol{W}^{(0)}_{L+1}$ and $\boldsymbol{W}^{(1)}_{L+1}$ are orthogonal matrices and where $\boldsymbol{D}^{(0)}_{K,L+1}$ and $\boldsymbol{D}^{(1)}_{K,L+1}$ are rectangular diagonal matrices. The diagonal entries of $\boldsymbol{D}^{(0)}_{K,L+1}$ are the singular values of (3.6) arranged in nonincreasing order

$$\sigma^{(0)}_1 \ge \sigma^{(0)}_2 \ge \ldots \ge \sigma^{(0)}_{M_0} \ge \sigma^{(0)}_{M_0+1} \ge \ldots \ge \sigma^{(0)}_{L+1} \ge 0\,.$$

We determine M_0 such that $\sigma^{(0)}_{M_0}/\sigma^{(0)}_1 \ge \varepsilon$, which is approximatively the rank of the matrix (3.6) and which coincides with the even Legendre sparsity M_0 of the polynomial (2.7).

Similarly, the diagonal entries of $\boldsymbol{D}^{(1)}_{K,L+1}$ are the singular values of (3.10) arranged in nonincreasing order

$$\sigma^{(1)}_1 \ge \sigma^{(1)}_2 \ge \ldots \ge \sigma^{(1)}_{M_1} \ge \sigma^{(1)}_{M_1+1} \ge \ldots \ge \sigma^{(1)}_{L+1} \ge 0\,.$$

We determine M_1 such that $\sigma^{(1)}_{M_1}/\sigma^{(1)}_1 \ge \varepsilon$, which is approximatively the rank of the matrix (3.10) and which coincides with the odd Legendre sparsity M_1 of the polynomial

(2.7). Note that there is often a gap in the singular values, such that we can choose $\varepsilon = 10^{-8}$ in general.
Introducing the matrices

$$\begin{aligned}
\boldsymbol{D}^{(0)}_{K,M_0} &:= \boldsymbol{D}^{(0)}_{K,L+1}(1:K,\,1:M_0) = \begin{pmatrix} \operatorname{diag}\,(\sigma^{(0)}_j)_{j=1}^{M_0} \\ \boldsymbol{O}_{K-M_0,M_0} \end{pmatrix}, \\
\boldsymbol{W}^{(0)}_{M_0,L+1} &:= \boldsymbol{W}^{(0)}_{L+1}(1:M_0,\,1:L+1)\,, \\
\boldsymbol{D}^{(1)}_{K,M_1} &:= \boldsymbol{D}^{(1)}_{K,L+1}(1:K,\,1:M_1) = \begin{pmatrix} \operatorname{diag}\,(\sigma^{(1)}_j)_{j=1}^{M_1} \\ \boldsymbol{O}_{K-M_1,M_1} \end{pmatrix}, \\
\boldsymbol{W}^{(1)}_{M_1,L+1} &:= \boldsymbol{W}^{(1)}_{L+1}(1:M_1,\,1:L+1)\,,
\end{aligned}$$

we can simplify the SVD of the Toeplitz–plus–Hankel matrices (3.6) and (3.10) as follows

$$\boldsymbol{H}^{(0)}_{K,L+1} = \boldsymbol{U}^{(0)}_K\,\boldsymbol{D}^{(0)}_{K,M_0}\,\boldsymbol{W}^{(0)}_{M_0,L+1}\,, \qquad \boldsymbol{H}^{(1)}_{K,L+1} = \boldsymbol{U}^{(1)}_K\,\boldsymbol{D}^{(1)}_{K,M_1}\,\boldsymbol{W}^{(1)}_{M_1,L+1}\,.$$

Using the know submatrix notation and setting

$$\boldsymbol{W}^{(0)}_{M_0,L}(s) := \boldsymbol{W}^{(0)}_{M_0,L+1}(1:M_0,\,1+s:L+s) \quad (s=0,\,1)\,, \tag{3.11}$$

$$\boldsymbol{W}^{(1)}_{M_1,L}(s) := \boldsymbol{W}^{(1)}_{M_1,L+1}(1:M_1,\,1+s:L+s) \quad (s=0,\,1)\,, \tag{3.12}$$

we form the matrices

$$\boldsymbol{F}^{(0)}_{M_0} := \big(\boldsymbol{W}^{(0)}_{M_0,L}(0)\big)^{\dagger}\,\boldsymbol{W}^{(0)}_{M_0,L}(1)\,, \tag{3.13}$$

$$\boldsymbol{F}^{(1)}_{M_1} := \big(\boldsymbol{W}^{(1)}_{M_1,L}(0)\big)^{\dagger}\,\boldsymbol{W}^{(1)}_{M_1,L}(1)\,, \tag{3.14}$$

where $\big(\boldsymbol{W}^{(0)}_{M_0,L}(0)\big)^{\dagger}$ denotes the Moore–Penrose pseudoinverse of $\boldsymbol{W}^{(0)}_{M_0,L}(0)$. Finally we determine the nodes $x_{0,j} \in [-1,\,1]$ $(j = 1,\ldots,M_0)$ and $x_{1,j} \in [-1,\,1]$ $(j = 1,\ldots,M_1)$ as eigenvalues of the matrix $\boldsymbol{F}^{(0)}_{M_0}$ and $\boldsymbol{F}^{(1)}_{M_1}$, respectively. Thus the algorithm reads as follows:

Algorithm 3.3 (Sparse Legendre interpolation based on SVD)
Input: $L,\,K,\,N \in \mathbb{N}$ $(N \gg 1,\ 3 \le L \le K \le N)$, L is upper bound of $\max\{M_0, M_1\}$, sampled values $H(\sin\frac{k\pi}{2N-1})$ $(k = 1-L-K,\ldots,L+K-1)$ of the polynomial (2.7) of degree at most $2N-1$.

1. Multiply

$$h_k := \sqrt{\frac{\pi}{2}}\,\sqrt{\cos\frac{k\pi}{2N-1}}\,H\big(\sin\frac{k\pi}{2N-1}\big) \quad (k = 1-L-K,\ldots,L+K-1)$$

and form

$$f_k := \frac{h_k + h_{-k}}{2}\,, \quad g_k := \frac{h_k - h_{-k}}{2} \quad (k = 0,\ldots,L+K-1)\,.$$

2. Compute the SVD of the rectangular Toeplitz–plus–Hankel matrices (3.6) and (3.10). Determine the approximative rank M_0 of (3.6) such that $\sigma^{(0)}_{M_0}/\sigma^{(0)}_1 > 10^{-8}$ and form the matrix (3.11). Determine the approximative rank M_1 of (3.10) such that $\sigma^{(1)}_{M_1}/\sigma^{(1)}_1 > 10^{-8}$ and form the matrix (3.12).
3. Compute all eigenvalues $x_{0,j} \in [-1, 1]$ $(j = 1, \ldots, M_0)$ of the square matrix (3.13). Assume that the eigenvalues are ordered in the following form $1 \geq x_{0,1} > x_{0,2} > \ldots > x_{0,M_0} \geq -1$. Calculate $n_{0,j} := [\frac{2N-1}{\pi} \arccos x_{0,j} - \frac{1}{2}]$ $(j = 1, \ldots, M_0)$, where $[x] := \lfloor x + 0.5 \rfloor$ means rounding of $x \in \mathbb{R}$ to the nearest integer.
4. Compute all eigenvalues $x_{1,j} \in [-1, 1]$ $(j = 1, \ldots, M_1)$ of the square matrix (3.14). Assume that the eigenvalues are ordered in the following form $1 \geq x_{1,1} > x_{1,2} > \ldots > x_{1,M_1} \geq -1$. Calculate $n_{1,j} := [\frac{2N-1}{\pi} \arccos x_{1,j} - \frac{1}{2}]$ $(j = 1, \ldots, M_1)$.
5. Compute the coefficients $c_{0,j} \in \mathbb{R}$ $(j = 1, \ldots, M_0)$ and $c_{1,j} \in \mathbb{R}$ $(j = 1, \ldots, M_1)$ as least squares solutions of the overdetermined linear Vandermonde–like systems

$$\begin{aligned} \sum_{j=1}^{M_0} c_{0,j}\, Q_{n_{0,j}}\big(\sin \frac{k\pi}{2N-1}\big) &= f_k \quad (k = 0, \ldots, L+K-1)\,, \\ \sum_{j=1}^{M_1} c_{1,j}\, Q_{n_{1,j}}\big(\sin \frac{k\pi}{2N-1}\big) &= g_k \quad (k = 0, \ldots, L+K-1)\,. \end{aligned}$$

Output: $M_0 \in \mathbb{N}_0$, $n_{0,j} \in \mathbb{N}_0$ $(0 \leq n_{0,1} < n_{0,2} < \ldots < n_{0,M_0} < 2N)$, $c_{0,j} \in \mathbb{R}$ $(j = 1, \ldots, M_0)$. $M_1 \in \mathbb{N}_0$, $n_{1,j} \in \mathbb{N}$ $(1 \leq n_{1,1} < n_{1,2} < \ldots < n_{1,M_1} < 2N)$, $c_{1,j} \in \mathbb{R}$ $(j = 1, \ldots, M_1)$.

Remark 3.4 The Algorithm 3.3 is very similar to the Algorithm 3.5 in [16]. Note that one can also use the QR decomposition of the rectangular Toeplitz–plus–Hankel matrices (3.6) and (3.10) instead of the SVD. In that case one obtains an algorithm similar to the Algorithm 3.4 in [16]. □

4 Extension to Gegenbauer polynomials

In this section we show that our reconstruction method can be generalized to sparse Gegenbauer extensions of low positive order. The *Gegenbauer polynomials* $C^{(\alpha)}_n$ of degree $n \in \mathbb{N}_0$ and fixed order $\alpha > 0$ can be defined by the recursion relation (see [19, p. 81]):

$$C^{(\alpha)}_{n+2}(x) := \frac{2\alpha + 2n + 2}{n+2}\, x\, C^{(\alpha)}_{n+1}(x) - \frac{2\alpha + n}{n+2}\, C^{(\alpha)}_n(x) \quad (n \in \mathbb{N}_0)$$

with $C^{(\alpha)}_0(x) := 1$ and $C^{(\alpha)}_1(x) := 2\alpha\, x$. Sometimes, $C^{(\alpha)}_n$ are called *ultraspherical polynomials* too. In the case $\alpha = \frac{1}{2}$, one obtains again the Legendre polynomials $P_n = C^{(1/2)}_n$. By [19, p. 84], an explicit representation of the Gegenbauer polynomial $C^{(\alpha)}_n$ reads as follows

$$C^{(\alpha)}_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j\, \Gamma(n - j + \alpha)}{\Gamma(\alpha)\, \Gamma(j+1)\, \Gamma(n - 2j + 1)}\, (2x)^{n-2j}\,.$$

Thus the Gegenbauer polynomials satisfy the symmetry relations

$$C_n^{(\alpha)}(-x) = (-1)^n \, C_n^{(\alpha)}(x)\,. \tag{4.1}$$

Further one obtains that for $m \in \mathbb{N}_0$

$$C_{2m}^{(\alpha)}(0) = \frac{(-1)^m \, \Gamma(m+\frac{1}{2})}{\Gamma(\alpha)\,\Gamma(m+1)}\,, \quad C_{2m+1}^{(\alpha)}(0) = 0\,, \tag{4.2}$$

$$\Big(\frac{\mathrm{d}}{\mathrm{d}x}\, C_{2m+1}^{(\alpha)}\Big)(0) = \frac{2\,(-1)^m \, \Gamma(\alpha+m+1)}{\Gamma(\alpha)\,\Gamma(m+1)}\,, \quad \Big(\frac{\mathrm{d}}{\mathrm{d}x}\, C_{2m}^{(\alpha)}\Big)(0) = 0\,. \tag{4.3}$$

Moreover, the Gegenbauer polynomial $C_n^{(\alpha)}$ satisfies the following homogeneous linear differential equation of second order (see [19, p. 80])

$$(1-x^2)\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\, C_n^{(\alpha)}(x) - (2\alpha+1)\,x\,\frac{\mathrm{d}}{\mathrm{d}x}\, C_n^{(\alpha)}(x) + n\,(n+2\alpha)\, C_n^{(\alpha)}(x) = 0\,. \tag{4.4}$$

Further, the Gegenbauer polynomials are orthogonal over the interval $[-1,\,1]$ with respect to the weight function

$$w^{(\alpha)}(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\,(1-x^2)^{\alpha-1/2} \quad (x \in (-1,\,1))\,,$$

i.e. more precisely by [19, p. 81]

$$\int_{-1}^{1} C_m^{(\alpha)}(x)\, C_n^{(\alpha)}(x)\, w^{(\alpha)}(x)\,\mathrm{d}x = \frac{\alpha\,\Gamma(2\alpha+n)}{(n+\alpha)\,\Gamma(n+1)\,\Gamma(2\alpha)}\,\delta_{m-n} \quad (m,\,n \in \mathbb{N}_0)\,.$$

Note that the weight function $w^{(\alpha)}$ is normalized by

$$\int_{-1}^{1} w^{(\alpha)}(x)\,\mathrm{d}x = 1\,.$$

Then the *normed Gegenbauer polynomials*

$$L_n^{(\alpha)}(x) := \sqrt{\frac{(n+\alpha)\,\Gamma(n+1)\,\Gamma(2\alpha)}{\alpha\,\Gamma(2\alpha+n)}}\; C_n^{(\alpha)}(x) \quad (n \in \mathbb{N}_0) \tag{4.5}$$

form an orthonormal basis in the weighted Hilbert space $L_{w^{(\alpha)}}([-1,\,1])$.
Let M be a positive integer. A polynomial

$$H(x) := \sum_{k=0}^{d} b_k\, L_k^{(\alpha)}(x)$$

of degree d with $d \gg M$ is called *M-sparse in the Gegenbauer basis* or simply a *sparse Gegenbauer expansion*, if M coefficients b_k are nonzero and if the other $d - M + 1$

coefficients vanish. Then such an M-sparse polynomial H can be represented in the form

$$H(x) = \sum_{j=1}^{M_0} c_{0,j}\, L_{n_{0,j}}^{(\alpha)}(x) + \sum_{k=1}^{M_1} c_{1,k}\, L_{n_{1,k}}^{(\alpha)}(x) \tag{4.6}$$

with $c_{0,j} := b_{n_{0,j}} \neq 0$ for all even $n_{0,j}$ with $0 \le n_{0,1} < n_{0,2} < \ldots < n_{0,M_0}$ and with $c_{1,k} := b_{n_{1,k}} \neq 0$ for all odd $n_{1,k}$ with $1 \le n_{1,1} < n_{1,2} < \ldots < n_{1,M_1}$. The positive integer $M = M_0 + M_1$ is called the *Gegenbauer sparsity* of the polynomial H. The integers M_0, M_1 are the *even* and *odd Gegenbauer sparsities*, respectively.
Now for each $n \in \mathbb{N}_0$, we introduce the functions $Q_n^{(\alpha)}$ by

$$Q_n^{(\alpha)}(x) := \sqrt{\frac{\Gamma(\alpha+1)\,\sqrt{\pi}}{\Gamma(\alpha+\frac{1}{2})}}\,(1-x^2)^{\alpha/2}\, L_n^{(\alpha)}(x) \quad (x \in [-1,\, 1]) \tag{4.7}$$

These functions $Q_n^{(\alpha)}$ possess the same symmetry properties (4.1) as the Gegenbauer polynomials, namely

$$Q_n^{(\alpha)}(-x) = (-1)^n\, Q_n^{(\alpha)}(x) \quad (x \in [-1,\, 1])\,. \tag{4.8}$$

Further the functions $Q_n^{(\alpha)}$ are orthonormal in the weighted Hilbert space $L_w^2([-1,\, 1])$ with the Chebyshev weight $w(x) = \frac{1}{\pi}\,(1-x^2)^{-1/2}$, since for all $m,\, n \in \mathbb{N}_0$

$$\int_{-1}^{1} Q_m^{(\alpha)}(x)\, Q_n^{(\alpha)}(x)\, w(x)\,\mathrm{d}x = \int_{-1}^{1} L_m^{(\alpha)}(x)\, L_n^{(\alpha)}(x)\, w^{(\alpha)}(x)\,\mathrm{d}x = \delta_{m-n}\,.$$

In the following, we use the standard substitution $x = \cos\theta$ ($\theta \in [0,\, \pi]$) and obtain

$$Q_n^{(\alpha)}(\cos\theta) := \sqrt{\frac{\Gamma(\alpha+1)\,\sqrt{\pi}}{\Gamma(\alpha+\frac{1}{2})}}\,(\sin\theta)^{\alpha}\, L_n^{(\alpha)}(\cos\theta)\,.$$

Lemma 4.1 *For all $n \in \mathbb{N}_0$ and $\alpha \in (0,\, 1)$, the functions $Q_n^{(\alpha)}(\cos\theta)$ are uniformly bounded on the interval $[0,\, \pi]$, i.e.*

$$|Q_n^{(\alpha)}(\cos\theta)| < 2 \quad (\theta \in [0,\, \pi])\,. \tag{4.9}$$

Proof. For $n \in \mathbb{N}_0$ and $\alpha \in (0,\, 1)$, we know by [12] that for all $\theta \in [0,\, \pi]$

$$(\sin\theta)^{\alpha}\, |C_n^{(\alpha)}(\cos\theta)| < \frac{2^{1-\alpha}}{\Gamma(\alpha)}\,(n+\alpha)^{\alpha-1}\,.$$

Then for the normed Gegenbauer polynomials $L_n^{(\alpha)}$, we obtain the estimate

$$(\sin\theta)^{\alpha}\, |L_n^{(\alpha)}(\cos\theta)| < \frac{2^{1-\alpha}}{\Gamma(\alpha)}\,\sqrt{\frac{(n+\alpha)\,\Gamma(n+1)\,\Gamma(2\alpha)}{\alpha\,\Gamma(2\alpha+n)}}\,(n+\alpha)^{\alpha-1}\,.$$

Using the duplication formula of the gamma function

$$\Gamma(\alpha)\,\Gamma(\alpha+\frac{1}{2}) = 2^{1-2\alpha}\,\sqrt{\pi}\,\Gamma(2\alpha)\,, \tag{4.10}$$

we can estimate

$$|Q_n^{(\alpha)}(\cos\theta)| < \sqrt{2}\,\sqrt{\frac{\Gamma(n+1)}{\Gamma(2\alpha+n)}}\,(n+\alpha)^{\alpha-1/2}\,.$$

For $\alpha = \frac{1}{2}$, we obtain $|Q_n^{(1/2)}(\cos\theta)| < \sqrt{2}$. In the following, we use the inequalities (see [8])

$$\big(n+\frac{\sigma}{2}\big)^{1-\sigma} < \frac{\Gamma(n+1)}{\Gamma(n+\sigma)} < \big(n-\frac{1}{2}+\sqrt{\sigma+\frac{1}{4}}\big)^{1-\sigma} \tag{4.11}$$

for all $n \in \mathbb{N}$ and $\sigma \in (0,\,1)$.
In the case $0 < \alpha < \frac{1}{2}$, the estimate (4.11) with $\sigma = 2\alpha$ implies that

$$|Q_n^{(\alpha)}(\cos\theta)| < \sqrt{2}\,\Big(\frac{n-\frac{1}{2}+\sqrt{2\alpha+\frac{1}{4}}}{n+\alpha}\Big)^{-\alpha+1/2}\,.$$

Since $n-\frac{1}{2}+\sqrt{2\alpha+\frac{1}{4}} < 2\,(n+\alpha)$ for all $n \in \mathbb{N}$, we conclude that

$$|Q_n^{(\alpha)}(\cos\theta)| < 2^{1-\alpha}\,.$$

In the case $\frac{1}{2} < \alpha < 1$, we set $\beta := 1-2\,\alpha \in (0,\,1)$. By (4.11) with $\sigma = \beta$, we can estimate

$$\frac{\Gamma(n+1)}{\Gamma(n+2\alpha)} = \frac{\Gamma(n+1)}{(n+\beta)\,\Gamma(n+\beta)} < \frac{1}{n+\beta}\,\big(n-\frac{1}{2}+\sqrt{\beta+\frac{1}{4}}\big)^{1-\beta}\,.$$

Hence we obtain by $n-\frac{1}{2}+\sqrt{\beta+\frac{1}{4}} < n+\beta$ that

$$\begin{aligned}|Q_n^{(\alpha)}(\cos\theta)| \;&<\; \frac{\sqrt{2}}{\sqrt{n+\beta}}\,\big(n-\frac{1}{2}+\sqrt{\beta+\frac{1}{4}}\big)^{(1-\beta)/2}\,(n+\alpha)^{\alpha-1/2}\\ &<\; \sqrt{2}\,\big(n-\frac{1}{2}+\sqrt{\beta+\frac{1}{4}}\big)^{-\beta/2}\,\big(n+\frac{1-\beta}{2}\big)^{\beta/2} < \sqrt{2}\,.\end{aligned}$$

Finally, by

$$Q_0^{(\alpha)}(\cos\theta) = \sqrt{\frac{\alpha\,\Gamma(\alpha)\,\sqrt{\pi}}{\Gamma(\alpha+\frac{1}{2})}}\,(\sin\theta)^{\alpha}$$

and

$$|Q_0^{(\alpha)}(\cos\theta)| \le \sqrt[4]{\pi}\,,$$

we see that the estimate (4.9) is also true for $n = 0$. ■

By (4.4), the function $Q_n^{(\alpha)}(\cos\theta)$ satisfies the following linear differential equation of second order (see [19, p. 81])

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\, Q_n^{(\alpha)}(\cos\theta) + \Big((n+\alpha)^2 + \frac{\alpha(1-\alpha)}{(\sin\theta)^2}\Big)\, Q_n^{(\alpha)}(\cos\theta) = 0 \quad (\theta \in (0,\,\pi))\,. \tag{4.12}$$

By the method of Liouville–Stekloff, see [19, p. 210 – 212], we show that for *arbitrary* $n \in \mathbb{N}_0$, the function $Q_n^{(\alpha)}(\cos\theta)$ is approximately equal to some multiple of $\cos[(n+\alpha)\,\theta - \frac{\alpha\pi}{2}]$ in a small neighborhood of $\theta = \frac{\pi}{2}$.

Theorem 4.2 *For each $n \in \mathbb{N}_0$ and $\alpha \in (0,\,1)$, the function $Q_n^{(\alpha)}(\cos\theta)$ can be represented by the asymptotic formula*

$$Q_n^{(\alpha)}(\cos\theta) = \lambda_n\, \cos[(n+\alpha)\,\theta - \frac{\alpha\pi}{2}] + R_n^{(\alpha)}(\theta) \quad (\theta \in [0,\,\pi]) \tag{4.13}$$

with the scaling factor

$$\lambda_n := \begin{cases} \sqrt{\frac{(2m+\alpha)\,\Gamma(2m+1)}{\Gamma(2\alpha+2m)}}\; \frac{2^{\alpha-1/2}\,\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} & n = 2m\,, \\ \sqrt{\frac{\Gamma(2m+2)}{(2m+1+\alpha)\,\Gamma(2\alpha+2m+1)}}\; \frac{2^{\alpha+1/2}\,\Gamma(\alpha+m+1)}{\Gamma(m+1)} & n = 2m+1 \end{cases}$$

and the error term

$$R_n^{(\alpha)}(\theta) := -\frac{\alpha(1-\alpha)}{n+\alpha} \int_{\pi/2}^{\theta} \frac{\sin[(n+\alpha)(\theta-\tau)]}{(\sin\tau)^2}\, Q_n^{(\alpha)}(\cos\tau)\,\mathrm{d}\tau \quad (\theta \in (0,\,\pi))\,. \tag{4.14}$$

The error term $R_n^{(\alpha)}(\theta)$ satisfies the conditions

$$R_n^{(\alpha)}(\frac{\pi}{2}) = \big(\frac{\mathrm{d}}{\mathrm{d}\theta}\, R_n^{(\alpha)}\big)(\frac{\pi}{2}) = 0$$

and has the symmetry property

$$R_n^{(\alpha)}(\pi - \theta) = (-1)^n\, R_n^{(\alpha)}(\theta)\,. \tag{4.15}$$

Further, the error term can be estimated by

$$|R_n^{(\alpha)}(\theta)| \le \frac{2\alpha(1-\alpha)}{n+\alpha}\, |\cot\theta|\,. \tag{4.16}$$

Proof. 1. Using the method of Liouville–Stekloff (see [19, p. 210 – 212]), we derive the asymptotic formula (4.13) from the differential equation (4.12), which can be written in the form

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\, Q_n^{(\alpha)}(\cos\theta) + (n+\alpha)^2\, Q_n^{(\alpha)}(\cos\theta) = -\frac{\alpha(1-\alpha)}{(\sin\theta)^2}\, Q_n^{(\alpha)}(\cos\theta) \quad (\theta \in (0,\,\pi))\,. \tag{4.17}$$

Since the homogeneous linear differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\, Q_n^{(\alpha)}(\cos\theta) + (n+\alpha)^2\, Q_n^{(\alpha)}(\cos\theta) = 0$$

has the fundamental system

$$\cos[(n+\alpha)\,\theta - \frac{\alpha\pi}{2}]\,, \quad \sin[(n+\alpha)\,\theta - \frac{\alpha\pi}{2}]\,,$$

the differential equation (4.17) can be transformed into the Volterra integral equation

$$\begin{aligned} Q_n^{(\alpha)}(\cos\theta) &= \lambda_n\, \cos[(n+\alpha)\,\theta - \frac{\alpha\pi}{2}] + \mu_n\, \sin[(n+\alpha)\,\theta - \frac{\alpha\pi}{2}] \\ &\quad - \frac{\alpha(1-\alpha)}{n+\alpha} \int_{\pi/2}^{\theta} \frac{\sin[(n+\alpha)(\theta-\tau)]}{(\sin\tau)^2}\, Q_n^{(\alpha)}(\cos\tau)\,\mathrm{d}\tau \quad (\theta \in (0,\,\pi)) \end{aligned}$$

with certain real constants λ_n and μ_n. The integral (4.14) and its derivative vanish for $\theta = \frac{\pi}{2}$.
2. Now we determine the constants λ_n and μ_n. For arbitrary even $n = 2m$ ($m \in \mathbb{N}_0$), the function $Q_{2m}^{(\alpha)}(\cos\theta)$ can be represented in the form

$$Q_{2m}^{(\alpha)}(\cos\theta) = \lambda_{2m}\, \cos[(2m+\alpha)\,\theta - \frac{\alpha\pi}{2}] + \mu_{2m}\, \sin[(2m+\alpha)\,\theta - \frac{\alpha\pi}{2}] + R_{2m}^{(\alpha)}(\theta)$$

Hence the condition $R_{2m}^{(\alpha)}(\frac{\pi}{2}) = 0$ means that $Q_{2m}^{(\alpha)}(0) = (-1)^m \lambda_{2m}$. Using (4.7), (4.5), (4.2), and the duplication formula (4.10), we obtain that

$$\lambda_{2m} = \sqrt{\frac{(2m+\alpha)\,\Gamma(2m+1)}{\Gamma(2\alpha+2m)}}\, \frac{2^{\alpha-1/2}\,\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}\,.$$

From $\big(\frac{\mathrm{d}}{\mathrm{d}x}\, C_{2m}^{(\alpha)}\big)(0) = 0$ by (4.3) it follows that the derivative of $Q_{2m}^{(\alpha)}(\cos\theta)$ vanishes for $\theta = \frac{\pi}{2}$. Thus the second condition $\big(\frac{\mathrm{d}}{\mathrm{d}\theta}\, R_{2m}^{(\alpha)}\big)(\frac{\pi}{2}) = 0$ implies that

$$0 = \mu_{2m}\,(2m+\alpha)\,(-1)^m\,,$$

i.e. $\mu_{2m} = 0$.
3. If $n = 2m+1$ ($m \in \mathbb{N}_0$) is odd, then

$$\begin{aligned} Q_{2m+1}^{(\alpha)}(\cos\theta) &= \lambda_{2m+1}\, \cos[(2m+1+\alpha)\theta - \frac{\alpha\pi}{2}] \\ &\quad + \mu_{2m+1}\, \sin[(2m+1+\alpha)\,\theta - \frac{\alpha\pi}{2}] + R_{2m+1}^{(\alpha)}(\theta)\,. \end{aligned}$$

Hence the condition $R_{2m+1}^{(\alpha)}(\frac{\pi}{2}) = 0$ implies by $C_{2m+1}^{(\alpha)}(0) = 0$ (see (4.2)) that

$$0 = \mu_{2m+1}\,(-1)^m\,,$$

i.e. $\mu_{2m+1} = 0$. The second condition $\big(\frac{\mathrm{d}}{\mathrm{d}\theta}\, R^{(\alpha)}_{2m+1}\big)(\frac{\pi}{2}) = 0$ reads as follows

$$-\sqrt{\frac{(n+\alpha)\,\Gamma(\alpha)\,\Gamma(2\alpha)\,\Gamma(2m+2)\,\sqrt{\pi}}{\Gamma(\alpha+\frac{1}{2})\,\Gamma(2\alpha+2m+1)\,\Gamma(\alpha)}}\,\big(\frac{\mathrm{d}}{\mathrm{d}x}\,C^{(\alpha)}_{2m+1}\big)(0) = -\lambda_{2m+1}\,(2m+1+\alpha)\,(-1)^m\,.$$

Thus we obtain by (4.3) and the duplication formula (4.10) that

$$\lambda_{2m+1} = \sqrt{\frac{\Gamma(2m+2)}{(2m+1+\alpha)\,\Gamma(2\alpha+2m+1)}\;\frac{2^{\alpha+1/2}\,\Gamma(\alpha+m+1)}{\Gamma(m+1)}}\,.$$

4. As shown, the error term $R^{(\alpha)}_n(\theta)$ has the explicit representation (4.14). Using (4.9), we estimate this integral and obtain

$$|R^{(\alpha)}_n(\theta)| \le \frac{2\alpha(1-\alpha)}{n+\alpha}\,\Big|\int_{\pi/2}^{\theta}\frac{1}{(\sin\tau)^2}\,\mathrm{d}\tau\Big| = \frac{2\alpha(1-\alpha)}{n+\alpha}\,|\cot\theta|\,.$$

The symmetry property (4.15) of the error term

$$R^{(\alpha)}_n(\theta) = Q^{(\alpha)}_n(\cos\theta) - \lambda_n\,\cos[(n+\alpha)\theta - \frac{\alpha\pi}{2}]$$

follows from the fact that $Q^{(\alpha)}_n(\cos\theta)$ and $\cos[(n+\alpha)\theta - \frac{\alpha\pi}{2}]$ possess the same symmetry properties as (4.8). This completes the proof. ■

Remark 4.3 The following result is stated in [9]: If $\alpha \ge \frac{1}{2}$, then

$$(\sin\theta)^\alpha\,|L^{(\alpha)}_n(\cos\theta)| \le 22\,\sqrt{\frac{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}}\,\big(\alpha-\frac{1}{2}\big)^{1/6}\,\big(1+\frac{2\alpha-1}{2n}\big)^{1/12}$$

for all $n \ge 6$ and $\theta \in [0,\,\pi]$. Using (4.11), we can see that the function $(\sin\theta)^\alpha\,|L^{(\alpha)}_n(\cos\theta)|$ is uniformly bounded for all $\alpha \ge \frac{1}{2}$, $n \ge 6$ and $\theta \in [0,\,\pi]$. Using above estimate, one can extend Theorem 4.2 to the case of moderately sized order $\alpha \ge \frac{1}{2}$. □

We observe that the approximation of $R^{(\alpha)}_n(\theta)$ in (4.16) is very accurate in a small neighborhood of $\theta = \frac{\pi}{2}$. By the substitution $t = \theta - \frac{\pi}{2} \in [-\frac{\pi}{2},\,\frac{\pi}{2}]$ and (4.8), we obtain

$$\begin{aligned} Q^{(\alpha)}_n(\sin t) &= (-1)^n\,\lambda_n\,\cos[(n+\alpha)\,t + \frac{n\pi}{2}] + (-1)^n\,R_n(t+\frac{\pi}{2}) \\ &= (-1)^n\,\lambda_n\,\cos(\frac{n\pi}{2})\cos[(n+\alpha)\,t] - (-1)^n\,\lambda_n\,\sin(\frac{n\pi}{2})\sin[(n+\alpha)\,t] \\ &\quad + (-1)^n\,R_n(t+\frac{\pi}{2})\,. \end{aligned}$$

Now the Algorithm 3.3 can be straightforward generalized to the case of a sparse Gegenbauer expansion (4.6):

Algorithm 4.4 (Sparse Gegenbauer interpolation based on SVD)

Input: $L,\ K,\ N \in \mathbb{N}$ ($N \gg 1,\ 3 \le L \le K \le N$), L is upper bound of $\max\{M_0, M_1\}$, sampled values $H(\sin \frac{k\pi}{2N-1})$ ($k = 1-L-K, \ldots, L+K-1$) of polynomial (4.6) of degree at most $2N-1$ and of low order $\alpha > 0$. ($k = 0, \ldots, L+K-1$).

1. Multiply

$$h_k := \sqrt{\frac{\Gamma(\alpha+1)\,\sqrt{\pi}}{\Gamma(\alpha+\frac{1}{2})}}\,\big(\cos\frac{k\pi}{2N-1}\big)^{\alpha}\,H\big(\sin\frac{k\pi}{2N-1}\big) \quad (k = 1-L-K, \ldots, L+K-1)$$

and form

$$f_k := \frac{h_k + h_{-k}}{2}\,, \quad g_k := \frac{h_k - h_{-k}}{2} \quad (k = 0, \ldots, L+K-1)\,.$$

2. Compute the SVD of the rectangular Toeplitz–plus–Hankel matrices (3.6) and (3.10). Determine the approximative rank M_0 of (3.6) such that $\sigma^{(0)}_{M_0}/\sigma^{(0)}_1 > 10^{-8}$ and form the matrix (3.11). Determine the approximative rank M_1 of (3.10) such that $\sigma^{(1)}_{M_1}/\sigma^{(1)}_1 > 10^{-8}$ and form the matrix (3.12).
3. Compute all eigenvalues $x_{0,j} \in [-1, 1]$ ($j = 1, \ldots, M_0$) of the square matrix (3.13). Assume that the eigenvalues are ordered in the following form $1 \ge x_{0,1} > x_{0,2} > \ldots > x_{0,M_0} \ge -1$. Calculate $n_{0,j} := [\frac{2N-1}{\pi} \arccos x_{0,j} - \alpha]$ ($j = 1, \ldots, M_0$).
4. Compute all eigenvalues $x_{1,j} \in [-1, 1]$ ($j = 1, \ldots, M_1$) of the square matrix (3.14). Assume that the eigenvalues are ordered in the following form $1 \ge x_{1,1} > x_{1,2} > \ldots > x_{1,M_1} \ge -1$. Calculate $n_{1,j} := [\frac{2N-1}{\pi} \arccos x_{1,j} - \alpha]$ ($j = 1, \ldots, M_1$).
5. Compute the coefficients $c_{0,j} \in \mathbb{R}$ ($j = 1, \ldots, M_0$) and $c_{1,j} \in \mathbb{R}$ ($j = 1, \ldots, M_1$) as least squares solutions of the overdetermined linear Vandermonde–like systems

$$\begin{aligned} \sum_{j=1}^{M_0} c_{0,j}\, Q^{(\alpha)}_{n_{0,j}}\big(\sin\frac{k\pi}{2N-1}\big) &= f_k \quad (k = 0, \ldots, L+K-1)\,, \\ \sum_{j=1}^{M_1} c_{1,j}\, Q^{(\alpha)}_{n_{1,j}}\big(\sin\frac{k\pi}{2N-1}\big) &= g_k \quad (k = 0, \ldots, L+K-1)\,. \end{aligned}$$

Output: $M_0 \in \mathbb{N}_0$, $n_{0,j} \in \mathbb{N}_0$ ($0 \le n_{0,1} < n_{0,2} < \ldots < n_{0,M_0} < 2N$), $c_{0,j} \in \mathbb{R}$ ($j = 1, \ldots, M_0$). $M_1 \in \mathbb{N}_0$, $n_{1,j} \in \mathbb{N}$ ($1 \le n_{1,1} < n_{1,2} < \ldots < n_{1,M_1} < 2N$), $c_{1,j} \in \mathbb{R}$ ($j = 1, \ldots, M_1$).

5 Numerical examples

Now we illustrate the behavior and the limits of the suggested algorithms. Using IEEE standard floating point arithmetic with double precision, we have implemented our algorithms in MATLAB. In Example 5.1, an M-sparse Legendre expansion is given in the form (2.7) with normed Legendre polynomials of even degree $n_{0,j}$ ($j = 1, \ldots, M_0$)

and odd degree $n_{1,k}$ $(k = 1, \ldots, M_1)$, respectively, and corresponding real non-vanishing coefficients $c_{0,j}$ and $c_{1,k}$, respectively. In Examples 5.2 and 5.3, an M-sparse Gegenbauer expansion is given in the form (4.6) with normed Gegenbauer polynomials (of even/odd degree $n_{0,j}$ resp. $n_{1,k}$ and order $\alpha > 0$) and corresponding real non-vanishing coefficients $c_{0,j}$ resp. $c_{1,k}$. We compute the absolute error of the coefficients by

$$e(\boldsymbol{c}) := \max_{\substack{j=1,\ldots,M_0 \\ k=1,\ldots,M_1}} \{|c_{0,j} - \tilde{c}_{0,j}|, |c_{1,k} - \tilde{c}_{1,k}|\} \quad (\boldsymbol{c} := (c_{0,1}, \ldots, c_{0,M_0}, c_{1,1}, \ldots, c_{1,M_1})^{\mathrm{T}}),$$

where $\tilde{c}_{0,j}$ and $\tilde{c}_{1,k}$ are the coefficients computed by our algorithms. The symbol + in the Tables 5.1 – 5.3 means that all degrees n_j are correctly reconstructed, the symbol $-$ indicates that the reconstruction of the degrees fails. We present the error $e(\boldsymbol{c})$ in the last column of the tables.

Example 5.1 We start with the reconstruction of a 5–sparse Legendre expansion (2.7) which is a polynomial of degree 200. We choose the even degrees $n_{0,1} = 6$, $n_{0,2} = 12$, $n_{0,3} = 200$ and the odd degrees $n_{1,1} = 175$, $n_{1,2} = 177$ in (2.7). The corresponding coefficients $c_{0,j}$ and $c_{1,k}$ are equal to 1. Note that for the parameters $N = 400$ and $K = L = 5$, due to roundoff errors, some eigenvalues $\tilde{x}_{0,j}$ resp. $\tilde{x}_{1,k}$ are not contained in $[-1, 1]$. But we can improve the stability by choosing more sampling values. In the case $N = 500$, $K = 9$ and $L = 5$, we need only $2\,(K + L) - 1 = 27$ sampled values of (3.1) for the exact reconstruction of the 5–sparse Legendre expansion (2.7). □

N	K	L	Algorithm 3.3	$e(\boldsymbol{c})$
101	5	5	+	3.3307e-15
200	5	5	+	5.5511e-16
300	5	5	+	1.5876e-14
400	5	5	$-$	$-$
400	6	5	+	1.6209e-14
500	6	5	$-$	$-$
500	7	5	$-$	$-$
500	9	5	+	2.4780e-13

Table 5.1: Results of Example 5.1.

Example 5.2 We consider now the reconstruction of a 5–sparse Gegenbauer expansion (4.6) of order $\alpha > 0$ which is a polynomial of degree 200. Similar as in Example 5.1, we choose the even degrees $n_{0,1} = 6$, $n_{0,2} = 12$, $n_{0,3} = 200$ and the odd degrees $n_{1,1} = 175$, $n_{1,2} = 177$ in (4.6). The corresponding coefficients $c_{0,j}$ and $c_{1,k}$ are equal to 1. Here we use only $2\,(L + K) - 1 = 19$ sampled values for the exact recovery of the 5–sparse Gegenbauer expansion (4.6) of degree 200. Note that we show also some examples for

$\alpha > 1$. But the suggested method fails for $\alpha = 3.5$. In this case our algorithm can not exactly detect the smallest degrees $n_{0,1} = 6$ and $n_{0,2} = 12$, but all the higher degrees are exactly detected. □

α	N	K	L	Algorithm 4.4	$e(\boldsymbol{c})$
0.1	101	5	5	+	5.5511e-16
0.2	101	5	5	+	2.2204e-16
0.4	101	5	5	−	–
0.4	200	5	5	+	1.0769e-14
0.5	200	5	5	+	8.8818e-16
0.9	200	5	5	+	7.5835e-16
1.5	200	5	5	+	1.3323e-15
2.5	200	5	5	+	1.1102e-16
3.5	200	5	5	−	–

Table 5.2: Results of Example 5.2.

Example 5.3 We consider the reconstruction of a 5–sparse Gegenbauer expansion (4.6) of order $\alpha > 0$ which does not consist of Gegenbauer polynomials of low degrees. Thus we choose the even degrees $n_{0,1} = 60$, $n_{0,2} = 120$, $n_{0,3} = 200$ and the odd degrees $n_{1,1} = 175$, $n_{1,2} = 177$ in (4.6). The corresponding coefficients $c_{0,j}$ and $c_{1,k}$ are equal to 1. In Table 5.3, we show also some examples for $\alpha \ge 2.5$. But the suggested method fails for $\alpha = 8$. In this case, Algorithm 4.4 can not exactly detect the smallest degree $n_{0,1} = 60$, but all the higher degrees are exactly recovered. This observation is in perfect accordance with the very good local approximation near by $\theta = \pi/2$, see Theorem 4.2 and Remark 4.3. □

Example 5.4 We stress again that the Prony–like methods are very powerful tools for the recovery of a sparse exponential sum

$$S(x) := \sum_{j=1}^{M} c_j \, \mathrm{e}^{f_j x} \quad (x \ge 0)$$

with distinct numbers $f_j \in (-\infty,\, 0] + \mathrm{i}\,[-\pi,\, \pi)$ and complex non–vanishing coefficients c_j, if only finitely many sampled data of S are given. In [16], we have presented a method to reconstruct functions of the form

$$F(\theta) = \sum_{j=1}^{M} \big(c_j \, \cos(\nu_j \theta) + d_j \, \sin(\mu_j \theta)\big) \quad (\theta \in [0,\, \pi])\,.$$

α	N	K	L	Algorithm 4.4	$e(\boldsymbol{c})$
0.1	101	5	5	+	1.2879e-14
0.2	101	5	5	+	1.1879e-14
0.4	101	5	5	−	−
0.4	200	5	5	+	3.1086e-15
0.9	200	5	5	+	1.3323e-14
2.5	200	5	5	+	7.7716e-16
3.5	200	5	5	+	5.4401e-15
4.5	200	5	5	+	3.3862e-14
7.0	200	5	5	+	2.2204e-16
7.5	200	5	5	+	3.3307e-16
8.0	200	5	5	−	−
9.0	200	5	5	−	−

Table 5.3: Results of Example 5.3.

with real coefficients c_j, d_j and distinct frequencies ν_j, $\mu_j > 0$ by sampling the function F, see [16, Example 5.5].

Now we reconstruct a sum of sparse Legendre and Chebyshev expansions

$$H(x) := \sum_{j=1}^{M} c_j \, L_{n_j}(x) + \sum_{k=1}^{M'} d_k \, T_{m_k}(x) \quad (x \in [-1,\, 1])\,. \tag{5.1}$$

Here we choose $c_j = d_k = 1$, $M = M' = 5$ and $(n_j)_{j=1}^5 = (6, 13, 165, 168, 190)^{\mathrm{T}}$ and $(m_k)_{k=1}^5 = (60, 120, 175, 178, 200)^{\mathrm{T}}$. We apply Algorithm 3.3 with $N = 200$, $K = L = 20$ and calculate in step 3 for the even polynomial degrees

$$(\frac{2N-1}{\pi} \arccos x_{0,j})_{j=1}^{7} = (199.999,\, 189.503,\, 178.002,\, 167.499,\, 120.000,\, 60.000,\, 5.519)^{\mathrm{T}}$$

and in step 4 for the odd polynomial degrees

$$(\frac{2N-1}{\pi} \arccos x_{1,j})_{j=1}^{3} = (164.500,\, 175.000,\, 12.509)^{\mathrm{T}}\,.$$

We use now the information that only polynomial degrees with the orders $\alpha = 0$ and $\alpha = \frac{1}{2}$ occur. So we infer that (5.1) contains Legendre polynomials (for $\alpha = \frac{1}{2}$) of degrees 190, 168, and 6 by step 3 and of degrees 165 and 13 by step 4. Similarly we find the Chebyshev polynomials (for $\alpha = 0$) of degrees 200, 178, 120, and 60 by step 3 and of degree 175 by step 4. Finally, we can compute the coefficients c_j and d_k. □

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