LOW LYING EIGENVALUES OF RANDOMLY CURVED QUANTUM WAVEGUIDES

DENIS BORISOV[1,2,3,4] AND IVAN VESELIĆ[1,5]

ABSTRACT. We consider the negative Dirichlet Laplacian on an infinite waveguide embedded in $\mathbb{R}^2$, and finite segments thereof. The waveguide is a perturbation of a periodic strip in terms of a sequence of independent identically distributed random variables which influence the curvature. We derive explicit lower bounds on the first eigenvalue of finite segments of the randomly curved waveguide in the small coupling (i.e. weak disorder) regime. This allows us to estimate the probability of low lying eigenvalues, a tool which is relevant in the context of Anderson localization for random Schrödinger operators.

(1) *Faculty of Mathematics, Technische Universität Chemnitz, 09107 Chemnitz, Germany*
(2) *Institute of Mathematics Of Ufa Scientific Center of RAS, Chernyshevskogo, 112, Ufa, 450008, Russia*
(3) *Department of Physics and Mathematics, Bashkir State Pedagogical University, October rev. st. 3a, Ufa, 450000, Russia*
(4) *E-mail:* `borisovdi@yandex.ru`, *URL:* `http://borisovdi.narod.ru/`
(5) *URL:* `http://www.tu-chemnitz.de/mathematik/stochastik/`

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1. Introduction

In this paper we study eigenvalues of finite segments of randomly curved (quantum) waveguides. More precisely we derive lower bounds on the bottom of the spectrum of a waveguide segment of length L with perturbation of (a small) size κ. The bounds are formulated in terms of the parameters L and κ. In this context lower bounds are more challenging than upper ones. The waveguide consists of the set of points in $\mathbb{R}^2$ which have a distance smaller than $\pi/2$ (half the width of the waveguide) to a random perturbation of a periodic curve in the plane. We consider the negative Laplace operator on this set with appropriate boundary conditions. As it has compact resolvent it has purely discrete, lower semi-bounded spectrum.

Since the perturbation in each periodicity cell of the waveguide is determined by an individual scalar random variable, the above bound can be considered as a multi-parameter variational problem. The main difficulty arises from the fact that the length of the waveguide is finite, but not fixed; indeed it tends to infinity. This also means that the number of variational parameters becomes arbitrarily large. To compensate this, we have to chose the size κ of the perturbations as a function of the length L. This is a special case of what is commonly called a *weak disorder* regime.

The estimates on lowest eigenvalues we derive are motivated by the question whether a random Hamiltonian on a non-compact space has localized or non-localized states. One specific question in this context is: Is there a interval containing the bottom of the spectrum of the random Hamiltonian where the eigenvalues lie dense and where there is no continuous spectral component almost surely. The two first models where this question has been answered positively is the so called Russian-school-model (in [19]) and the Anderson model (in [16, 15]). In the later papers the so-called *multiscale analysis* was introduced as a method for proving (Anderson) localization, i.e. dense pure-point spectrum, with exponentially decaying eigenfunctions, almost surely. This method is an induction over increasing finite length scales. On each scale one considers a restricted Hamiltonian which lives on a finite volume. An important input for the multiscale analysis is the induction anchor, which is called *initial scale estimate*. It is this estimate which follows from our eigenvalue bounds, for the case that the random Hamiltonian is the negative Laplacian on a infinite randomly curved waveguide. The mentioned finite volume restrictions correspond to finite segments of the random waveguide.

In a previous paper [3] we have studied the analogous problem for a different type of random waveguide, which we will call for definitness *randomly shifted waveguide*, cf. Section 4 for precise definitions and a comparison of the two models. Somewhat surprisingly, although the two types or random waveguides seem very similar, a very basic feature is different: namely, the configuration, which produces the lowest first eigenvalue. In one case the optimal configuration corresponds to zero randomness, in the other to maximal randomness. The model studied here is more difficult to analyse than the one of [3]. On the other hand it is also more natural, since it corresponds to the standard way how local curvature perturbations are introduced for waveguides, cf. e.g. [14].

An interesting feature of our model in comparison to other extensively studied random Schrödinger operators, e.g. the above-mentioned Anderson-model, is the geometric influence of the randomness. Another model with geometric randomness, which as recently attracted much attention, is the random displacement model, cf. e.g. [26, 1, 18, 29]. One important feature of models with geometric disorder is that they exhibit no obvious *monotonicity with respect to the individual random variables*. This is the case for the random displacement model, for the randomly shifted waveguide (studied in [3]), as well as for the randomly curved

waveguide, to which the present paper is devoted. We refer the interested reader to the Introduction of [3], where we have discussed in some detail the challanges non-monotonicity poses in the spectral analysis of random Hamiltonians.

There are also differences in the difficulties these models pose: for the random displacement model it is non-trivial to identify (all) optimizing configurations of the random parameters. (Here optimizing means: minimizing the first eigenvalue.) This is easier for the two mentioned models of random waveguides, although the optimal configurations turn out to be different in the two cases, cf. Section 4. On the other hand, in the random displacement model the random variables enter the potential, i.e. the zero order term, only, where in the waveguide models the higher order terms of the differential operator depend on the randomness.

We would like to stress that localization for a certain type of random waveguides was established already in [25]. There the randomness enters via a variation of the width of the waveguide. This type of perturbation leads to a quadratic form which depends monotonously on the random variables and is thus structurally different from our model.

As already mentioned, the main challange in our analysis is the non-monotone dependence of the Hamiltonian on the random variables. We have thus to overcome difficulties as they are encountered in other types of random Hamiltonians which exhibit a non-monotone dependence on the randomness. The model of this type to which most attention was devoted so far is the alloy-type random Schrödinger operators with single site potentials of changing sign, see e.g. [27], [40], [47], [28], [48], [20], [32], [30], [50]. More recently also the discrete analog of this model was studied in [9], [49], [10], [42], [33], [5], [8], [41]. Electromagnetic Schrödinger operators with random magnetic field [43], [44], [20], [31], [46], [4], [13], [12], [11], as well as Laplace-Beltrami operators with random metrics [37], [35], [36] exhibit a non-monotonous parameter dependence which affects the higher order terms of the differential operator. Another relevant model (although not defined in terms of a countable family of random parameters) without obvious monotonicity is a random potential given by a Gaussian stochastic field with sign-changing covariance function, c.f. [22], [45], [51].

2. Main deterministic results

Let $l \geq 1$, $l \geq a > 0$ and $g \in C_0^4(\mathbb{R})$ be an function with support in $[0, a]$. Let $\kappa \geqslant 0$ and $\rho := (\rho_j)_{j \in \mathbb{Z}}$ be a sequence of reals $\rho_j \in [0, \kappa]$. We introduce the function

$$G(\cdot, \rho) \colon \mathbb{R} \to \mathbb{R}, \quad G(x_1, \rho) := \sum_{j \in \mathbb{Z}} \rho_j \, g_j(x_1),$$

where $g_j(x_1) := g(x_1 - jl)$ is the translate of the function g by jl to the right. Consider the curve $\Upsilon_\rho := \{r_\rho(t) \mid t \in \mathbb{R}\} \subset \mathbb{R}^2$ determined by the equation

$$r_\rho(t) := \big(t, G(t, \rho)\big), \quad t \in \mathbb{R}.$$

By $\nu_\rho = \nu_\rho(t)$ we denote the unit normal vector to Υ_ρ at the point $r_\rho(t)$ defined by

$$\nu_\rho(t) = \left(-\frac{G'(t, \rho)}{P_1^{1/2}(t, \rho)}, \frac{1}{P_1^{1/2}(t, \rho)} \right), \quad P_1(t, \rho) := 1 + \big(G'(t, \rho)\big)^2,$$

for $t \in \mathbb{R}$. Here $'$ denotes the derivative w.r.t. t. We suppose that

$$(2.1) \qquad \qquad \|g\|_{C^4[0,l]} = 1.$$

In the following when we say that a quantity does not depend on g or a, we mean that it can be chosen uniformly for all elements satisfying conditions (2.1).

In a vicinity of Υ_ρ we introduce new coordinates $\xi := (\xi_1, \xi_2)$ by the formula $x = x(\xi) = r_\rho(\xi_1) + \xi_2 \nu_\rho(\xi_1)$. In other words, given ξ_1, we take a point at Υ_ρ corresponding to $t = \xi_1$, and then shift along the direction of $\nu_\rho(\xi_1)$ by the distance ξ_2. We note that the coordinates ξ depend on ρ, and are well-defined, if κ is small enough.

We introduce a randomly curved waveguide segment

$$D_\rho(N, j) := \{x : jl < \xi_1 < (j + N)l, \ 0 < \xi_2 < \pi\}.$$

Denote by $\Gamma_\rho(N, j)$ the upper and lower part of the boundary of $D_\rho(N, j)$, i.e.,

$$\Gamma_\rho(N, j) := \{x : jl < \xi_1 < (j + N)l, \xi_2 = 0\} \cup \{x : jl < \xi_1 < (j + N)l, \xi_2 = \pi\}.$$

The remaining part of the boundary $\partial D_\rho(N, j) \setminus \overline{\Gamma_\rho(N, j)}$ is denoted by $\gamma_\rho(N, j)$. If $j = 0$, we will use the following shorthand notation: $D_\rho(N) := D_\rho(N, 0)$, $\Gamma_\rho(N) := \Gamma_\rho(N, 0)$, $\gamma_\rho(N) := \gamma_\rho(N, 0)$. Note that the length of $D_\rho(N)$ is $L = N \cdot l$.

In order to state the deterministic lower bound on the first eigenvalue, as a function of the parameters $\rho = (\rho_j)_{j \in \mathbb{Z}}$, we first have to introduce a reference energy. This will be the spectral bottom of a comparison operator. More precisely, we will use the symbol $\mathcal{H}^{\mathrm{per}}(\rho)$ to denote the negative Laplacian in $L_2(D_\rho(1))$ with Dirichlet boundary condition on $\Gamma_\rho(1)$ and periodic boundary condition on $\gamma_\rho(1)$. Note that is this situation the array $\rho = (\rho_j)_j$ consists just of the single value ρ_0. So we identify here ρ_0 with ρ. Let $\lambda^{\mathrm{per}}(\rho)$ be the lowest eigenvalue of $\mathcal{H}^{\mathrm{per}}(\rho)$ and $\psi^{\mathrm{per}} = \psi^{\mathrm{per}}(x, \rho)$ be the associated normalized eigenfunction. We extend the function ψ^{per} l-periodically w.r.t. ξ_1 and denote the extension by the same symbol.

We denote by $\mathcal{H}(\rho, N, j)$ the negative Laplacian on $L_2(D_\rho(N, j))$ with Dirichlet boundary conditions on $\Gamma_\rho(N, j)$ and with Robin boundary condition

$$
(2.2) \quad
\begin{aligned}
&\left(\frac{\partial}{\partial x_1} - h(\cdot, \kappa) \right) u = 0 \quad \text{on} \quad \gamma_\rho(N, j), \quad \text{where} \\
&h(\cdot, \kappa) \colon [0, \pi] \to \mathbb{R}, \quad h(x_2, \kappa) := \frac{1}{\psi^{\mathrm{per}}(0, x_2, \kappa)} \frac{\partial \psi^{\mathrm{per}}}{\partial x_1}(0, x_2, \kappa),
\end{aligned}
$$

is the logarithmic derivative in x_2-direction. We shall show in Sec. 7 that the function h is well-defined. In the following we denote for some $s \in \mathbb{R}$ by $s \cdot \mathbf{1}$ the constant sequence $\rho_j = s$ $(j \in \mathbb{Z})$. The important point is that $\psi^{\mathrm{per}}(\cdot, \rho)$ is still an eigenfunction of the operator $\mathcal{H}(\rho \cdot \mathbf{1}, N, j)$ with the new boundary conditions, i.e.

$$\mathcal{H}(\rho \cdot \mathbf{1}, N, j)\psi^{\mathrm{per}} = \lambda(\rho)\psi^{\mathrm{per}},$$

where we use for the restriction of ψ^{per} to the finite waveguide segment $D_{\rho \cdot \mathbf{1}}(N, j)$ the same symbol. This relation was first exploited in the present context by Mezincescu [38]. For this reason we call (2.2) in the sequel Mezincescu boundary conditions. We denote the lowest eigenvalue of $\mathcal{H}(\rho, N) = \mathcal{H}(\rho, N, 0)$ by $\lambda(\rho, N)$.

Theorem 2.1. *Assume* $\alpha := \frac{\pi^3}{32} - a > 0$. *There exists a constant* $\delta = \delta(a, g) > 0$ *independent of* ρ, l, N *such that for*

$$(2.3) \qquad\qquad \kappa \leqslant \delta L^{-11/2}, \ \text{and} \ 0 \leqslant \rho_i \leqslant \kappa \ \text{for all} \ i \in 0, \ldots N - 1$$

the estimate

$$(2.4) \qquad\qquad \lambda(\rho, N) - \lambda^{\mathrm{per}}(\kappa) \geqslant C_{lb} \frac{\kappa}{L} \sum_{j=0}^{N-1} (\kappa - \rho_j)$$

holds true with

$$C_{lb} := \frac{\|g'\|^2_{L_2(0,l)}}{4} \left(\frac{\pi}{2} - \frac{16a}{\pi^2} \right) = \frac{4}{\pi^2} \|g'\|^2_{L_2(0,l)} \, \alpha.$$

In view of the well known result that curvature induces eigenvalues below the bottom of the essential spectrum of the straight quantum waveguide, cf.[14], it can be considered natural that maximal coupling constants produce the lowest first eigenvalue in the case of randomly curved waveguides

3. Probability of low lying eigenvalues

In this section we want to formulate an upper bound on the (small) probability that a randomly perturbed, finite waveguide segment has an eigenvalue close to the overal possible minimal spectral value. For this we use the deterministic results of perturbation theory, formulated in Section 2. The probabilistic results we obtain hold in the so-called weak disorder regime. To make this explicit, we use in this section a global (scalar) disorder parameter: For $\kappa > 0$ we set

$$\omega_j := \rho_j/\kappa \quad (j \in \mathbb{Z}).$$

Then we have the following equivalence for the condition used in Theorem 2.1

$$\rho_j \in [0,\kappa] \quad (j \in \mathbb{Z}) \Longleftrightarrow \omega_j \in [0,1] \quad (j \in \mathbb{Z})$$

We consider the parameters $\omega_j, j \in \mathbb{Z}$ as a sequence of non-trivial independent, identically distributed random variables taking values in the unit interval $[0,1]$. We denote the distribution of the single random variable ω_j by μ, and by $\mathbb{P}$ the product probability measure on $\Omega := [0,1]^{\mathbb{Z}}$ which is the distribution of the random sequence $\omega_j, j \in \mathbb{Z}$. Of course Ω is equipped with the natural product sigma-field. Now the deviations of the parameters from the extremal value, which determines the lower bound in Theorem 2.1, can be rewritten as follows:

$$\kappa - \rho_j = \kappa(1 - \omega_j) = \kappa \cdot \tilde{\omega}_j \text{ for all } j \in \mathbb{Z}$$

Here we denote by $\tilde{\omega}_j = (1 - \omega_j) \in [0,1]$ the flipped random variables, and the multiplication of a sequence $\omega = (\omega_j)_{j \in \mathbb{Z}}$ with a scalar $s \in \mathbb{R}$ by $s \cdot \omega$.

Now the result of Theorem 6.1 can be formulated in the new parameters as follows: If $\alpha := \frac{\pi^3}{32} - a > 0$, then there exists a positive constant δ independent of l, N, L and of the sequence $\omega = (\omega_j)_{j \in \mathbb{Z}}$, such that for all $\kappa \leq \delta L^{-11/2}$ we have

$$(3.1) \qquad \lambda(\kappa \cdot \omega, N) - \lambda^{\mathrm{per}}(\kappa) \geqslant C_{lb} \frac{\kappa^2}{L} \sum_{j=0}^{N-1} \tilde{\omega}_j \quad \text{where } C_{lb} = \frac{4}{\pi^2} \|g'\|^2_{L_2(0,l)} \, \alpha$$

Here we denote the lowest eigenvalue of $\mathcal{H}(\kappa \cdot \omega, N) = \mathcal{H}(\rho, N)$ by $\lambda(\kappa \cdot \omega, N) = \lambda(\rho, N)$. Recall that $\lambda^{\mathrm{per}}(\kappa)$ is the lowest eigenvalue of $\mathcal{H}^{\mathrm{per}}(\rho)$, i.e. the negative Laplacian in $L_2(D_\rho(1))$ with Dirichlet boundary condition on $\Gamma_\rho(1)$ and periodic boundary condition on $\gamma_\rho(1)$. This is the lowest achieveable spectral value, if all perturbation parameters are bounded by κ.

The announced bound on the probability of low-lying eigenvalues is:

Theorem 3.1. *Let $\gamma > 22, \gamma \in \mathbb{N}$. Then there exists $N_1 = N_1(\gamma, \mu, l, \delta, g, a) \in (0, \infty)$, such that for all $N \geqslant N_1$ the interval*

$$I_N := \left[\tilde{C} \, N^{-1/4}, \delta \, l^{-11/2} N^{-\frac{11}{2\gamma}} \right]$$

is non-empty. For $N \geqslant N_1$ and $\kappa \in I_N$, we have

$$(3.2) \qquad \mathbb{P}\left(\omega \in \Omega \mid \lambda(\kappa \cdot \omega, N) - \lambda^{\mathrm{per}}(\kappa) \leqslant N^{-\frac{1}{2}}\right) \leqslant N^{1-\frac{1}{\gamma}} \, \mathrm{e}^{-C_{\mathrm{LDP}} \, N^{1/\gamma}}.$$

The constants $\tilde{C} = \tilde{C}(\mu, l, g, a)$ and $C_{\mathrm{LDP}} = C_{\mathrm{LDP}}(\mu) > 0$, as well as N_1 are given explicitly in Section 5

Remark 3.2. The requirement $\kappa \in I_N$ encodes how our weak-disorder regime depends on the length scale N. For purpose of illustration, let us choose $\gamma = 33$. Then $\frac{1}{4} > \frac{11}{2 \cdot \gamma} = \frac{1}{6}$, thus it is possible to choose $\kappa = const. \, N^{-\frac{1}{4}}$, which means that the allowed disorder regime does not shrink too fast for $N \to \infty$.

For the purposes of a localization proof, for instance using the multiscale analysis or the fractional moment method, the following initial length scale estimate or finite volume criterion is of interest. It can be derived from Theorem 3.1 using a Combes-Thomas estimate [6]. This is a well established tool in the theory or random (Schrödinger) operators. An explicit derivation adapted to the setting of quantum waveguides can be found, e. g., in Section 4 in [3].

The mentioned initial length scale estimate is a bound on the probability that the Green's function exhibits exponential off-diagonal decay for energies in a small interval at the bottom of the spectrum, i.e. the overall minimum of the spectrum $\lambda^{\mathrm{per}}(\kappa)$. For the following statement note that for N large $I_N \subset [0, 1]$.

Corollary 3.3. *Let $l, a, g, \mu, I_N, \gamma, N_1$ and C_{LDP} be as in Theorem 3.1. Let $\kappa \in I_N \cap [0, 1]$, $\alpha, \beta \geqslant 2$ and set*

$$A := \{x \in D_{\kappa \cdot \omega}(N) \mid 0 \leqslant x_1 \leqslant \alpha\},$$
$$B := \{x \in D_{\kappa \cdot \omega}(N) \mid L - \beta \leqslant x_1 \leqslant L\}.$$

Then there exists an absolute constant c such that for any $N \geqslant N_1$

$$\mathbb{P}\left(\omega \in \Omega \mid \forall \, \lambda \leq \lambda^{\mathrm{per}}(\kappa) + 1/(2\sqrt{N}) \colon \, \left\|\chi_A(\mathcal{H}(\kappa \cdot \omega, N) - \lambda)^{-1}\chi_B\right\| \leqslant \sqrt{N} \, \mathrm{e}^{-c \, \mathrm{dist}(A,B)/\sqrt{N}}\right)$$
$$\geqslant 1 - N^{1-\frac{1}{\gamma}} \, \mathrm{e}^{-c_{\mathrm{LDP}} \, N^{1/\gamma}}.$$

4. COMPARISON WITH THE RANDOMLY WIGGLED OR SHIFTED WAVEGUIDE

In this section we compare the results obtained in the present paper to those of [3], whether a related problem was studied. There we derived deterministic and probabilistic lower bounds on the lowest eigenvalue of finite segments of a randomly perturbed waveguide. In contrast to the model studied here, the random geometric perturbation was introduced by locally shifting or wiggling the waveguide. To explain the difference between the two models we recall the mathematical definition of the Hamiltonians studied in [3].

4.1. **Statement of the results of [3] on randomly shifted waveguides.** Similarly as in the model studied in this paper we consider random quantum waveguides in $\mathbb{R}^2$, determined by the following data: Let $(\omega_k)_{k \in \mathbb{Z}}$ be a sequence of independent, identically distributed, non-trivial random variables with values in $[0, 1]$, $\kappa > 0$ a global coupling constant, $l \geqslant 1$ the length of one (periodicity) cell of the waveguide, and $g \in C_0^2(0, l)$ a single bump function. The following function $G \colon \mathbb{R} \times \Omega \to \mathbb{R}$ determines the shape of the waveguide

$$G(x_1, \omega) := \sum_{k \in \mathbb{Z}} \omega_k \, g(x_1 - kl).$$

Note that $\kappa G(x_1, \omega) = G(x_1, \kappa\omega)$. For a global coupling constant $\kappa > 0$, $N \in \mathbb{N}$ and $j \in \mathbb{Z}$ we define finite segments of a infinite waveguide

$$D_{\kappa,\omega}(N,j) := \{x \in \mathbb{R}^2 \mid jl < x_1 < (j+N)l, \kappa G(x_1, \omega) < x_2 < \kappa G(x_1, \omega) + \pi\}.$$

The upper and lower part of the boundary of $D_{\kappa,\omega}(N,j)$, we denote by

$$\begin{aligned}
\Gamma_{\kappa,\omega}(N,j) := & \{x \in \mathbb{R}^2 \mid jl < x_1 < (j+N)l, x_2 = \kappa G(x_1, \omega)\} \\
& \cup \{x \in \mathbb{R}^2 \mid jl < x_1 < (j+N)l, x_2 = \kappa G(x_1, \omega) + \pi\}.
\end{aligned}$$

while $\partial D_{\kappa,\omega}(N,j) \setminus \Gamma_{\kappa,\omega}(N,j)$ is denoted by $\gamma_{\kappa,\omega}(N,j)$. Denote the negative Laplace operator on $D_{\kappa,\omega}(N,j)$ with Dirichlet boundary conditions on $\Gamma_{\kappa,\omega}(N,j)$ and Neumann b.c. on $\gamma_{\kappa,\omega}(N,j)$ by $\mathcal{H}_{\kappa,\omega}(N,j)$, and its lowest eigenvalue by $\lambda_{\kappa,\omega}(N,j)$. We use the following abbreviation:

$$\begin{aligned}
D_{\kappa,\omega}(N) &:= D_{\kappa,\omega}(N,0), & \Gamma_{\kappa,\omega}(N) &:= \Gamma_{\kappa,\omega}(N,0), & \gamma_{\kappa,\omega}(N) &:= \gamma_{\kappa,\omega}(N,0), \\
\mathcal{H}_{\kappa,\omega}(N) &:= \mathcal{H}_{\kappa,\omega}(N,0), & \lambda_{\kappa,\omega}(N) &:= \lambda_{\kappa,\omega}(N,0).
\end{aligned}$$

Note that $D_{\kappa,\omega}(N,j) = D_{1,\kappa\omega}(N,j)$. Since $\kappa > 0$ is arbitrary we may assume without restricting the model

$$(4.1) \qquad\qquad \max\{\|g\|_{C[0,l]}, \|\|g'\|_{C[0,l]}, \|g''\|_{C[0,l]}\} = 1.$$

Denote the distribution measure of ω_k by μ and by $\mathbb{P} = \bigotimes_{k \in \mathbb{Z}} \mu$ the product measure on the configuration space $\Omega = \times_{k \in \mathbb{Z}}[0,1]$ whose elements we denote by $\omega = (\omega_k)_{k \in \mathbb{Z}}$.

The minimum of the spectrum of the Laplacian on an straight waveguide segment $\mathcal{H}_{0,\omega}(N,j)$ is equal to one, and no operator $\mathcal{H}_{\kappa,\omega}(N,j)$ has spectrum below one. In this sense, one is the minimal spectral value for all random configurations.

Theorem 4.1. *Let $\gamma > 34$. Let g and μ be as above and set*

$$\widetilde{g} := g - \frac{1}{l}\int_0^l g(t)\,\mathrm{d}t, \quad c_2 = \frac{3\,\|\widetilde{g}\|_{L_2(0,l)}}{2\,l^3}, \quad c_3 = \frac{3\,\|\widetilde{g}\|_{L_2(0,l)}^2}{5000\,l^7}.$$

Then there exists an initial scale N_1 such that if $N \geqslant N_1$ the interval

$$I_N := \left[\frac{2N^{\frac{1}{\gamma}-\frac{1}{4}}}{\mathbb{E}\{\omega_k\}\sqrt{c_2}}, c_3 N^{-\frac{15}{2\gamma}}\right]$$

is non-empty. If $N \geqslant N_1$ and $\kappa \in I_N$, then

$$(4.2) \qquad\qquad \mathbb{P}\left(\omega \in \Omega \mid \lambda_{\kappa,\omega}(N) - 1 \leqslant N^{-\frac{1}{2}}\right) \leqslant N^{1-\frac{1}{\gamma}}\,\mathrm{e}^{-C_{\mathrm{LDP}}N^{1/\gamma}}$$

for the constant $C_{\mathrm{LDP}} > 0$ of Lemma 5.1, which depends only on μ.

The corresponding deterministic lower bound on the first eigenvalue is:

Theorem 4.2. *Let $l \geqslant 1$, $g: \mathbb{R} \to \mathbb{R}$, $\widetilde{g} \in \mathbb{R}$ as above, and $L := Nl$. For*

$$(4.3) \qquad\qquad \kappa\left(\sum_{j=1}^N \omega_j^2\right)^{\frac{1}{2}} \leqslant \frac{3}{5000}\,\|\widetilde{g}\|_{L_2(0,l)}^2\,\frac{1}{L^7}$$

we have the bound

$$(4.4) \qquad \lambda_{\kappa,\omega}(N) - 1 \geqslant \frac{3}{2} \|\widetilde{g}\|_{L_2(0,l)}^2 \frac{\kappa}{L^3} \sum_{j=0}^{N-1} \omega_j^2$$

4.2. **Comparison of the results on randomly shifted and curved waveguides.** For definitness we will call the model of [3] *randomly shifted waveguide* and the model studied in the present paper *randomly curved waveguide*.

(i) In the first mentioned model the configuration of parameters which produces the minimal first eigenvalue corresponds to a straight waveguide, where the perturbation is switched off. In contrast to this, the configuration of the randomly curved waveguide yielding the minimal first eigenvalue corresponds to a periodic waveguide with maximal possible curvature. This difference between the two considered problems was somewhat surprising for us, since, naively, the two models look very similar.

(ii) Since for the randomly curved waveguide the bottom of the spectrum is achieved as the perturbation is maximal, the analysis of the model is more complicated. For we have to perform an asymptotic expansion not around the straight waveguide, but around the periodic waveguide $\mathcal{H}(\rho \cdot \mathbf{1}, N)$ with maximal allowed curvature, which arises itself as a result of the perturbation.

(iii) The smallness condition on the coupling constants — formulae (2.3) and (4.3) — and the lower bounds on the first eigenvalue — formulae (2.4) and (4.4) — in Theorems 2.1 and 4.2 exhibit a power-like, i.e. polynomial behaviour. The exponents do not coincide, but are of the same order of magnitude.

(iv) In both models we employ a change of the variables $x \to \xi$ straightening the waveguide. After this change the operator depends on the variables ρ_i. For the randomly curved waveguide the dependence on the ρ_i's is not polynomial, but irrational. For the randomly shifted waveguide considered in [3] the relevant differential operator contains only terms which were constant, linear or quadratic functions of the variables ρ_i.

(v) In Theorem 2.1 we do not specify an explicit value of δ as in the corresponding result in [3] on the randomly shifted waveguide. Although our technique allows us to obtain this constant explicitly, the required calculations are cumbersome and tedious. The main reason why the control of constants for the randomly curved waveguide is harder than for the randomly shifted one is the aforementioned irrational dependence of the perturbation on the random variables.

5. PROBABILISTIC ESTIMATES: PROOF OF THEOREM 3.1

We will use the following *large deviations principle* in the proof of Theorem 3.1. For a reference see for instance [7].

Lemma 5.1. *Let $\omega_k, k \in \mathbb{Z}$ be an i.i.d. sequence of non-trivial, non-negative, bounded random variables. Then there exists a constant $C_{\mathrm{LDP}} > 0$ depending only on μ such that*

$$\forall \, n \in \mathbb{N} \; : \; \mathbb{P}\left(\omega \mid \frac{1}{n} \sum_{k=1}^{n} \omega_k \leqslant \frac{\mathbb{E}\{\omega_k\}}{2} \right) \leqslant \mathrm{e}^{-C_{\mathrm{LDP}}n}.$$

For the *proof of Theorem 3.1* we choose $K \in 2\mathbb{N}$ and $\gamma \in \mathbb{N}$ and set $N := K^{\gamma}$. Thus $N \in \mathbb{N}$. Set $J = N/K = K^{\gamma-1} = N^{1-\frac{1}{\gamma}}$. Following the same ideas as in [21, 24, 3] we decompose the

waveguide segment $D_{\kappa \cdot \omega}(N)$ into smaller parts

$$(5.1) \qquad \overset{\bullet}{\bigcup_{j=0,\ldots,J-1}} D_{\kappa \cdot \omega}(K,j)$$

where $\overset{\bullet}{\bigcup}$ denotes a disjoint union, up to a set of measure zero. According to this decomposition we introduce new Mezincescu boundary conditions on the interfaces joining the shorter segments (5.1). As in the original paper [38] we conclude that in the sense of quadratic forms

$$\mathcal{H}(\kappa \cdot \omega, N) \geqslant \bigoplus_{j=0}^{J-1} \mathcal{H}(\kappa \cdot \omega, K, j).$$

If we denote the lowest eigenvalue of $\mathcal{H}(\kappa \cdot \omega, N, j)$ by $\lambda(\kappa \cdot \omega, N, j)$, it follows

$$(5.2) \qquad \lambda(\kappa \cdot \omega, N) \geqslant \min_{j=0}^{J-1} \lambda(\kappa \cdot \omega, K, j)$$

In particular, we have the inclusion

$$\left\{ \omega \in \Omega \mid \lambda(\kappa \cdot \omega, N) - \lambda^{\mathrm{per}}(\kappa) \leqslant N^{-\frac{1}{2}} \right\} \subset \bigcup_{j=0}^{J-1} \left\{ \omega \in \Omega \mid \lambda(\kappa \cdot \omega, K, j) - \lambda^{\mathrm{per}}(\kappa) \leqslant K^{-\frac{\gamma}{2}} \right\}$$

Since the random variables $\omega_k, k \in \mathbb{Z}$ are independent and identically distributed, we obtain

$$\sum_{j=0}^{J-1} \mathbb{P}\left(\omega \mid \lambda(\kappa \cdot \omega, K, j) - \lambda^{\mathrm{per}}(\kappa) \leqslant K^{-\frac{\gamma}{2}} \right) \leqslant N^{1-\frac{1}{\gamma}} \, \mathbb{P}\left(\omega \mid \lambda(\kappa \cdot \omega, K) - \lambda^{\mathrm{per}}(\kappa) \leqslant K^{-\frac{\gamma}{2}} \right).$$

For $\kappa \leq \delta L^{-11/2}$ we have $\lambda(\kappa \cdot \omega, K) - \lambda^{\mathrm{per}}(\kappa) \geqslant C_{lb}\frac{\kappa^2}{lK}\sum_{j=0}^{K-1} \tilde{\omega}_j$ and thus the inclusion

$$\left\{ \omega \mid \lambda(\kappa \cdot \omega, K) - \lambda^{\mathrm{per}}(\kappa) \leqslant K^{-\frac{\gamma}{2}} \right\} \subset \left\{ \omega \mid C_{lb}\frac{\kappa^2}{l \cdot K} \sum_{j=0}^{K-1} \tilde{\omega}_j \leqslant K^{-\frac{\gamma}{2}} \right\}$$

$$= \left\{ \omega \mid \frac{1}{K} \sum_{j=0}^{K-1} \tilde{\omega}_j \leqslant \frac{l}{C_{lb}\kappa^2} K^{-\frac{\gamma}{2}} \right\}$$

Denote by $\mathbb{E}\{\tilde{\omega}_0\} = \mathbb{E}\{1 - \omega_0\}$ the expectation value of (any) $\tilde{\omega}_k$. Choose now κ such that

$$(5.3) \qquad \frac{l}{C_{lb}\kappa^2} K^{-\frac{\gamma}{2}} \leqslant \frac{\mathbb{E}\{\tilde{\omega}_0\}}{2} \quad \text{i.e.} \quad \tilde{C}K^{-\frac{\gamma}{4}} \leqslant \kappa, \text{ where } \tilde{C} := \sqrt{\frac{2\,l}{C_{lb}\mathbb{E}\{\tilde{\omega}_k\}}}.$$

On the other we also need to satisfy the restriction $\kappa \leq \delta(l\,K)^{-11/2} = \delta l^{-11/2} N^{-11/(2\gamma)}$. The upper and the lower bound for κ can be reconciled if $\gamma > 22$ and

$$K \geqslant K_1 := \left(\frac{\tilde{C}\,l^{11/2}}{\delta} \right)^{\frac{4}{\gamma-22}}.$$

The last inequality is equivalent to

$$N \geqslant N_1 := \left(\frac{2\,l^{12}}{C_{lb}\mathbb{E}\{\tilde{\omega}_0\}\delta^2} \right)^{\frac{2\gamma}{\gamma-22}}$$

For κ satisfying (5.3) we are able to apply the large deviations principle of Lemma 5.1 and thus obtain

$$\mathbb{P}\left\{\omega \mid \frac{1}{K}\sum_{j=0}^{K-1}\tilde{\omega}_j \leqslant \frac{l}{C_{lb}\kappa^2}K^{-\frac{\gamma}{2}}\right\} \leqslant \mathbb{P}\left\{\omega \mid \frac{1}{K}\sum_{j=0}^{K-1}\tilde{\omega}_j \leqslant \frac{\mathbb{E}\{\tilde{\omega}_0\}}{2}\right\} \leqslant \mathrm{e}^{-C_{\mathrm{LDP}}K}$$

for $K \geqslant K_1$. Hence, for $N \geqslant N_1$

$$\mathbb{P}\left\{\omega \mid \lambda(\kappa\cdot\omega, N) - \lambda^{\mathrm{per}}(\kappa) \leqslant N^{-\frac{1}{2}}\right\} \leqslant N^{1-\frac{1}{\gamma}}\,\mathrm{e}^{-C_{\mathrm{LDP}}N^{1/\gamma}}.$$

6. New parameters corresponding to a linearization around the optimal configuration

Here we reformulate our main deterministic result, i.e. Theorem 2.1 in a different parametrisation, which reflects the perturbation theoretic proof. In what follows we want to study waveguide segments which correspond to configurations ρ where all coefficients ρ_i are close to κ. In other words, we perform an asymptotic analysis around a linearisation of the operator family, not around the point $\rho = 0 \cdot \mathbf{1}$ in parameter space, but rather around $\rho = \kappa \cdot \mathbf{1}$. Thus the natural parameters for perturbation theory are

$$\epsilon := (\epsilon_j)_{j\in\mathbb{Z}}, \epsilon_j := \kappa - \rho_j$$

In this and the following sections we denote by $\mathcal{H}_\kappa(\epsilon, N)$ the negative Laplacian on $L_2(D_\rho(N))$ with Dirichlet boundary condition on $\Gamma_\rho(N)$ and with Robin boundary condition, cf. (2.2),

(6.1)
$$\left(\frac{\partial}{\partial x_1} - h(\cdot, \kappa)\right)u = 0 \quad \text{on} \quad \gamma_\rho(N),$$
$$\text{where} \quad h(\cdot, \kappa)\colon [0, \pi] \to \mathbb{R},$$
$$h(x_2, \kappa) := \frac{1}{\psi^{\mathrm{per}}(0, x_2, \kappa)}\frac{\partial\psi^{\mathrm{per}}}{\partial x_1}(0, x_2, \kappa).$$

is the logarithmic derivative in x_2-direction. Recall that $\psi^{\mathrm{per}}(\cdot, \kappa)$ still satisfies the eigenvalue equation of the operator $\mathcal{H}_\kappa(0, N)$ with the new boundary conditions, i.e.

$$\mathcal{H}_\kappa(0, N)\psi^{\mathrm{per}} = \lambda(\kappa)\psi^{\mathrm{per}}$$

In the following we denote the lowest eigenvalue of $\mathcal{H}_\kappa(\epsilon, N)$ by $\lambda(\epsilon, \kappa)$ and suppress here the N-dependence in this notation. Here is the announced reformulation of Theorem 6.1.

Theorem 6.1. *Suppose $a < \frac{\pi^3}{32}$. There exists a constant $\delta > 0$ independent of ρ, l, N such that for*

(6.2)
$$\kappa \leqslant \delta L^{-11/2}, \text{ and } \rho_i \leqslant \kappa \text{ for all } i \in 0, \ldots N - 1$$

the estimate

(6.3)
$$\lambda(\epsilon, \kappa) - \lambda^{\mathrm{per}}(\kappa) \geqslant \frac{\|g'\|_{L_2(0,l)}^2}{4L}\left(\frac{\pi}{2} - \frac{16a}{\pi^2}\right)\kappa\sum_{j=0}^{N-1}\epsilon_j$$

holds true. The value of δ depends on g and a.

7. DETERMINISTIC LOWER BOUNDS: PRELIMINARIES

In this section we present certain less technical preliminaries which are necessary for the setup of perturbation theory and the proof of Theorem 6.1. First we introduce the necessary notation. Given ρ, we define the operator

$$\mathcal{T}(\rho)\colon L_2(D_\rho(N)) \to L_2(D_0(N)), \quad (\mathcal{T}(\rho)u)(\xi) := u(x(\xi)).$$

corresponding to the change of the variables $x \mapsto \xi$, where ξ is associated with ρ. If $\rho_i = \kappa$ for all i, we will write shortly $\mathcal{T}(\kappa)$ instead of $\mathcal{T}(\kappa, \kappa, \ldots)$. Let us study in more detail the change $x \mapsto \xi$. It is easy to check that

$$\nabla_x = \mathrm{A}\nabla_\xi,$$

$$\mathrm{A}^{-1} := \begin{pmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_1} \\ \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_2} \end{pmatrix} = \begin{pmatrix} 1 - \xi_2(G'P_1^{-1/2})' & G' + \xi_2(P_1^{-1/2})' \\ -G'P_1^{-1/2} & P_1^{-1/2} \end{pmatrix}.$$

We set

$$P_2(\xi, \rho) := 1 - \xi_2 K(\xi_1, \rho), \quad K(\xi_1, \rho) := G''(\xi_1, \rho)P_1^{-3/2}(\xi_1, \rho)$$

and obtain by direct calculation

$$(7.1) \qquad P_3 := \det^{-1} \mathrm{A} = P_1^{-1/2}(1 - \xi_2 G'' P_1^{-1/2} + G'^2) = P_1^{1/2} P_2,$$

where $'$ denotes the derivative w.r.t. ξ_1. Hence,

$$(7.2) \qquad \|u\|_{L_2(D_\rho(N))}^2 = \langle \mathcal{T}(\rho)u, P_3 \mathcal{T}(\rho)u \rangle_{L_2(D_0(N))}.$$

Thus the operator

$$\sqrt{P_3}\mathcal{T}(\rho)\colon L_2(D_\rho(N)) \to L_2(D_0(N)), \quad (\sqrt{P_3}\mathcal{T}(\rho)u)(\xi) := \sqrt{P_3}u(x(\xi))$$

is unitary. We observe that $P_i(\xi, \rho) = 1 + \mathcal{O}(\kappa)$, $i = 1, 2, 3$, for $\rho_j \le \kappa$. Thus for sufficiently small values of κ, non of these functions vanish. One can calculate directly that

$$(7.3) \qquad P_3 \mathrm{A}^* \mathrm{A} = \begin{pmatrix} P_3^{-1} & 0 \\ 0 & P_3 \end{pmatrix}$$

$$(7.4) \qquad \begin{aligned} \|\nabla_x u\|_{L_2(D_\rho(N))}^2 &= \langle \mathrm{A}\nabla_\xi \mathcal{T}(\rho)u, P_3 \mathrm{A}\nabla_\xi \mathcal{T}(\rho)u \rangle_{L_2(D_0(N))} \\ &= \left\| P_3^{-1/2} \frac{\partial \mathcal{T}(\rho)u}{\partial \xi_1} \right\|_{L_2(D_0(N))}^2 + \left\| P_3^{1/2} \frac{\partial \mathcal{T}(\rho)u}{\partial \xi_2} \right\|_{L_2(D_0(N))}^2, \end{aligned}$$

$$(7.5) \qquad \Delta_x = \frac{1}{P_3}\left(\frac{\partial}{\partial \xi_1} \frac{1}{P_3} \frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2} P_3 \frac{\partial}{\partial \xi_2} \right) = \Delta_\xi - \mathcal{Q}_\rho = \mathcal{H}^{\mathrm{per}}(0) + \mathcal{Q}_\rho,$$

$$(7.6) \qquad \mathcal{Q}_\rho := Q_{11}\frac{\partial^2}{\partial \xi_1^2} + Q_1\frac{\partial}{\partial \xi_1} + Q_2\frac{\partial}{\partial \xi_2},$$

$$\mathcal{Q}_{11}(\xi, \rho) := 1 - \frac{1}{P_3^2} = -\frac{2\xi_2 K(\xi_1, \rho) - \xi_2^2 K^2(\xi_1, \rho)}{P_2^2(\xi, \rho)} + \frac{\left(G'(\xi_1, \rho)\right)^2}{P_1(\xi_1, \rho)P_2^2(\xi, \rho)},$$

$$\mathcal{Q}_1(\xi, \rho) := -\frac{1}{P_3}\frac{\partial}{\partial \xi_1}\frac{1}{P_3} = -\frac{\xi_2 K'(\xi_1, \rho)}{P_1(\xi, \rho)P_2^3(\xi_1, \rho)} + \frac{G'(\xi_1, \rho)K(\xi_1, \rho)}{P_1^{1/2}(\xi_1, \rho)P_2^2(\xi_1, \rho)},$$

$$\mathcal{Q}_2(\xi, \rho) := -\frac{1}{P_3}\frac{\partial P_3}{\partial \xi_2} = \frac{K(\xi_1, \rho)}{P_2(\xi, \rho)}.$$

Under the standing assumption (2.1) we can find a constant C independent of l, a, g, N, and κ such that

$$\text{(7.7)} \qquad \|\mathcal{Q}_{11}\|_{C(\overline{D_0(N)})} + \|\mathcal{Q}_1\|_{C(\overline{D_0(N)})} + \|\mathcal{Q}_2\|_{C(\overline{D_0(N)})} \leqslant C\kappa$$

uniformly for all configurations satisfying $\rho_i \leqslant \kappa$ for all $i \in 0, \dots N-1$.

Our final aim is to have a lower bound on the movement of eigenvalues under a perturbation. However, first we need some rough a-priori information about the position of eigenvalues. This amounts of showing that the relevant eigenvalue is not too far from the one of a simple reference operator. So an upper bound on the distance is in question. For this preliminary considerations it will be sufficient to consider the operator on a waveguide segment of length l, i.e. the case $N = 1$. Before we state the next Lemma, we collect some simplifications which will be useful here and in subsequent considerations.

Remark 7.1 (Simplified expressions for the case $N = 1$). Here we consider the case $N = 1$, i.e. the operator $\mathcal{H}^{\mathrm{per}}(\kappa)$ on the unit cell $D_\kappa(1)$. Note that in this situation there is only one coupling constant ρ_0 in the game; for simplicity we will write here $\rho = \rho_0$ and similarly $g(\xi_1) = g_0(\xi_1)$. Furthermore the following functions take on a simple, explicit form:

$$P_1(\xi_1, \rho) = 1 + \big(G'(\xi_1, \rho)\big)^2 = 1 + \big(\rho g'(\xi_1)\big)^2$$

$$K(\xi_1, \rho) = \rho g''(\xi_1) P_1^{-3/2}(\xi_1, \rho) = \rho g''(\xi_1)\big(1 + (\rho g'(\xi_1))^2\big)^{-3/2}$$

$$P_2(\xi, \rho) = 1 - \xi_2 K(\xi_1, \rho) = 1 - \xi_2 \rho g''(\xi_1) P_1^{-3/2}(\xi_1, \rho)$$

$$P_3(\xi, \rho) = P_1^{1/2} P_2 = P_1^{1/2}(\xi_1, \rho) - \xi_2 \rho g''(\xi_1) P_1^{-1}(\xi_1, \rho)$$

$$\mathcal{Q}_2(\xi, \rho) = \frac{K(\xi_1, \rho)}{P_2(\xi, \rho)} = \frac{\rho g''(\xi_1)}{P_2(\xi, \rho) P_1^{3/2}(\xi_1, \rho)}$$

In the following we will make use of the following error term estimates on the functions above:

$$P_1^{1/2}(\xi_1, \rho) = 1 + \frac{\rho^2}{2} g'(\xi_1)^2 + \mathcal{O}(\rho^4), \quad P_1^{-1}(\xi_1, \rho) = 1 - \rho^2 g'(\xi_1)^2 + \mathcal{O}(\rho^4),$$

$$P_3(\xi, \rho) = P_1^{1/2}(\xi_1, \rho) - \rho \xi_2 g''(\xi_1) P_1^{-1}(\xi_1, \rho)$$

$$= 1 - \rho \xi_2 g''(\xi_1) + \frac{\rho^2}{2} g'(\xi_1)^2 + \rho^3 \xi_2 g''(\xi_1) g'(\xi_1)^2 + \mathcal{O}(\rho^4).$$

in particular $\mathcal{Q}_\rho = \mathcal{O}(\rho)$.

Lemma 7.2. *There is a constant c independent of a, l, g such that for $\rho < c$ the eigenpair $\lambda^{\mathrm{per}}(\rho)$, $\psi^{\mathrm{per}}(\cdot, \rho)$ of the operator $\mathcal{H}^{\mathrm{per}}(\rho)$ on the unit cell $D_\rho(1)$ satisfies the relations*

$$|\lambda^{\mathrm{per}}(\rho) - 1| \leqslant C\rho^2, \quad \|\mathcal{T}(\rho)\psi^{\mathrm{per}}(\cdot, \rho) - \psi_0^{\mathrm{per}} - \rho \psi_1^{\mathrm{per}}\|_{C^3(\overline{D_0(1)})} \leqslant C\rho^2,$$

where the constant C is independent of ρ, a, l, and g. Here $\psi_0^{\mathrm{per}}(\xi) := \sqrt{2/(\pi l)} \sin \xi_2$ is the ground state of $\mathcal{H}^{\mathrm{per}}(0)$, and ψ_1^{per} is orthogonal to ψ_0^{per} in $L_2(D_0(1))$ and solves the equation

$$\text{(7.8)} \qquad (\mathcal{H}^{\mathrm{per}}(0) - 1)\psi_1^{\mathrm{per}} = -g'' \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2}.$$

Remark 7.3. The fact that the function ψ_0^{per} depends only on the variable ξ_2 leads to several simplifications. On the one hand, $\mathcal{Q}_\rho \psi_0^{\mathrm{per}} = \mathcal{Q}_2(\rho) \frac{\partial}{\partial \xi_2} \psi_0^{\mathrm{per}}$. On the other, integrals involving the functions g and ψ_0 can be often simplified by separation of variables, e.g.:

$$\langle g'' \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2}, \psi_0^{\mathrm{per}} \rangle_{L_2(D_0(1))} = \frac{2}{l\pi} \int_0^l d\xi_1 \, g''(\xi_1) \int_0^\pi d\xi_2 \, \cos(\xi_2)\sin(\xi_2) = 0.$$

In the following this will be used repeatedly.

Proof. The operator $\mathcal{T}(\rho)\mathcal{H}^{\mathrm{per}}(\rho)\mathcal{T}^{-1}(\rho)$ can be regarded as a perturbation of $\mathcal{H}^{\mathrm{per}}(0)$, since $\mathcal{T}(\rho)\mathcal{H}^{\mathrm{per}}(\rho)\mathcal{T}(\rho)^{-1} = \mathcal{H}^{\mathrm{per}}(0) + \mathcal{Q}_\rho$. Note that $\mathcal{Q}_\rho$ is relatively bounded w.r.t. the operator $\mathcal{H}^{\mathrm{per}}(0)$. This follows, since the boundary conditions of the functions in the domain of $\mathcal{H}^{\mathrm{per}}(0)$ imply that the boundary integral terms resulting from partial integration vanish. The bound (7.7) implies that for sufficiently small ρ, $\mathcal{Q}_\rho$ is relatively bounded w.r.t. $\mathcal{H}^{\mathrm{per}}(0)$ with relative bound strictly smaller than one. Consequentyl the sum $\mathcal{H}^{\mathrm{per}}(0) + \mathcal{Q}_\rho$ is a closed operator on the domain of $\mathcal{H}^{\mathrm{per}}(0)$. For a ψ in this domain and every $\phi \in L^2(D_0(1))$ the function $\rho \mapsto \langle \phi, \mathcal{Q}_\rho \psi \rangle$ is holomorphic for ρ in a small complex neighbourhood of zero. Thus $\rho \mapsto \mathcal{H}^{\mathrm{per}}(0) + \mathcal{Q}_\rho$ is a holomorphic family of operators of type (A) in the terminology of [23, VII.§2]. Note that one is the lowest eigenvalue of $\mathcal{H}^{\mathrm{per}}(0)$ and ψ_0^{per} is the associated normalized eigenfunction. Thus, e.g., Theorem II.5.11 of [23] implies that there is a constant C such that

(7.9)
$$\lambda^{\mathrm{per}}(\rho) = 1 + \rho\lambda_1^{\mathrm{per}} + \rho^2 \lambda_2^{\mathrm{per}}(\rho), \qquad\qquad |\lambda_2^{\mathrm{per}}(\rho)| \leqslant C,$$
$$\mathcal{T}(\rho)\psi^{\mathrm{per}}(\cdot, \rho) = \psi_0^{\mathrm{per}} + \rho\psi_1^{\mathrm{per}} + \rho^2 \psi_2^{\mathrm{per}}(\cdot, \rho) \quad \|\psi_2^{\mathrm{per}}\|_{L_2(D_0(1))} \leqslant C,$$

where $\lambda_1^{\mathrm{per}} \in \mathbb{R}$ and $\psi_1^{\mathrm{per}}: D_0(1) \to \mathbb{R}$ are independent of ρ. The constant C is independent of ρ, a, l, and g since the perturbation $\mathcal{Q}_\rho$ is uniformly bounded in these parameters, provided the normalization condition (2.1) holds.

We substitute (7.9) into the eigenvalue equation $\mathcal{H}^{\mathrm{per}}(\rho)\psi^{\mathrm{per}}(\cdot, \rho) = \lambda^{\mathrm{per}}(\rho)\psi^{\mathrm{per}}(\cdot, \rho)$ and obtain

(7.10)
$$\mathcal{T}(\rho)\big(\mathcal{H}^{\mathrm{per}}(0) + \mathcal{Q}_\rho\big)(\psi_0^{\mathrm{per}} + \rho\psi_1^{\mathrm{per}} + \rho^2\psi_2^{\mathrm{per}}) = \mathcal{H}^{\mathrm{per}}(\rho)\psi^{\mathrm{per}}(\cdot, \rho) = \lambda^{\mathrm{per}}(\rho)\psi^{\mathrm{per}}(\cdot, \rho)$$
$$= \mathcal{T}(\rho)\left(1 + \rho\lambda_1^{\mathrm{per}} + \rho^2\lambda_2^{\mathrm{per}}\right)\left(\psi_0^{\mathrm{per}} + \rho\psi_1^{\mathrm{per}} + \rho^2\psi_2^{\mathrm{per}}\right).$$

Thus,

$$\mathcal{H}^{\mathrm{per}}(0)\psi_0^{\mathrm{per}} + \rho\mathcal{H}^{\mathrm{per}}(0)\psi_1^{\mathrm{per}} + \mathcal{Q}_\rho\psi_0^{\mathrm{per}} + \rho\mathcal{Q}_\rho\psi_1^{\mathrm{per}} + \mathcal{O}(\rho^2) = \psi_0^{\mathrm{per}} + \rho\psi_1^{\mathrm{per}} + \rho\lambda_1^{\mathrm{per}}\psi_0^{\mathrm{per}} + \mathcal{O}(\rho^2).$$

We want to isolate the terms which are linear in the perturbation parameter ρ. For this reason we substract the eigenvalue equation $\mathcal{H}^{\mathrm{per}}(0)\psi_0^{\mathrm{per}} = \psi_0^{\mathrm{per}}$ for the unperturbed operator,

$$\rho\mathcal{H}^{\mathrm{per}}(0)\psi_1^{\mathrm{per}} + \mathcal{Q}_\rho\psi_0^{\mathrm{per}} + \rho\mathcal{Q}_\rho\psi_1^{\mathrm{per}} = \rho\mathcal{H}^{\mathrm{per}}(0)\psi_1^{\mathrm{per}} + \mathcal{Q}_2(\xi, \rho)\frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} + \mathcal{O}(\rho^2)$$
$$= \rho\psi_1^{\mathrm{per}} + \rho\lambda_1^{\mathrm{per}}\psi_0^{\mathrm{per}} + \mathcal{O}(\rho^2),$$

divide by ρ, and finaly take the limit $\rho \to 0$ resulting in:

$$(\mathcal{H}^{\mathrm{per}}(0) - 1)\psi_1^{\mathrm{per}} = -g''\frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} + \lambda_1^{\mathrm{per}}\psi_0^{\mathrm{per}}.$$

This equation is solvable, if and only if the right hand side is orthogonal to ψ_0^{per} in $L_2(D_0(1))$, i.e.,

$$\lambda_1^{\mathrm{per}} = \lambda_1^{\mathrm{per}}\langle \psi_0^{\mathrm{per}}, \psi_0^{\mathrm{per}} \rangle_{L_2(D_0(1))} = \langle g''\frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2}, \psi_0^{\mathrm{per}} \rangle_{L_2(D_0(1))} = 0$$

This implies (7.8). The orthogonality condition for ψ_1^{per} is implied by the identities

$$
\begin{aligned}
1 &= \|\psi^{\mathrm{per}}\|^2_{L_2(D_\rho(1))} = \langle \mathcal{T}(\rho)\psi^{\mathrm{per}}, P_1^{1/2} P_2 \mathcal{T}(\rho)\psi^{\mathrm{per}}\rangle_{L_2(D_0(1))} \\
&= \langle \psi_0^{\mathrm{per}}, P_3 \psi_0^{\mathrm{per}}\rangle_{L_2(D_0(1))} + 2\rho\langle \psi_0^{\mathrm{per}}, P_3 \psi_1^{\mathrm{per}}\rangle_{L_2(D_0(1))} + \mathcal{O}(\rho^2) \\
&= \langle \psi_0^{\mathrm{per}}, (1 - \rho\xi_2 g_1'')\psi_0^{\mathrm{per}}\rangle_{L_2(D_0(1))} + 2\rho\langle \psi_0^{\mathrm{per}}, \psi_1^{\mathrm{per}}\rangle_{L_2(D_0(1))} + \mathcal{O}(\rho^2) \\
&= 1 + 2\rho\langle \psi_0^{\mathrm{per}}, \psi_1^{\mathrm{per}}\rangle_{L_2(D_0(1))} + \mathcal{O}(\rho^2)
\end{aligned}
$$

since $\int_0^l d\xi_1 g''(\xi_1) = 0$.

The functions $\mathcal{T}(\rho)\psi^{\mathrm{per}}$ and ψ_1^{per} belong to $W_2^1(D_0(1))$. By the standard smoothness improving theorems (see, for instance, [39, Ch. IV, Sec. 2]) and by the smoothness of the function g we obtain that $\mathcal{T}(\rho)\psi^{\mathrm{per}}$ and ψ_1^{per} are also elements of $W_2^4(D_0(1))$. In view of the embedding $W_2^4(D_0(1)) \subset C^2(\overline{D_0(1)})$ [39, Ch III, Sec. 6] the functions $\mathcal{T}(\rho)\psi^{\mathrm{per}}$ and ψ_1^{per} are the classical solutions to the eigenvalue problem for ψ_1^{per} and to (7.8). By applying Schauder estimates (see [34, Ch. III, Sec. 1-3]) we conclude that $\mathcal{T}(\rho)\psi^{\mathrm{per}}$ and ψ_1^{per} belong to $C^3(\overline{D_0(1)})$.

It remains to prove that the asymptotics (7.9) for $\mathcal{T}(\rho)\psi^{\mathrm{per}}$ holds true also in $C^3(\overline{D_0(1)})$-norm. In order to do it, by (7.10) we write first the equation for ψ_2^{per}

$$
\begin{aligned}
\mathcal{H}^{\mathrm{per}}(0)\psi_2^{\mathrm{per}} =& (-\mathcal{Q}_\rho + \lambda^{\mathrm{per}})\psi_2^{\mathrm{per}} \\
&+ \mathcal{T}^{-1}(\rho)\left[\lambda_2^{\mathrm{per}}(\psi_0^{\mathrm{per}} + \rho\psi_1^{\mathrm{per}}) + \lambda_1^{\mathrm{per}}\psi_1^{\mathrm{per}} + \rho^{-1}\mathcal{Q}_\rho\psi_1^{\mathrm{per}} + \rho^{-2}\left(\mathcal{Q}_\rho - \rho g''\frac{\partial}{\partial\xi_2}\right)\psi_0^{\mathrm{per}}\right].
\end{aligned}
$$

As above we again apply smoothness improving theorems from [39], [34] and obtain the uniform in ρ estimate

$$
\|\psi_2^{\mathrm{per}}\|_{C^3(\overline{D_0(1)})} \leqslant C,
$$

which yields the desired asymptotics for ψ^{per}. $\square$

Since ψ^{per} is the ground state, it is positive in $D_\rho(1)$. Hence, h in (6.1) is well-defined. Let us check its behaviour near the Dirichlet boundaries. It follows from the definition of $\mathcal{T}(\rho)$ that

$$
\big(\mathcal{T}(\rho)\psi^{\mathrm{per}}\big)(0, \xi_2, \rho) = \psi^{\mathrm{per}}(0, \xi_2, \rho).
$$

In view of the smoothness of ψ^{per} we can apply Taylor formula as $\xi_2 \to +0$,

$$
\psi^{\mathrm{per}}(0, \xi_2, \rho) = \xi_2 \left.\frac{\partial\mathcal{T}(\rho)\psi^{\mathrm{per}}}{\partial\xi_2}\right|_{\xi=0} + \xi_2^2 \left.\frac{\partial^2\mathcal{T}(\rho)\psi^{\mathrm{per}}}{\partial\xi_2^2}\right|_{\substack{\xi_1=0 \\ \xi_2=\tilde{\xi}_2}},
$$

for some $\tilde{\xi}_2 \in [0, \xi_2]$. Here we have also employed the Dirichlet condition for ψ^{per} at $\xi_2 = 0$. We substitute the asymptotics for $\mathcal{T}(\rho)\psi^{\mathrm{per}}$ from Lemma 7.2 into the last formula and arrive at

$$
\psi^{\mathrm{per}}(x_1, 0, \rho) - \sqrt{\frac{2}{\pi l}}x_2 = \mathcal{O}(x_2\rho + x_2^2) \quad \text{as} \quad x_2 \to +0, \quad \rho \to 0.
$$

In the same way we obtain

$$
\psi^{\mathrm{per}}(x_1, 0, \rho) - \sqrt{\frac{2}{\pi l}}x_2 = \mathcal{O}\big((\pi - x_2)\rho + (\pi - x_2)^2\big) \quad \text{as} \quad x_2 \to +0, \quad \rho \to 0.
$$

Therefore, for small ρ this function satisfies the estimate

(7.11) $$\|h\|_{C^2[0,\pi]} \leqslant C\rho,$$

where the constant C is independent of l, ρ, and g.

After this preliminary analysis of the operator on the unit cell, i.e. a waveguide segment of length l, we turn now to waveguide segments composed of N cells. In what follows we denote all numerical constants by C. In different formulas this symbol will denote different positive numbers independent of L, N, l, ρ, a, and g. Let $\mathcal{H}_0(0, N)$ be the operator $\mathcal{H}_\kappa(0, N)$ taken for $\kappa = 0$. In fact, $\mathcal{H}_0(0, N)$ is the Laplacian on $D_0(N)$ with Dirichlet condition on $\Gamma_0(N)$ and with Neumann condition on $\gamma_0(N)$. Note that the ground state ψ_0^{per} of $\mathcal{H}_0(0, N)$ is a function of ξ_2 only. Thus the function H defining the Mezincescu b.c. vanishes.

The lowest eigenvalue of $\mathcal{H}_\kappa(0, N)$ is $\lambda^{\mathrm{per}}(\kappa)$; the corresponding eigenfunction is the function ψ^{per}. Let $\widehat{\lambda}^{\mathrm{per}}(\kappa)$ be the second eigenvalue of $\mathcal{H}_\kappa(0, N)$. For the purposes of perturbation theory we need to establish a minimal distance between the two lowest eigenvalues $\lambda^{\mathrm{per}}(\kappa)$ and $\widehat{\lambda}^{\mathrm{per}}(\kappa)$ For this purpose we need

Lemma 7.4. *Under the assumption (2.1) there exists a constant C independent of l, N, a, g such that the eigenvalue $\widehat{\lambda}^p$ obeys the inequality*

$$|\widehat{\lambda}^p(\kappa) - 1 - 4\pi^2 L^{-2}| \leqslant C\kappa.$$

Proof. The quadratic form associated with $\mathcal{H}_\kappa(0, N)$ reads as follows

$$\langle \mathcal{H}_\kappa(0, N)u, u \rangle_{L_2(D_\kappa(N))} = \|\nabla u\|_{L_2(D_\kappa(N))}^2 + \int_{x_1=0} h|u|^2 \, \mathrm{d}x_2 - \int_{x_1=L} h|u|^2 \, \mathrm{d}x_2.$$

Denoting $v := \mathcal{T}(\kappa)u$, by (7.4), (7.3) we rewrite the form as follows

$$\langle \mathcal{H}_\kappa(0, N)u, u \rangle_{L_2(D_\kappa(N))} = \left\| P_3^{-1/2} \frac{\partial v}{\partial \xi_1} \right\|_{L_2(D_0(N))}^2 + \left\| P_3^{1/2} \frac{\partial v}{\partial \xi_2} \right\|_{L_2(D_0(N))}^2$$
$$+ \int_0^\pi h(\xi_2, \kappa)(|v(0, \xi_2)|^2 - |v(L, \xi_2)|^2) \, \mathrm{d}\xi_2.$$
$$= \langle \mathcal{H}_0(0, N)v, v \rangle_{L_2(D_0(N))} + \langle \mathcal{Q}_\kappa v, v \rangle_{L_2(D_0(N))}$$

Since the two lowest eigenvalues of $\mathcal{H}_0(0, N)$ are 1 and $1 + 4\pi^2 L^{-2}$, it is sufficient to estimate the quadratic form of $\mathcal{Q}_\kappa$. It will be convenient to use the abbreviation $\langle \cdot, \cdot \rangle_0 := \langle \cdot, \cdot \rangle_{L_2(D_0(N))}$ and $\| \cdot \|_0 := \| \cdot \|_{L_2(D_0(N))}$, in the sequel. The estimate (7.11) and standard embedding theorems yield

$$(7.12) \qquad \left| \int_0^\pi h(\xi, \kappa)(|v(0, \xi_2)|^2 - |v(L, \xi_2)|^2) \, \mathrm{d}\xi_2 \right| \leqslant C\kappa \|\nabla_\xi v\|_0^2.$$

Thus we obtain

$$\langle \mathcal{Q}_\kappa v, v \rangle_{L_2(D_0(N))} = \left\langle \frac{\partial v}{\partial \xi_1}, (1 - P_3^{-1}) \frac{\partial v}{\partial \xi_1} \right\rangle_0 + \left\langle \frac{\partial v}{\partial \xi_2}, (1 - P_3) \frac{\partial v}{\partial \xi_2} \right\rangle_0$$
$$+ \int_0^\pi h(\xi, \kappa)(|v(0, \xi_2)|^2 - |v(L, \xi_2)|^2) \, \mathrm{d}\xi_2$$
$$= \left\langle \frac{\partial v}{\partial \xi_1}, (\kappa \xi_2 g'' + \mathcal{O}(\kappa^2)) \frac{\partial v}{\partial \xi_1} \right\rangle_0 + \left\langle \frac{\partial v}{\partial \xi_2}, (-\kappa \xi_2 g'' + \mathcal{O}(\kappa^2)) \frac{\partial v}{\partial \xi_2} \right\rangle_0$$
$$+ \mathcal{O}\left(\kappa \|\nabla v\|_0^2\right)$$

$$=\kappa \left\langle \frac{\partial v}{\partial \xi_1}, \xi_2 g'' \frac{\partial v}{\partial \xi_1} \right\rangle_0 - \kappa \left\langle \frac{\partial v}{\partial \xi_2}, \xi_2 g'' \frac{\partial v}{\partial \xi_2} \right\rangle_0 + \mathcal{O}\left(\kappa \|\nabla v\|_0^2\right)$$
$$=\mathcal{O}\left(\kappa \|\nabla v\|_0^2\right)$$

Thus we have the estimate

$$\langle \mathcal{H}_0(0, N)v, v \rangle_0 + \langle \mathcal{Q}_\kappa v, v \rangle_0 = (1 + \mathcal{O}(\kappa)) \langle \mathcal{H}_0(0, N)v, v \rangle_0$$

Now we apply the minimax principle and obtain

$$\widehat{\lambda}^p(\kappa) = \max_{w \in L_2(D_0(N))} \min_{v \in W_2^1(D_0(N)), v \perp w, \|v\|=1} \langle (\mathcal{H}_0(0, N) + \mathcal{Q}_\kappa)v, v \rangle_0$$
$$= (1 + \mathcal{O}(\kappa)) \max_{w \in L_2(D_0(N))} \min_{v \in W_2^1(D_0(N)), v \perp w, \|v\|=1} \langle \mathcal{H}_0(0, N)v, v \rangle_0$$
$$= (1 + \mathcal{O}(\kappa)) \left(1 + (\frac{2\pi}{L})^2\right) = 1 + \left(\frac{2\pi}{L}\right)^2 + \mathcal{O}(\kappa)$$

since $L \geqslant 1$. $\qquad\square$

The proven lemma implies that there exists $C > 0$ such that the set $\Xi := \{\lambda \in \mathbb{C} : |\lambda - 1| < C\kappa\}$ contains no eigenvalues of $\mathcal{H}_\kappa(0, N)$ except $\lambda^{\mathrm{per}}(\kappa)$. Now for $\lambda \in \Xi$:

$$|\widehat{\lambda}^p(\kappa) - \lambda| \geqslant |1 + 4\pi^2 L^{-2} - 1| + |1 + 4\pi^2 L^{-2} - \widehat{\lambda}^p(\kappa)| + |\lambda - 1|$$
$$\geqslant 4\pi^2 L^{-2} - 2C\kappa$$

Thus the distance from this set Ξ to the remaining part of the spectrum $\sigma(\mathcal{H}_\kappa(0, N)) \setminus \{\lambda^p(\kappa)\}$ is estimated from below by $4\pi^2 L^{-2} - 2C\kappa$. Theorem 6.1 concerns values of κ which are much smaller than the inverse length $1/L$ of the waveguide segment. Thus we can certainly assume $\kappa \leqslant \pi^2/(C L^2)$ which means that $4\pi^2 L^{-2} - 2C\kappa \geqslant 3\pi^2 L^{-2}$ is positive. By [23, Ch. V, Sec. 3.5] it implies that for $\lambda \in \Xi$ the identity

$$(7.13) \qquad (\mathcal{H}_\kappa(0, N) - \lambda)^{-1} = \frac{\langle \cdot, \widetilde{\psi}^{\mathrm{per}} \rangle_\kappa}{\lambda^{\mathrm{per}}(\kappa) - \lambda} \widetilde{\psi}^{\mathrm{per}} + \mathcal{R}_\kappa(\lambda)$$

holds true, where $\widetilde{\psi}^{\mathrm{per}} := N^{-1/2} \psi^{\mathrm{per}}$, $\langle \cdot, \cdot \rangle_\kappa := \langle \cdot, \cdot \rangle_{L_2(D_\kappa(N))}$, $\mathcal{R}_\kappa$ is holomorphic w.r.t. to $\lambda \in \Xi$ as an operator from $L_2(D_\rho(N))$ to $W_2^2(D_\rho(N))$. The factor $N^{-1/2}$ in the definition of $\widetilde{\psi}^{\mathrm{per}}$ provides the normalization $\|\widetilde{\psi}^{\mathrm{per}}\|_{L_2(D_0(N))} = 1$. For $f \in L_2(D_\kappa(N))$ the function $u = \mathcal{R}_\kappa(\lambda)f$ solves the equation

$$(7.14) \qquad \left(\mathcal{H}_\kappa(0, N) - \lambda^{\mathrm{per}}(\kappa)\right)u = \widehat{f}_\kappa, \quad \widehat{f}_\kappa := f - \widetilde{\psi}^{\mathrm{per}} \langle f, \widetilde{\psi}^{\mathrm{per}} \rangle_{L_2(D_\kappa(N))},$$
$$(7.15) \qquad \|f_\kappa\|_{L_2(D_\kappa(N))} \lesssim \|f\|_{L_2(D_\kappa(N))}.$$

For each $f \in L_2(D_\kappa(N))$ the function $\mathcal{R}_\kappa(\lambda)f$ is orthogonal to $\widetilde{\psi}_0^{\mathrm{per}}$ in $L_2(D_\kappa(N))$.

The formulas (7.2), (7.13) imply

$$(7.16) \qquad (\mathcal{T}(\kappa)\mathcal{H}_\kappa(0, N)\mathcal{T}^{-1}(\kappa) - \lambda)^{-1} = \frac{\langle \cdot, P_3 \mathcal{T}(\kappa)\widetilde{\psi}^{\mathrm{per}} \rangle_0}{\lambda^{\mathrm{per}}(\kappa) - \lambda} \mathcal{T}(\kappa)\widetilde{\psi}^{\mathrm{per}} + \widehat{\mathcal{R}}_\kappa(\lambda),$$

$$\widehat{\mathcal{R}}_\kappa(\lambda) := \mathcal{T}(\kappa)\mathcal{R}_\kappa(\lambda)\mathcal{T}^{-1}(\kappa).$$

Let $\mathcal{H}_0(0, N)$ be the Laplacian on $D_0(N)$ subject to Dirichlet boundary condition on $\Gamma_0(N)$ and to Neumann one on $\gamma_0(N)$.

For the subsequent estimates in the perturbation theory we will need some a-priori bounds on several 'tame' operators, like resolvents. These are gathered in the next

Lemma 7.5. *For all $\lambda \in \Xi$ and all $f \in L_2(D_\kappa(N))$ the estimates*

$$(7.17) \qquad \|\mathcal{R}_\kappa(\lambda)f\|_{W_2^2(D_\rho(N))} \leqslant CL^2\|f\|_{L_2(D_\kappa(N))},$$

$$(7.18) \qquad \|\widehat{\mathcal{R}}_\kappa(\lambda)f - \mathcal{R}_0(1)\mathcal{T}(\kappa)f\|_{W_2^2(D_0(N))} \leqslant C\kappa L^4\|f\|_{L_2(D_\kappa(N))}$$

hold true, where the constants C are independent of L, l, N, a, g, and f.

Proof. Given and $f \in L_2(D_\kappa(N))$, we introduce f_κ and u by (7.14). It follows from the definition of Ξ and $\mathcal{R}_\kappa(\lambda)$ and (7.15) that

$$(7.19) \qquad \|u\|_{L_2(D_\kappa(N))} \leqslant CL^2\|f\|_{L_2(D_\kappa(N))}.$$

Let $\chi = \chi(t)$ be an infinitely differentiable cut-off function equalling one for $t < 1/4$ and zero for $t > 1/2$. We denote

$$\phi(\xi, \kappa) := \chi(\xi_1)e^{\xi_1 h(\xi_2, \kappa)} + \chi(L - \xi_1)e^{(L-\xi_1)h(\xi_2,\kappa)} + 2 - \chi(\xi_1) - \chi(L - \xi_1).$$

By (7.11) we have

$$(7.20) \qquad \|\phi - 1\|_{C^2(\overline{D_0(N)})} \leqslant C\kappa.$$

We construct u as $u = \mathcal{T}^{-1}(\kappa)\phi\widetilde{u}$. We substitute this identity and (7.5) into the equation (7.14) which implies that for $\widetilde{u}$

$$(7.21) \qquad (\mathcal{H}_0(0, N) + \mathcal{L}_\kappa)\widetilde{u} = \widetilde{u} + \phi^{-1}\mathcal{T}(\kappa)f_\kappa,$$

while $\mathcal{L}_\kappa$ is a second order differential operator such that

$$(7.22) \qquad \|\mathcal{L}_\kappa v\|_{W_2^2(D_0(N))} \leqslant C\kappa\|v\|_{W_2^2(D_0(N))}.$$

It follows from (7.19) that

$$(7.23) \qquad \|\widetilde{u}\|_{L_2(D_0(N))} \leqslant CL^2\|f\|_{L_2(D_0(N))}.$$

Reproducing word by word the proof of Lemma 7.1 in [34, Ch. Ⅲ, Sec. 7] and employing (7.23), one can prove easily an estimate

$$(7.24) \qquad \|v\|_{W_2^2(D_0(N))} \leqslant C\|\mathcal{H}_0(0, N)v\|_{L_2(D_0(N))}.$$

Combining this estimate with (7.22), we can solve the equation for $\widetilde{u}$ as follows,

$$\widetilde{u} = \mathcal{H}_0(0, N)^{-1}(I - \mathcal{L}_\kappa\mathcal{H}_0(0, N)^{-1})^{-1}(\widetilde{u} + \phi^{-1}\mathcal{T}(\kappa)f_\kappa),$$

where all the operators are well-defined. Applying the estimates (7.23), (7.24) once again, we arrive at the inequality

$$\|\widetilde{u}\|_{W_2^2(D_0(N))} \leqslant CL^2\|f\|_{L_2(D_0(N))},$$

which implies (7.17).

We proceed to the proof of (7.18). We rewrite (7.21) as

$$(7.25) \qquad (\mathcal{H}_0(0, N) - 1)\widetilde{u} = \phi^{-1}\mathcal{T}(\kappa)f_\kappa + \mathcal{L}_\kappa\widetilde{u}.$$

In accordance with (7.14) the function $u_0 := \mathcal{R}_0(1)\mathcal{T}(\kappa)f$ is the solution to the equation

$$(7.26) \qquad (\mathcal{H}_0(0, N) - 1)u_0 = f_0,$$

$$f_0 := \mathcal{T}(\kappa)f - (\mathcal{T}(\kappa)f, \widetilde{\psi}_0^{\mathrm{per}})_{L_2(D_0(N))}\widetilde{\psi}_0^{\mathrm{per}}, \quad \widetilde{\psi}_0^{\mathrm{per}} := N^{-1/2}\psi_0.$$

By the definition of $\mathcal{R}_\kappa(\lambda)$ and $\mathcal{R}_0(1)$, the function $u = \mathcal{T}^{-1}(\kappa)\phi\widetilde{u}$ is orthogonal to $\widetilde{\psi}^{\mathrm{per}}$ in $L_2(D_\kappa(N))$, while u_0 is orthogonal to $\widetilde{\psi}_0^{\mathrm{per}}$ in $L_2(D_0(N))$. Hence, by the definition of ϕ and Lemma 7.2,

$$(7.27) \qquad \begin{aligned} \widetilde{u} &= \widehat{u} + \widetilde{\psi}_0^{\mathrm{per}}(\widetilde{u}, \widetilde{\psi}_0^{\mathrm{per}})_{L_2(D_0(N))}, \\ (\widehat{u}, \widetilde{\psi}_0^{\mathrm{per}})_{L_2(D_0(N))} &= 0, \quad |(\widetilde{u}, \widetilde{\psi}_0^{\mathrm{per}})_{L_2(D_0(N))}| \leqslant C\kappa. \end{aligned}$$

The equations (7.25), (7.26) imply

$$(\mathcal{H}_0(0, N) - 1)(\widehat{u} - u_0) = \phi^{-1}\mathcal{T}(\kappa)f_\kappa - \mathcal{L}_\kappa\widetilde{u} - f_0,$$

where $(\widehat{u} - u_0)$ is orthogonal to $\widetilde{\psi}_0^{\mathrm{per}}$. Since the function $\widehat{u} - u_0$ is well-defined, the right hand side of the last equation is orthogonal to $\widetilde{\psi}_0^{\mathrm{per}}$ in $L_2(D_0(N))$. Moreover, the definition of ϕ, f_κ, f_0, Lemma 7.2, and (7.22) yields

$$\|\phi^{-1}\mathcal{T}(\kappa)f_\kappa - \mathcal{L}_\kappa\widetilde{u} - f_0\|_{L_2(D_0(N))} \leqslant C\kappa L^2\|f\|_{L_2(D_\kappa(N))}.$$

Employing the aforementioned facts, by analogy with (7.19), (7.24) we get

$$\|\widehat{u} - u_0\|_{W_2{}^2(D_0(N))} \leqslant C\kappa L^2\|f\|_{L_2(D_\kappa(N))}.$$

Since $\mathcal{T}(\kappa)\mathcal{R}_\kappa(\lambda)f = \mathcal{T}(\kappa)u = \widetilde{u}\phi$, the last inequality and (7.27) imply (7.18). $\qquad\square$

8. Deterministic lower bounds: Proof of Theorem 6.1

After these preparatory lemmata we turn to the heart of the proof of of Theorem 6.1. It consists in deteremining the sign and estimating the three terms of the perturbation $\mathcal{Q}_\rho = \mathcal{Q}_{11}\frac{\partial^2}{\partial\xi_1^2} + \mathcal{Q}_1\frac{\partial}{\partial\xi_1} + \mathcal{Q}_2\frac{\partial}{\partial\xi_2}$. In fact, since we have to perform a perturbative analysis not around the Laplacian $\mathcal{H}_0(0, N)$ of the straight waveguide segment but rather around the one $\mathcal{H}_\kappa(0, N)$ with the maximal possible curvature, the effective perturbation will be $\mathcal{H}_\kappa(\epsilon, N) - \mathcal{H}_\kappa(0, N) = \mathcal{Q}_\rho - \mathcal{Q}_\kappa$.

As before we will work with the change of the variables $x \mapsto \xi$ introduced above to transform the operator $\mathcal{H}_\kappa(\epsilon, N)$,

$$(8.1) \qquad \mathcal{T}(\rho)\mathcal{H}_\kappa(\epsilon, N)\mathcal{T}(\rho)^{-1} = \mathcal{T}(\kappa)\mathcal{H}_\kappa(0, N)\mathcal{T}(\kappa)^{-1} + \mathcal{M}(\epsilon, \kappa)$$

$$\mathcal{M}(\epsilon, \kappa) = M_{11}\frac{\partial^2}{\partial\xi_1^2} + M_1\frac{\partial}{\partial\xi_1} + M_2\frac{\partial}{\partial\xi_2},$$

$$(8.2) \qquad M_{11} := \mathcal{Q}_{11}(\xi, \rho) - \mathcal{Q}_{11}(\xi, \kappa), \quad M_i := \mathcal{Q}_i(\xi, \rho) - \mathcal{Q}_i(\xi, \kappa), \quad i = 1, 2.$$

where the functions $\mathcal{Q}_{11}$, $\mathcal{Q}_i$ have been introduced in (7.6). Let us calculate explicitly $\mathcal{Q}_2$ for values $\xi_1 \in [0, l]$. First we expand the function $\rho \mapsto \mathcal{Q}_2(\xi, \rho)$ around $\rho = 0$:

$$(8.3) \qquad \mathcal{Q}_2(\xi, \rho) = g''(\xi_1)\rho\left(1 + \xi_2 g''(\xi_1)\rho + \mathcal{O}(\rho^2)\right) = g''(\xi_1)\rho + \xi_2(g''(\xi_1))^2\rho^2 + \mathcal{O}(\rho^3).$$

Suppressing for the moment the ξ-dependence and using here the prime $'$ to denote derivatives w.r.t. the variable ρ we have

$$\mathcal{Q}_2'(\rho) = \mathcal{Q}_2'(0) + \mathcal{Q}_2''(0)\rho + \mathcal{O}(\rho^2)$$
$$\mathcal{Q}_2''(\rho) = \mathcal{Q}_2''(0) + \mathcal{O}(\rho).$$

Thus for some intermediate value $\tilde{\rho} \in [\rho, \kappa]$ we obtain

$$\mathcal{Q}_2(\rho) - \mathcal{Q}_2(\kappa) = \mathcal{Q}_2'(\kappa)(\rho - \kappa) + \frac{1}{2}\mathcal{Q}_2''(\kappa)(\rho - \kappa)^2 + \frac{1}{6}\mathcal{Q}_2'''(\tilde{\rho})(\rho - \kappa)^3$$

$$= -\mathcal{Q}_2'(0)\epsilon - \mathcal{Q}_2''(0)\epsilon\kappa + \frac{1}{2}\mathcal{Q}_2''(0)\epsilon^2 + \mathcal{O}(\kappa^2\epsilon)$$

and after evaluating (8.3) and inserting the value at $\rho = 0$,

$$= -\frac{\partial^2 g(\xi_1)}{\partial\xi_1^2}\epsilon - \xi_2\left(\frac{\partial^2 g(\xi_1)}{\partial\xi_1^2}\right)^2\epsilon(\kappa + \kappa) + \mathcal{O}(\kappa^2\epsilon)$$

follows, where we have made the dependence on ξ explicit, again. Direct calculations of similar type show that

(8.4)
$$M_{11} := M_{11}^{(0)} + M_{11}^{(1)} + \epsilon_j\kappa^2 M_{11}^{(2)}, \quad M_i := M_i^{(0)} + M_i^{(1)} + \epsilon_j\kappa^2 M_i^{(2)},$$

$$M_{11}^{(0)} = 2\epsilon_j\xi_2 g_j'', \quad M_{11}^{(1)} = -\epsilon_j(\kappa + \rho_j)\big((g_j')^2 - 3\xi_2^2(g_j'')^2\big),$$

$$M_1^{(0)} = \epsilon_j\xi_2 g_j''', \quad M_1^{(1)} = -\epsilon_j(\kappa + \rho_j)(g_j' - 3\xi_2^2 g_j''')g_j'',$$

$$M_2^{(0)} = -\epsilon_j g_j'', \quad M_2^{(1)} = -\epsilon_j(\kappa + \rho_j)\,\xi_2(g_j'')^2,$$

for $\xi_1 \in [(j-1)l, jl]$, and

(8.5)
$$|M_{11}^{(2)}| + |M_1^{(2)}| + |M_2^{(2)}| \leqslant C,$$

where the constant C is independent of L, N, l, j, a, ξ, and g.

Consider $\lambda \in \Theta$. Employing the technique used in [2, Sect. 4], [17] and (7.13), (8.1), one can show easily that $\lambda = \lambda(\rho) \in \Theta$ is an eigenvalue of $\mathcal{H}_\kappa(\epsilon, N)$ if and only if it solves the equation

$$\lambda - \lambda^{\mathrm{per}}(\rho) = \langle(I + \mathcal{M}(\epsilon, \kappa)\widehat{\mathcal{R}}_\kappa(\lambda))^{-1}\mathcal{M}(\epsilon, \kappa)\widehat{\psi}^{\mathrm{per}}, P_3\widehat{\psi}^{\mathrm{per}}\rangle,$$

where $\widehat{\psi}^{\mathrm{per}} := \mathcal{T}(\kappa)\widetilde{\psi}^{\mathrm{per}}$, $\langle\cdot, \cdot\rangle := \langle\cdot, \cdot\rangle_0$, P_3 is taken for $\rho = (\kappa, \ldots, \kappa)$. We rewrite this equation as

(8.6)
$$\lambda(\rho) - \lambda^{\mathrm{per}}(\kappa) = \langle\mathcal{M}(\epsilon, \kappa)\widehat{\psi}^{\mathrm{per}}, P_3\widehat{\psi}^{\mathrm{per}}\rangle - \langle\mathcal{M}(\epsilon, \kappa)\widehat{\mathcal{R}}_\kappa(\lambda(\rho))\mathcal{M}(\epsilon, \kappa)\widehat{\psi}^{\mathrm{per}}, P_3\widehat{\psi}^{\mathrm{per}}\rangle$$
$$+ \langle\big(\mathcal{M}(\epsilon, \kappa)\widehat{\mathcal{R}}_\kappa(\lambda(\rho))\big)^2(I + \mathcal{M}(\epsilon, \kappa)\widehat{\mathcal{R}}_\kappa(\lambda))^{-1}\mathcal{M}(\epsilon, \kappa)\widehat{\psi}^{\mathrm{per}}, P_3\widehat{\psi}^{\mathrm{per}}\rangle.$$

The reduced resolvent $\widehat{\mathcal{R}}_\kappa$ satisfies the resolvent identity

$$\widehat{\mathcal{R}}_\kappa(\lambda) - \widehat{\mathcal{R}}_\kappa(\mu) = (\lambda - \mu)\widehat{\mathcal{R}}_\kappa(\mu)\widehat{\mathcal{R}}_\kappa(\lambda).$$

Substituting this identity with $\lambda = \lambda(\rho)$, $\mu = 1$ and (8.4) into (8.6), and using the fact that $\widehat{\mathcal{R}}_\kappa(1) = \mathcal{R}_0(1)$, we obtain

(8.7)
$$\lambda(\rho) - \lambda^{\mathrm{per}}(\kappa) = S_1(\epsilon, \kappa) + (\lambda(\rho) - \lambda^{\mathrm{per}}(\kappa))S_2(\epsilon, \kappa) + S_3(\epsilon, \kappa),$$

$$S_1(\epsilon, \kappa) = \langle\widehat{\mathcal{M}}(\epsilon, \kappa)\widehat{\psi}^{\mathrm{per}}, \widehat{P}_3\widehat{\psi}^{\mathrm{per}}\rangle - \langle\widehat{\mathcal{M}}(\epsilon, \kappa)\mathcal{R}_0(1)\widehat{\mathcal{M}}(\epsilon, \kappa)\widehat{\psi}^{\mathrm{per}}, \widehat{\psi}^{\mathrm{per}}\rangle,$$

$$S_2(\epsilon, \kappa) = -\langle\mathcal{M}(\epsilon, \kappa)\widehat{\mathcal{R}}_\kappa(\lambda^{\mathrm{per}}(\kappa))\widehat{\mathcal{R}}_\kappa(\lambda(\rho))\mathcal{M}(\epsilon, \kappa)\widehat{\psi}^{\mathrm{per}}, P_3\widehat{\psi}^{\mathrm{per}}\rangle$$

$$S_3(\epsilon, \kappa) = -\langle\mathcal{M}^{(2)}(\epsilon, \kappa)\big(I - \widehat{\mathcal{R}}_\kappa(\lambda^{\mathrm{per}}(\kappa))\mathcal{M}(\epsilon, \kappa)\big)\widehat{\psi}^{\mathrm{per}}, P_3\widehat{\psi}^{\mathrm{per}}\rangle$$

$$- \langle\widehat{\mathcal{M}}(\epsilon, \kappa)\widehat{\mathcal{R}}_\kappa(\lambda^{\mathrm{per}}(\kappa))\mathcal{M}^{(2)}(\epsilon, \kappa)\widehat{\psi}^{\mathrm{per}}, P_3\psi^{\mathrm{per}}\rangle$$

$$+ \langle\big(\mathcal{M}(\epsilon, \kappa)\widehat{\mathcal{R}}_\kappa(\lambda(\rho))\big)^2(I + \mathcal{M}(\epsilon, \kappa)\widehat{\mathcal{R}}_\kappa(\lambda))^{-1}\mathcal{M}(\epsilon, \kappa)\widehat{\psi}^{\mathrm{per}}, P_3\widehat{\psi}^{\mathrm{per}}\rangle,$$

$$- \langle\widehat{\mathcal{M}}(\epsilon, \kappa)\big(\widehat{\mathcal{R}}_\kappa(\lambda(\rho)) - \mathcal{R}_0(1)\big)\widehat{\mathcal{M}}(\epsilon, \kappa)\widehat{\psi}^{\mathrm{per}}, P_3\widehat{\psi}^{\mathrm{per}}\rangle$$

$$+ \langle\widehat{\mathcal{M}}(\epsilon, \kappa)\widehat{\psi}^{\mathrm{per}}, (P_3 - \widehat{P}_3)\psi^{\mathrm{per}}\rangle$$

$$- \langle\widehat{\mathcal{M}}(\epsilon, \kappa)\mathcal{R}_0(1)\widehat{\mathcal{M}}(\epsilon, \kappa)\widehat{\psi}^{\mathrm{per}}, (P_3 - 1)\widehat{\psi}^{\mathrm{per}}\rangle,$$

$$\widehat{P}_3(\xi) := 1 - \xi_2 G''(\xi_1, \kappa, \dots, \kappa),$$

$$\widehat{\mathcal{M}}(\epsilon, \kappa) := \mathcal{M}^{(0)}(\epsilon, \kappa) + \mathcal{M}^{(1)}(\epsilon, \kappa),$$

$$\mathcal{M}^{(i)}(\epsilon, \kappa) := M_{11}^{(i)} \frac{\partial^2}{\partial \xi_1^2} + M_1^{(i)} \frac{\partial}{\partial \xi_1} + M_2^{(i)} \frac{\partial}{\partial \xi_2}, \quad i = 0, 1, 2.$$

By (8.5), (6.2) and Lemmas 7.2, 7.5 we get the estimate for $S_3(\epsilon, \kappa)$:

$$(8.8) \qquad |S_3(\epsilon, \kappa)| \leqslant C \kappa^2 |\epsilon|_2 L^{9/2}, \quad \text{where} \quad |\epsilon|_2 := \left(\sum_{i=1}^{N} \epsilon_i^2 \right)^{1/2}.$$

It follows from (2.1), (6.2), the definition of $\mathcal{M}(\epsilon, \kappa)$ and Lemma 7.5 that

$$|S_2| \leqslant C \kappa^2 L^2 \leqslant C c L^{-7}.$$

Hence, for δ small enough we have $|S_2| < 1/2$, and by (8.7), (8.6) we get

$$(8.9) \qquad \lambda(\rho) - \lambda^{\mathrm{per}}(\kappa) \geqslant \frac{S_1(\epsilon, \kappa)}{2} - C \kappa^2 |\epsilon|_2 L^{9/2}.$$

In order to control the contribution S_1 we split it into two parts, using thereby identity (8.4) and Lemmas 7.2

$$(8.10) \qquad S_1(\epsilon, \kappa) = S_4(\epsilon, \kappa) + S_5(\epsilon, \kappa),$$

$$S_4(\epsilon, \kappa) := N^{-1} \langle \widehat{\mathcal{M}}(\epsilon, \kappa) \psi_0^{\mathrm{per}}, \psi_0^{\mathrm{per}} \rangle + \kappa N^{-1} \langle \mathcal{M}^{(0)} \psi_1^{\mathrm{per}}, \psi_0^{\mathrm{per}} \rangle$$

$$+ \kappa N^{-1} \langle \mathcal{M}^{(0)} \psi_0^{\mathrm{per}}, \psi_1^{\mathrm{per}} \rangle - N^{-1} \langle \mathcal{M}^{(0)}(\epsilon, \kappa) \psi_0^{\mathrm{per}}, \xi_2 G'' \psi_0^{\mathrm{per}} \rangle$$

$$- N^{-1} \langle \mathcal{M}^{(0)}(\epsilon, \kappa) \mathcal{R}_0(1) \mathcal{M}^{(0)}(\epsilon, \kappa) \psi_0^{\mathrm{per}}, \psi_0^{\mathrm{per}} \rangle,$$

$$S_5(\epsilon, \kappa) := S_1(\epsilon, \kappa) - S_4(\epsilon, \kappa),$$

where G'' is taken for $\rho = (\kappa, \dots, \kappa)$. By Lemma 7.5 and (8.5) we can estimate S_5,

$$(8.11) \qquad |S_5(\epsilon, \kappa)| \leqslant C \kappa^2 |\epsilon|_2 L^{7/2}.$$

We shall show that S_5 can be regarded as a small order perturbation of the main term S_4. In order to it, we need to calculate S_4. It is easy to check that

$$N^{-1} \langle \widehat{\mathcal{M}}(\epsilon, \kappa) \psi_0^{\mathrm{per}}, \psi_0^{\mathrm{per}} \rangle = \frac{\|g''\|_{L_2(0,l)}^2}{2L} \sum_{j=1}^{N} \epsilon_j (\kappa + \rho_j),$$

$$(8.12)$$

$$-N^{-1} \langle \mathcal{M}^{(0)}(\epsilon, \kappa) \psi_0^{\mathrm{per}}, \xi_2 G'' \psi_0^{\mathrm{per}} \rangle = -\frac{\|g''\|_{L_2(0,l)}^2}{2L} \kappa \sum_{j=1}^{N} \epsilon_j.$$

By (8.4) we obtain

$$\langle \mathcal{M}^{(0)}(\epsilon, \kappa) \psi_1^{\mathrm{per}}, \psi_0^{\mathrm{per}} \rangle + \langle \mathcal{M}^{(0)}(\epsilon, \kappa) \psi_0^{\mathrm{per}}, \psi_1^{\mathrm{per}} \rangle$$

$$= \sum_{j=1}^{N} \epsilon_j \left\langle 2\xi_2 g_j'' \frac{\partial^2 \psi_1^{\mathrm{per}}}{\partial \xi_1^2} + \xi_2 g_j''' \frac{\partial \psi_1^{\mathrm{per}}}{\partial \xi_1} - g_j'' \frac{\partial \psi_1^{\mathrm{per}}}{\partial \xi_2}, \psi_0^{\mathrm{per}} \right\rangle_{L_2(D_0(j-1,j))}$$

$$- \sum_{j=1}^{N} \epsilon_j \left\langle \xi_2 g_j'' \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2}, \psi_1^{\mathrm{per}} \right\rangle_{L_2(D_0(j-1,j))}$$

$$= \left(\left\langle 2\xi_2 g'' \frac{\partial^2 \psi_1^{\mathrm{per}}}{\partial \xi_1^2} + \xi_2 g''' \frac{\partial \psi_1^{\mathrm{per}}}{\partial \xi_1} - \xi_2 g'' \frac{\partial \psi_1^{\mathrm{per}}}{\partial \xi_2}, \psi_0^{\mathrm{per}} \right\rangle_{L_2(D_0(1))} \right.$$

$$\left. - \left\langle \psi_1^{\mathrm{per}}, \xi_2 g'' \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(1))} \right) \sum_{j=1}^{N} \epsilon_j.$$

We take into the account the equation (7.8) for ψ_1^{per} and integrate by parts,
(8.13)
$$\left\langle 2\xi_2 g'' \frac{\partial^2 \psi_1^{\mathrm{per}}}{\partial \xi_1^2} + \xi_2 g''' \frac{\partial \psi_1^{\mathrm{per}}}{\partial \xi_1} - \xi_2 g'' \frac{\partial \psi_1^{\mathrm{per}}}{\partial \xi_2}, \psi_0^{\mathrm{per}} \right\rangle_{L_2(D_0(1))} - \left\langle \psi_1^{\mathrm{per}}, \xi_2 g'' \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(1))}$$

$$= \left\langle \xi_2 g'' \frac{\partial^2 \psi_1^{\mathrm{per}}}{\partial \xi_1^2}, \psi_0^{\mathrm{per}} \right\rangle_{L_2(D_0(1))} - \left\langle \xi_2 g'', \frac{\partial}{\partial \xi_2} \psi_0^{\mathrm{per}} \psi_1^{\mathrm{per}} \right\rangle_{L_2(D_0(1))}$$

$$= - \left\langle \xi_2 g'' \left(\frac{\partial^2 \psi_1^{\mathrm{per}}}{\partial \xi_2^2} + \psi_1^{\mathrm{per}} - g'' \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right), \psi_0^{\mathrm{per}} \right\rangle_{L_2(D_0(1))} + \langle g'' \psi_1^{\mathrm{per}}, \psi_0^{\mathrm{per}} \rangle_{L_2(D_0(1))}$$

$$= - \frac{\|g''\|_{L_2(0,l)}^2}{2l} - 2 \left\langle g'' \psi_1^{\mathrm{per}}, \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(1))} + \langle g'' \psi_1^{\mathrm{per}}, \psi_0^{\mathrm{per}} \rangle_{L_2(D_0(1))}.$$

Now consider the last contribution to S_4. Denote $u := \mathcal{R}_0(1)\mathcal{M}^{(0)}(\epsilon, \kappa)\psi_0$. This function solves the equation

$$(8.14) \qquad\qquad (\mathcal{H}_0(0, N) - 1)u = - \sum_{j=1}^{N} \epsilon_j g_j'' \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2}$$

and is orthogonal to ψ_0^{per} in $L_2(D_0(L))$. We use this equation and proceed as in (8.13), to calculate

$$- \langle \mathcal{M}^{(0)}(\epsilon, \kappa) \mathcal{R}_0(1) \mathcal{M}^{(0)}(\epsilon, \kappa) \psi_0^{\mathrm{per}}, \psi_0^{\mathrm{per}} \rangle$$

$$= - \sum_{j=1}^{N} \epsilon_j \left\langle 2\xi_2 g_j'' \frac{\partial^2 u}{\partial \xi_1^2} + \xi_2 g_j''' \frac{\partial u}{\partial \xi_1} - \xi_2 g_j'' \frac{\partial u}{\partial \xi_2}, \psi_0^{\mathrm{per}} \right\rangle_{L_2(D_0(j-1,j))}$$

$$= - \sum_{j=1}^{N} \epsilon_j \left\langle \xi_2 g_j'' \frac{\partial^2 u}{\partial \xi_1^2} - \xi_2 g_j'' \frac{\partial u}{\partial \xi_2}, \psi_0^{\mathrm{per}} \right\rangle_{L_2(D_0(j-1,j))}$$

$$= \sum_{j=1}^{N} \epsilon_j \left\langle \xi_2 g_j'' \left(- \epsilon_j g_j'' \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} + \frac{\partial^2 u}{\partial \xi_2^2} + \frac{\partial u}{\partial \xi_2} + u \right), \psi_0^{\mathrm{per}} \right\rangle_{L_2(D_0(j-1,j))}$$

$$= \frac{\|g''\|_{L_2(0,l)}^2}{2l} \sum_{j=1}^{N} \epsilon_j^2 + 2 \sum_{j=1}^{N} \epsilon_j \left\langle g_j'' u, \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(j-1,j))}$$

$$- \sum_{j=1}^{N} \epsilon_j \left\langle g_j'' u, \frac{\partial}{\partial \xi_2} (\xi_2 \psi_0^{\mathrm{per}}) \right\rangle_{L_2(D_0(j-1,j))}.$$

We substitute the last identity and (8.12), (8.13) into (8.10),

$$S_4(\epsilon, \kappa) = 2N^{-1} \sum_{j=1}^{N} \epsilon_j \left\langle g_j'' u, \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(j-1,j))}$$

(8.15)
$$- N^{-1}\kappa \left(2 \left\langle g'' \psi_1^{\mathrm{per}}, \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(1))} - \left\langle g'' \psi_1^{\mathrm{per}}, \psi_0^{\mathrm{per}} \right\rangle_{L_2(D_0(1))} \right) \sum_{j=1}^{N} \epsilon_j$$

$$- N^{-1} \sum_{j=1}^{N} \epsilon_j \left\langle g_j'' u, \frac{\partial}{\partial \xi_2}(\xi_2 \psi_0^{\mathrm{per}}) \right\rangle_{L_2(D_0(j-1,j))}.$$

Note that terms involving $\|g''\|_{L_2(0,l)}$ cancel. Next we study each term in the obtained expression for S_4.

Lemma 8.1. *The inequality*

(8.16)
$$S_4(\epsilon, \kappa) \geqslant \frac{\|g'\|_{L_2(0,l)}^2}{L} \left(\frac{\pi}{2} - \frac{16a}{\pi^2} \right) \kappa \sum_{j=1}^{N} \epsilon_j$$

holds true.

Proof. Since

(8.17)
$$\frac{\partial \psi_0}{\partial \xi_2}(\xi_1, \pi - \xi_2) = -\frac{\partial \psi_0}{\partial \xi_2}(\xi_1, \xi_2),$$

the solutions ψ_1^{per}, u to the equations (7.8), (8.14) satisfy the same identity,

(8.18)
$$\psi_1^{\mathrm{per}}(\xi_1, \pi - \xi_2) = -\psi_1^{\mathrm{per}}(\xi_1, \xi_2), \quad u(\xi_1, \pi - \xi_2) = -u(\xi_1, \xi_2).$$

It implies

$$\sum_{j=1}^{N} \epsilon_j \left\langle g_j'' u, \frac{\partial}{\partial \xi_2}(\xi_2 \psi_0^{\mathrm{per}}) \right\rangle_{L_2(D_0(j-1,j))} = \frac{\pi}{2} \sum_{j=1}^{N} \epsilon_j \left\langle g_j'' u, \frac{\partial \psi_0^{\mathrm{per}}}{\partial \psi_0^{\mathrm{per}}} \right\rangle_{L_2(D_0(j-1,j))},$$

(8.19)
$$S_4(\epsilon, \kappa) = \left(2 - \frac{\pi}{2} \right) N^{-1} \sum_{j=1}^{N} \epsilon_j \left\langle g_j'' u, \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(j-1,j))}$$

$$- 2N^{-1}\kappa \left\langle \psi_1^{\mathrm{per}}, g'' \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(1))} \sum_{j=1}^{N} \epsilon_j.$$

Let us estimate the first term in (8.18). We integrate by parts taking into consideration (8.17), (8.18),

$$\sum_{j=1}^{N} \epsilon_j \left\langle u, g_j'' \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(j-1,j))} = \sum_{j=1}^{N} \epsilon_j \left\langle \frac{\partial^2 u}{\partial \xi_1^2}, g_j \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(j-1,j))}$$

(8.20)
$$= \sum_{j=1}^{N} \epsilon_j^2 \left\langle g_j'' \frac{\partial \psi_0 u}{\partial \xi_2}, g_j \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(j-1,j))} - \sum_{j=1}^{N} \epsilon_j^2 \left\langle \frac{\partial^2 u}{\partial \xi_2} + u, g_j \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(j-1,j))}$$

$$= -\frac{\|g'\|_{L_2(0,l)}^2}{l} \sum_{j=1}^{N} \epsilon_j^2 + 2 \left(\frac{\partial u}{\partial \xi_2}(\cdot, \pi), \sum_{j=1}^{N} \epsilon_j g_j \right)_{L_2(0,L)}.$$

We construct the function u by the separation of variables,

$$\frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} = \sum_{m=1}^{\infty} \alpha_m \sin 2m\xi_2, \quad \alpha_m = \sqrt{\frac{2}{\pi l}} \frac{4m}{4m^2 - 1},,$$

(8.21)
$$u(\xi) = \sum_{m=1}^{\infty} \alpha_m U_m(\xi_2) \sin 2m\xi_1,$$

where the functions U_m solve the boundary value problems,

$$-U_m'' + (4m^2 - 1)U_m = -\sum_{j=1}^{N} \epsilon_j g_j'' \quad \text{in} \quad L_2(0, L), \qquad U_m'(0) = U_m'(L) = 0.$$

It follows from (8.20), (8.21) that

(8.22)
$$\sum_{j=1}^{N} \epsilon_j \Big\langle u, g_j'' \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \Big\rangle = -\frac{\|g'\|_{L_2(0,l)}^2}{l} \sum_{j=1}^{N} \epsilon_j^2 + 4 \sum_{m=1}^{\infty} m\alpha_m \Big(\sum_{j=1}^{N} \varepsilon_j g_j, U_m \Big)_{L_2(0,L)}.$$

We represent U_m as

$$U_m = \sum_{j=1}^{N} \epsilon_j g_j + (4m^2 - 1)\widetilde{U}_m,$$

where $\widetilde{U}_m$ is the solution to the problem

$$-\widetilde{U}_m'' + (4m^2 - 1)\widetilde{U}_m = -\sum_{j=1}^{N} \epsilon_j g_j \quad \text{in} \quad L_2(0, L), \qquad \widetilde{U}_m'(0) = \widetilde{U}_m'(L) = 0.$$

Thus,

$$\|\widetilde{U}_m'\|_{L_2(0,l)}^2 + (4m^2 - 1)\|\widetilde{U}_m\|_{L_2(0,l)}^2 = -\Big(\sum_{j=1}^{N} \epsilon_j g_j, \widetilde{U}_m \Big)_{L_2(0,L)},$$

$$\Big(\sum_{j=1}^{N} \varepsilon_j g_j, U_m \Big)_{L_2(0,L)} = \|g\|_{L_2(0,l)}^2 \sum_{j=1}^{N} \epsilon_j^2 + (4m^2 - 1)\Big(\sum_{j=1}^{N} \varepsilon_j g_j, \widetilde{U}_m \Big)_{L_2(0,L)},$$

$$(4m^2 - 1)\|\widetilde{U}_m\|_{L_2(0,L)} \leqslant \Big\| \sum_{j=1}^{N} \epsilon_j g_j \Big\|_{L_2(0,L)} = \|g\|_{L_2(0,l)} \sum_{j=1}^{N} \epsilon_j^2.$$

Two last relations imply

$$\Big(\sum_{j=1}^{N} \varepsilon_j g_j, U_m \Big)_{L_2(0,L)} \geqslant 0,$$

and by (8.22) it yields

(8.23)
$$\sum_{j=1}^{N} \epsilon_j \Big\langle u, g_j'' \frac{\partial \psi_0}{\partial \xi_2} \Big\rangle_{L_2(D_0(j-1,j))} \geqslant -\frac{\|g\|_{L_2(0,l)}^2}{l} \sum_{j=1}^{N} \epsilon_j^2.$$

We proceed to the second term in the right hand side of (8.18). We construct ψ_1^{per} as

(8.24)
$$\psi_1^{\mathrm{per}} = g\frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} + \sqrt{\frac{2}{\pi l}}\widetilde{\psi}_1^{\mathrm{per}},$$

where $\widetilde{\psi}_1^{\mathrm{per}}$ solves the boundary value problem

(8.25) $(\Delta+1)\widetilde{\psi}_1^{\mathrm{per}} = 0$ in $D_0(1)$, $\qquad \widetilde{\psi}_1^{\mathrm{per}} = g$ as $\xi_2 = \pi$, $\qquad \widetilde{\psi}_1^{\mathrm{per}} = -g$ as $\xi_2 = 0$,

and satisfies periodic boundary condition on $\xi_1 = 0$ and $\xi_1 = l$. Hence,

(8.26)
$$-\left\langle \psi_1^{\mathrm{per}}, g'' \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(1))} = \frac{\|g'\|_{L_2(0,l)}^2}{l} - \sqrt{\frac{2}{\pi l}} \left\langle \widetilde{\psi}_1^{\mathrm{per}}, g'' \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(1))}$$
$$= \frac{\|g'\|_{L_2(0,l)}^2}{l} + \sqrt{\frac{2}{\pi l}} \left\langle \frac{\partial \widetilde{\psi}_1^{\mathrm{per}}}{\partial \xi_1}, g' \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(1))}.$$

Now we construct $\widetilde{\psi}_1^{\mathrm{per}}$ by the separation of variables,

(8.27)
$$\frac{\partial \widetilde{\psi}_1^{\mathrm{per}}}{\partial \xi_1} = \sum_{m=1}^{\infty} \left(\beta_m^{(+)} \cos \frac{2\pi m}{l} \xi_1 + \beta_m^{(-)} \sin \frac{2\pi m}{l} \xi_1 \right) V_m(\xi_2),$$

where $\beta_m^{(\pm)}$ are introduced as the coefficients of the Fourier series for g',

(8.28)
$$g'(\xi_1) = \sum_{m=1}^{\infty} \left(\beta_m^{(+)} \cos \frac{2\pi m}{l} \xi_1 + \beta_m^{(-)} \sin \frac{2\pi m}{l} \xi_1 \right).$$

The functions V_m are defined as the solutions to the boundary value problems

(8.29) $\qquad -V_m'' + \left(\frac{4\pi^2 m^2}{l^2} - 1 \right) V_m = 0$ in $(0, \pi)$, $\qquad V_m(0) = -1$, $\quad V_m(\pi) = 1$.

The functions V_m can be found explicitly,

(8.30)
$$V_m(\xi_2) = \frac{\sinh A_m \left(\xi_2 - \frac{\pi}{2} \right)}{\sinh \frac{\pi A_m}{2}}, \quad A_m := \sqrt{\frac{4\pi^2 m^2}{l^2} - 1}, \;\; \text{if} \;\; \frac{4\pi^2 m^2}{l^2} > 1,$$
$$V_m(\xi_2) = \frac{\sin A_m \left(\xi_2 - \frac{\pi}{2} \right)}{\sin \frac{\pi A_m}{2}}, \quad A_m := \sqrt{1 - \frac{4\pi^2 m^2}{l^2}}, \quad \text{if} \;\; \frac{4\pi^2 m^2}{l^2} < 1,$$
$$V_m(\xi_2) = \frac{2}{\pi} \xi_2 - 1, \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \;\; \frac{4\pi^2 m^2}{l^2} = 1.$$

We substitute (8.27) and (8.28) into (8.26),

(8.31)
$$-2\left\langle \psi_1^{\mathrm{per}}, g'' \frac{\partial \psi_0^{\mathrm{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(1))} = \frac{2\|g'\|_{L_2(0,l)}^2}{l} + \sum_{m=1}^{\infty} \left(\left(\beta_m^{(+)} \right)^2 + \left(\beta_m^{(-)} \right)^2 \right) B_m,$$

where

(8.32)
$$B_m := -\frac{4 A_m \coth \frac{\pi A_m}{2}}{\pi (A_m^2 + 1)}, \;\; \text{if} \;\; \frac{4\pi^2 m^2}{l^2} > 1,$$
$$B_m := -\frac{4 A_m \cot \frac{\pi A_m}{2}}{\pi (1 - A_m^2)}, \;\; \text{if} \;\; \frac{4\pi^2 m^2}{l^2} < 1,$$
$$B_m := -\frac{8}{\pi^2}, \qquad\qquad\quad \text{if} \;\; \frac{4\pi^2 m^2}{l^2} = 1.$$

In view of the estimates
$$\frac{4z \coth \frac{\pi z}{2}}{\pi \sqrt{z^2 + 1}} \leqslant \frac{4}{\pi}, \quad z \geqslant 0, \qquad \frac{4z \cot \frac{\pi z}{2}}{\pi \sqrt{1 - z^2}} \leqslant \frac{8}{\pi^2}, \quad z \in [0, 1],$$

the coefficients B_m can be estimated as follows,

$$0 > B_m > -\frac{2l}{\pi^2 m}, \quad \text{if} \quad 2m > \frac{l}{\pi}, \qquad 0 > B_m > -\frac{4l}{\pi^3 m}, \quad \text{if} \quad 2m \leqslant \frac{l}{\pi}.$$

We substitute these estimate in (8.31),

$$-2\left\langle \psi_1^{\text{per}}, g'' \frac{\partial \psi_0^{\text{per}}}{\partial \xi_2} \right\rangle_{L_2(D_0(1))} \geqslant \frac{2\|g'\|_{L_2(0,l)^2}}{l} - \frac{4l}{\pi^3} \sum_{2m \leqslant \frac{l}{\pi}} \frac{\left(\beta_m^{(+)}\right)2 + \left(\beta_m^{(-)}\right)^2}{m}$$

$$-\frac{2l}{\pi^2} \sum_{2m > \frac{l}{\pi}} \frac{\left(\beta_m^{(+)}\right)2 + \left(\beta_m^{(-)}\right)^2}{m}$$

$$\geqslant \frac{2\|g'\|_{L_2(0,l)}^2}{l} - \frac{2l}{\pi^2} \sum_{m=1}^{\infty} \frac{\left(\beta_m^{(+)}\right)^2 + \left(\beta_m^{(-)}\right)^2}{m}.$$

Together with (8.23), (8.19) and an obvious estimate $\epsilon_j \leqslant \kappa$ it yields

(8.33)
$$S_4(\epsilon, \kappa) \geqslant \left(\frac{\pi}{2} \|g'\|_{L_2(0,l)}^2 - \frac{4l^2}{\pi^2} \sum_{m=1}^{\infty} \frac{\left(\beta_m^{(+)}\right)^2 + \left(\beta_m^{(-)}\right)^2}{m} \right) \frac{\kappa}{L} \sum_{j=1}^{N} \epsilon_j$$

It remains to estimate the last sum in this expression.

We introduce the function

$$h(\xi_1) := \sum_{m=1}^{\infty} \frac{1}{m} \left(\beta_m^{(+)} \cos \frac{2\pi m}{l} \xi_1 + \beta_m^{(-)} \sin \frac{2\pi m}{l} \xi_1 \right) + \frac{2\pi \beta_0^+}{l},$$

$$\beta_0^+ := \frac{1}{l} \int_0^l g(\xi_1) \, d\xi_1,$$

and by the definition (8.28) of the coefficients $\beta_m^{(\pm)}$ we see that

$$\|g\|_{L_2(0,l)}^2 = \frac{l}{2} \sum_{m=1}^{\infty} \left(\frac{l}{2\pi m} \right)^2 \left((\beta_m^{(+)})^2 + (\beta_m^{(-)})^2 \right) + (\beta_0^{(+)})^2 l,$$

(8.34)
$$\|h\|_{L_2(0,l)}^2 = \frac{l}{2} \sum_{m=1}^{\infty} \frac{\left(\beta_m^{(+)}\right)2 + \left(\beta_m^{(-)}\right)^2}{m} + \left(\frac{2\pi \beta_0^{(+)}}{l} \right)^2 = \frac{4\pi^2}{l^2} \|g\|_{L_2(0,l)}^2,$$

(8.35)
$$\frac{4l^2}{\pi^2} \sum_{m=1}^{\infty} \frac{\left(\beta_m^{(+)}\right)^2 + \left(\beta_m^{(-)}\right)^2}{m} = \frac{8l}{\pi^2} (h, g')_{L_2(0,l)}.$$

Now we use the fact that $\text{supp}\, g$ lies in a segment of the size a and an obvious estimate

$$\|g'\|_{L_2(0,l)} \geqslant \frac{\pi}{a} \|g\|_{L_2(0,l)},$$

to obtain

$$-\frac{4l^2}{\pi^2} \sum_{m=1}^{\infty} \frac{\left(\beta_m^{(+)}\right)^2 + \left(\beta_m^{(-)}\right)^2}{m} \geqslant -\frac{8l}{\pi^2} \|h\|_{L_2(0,l)} \|g'\|_{L_2(0,l)}$$

$$\geqslant -\frac{16}{\pi} \|g\|_{L_2(0,l)} \|g'\|_{L_2(0,l)} \geqslant -\frac{16a}{\pi^2} \|g'\|_{L_2(0,l)}^2.$$

We substitute this inequality into (8.33) and complete the proof.   □

We substitute the inequalities (8.16), (8.10), (8.11), and a trivial one

$$|\epsilon|_2 \leqslant \sum_{j=1}^{N} \epsilon_j$$

into (8.9),

$$\lambda(\rho) - \lambda^{\mathrm{per}}(\kappa) \geqslant \left(\frac{\|g'\|^2_{L_2(0,l)}}{2L} \left(\frac{\pi}{2} - \frac{16a}{\pi^2} \right) - C\kappa L^{9/2} \right) \kappa \sum_{j=1}^{N} \epsilon_j.$$

Choosing the constant δ in (6.2) small enough, we arrive at the estimate (6.3).

Acknowledments

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