A sparse Prony FFT

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***Abstract*—We describe the application of Prony-like reconstruction methods to the problem of the sparse Fast Fourier transform (sFFT) [5]. In particular, we adapt both important parts of the sFFT, quasi random sampling and filtering techniques, to Prony-like methods.**

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I. Introduction

Computing the discrete Fourier transform of a vector of size N, requires $\mathcal{O}(N \log N)$ arithmetical operations. The problem of a sparse Fourier transform (sFFT) now reads as follows: For a vector $\boldsymbol{x} \in \mathbb{C}^N$, assume that its Fourier representation

$$x_l = \frac{1}{N} \sum_{j=0}^{N-1} \hat{x}_j \, \mathrm{e}^{2\pi \mathrm{i} l j/N}, \quad l = 0, \ldots, N-1,$$

has only $K \ll N$ non-vanishing Fourier coefficients $\hat{x}_{j_k} \in \mathbb{C}$, $j_k \in \{0, \ldots, N-1\}$, $k = 1, \ldots, K$. Now given part of the vector of samples $\boldsymbol{x} \in \mathbb{C}^N$, determine the non-vanishing Fourier coefficients $\hat{x}_{j_k} \in \mathbb{C}$ and their support $j_k \in \{0, \ldots, N-1\}$, $k = 1, \ldots, K$.

This problem has recently attracted much attention in the field of compressed sensing [1], [3], where one generally aims to reconstruct a vector with few non-vanishing coefficients from a relatively small number of linear measurements. Besides measurement matrices with independent random entries, structured matrices generated by a smaller number of random variables have been studied over the last years. Here, the most prominent example is given by a random selection of K rows of the N-th Fourier matrix, see e.g. [14], [10]. For this particular setting, the class of sublinear-time Fourier algorithms [4], [9] with a runtime that is polynomial in $\log N$ and K received much attention. The key idea, as outlined recently in [7], [5], [6] is the use of quasi random sampling and a band pass filter.

On the other hand, Prony-like methods are known for a long time in parameter estimation, in particular for exponential sums, see e.g. [12], [13] and references therein. In this note, we combine Prony-like methods with the above quasi random sampling and band pass filtering techniques.

II. Prony method

Let $K \geq 1$ be an integer, $f_k \in (-\infty, 0] + \mathrm{i}\,[-\pi, \pi)$ $(k = 1, \ldots, K)$ be distinct complex numbers and $c_k \in \mathbb{C} \setminus \{0\}$ $(k = 1, \ldots, K)$. We assume that $|c_k| > \varepsilon$ for a convenient bound $0 < \varepsilon \ll 1$ and consider the exponential sum of order K,

$$h(x) := \sum_{k=1}^{K} c_k \, \mathrm{e}^{f_k x} \quad (x \geq 0), \tag{II.1}$$

where the nodes $z_k := \mathrm{e}^{f_k}$ $(k = 1, \ldots, K)$ are distinct values in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : 0 < |z| \leq 1\}$ without zero. The well known Prony method recovers all parameters of the exponential sum (II.1), if sampled data

$$h(m) = \sum_{k=1}^{K} c_k \, \mathrm{e}^{f_k m} = \sum_{k=1}^{K} c_k \, z_k^m \in \mathbb{C} \quad (m = 0, \ldots, M-1) \tag{II.2}$$

with $M \geq 2K$ are given. This problem is known as frequency analysis problem, which is important within many disciplines in sciences and engineering (see [12]). For a survey of the most successful methods for the data fitting problem with linear combinations of complex exponentials, we refer to [11]. We follow the lines in [13] and consider the case of an unknown order K for the exponential sum (II.1) and given noiseless sampled data $h(m)$ $(m = 0, \ldots, M-1)$. Let $K_0 \in \mathbb{N}$ be a convenient upper bound of K, i.e. $K \leq K_0 \leq M/2$. With the M sampled data $h(m) \in \mathbb{C}$ $(m = 0, \ldots, M-1)$ we form the rectangular Hankel matrix

$$\boldsymbol{H}_{M-K_0, K_0+1} \;:=\; \big(h(m+k)\big)_{m,k=0}^{M-K_0-1, K_0}, \tag{II.3}$$

and compute the singular value decomposition (SVD)

$$\boldsymbol{H}_{M-K_0, K_0+1} = \boldsymbol{U}_{M-K_0} \, \boldsymbol{D}_{M-K_0, K_0+1} \, \boldsymbol{W}_{K_0+1}, \tag{II.4}$$

where $\boldsymbol{U}_{M-K_0}$ and $\boldsymbol{W}_{K_0+1}$ are unitary matrices and where $\boldsymbol{D}_{M-K_0, K_0+1}$ is a rectangular diagonal matrix. The diagonal entries of $\boldsymbol{D}_{M-K_0, K_0+1}$ are the singular values of (II.3) arranged in nonincreasing order $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_K > \sigma_{K_0+1} = \ldots = \sigma_{K_0+1} = 0$. Thus we can determine the rank K of the Hankel matrix (II.3) which coincides with the order of the exponential sum (II.1). Introducing the matrices

$$\begin{aligned}\boldsymbol{D}_{M-K_0,K} &:= \boldsymbol{D}_{M-K_0,K_0+1}(1:M-K_0,\,1:K)\\ &= \begin{pmatrix}\operatorname{diag}(\sigma_k)_{k=1}^K\\ \boldsymbol{O}_{M-K_0-K,K}\end{pmatrix},\\ \boldsymbol{W}_{K,K_0+1} &:= \boldsymbol{W}_{K_0+1}(1:K,\,1:K_0+1),\end{aligned}$$

we can simplify the SVD of the Hankel matrix (II.3) as follows $\boldsymbol{H}_{M-K_0,K_0+1} = \boldsymbol{U}_{M-K_0}\,\boldsymbol{D}_{M-K_0,K}\,\boldsymbol{W}_{K,K_0+1}$. Setting

$$\boldsymbol{W}_{K,K_0}(s) = \boldsymbol{W}_{K,K_0+1}(1:K,\,1+s:K_0+s)\quad (s=0,\,1),\tag{II.5}$$

we determine the nodes $z_k \in \mathbb{D}$ $(k=1,\ldots,K)$ as eigenvalues of the matrix

$$\boldsymbol{F}_K^{SVD} := \big(\boldsymbol{W}_{K,K_0}(0)^{\mathrm{T}}\big)^{\dagger}\,\boldsymbol{W}_{K,K_0}(1)^{\mathrm{T}},\tag{II.6}$$

where † denotes the Moore-Penrose-Inverse. Thus the ESPRIT algorithm reads as follows:

Algorithm II.1 (ESPRIT method)

Input: K_0, $M \in \mathbb{N}$ $(M \gg 2,\ 3 \le K_0 \le M/2,\ K_0$ is upper bound of the order K of (II.1)), $h(m) \in \mathbb{C}$ $(m = 0,\ldots,M-1)$, $0 < \varepsilon \ll 1$.

1. Compute the SVD of the rectangular Hankel matrix (II.4). Determine the rank K of (II.3) such that $\sigma_{K+1} < \varepsilon\sigma_1$ and form the matrices (II.5).
2. Compute all eigenvalues $z_k \in \mathbb{D}$ $(k=1,\ldots,K)$ of the square matrix (II.6). Set $f_k := \log z_k$ $(k=1,\ldots,K)$.
3. Compute the coefficients $c_k \in \mathbb{C}$ $(k=1,\ldots,K)$ as least squares solution of the overdetermined linear Vandermonde–type system

$$(z_k^m)_{m=0,k=1}^{M-1,K}\,\boldsymbol{c} = \big(h(m)\big)_{m=0}^{M-1}\tag{II.7}$$

with $\boldsymbol{z} := (z_k)_{k=1}^K$ and $\boldsymbol{c} := (c_k)_{k=1}^K$

Output: $K \in \mathbb{N}$, $f_k \in (-\infty,\,0] + \mathrm{i}\,[-\pi,\pi)$, $c_k \in \mathbb{C}$ $(k = 1,\ldots,K)$.

Remark II.2 For noiseless sampled data, the authors in [13] describe the close connections between the classical Prony method, the matrix pencil method based on a QR decomposition, and the ESPRIT method. □

III. Random sampling and integer frequencies

We consider the sparse Fourier approximation problem. For a vector $\boldsymbol{x} \in \mathbb{C}^N$, we assume that its Fourier representation

$$x_l = \frac{1}{N}\sum_{j=0}^{N-1}\hat{x}_j\,\mathrm{e}^{2\pi\mathrm{i}lj/N},\quad l=0,\ldots,N-1,$$

has only $K \ll N$ non-vanishing Fourier coefficients $\hat{x}_{j_k}$, $j_k \in \{0,\ldots,N-1\}$, $k=1,\ldots,K$. That is,

$$x_l = \frac{1}{N}\sum_{k=1}^{K}\hat{x}_{j_k}\,\mathrm{e}^{2\pi\mathrm{i}lj_k/N} = \sum_{k=1}^{K} c_k\,\mathrm{e}^{\tilde{f}_k l},\quad l=0,\ldots,N-1,$$

with $c_k = \frac{1}{N}\hat{x}_{j_k} \in \mathbb{C}\backslash\{0\}$ and $\tilde{f}_k = 2\pi\mathrm{i}j_k/N \in \mathbb{C}$ satisfying $\operatorname{Re}\tilde{f}_k = 0$ and $\operatorname{Im}\tilde{f}_k \in [0,2\pi)$, $k=1,\ldots,K$. Applying e.g. the ESPRIT method to the first M entries $x_0,\ldots,x_{M-1}$ of $\boldsymbol{x}$ would yield coefficients $c_k \in \mathbb{C}\backslash\{0\}$ and frequencies $f_k \in \mathbb{C}$ with $\operatorname{Re} f_k = 0$ and $\operatorname{Im} f_k \in [-\pi,\pi)$, $k=1,\ldots,K$. By

$$\tilde{f}_k = \begin{cases} f_k, & \operatorname{Im} f_k \ge 0,\\ f_k + 2\pi\mathrm{i}, & \operatorname{Im} f_k < 0,\end{cases}$$

and

$$\begin{aligned} j_k &= \operatorname{round}\big(\frac{N}{2\pi}\operatorname{Im}\tilde{f}_k\big),\\ \hat{x}_{j_k} &= Nc_k,\end{aligned}$$

$k=1,\ldots,K$, we could accomplish the computation of the K-sparse Fourier transform $\hat{\boldsymbol{x}} \in \mathbb{C}^N$ that way.

However, we do not intend to take necessarily the first M entries $x_0,\ldots,x_{M-1}$ as input for the Prony-like method but (to a certain extend) random M entries of the vector $\boldsymbol{x}$. We use a random parameter $\sigma \in \{1,\ldots,N-1\}$ being invertible modulo N and a random shift parameter $\tau \in \{0,\ldots,N-1\}$ similar as used in [7]. The following theorem confirms the possibility to connect randomized signal samples and a Prony-like algorithm for computation as suggested in [15].

Theorem III.1 *Let the vector $\boldsymbol{x} = (x_l)_{l=0}^{N-1} \in \mathbb{C}^N$ with a K-sparse Fourier representation*

$$x_l = \frac{1}{N}\sum_{k=1}^{K}\hat{x}_{j_k}\,\mathrm{e}^{2\pi\mathrm{i}lj_k/N},\quad l=0,\ldots,N-1,\tag{III.1}$$

and two integers σ, $\tau \in \{0,\ldots,N-1\}$, σ being invertible, be given. Then we have

$$x_{\sigma l+\tau} = \sum_{k=1}^{K} c_k\,\mathrm{e}^{\tilde{f}_k l},\quad l=0,\ldots,N-1,$$

with coefficients

$$c_k = \frac{1}{N}\hat{x}_{j_k}\omega_N^{j_k\tau} \in \mathbb{C}\backslash\{0\}$$

and frequencies

$$\tilde{f}_k = \mathrm{i}\frac{2\pi}{N}\big((j_k\sigma) \bmod N\big) \in \mathbb{C}$$

such that $\operatorname{Re}\tilde{f}_k = 0$ and $\operatorname{Im}\tilde{f}_k \in [0,2\pi)$, $k=1,\ldots,K$. Here, $\omega_N = \mathrm{e}^{2\pi\mathrm{i}/N}$ denotes the principal N-th primitive root of unity.

A simple consequence is the following: Let two integers σ, $\tau \in \{0,\ldots,N-1\}$, σ invertible modulo N, and a sufficiently big number of samples $M \ge 2K$ be given.

One can determine the K non-zero Fourier coefficients $\hat{x}_{j_k} \in \mathbb{C}$ and integer frequencies $j_k \in \{0, \dots, N-1\}$ of the vector $\boldsymbol{x} \in \mathbb{C}^N$ with entries (III.1) using the samples $x_{\sigma l+\tau}$, $l = 0, \dots, M-1$, by

$$j_k = (\text{round}(\frac{N}{2\pi}\text{Im}\, f_k)\sigma^{-1}) \bmod N, \quad k = 1, \dots, K,$$

and

$$\hat{x}_{j_k} = N c_k \omega_N^{-j_k\tau}, \quad k = 1, \dots, K,$$

where σ^{-1} denotes the inverse of σ modulo N and c_k, f_k, $k = 1, \dots, K$, the output of a Prony-like reconstruction method. Hence, the Prony-like methods are well-suited for the computation of sparse Fourier transforms. After applying the Algorithm II.1 to the permuted signal samples, we obtain the integer frequencies by rounding. Further, we need to invert the random separation and take the modulo of the result in order to guarantee that $j_k \in \{0, \dots, N-1\}$. The random shift in the sampling index causes a modulation of the Fourier coefficients which can be easily corrected. Assigning such a quasi-random sign is intended to prevent cancellations of the coefficients. A more detailed analysis follows for the expected separation of nearby frequencies [8] which leads to a stabilization of the Prony method, see [13].

Theorem III.2 *Let $N \in \mathbb{N}$ be prime, the vector $\hat{\boldsymbol{x}} \in \mathbb{C}^N$ contain K nonzeros and choose $\sigma \in \{1, \dots, N-1\}$ uniformly distributed at random. Then the separation distance of the vector $(\hat{x}_{\sigma j})_{j=0,\dots,N-1} \in \mathbb{C}^N$ fulfils*

$$\mathbb{P}\left(\min_{k \neq l} |\sigma j_k - \sigma j_l| \geq \frac{N-1}{2K(K-1)}\right) \geq \frac{1}{2}. \quad \text{(III.2)}$$

This result becomes effective for $K \in o(\sqrt{N})$ and gives high probability of success for the Prony method by independent repetition.

IV. A splitting approach

The most time-consuming steps of Prony-like recovery methods are the factorization of the Hankel matrix (II.4) and the least squares solution of the system (II.7). Hence, the computational costs for the Prony-like methods are $\mathcal{O}(K^2 M)$ in general and $\mathcal{O}(K^3 \log(\frac{N}{K}))$ if we choose $M = \mathcal{O}(K \log(\frac{N}{K}))$ samples. This recovery method is sublinear in the problem size N but scales cubic in the number K of non-zeros, such that only very sparse Fourier transforms can be computed in an efficient way. We proceed by a modification of the Prony-like methods adapted to the sparse Fourier transform problem to further reduce computational costs.

Let a number $B \in \mathbb{N}$, $B \leq K$, of frequency bands be chosen. Instead of recovering all of the K non-vanishing Fourier coefficients at once, we split the frequency set $\{0, 1, \dots, N-1\}$ into the disjoint subsets $\{\frac{b-1}{B}N, \frac{b-1}{B}N + 1, \dots, \frac{b}{B}N - 1\}$, $b = 1, \dots, B$, and determine only coefficients with frequencies in such a subset in each recovery step.

In order to do so, we use a filter that is concentrated both in time and frequency. Let $\varepsilon > \varepsilon' > 0$ be two parameters and set $N_1 = \lceil \varepsilon' N/2 \rceil$ and $N_2 = \lfloor \varepsilon N/2 \rfloor$. We define the auxiliary function $a : [N_1, N_2] \to \mathbb{R}$ using $a_1 : \mathbb{R} \to \mathbb{R}$ and $a_2 : [-1, 1] \to \mathbb{R}$ via

$$\begin{aligned} a_1(x) &= \mathrm{e}^{-1/x^2},\ x \in \mathbb{R}\backslash\{0\},\ a_1(0) = 0, \\ a_2(x) &= \frac{a_1(1-x)}{a_1(1-x) + a_1(1+x)}, \\ a(x) &= a_2(2/(N_2 - N_1)(x - N_1) - 1). \end{aligned} \quad \text{(IV.1)}$$

Then, the function a is smooth in $[N_1, N_2]$, $a(N_1) = 1$, $a(N_2) = 0$, and all derivatives of a vanish at N_1 and N_2. We now set $\hat{\boldsymbol{g}} = (\hat{g}_j)_{j=0}^{N-1} \in \mathbb{R}^N$,

$$\hat{g}_j = \begin{cases} 1, & j < N_1 \text{ and } j > N - N_1, \\ 0, & N_2 < j < N - N_2, \\ a(j), & j \in [N_1, N_2], \\ a(N-j), & j \in [N - N_2, N - N_1]. \end{cases}$$

and define the final filter for a spatial cut-off $W \in \mathbb{N}$, $W < N$, by $\boldsymbol{h} = (h_l)_{l=0}^{N-1} \in \mathbb{C}^N$,

$$h_l = \begin{cases} g_l, & l \in [-\frac{W}{2}, \frac{W}{2}], \\ 0, & \text{elsewise}. \end{cases}$$

Instead of the signal $\boldsymbol{x}$, we use the convolved vector

$$\boldsymbol{x} * \boldsymbol{h} = (\sum_{k=0}^{N-1} x_k h_{l-k})_{l=0}^{N-1} \in \mathbb{C}^N \quad \text{(IV.2)}$$

in the computation of the sparse Fourier transform. We have

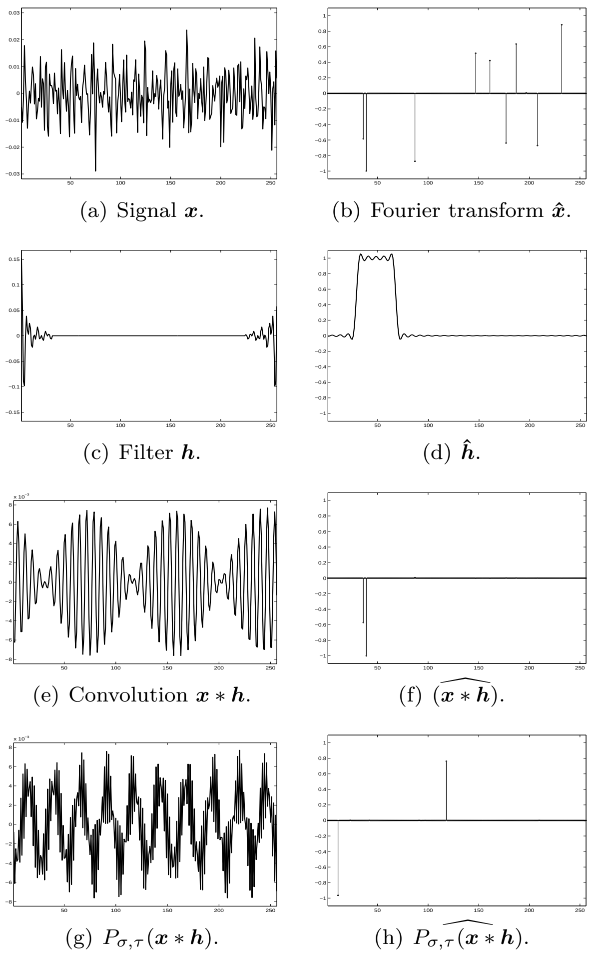
$$\widehat{(\boldsymbol{x} * \boldsymbol{h})} = \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{h}} = (\hat{x}_j \hat{h}_j)_{j=0}^{N-1}.$$

Therefore, it is likely that the number of non-zero Fourier coefficients of $\boldsymbol{x} * \boldsymbol{h}$ is smaller than before since most of the coefficients $\hat{h}_j$, $j \notin \{\frac{b-1}{B}N, \dots, \frac{b}{B}N - 1\}$, are (almost) zero. In case the randomized Prony-like method outputs a frequency which is not in the currently considered subset $\{\frac{b-1}{B}N, \dots, \frac{b}{B}N - 1\}$, we discard the corresponding coefficient, such that the expected number of non-zero coefficients which we seek to identify in each of the B steps is $\frac{K}{B}$. We employing the randomization in each step as well and use $M_1 = \mathcal{O}(\frac{K}{B} \log(\frac{NB}{K}))$ samples

$$(P_{\sigma,\tau}(\boldsymbol{x} * \boldsymbol{h}))_l := (\boldsymbol{x} * \boldsymbol{h})_{\sigma l + \tau}, \quad \text{(IV.3)}$$

$l = 0, \dots, M_1 - 1$, in each recovery step. Figure IV.1, see also [15], serves as an illustration for this splitting approach. The effects of convolving the signal $\boldsymbol{x}$ with the filter $\boldsymbol{h}$ and applying the randomization afterwards are pictured both in the time domain and in the frequency domain for a particular example.

Finally, we shortly analyse the expected computational complexity of this splitted Prony method. As argued above, the expected number of frequencies is $\mathcal{O}(\frac{K}{B})$ and we thus choose $M_1 = \mathcal{O}(\frac{K}{B} \log(\frac{NB}{K}))$ samples per recovery step. The computationally most expensive parts of



(a) Signal $\boldsymbol{x}$. (b) Fourier transform $\hat{\boldsymbol{x}}$.
(c) Filter $\boldsymbol{h}$. (d) $\hat{\boldsymbol{h}}$.
(e) Convolution $\boldsymbol{x} * \boldsymbol{h}$. (f) $\widehat{(\boldsymbol{x} * \boldsymbol{h})}$.
(g) $P_{\sigma,\tau}(\boldsymbol{x} * \boldsymbol{h})$. (h) $\widehat{P_{\sigma,\tau}(\boldsymbol{x} * \boldsymbol{h})}$.

Figure IV.1. **Example step of the split Prony-like method.** In step $b = 2$ of $B = 8$ computation steps, the signal $\boldsymbol{x}$ of length $N = 2^8 = 256$ (a) with $K = 10$ non-zeros in its Fourier transform $\hat{\boldsymbol{x}}$ (b) is convolved with the filter $\boldsymbol{h}$ (c). The resulting vector $\boldsymbol{x} * \boldsymbol{h}$ (e) is randomly permuted. In all the subfigures only the real parts of the complex vectors are plotted.

one step then is spatial filtering which takes $\mathcal{O}(M_1 W)$ arithmetic operations and the Prony-like method requiring $\mathcal{O}(M_1 \frac{K^2}{B^2})$ arithmetic operations. Moreover, we assume a spatial filter length $W = \mathcal{O}(B \log N)$, which is supported for a particular error measure in [7], [5], [6], and choose the optimal value $B = \mathcal{O}(K^{\frac{2}{3}})$ of recovery steps. In total, this leads to a complexity of $\mathcal{O}(K^{\frac{5}{3}} \log^2 N)$. While this is beyond the recently achieved bounds for sparse FFTs we nevertheless expect a wider applicability due to the fact that Prony-like methods seem to be more stable with respect to off-grid frequencies, cf. [2].

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