A NOTE ON REGULARITY FOR DISCRETE ALLOY-TYPE MODELS II

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Abstract. Alloy-type potentials on the lattice $\mathbb{Z}^d$ give rise to a correlated random field. Depending on the regularity properties of the conditional distributions (or conditional densities — if they exist) standard methods developed for the i.i.d. Anderson model can be applied or not. This refers to Wegner estimates, fractional moment bounds, Minami estimates, and other estimates obtained by averaging procedures. In [5] we studied a (quite large) class of alloy-type potentials on the lattice $\mathbb{Z}$ and showed that certain conditional probabilities exhibit a bad behavior. Consequently, a regularity condition spelled out in [1] is not satisfied in this case. We revisit in this note the question of regularity properties of the conditional distribution of the potential values and discuss certain consequences for the recent preprint [4].

Contents

1. Discrete alloy type model and purpose of [5]

We consider for each $k \in \mathbb{Z}^d$ the probability space $(\Omega_k, \mathcal{B}, \mu_k)$, where $\Omega_k = \mathbb{R}$, $\mathcal{B}$ is the Borel sigma algebra on $\mathbb{R}$ and $\mu_k \colon \mathcal{B} \to [0,1]$ a probability measure. For each $\Lambda \subset \mathbb{Z}^d$ we introduce the product probability space $(\Omega_\Lambda, \mathcal{A}_\Lambda, \mathbb{P}_\Lambda)$, where $\Omega_\Lambda := \times_{k\in\Lambda} \Omega_k$, $\mathcal{A}_\Lambda := \bigotimes_{k\in\Lambda} \mathcal{B}$ and $\mathbb{P}_\Lambda := \bigotimes_{k\in\Lambda} \mu_k$. We will use the abbreviation $\Omega := \Omega_{\mathbb{Z}^d}$, $\mathcal{A} := \mathcal{A}_{\mathbb{Z}^d}$, $\mathbb{P} := \mathbb{P}_{\mathbb{Z}^d}$, $\mathbb{Z}^d_m := \mathbb{Z}^d \setminus \{m\}$, $\Omega_m^\perp = \Omega_{\mathbb{Z}^d\setminus\{m\}}$ and $\mathcal{A}_m^\perp = \mathcal{A}_{\mathbb{Z}^d\setminus\{m\}}$. Elements of Ω are multidimensional sequences and will be denoted by $\omega = (\omega_k)_{k\in\mathbb{Z}^d}$.

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The expectation with respect to the probability measure $\mathbb{P}$ will be denoted by $\mathbb{E}$. Projection maps are defined as follows:

$$\pi_k \colon \Omega \to \Omega_k, \qquad \pi_k(\omega) = \omega_k,$$

and

$$\pi_k^\perp \colon \Omega \to \Omega_{\mathbb{Z}^d \setminus \{k\}}, \qquad \pi_k^\perp(\omega) = (\omega_j)_{j \in \mathbb{Z}^d_k}.$$

Let $u \colon \mathbb{Z}^d \to \mathbb{R}$ be a summable sequence. We consider the random field given by the *discrete alloy-type potential*

$$V_\omega \colon \mathbb{Z}^d \to \mathbb{R}, \qquad V_\omega(x) = \sum_{k \in \mathbb{Z}^d} \omega_k u(x-k). \tag{1}$$

This function is certainly well defined if there is a compact subset $K \subset \mathbb{R}$ such that the support of all μ_k is contained in K. This class of models (and its reformulation as a correlated random field) has been first considered in [7].

Such random fields V may be considered as the potential of a random Schrödinger operator. In the special case $u = \delta_0$ it is a field of independent random variables. If in addition all μ_k coincide, we are in fact dealing with the potential of the well known Anderson model. For this type of discrete random Schrödinger operators a variety of results concerning some form of averaging of spectral quantities have been derived. Instances of such results are: Wegner estimates, fractional moment bounds, Minami estimates, and generalizations thereof. A natural question is, whether these results extend to the case that u is more complicated than δ_0. This induces correlations (or at least dependence) between values of the potential V at different sites $x, y \in \mathbb{Z}^d$.

In this note, we consider a field of random variables

$$\eta_x \colon \Omega \to \mathbb{R}, \quad x \in \mathbb{Z}^d. \tag{2}$$

The collection $(\eta_k)_{k \in \mathbb{Z}^d}$ will be denoted by

$$\eta := (\eta_k)_{k \in \mathbb{Z}^d} : \Omega \to \Omega.$$

Of particular interest is the case where η is given by a linear transformation of the i.i.d. random field ω, i.e.

$$\eta(\omega) = A\omega, \tag{3}$$

where $A : \Omega \to \Omega$ is a bounded (with respect to the ℓ^∞-norm) linear operator. In the case where $A := (a_{i,j})_{i,j \in \mathbb{Z}^d}$ has the Toeplitz structure $a_{i,j} = u(i-j)$, we obtain the discrete alloy-type model

$$\eta_x : \Omega \to \mathbb{R}, \quad \eta_x(\omega) = \sum_{k \in \mathbb{Z}^d} \omega_k u(x-k), \quad x \in \mathbb{Z}^d. \tag{4}$$

It is possible to formulate sufficiently strong regularity assumptions on the conditional distribution of the individual random variables η_x (conditioned on all other random variables $\eta_y, y \in \mathbb{Z}^d_x$) such that the methods developed for the i.i.d. model apply to the correlated one as well. Instances of such a condition can be found in [3] or [1]. In [5] we have identified a (quite large class) of discrete alloy type potentials for which the regularity conditions of [1] do *not* apply. In fact, we conjecture, that for *no* potential (1) with compactly supported u and uniformly bounded random variables $\eta_x\colon \Omega \to \mathbb{R}, x \in \mathbb{Z}^d$, the regularity conditions of [1] will hold.

One clarification is in order. While the regularity condition of [1] is formulated in terms of *conditional distributions* we have studied in [5] certain *conditional probabilities*. Our results in [5] show *intuitively* that the regularity condition of [1] are not satisfied. Nevertheless, we revisit here the same topic, *calculate* for a certain class of random fields $\eta : \Omega \to \Omega$ the quantity appearing in the regularity condition of [1] and thereby show that regularity assumptions and thus the fractional moment results of [1] do not apply to this class.

The same class of examples shows that several statements in Section 3 of [4] are not correct.

2. Conditional distributions and modulus of continuity

Let $m \in \mathbb{Z}^d$. We introduce the random variable

$$\eta_m^\perp : (\Omega, \mathcal{A}) \to (\Omega_m^\perp, \mathcal{A}_m^\perp), \quad \eta_m^\perp(\omega) = \pi_m^\perp(\eta(\omega)) = (\eta_k(\omega))_{k \neq m}.$$

We denote by $\mathbb{P}_{\eta_m^\perp} : \mathcal{A}_m^\perp \to [0,1]$ the push-forward measure of $\mathbb{P}$, i.e. $\mathbb{P}_{\eta_m^\perp}(B) := \mathbb{P}(\{\omega \in \Omega\colon \eta_m^\perp(\omega) \in B\})$. For $m \in \mathbb{Z}^d$, $a \in \mathbb{R}$ and $\varepsilon > 0$ let

$$Y_m^{\varepsilon,a} := \mathbb{P}\big(\eta_m \in [a, a+\varepsilon] \mid \eta_m^\perp\big) := \mathbb{E}\big(\mathbb{1}_{\eta_m \in [a,a+\varepsilon]} \mid \eta_m^\perp\big).$$

A conditional expectation $Y_m^{\varepsilon,a} = \mathbb{E}(\mathbb{1}_{\{\eta_m \in [a,a+\varepsilon]\}} \mid \eta_m^\perp)$ is a random variable $Y_m^{\varepsilon,a} : \Omega \to [0,1]$ with the property that

(i) $Y_m^{\varepsilon,a}$ is $\mathcal{F}$-measurable, where $\mathcal{F} = \sigma(\eta_m^\perp)$, and that
(ii) for all $A \in \mathcal{F}$ we have $\mathbb{E}(\mathbb{1}_{\{\eta_m \in [a,a+\varepsilon]\}} \mathbb{1}_A) = \mathbb{E}(Y_m^{\varepsilon,a} \mathbb{1}_A)$.

There may exist several functions $Y_m^{\varepsilon,a}$ which satisfy conditions (i) and (ii). They are called *versions* of $\mathbb{E}(\mathbb{1}_{\{\eta_m \in [a,a+\varepsilon]\}} \mid \eta_m^\perp)$. Two such versions $Y_m^{\varepsilon,a}$ and $\tilde{Y}_m^{\varepsilon,a}$ coincide $\mathbb{P}$-almost everywhere. For convenience, for each $a \in \mathbb{R}$ and $\varepsilon > 0$ we fix one version $Y_m^{\varepsilon,a}$ of the conditional expectation. Since $Y_m^{\varepsilon,a}$ is $\mathcal{F}$-measurable, the factorization lemma tells us that (for each a, ε) there is a measurable function $g_m^{\varepsilon,a} : (\Omega_m^\perp, \mathcal{A}_m^\perp) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $Y_m^{\varepsilon,a} = g \circ \eta_m^\perp$, i.e. for all $\omega \in \Omega$ we have

$$Y_m^{\varepsilon,a}(\omega) = g_m^{\varepsilon,a}(\eta_m^\perp(\omega)). \tag{5}$$

We introduce several quantities used in the literature to describe the regularity of (the conditional distribution) of the random fields ω_k and η_k, $k \in \mathbb{Z}^d$. For $k \in \mathbb{Z}^d$ we denote by $S_k\colon [0,\infty) \to [0,1]$,

$$S_k(\varepsilon) := \sup_{a\in\mathbb{R}} \mu_k([a, a+\varepsilon]) = \sup_{a\in\mathbb{R}} \mathbb{P}(\{\omega \in \Omega\colon \pi_k(\omega) \in [a, a+\varepsilon]\})$$

the global modulus of continuity or the concentration function of the measure μ_k. For $\Lambda \subset \mathbb{Z}^d$ and $\varepsilon > 0$ we define

$$\hat{S}_\Lambda(\varepsilon) := \sup_{m\in\Lambda} \sup_{a\in\mathbb{R}} \operatorname*{ess\,sup}_{\eta_m^\perp \in \Omega_m^\perp} g_m^{\varepsilon,a}(\eta_m^\perp).$$

Here, the essential supremum refers to the measure $\mathbb{P}_{\eta_m^\perp}$, that is,

$$\operatorname*{ess\,sup}_{\eta_m^\perp\in\Omega_m^\perp} g_m^{\varepsilon,a}(\eta_m^\perp) = \inf\Big\{b \in \mathbb{R} \colon \mathbb{P}_{\eta_m^\perp}\big(\{\eta_m^\perp \in \Omega_m^\perp \colon g_m^{\varepsilon,a}(\eta_m^\perp) > b\}\big) = 0\Big\}.$$

Denote by $\tilde{S}_m$ the conditional global modulus of continuity or the conditional concentration function of the distribution of η_m, i.e.

$$\tilde{S}_m^\varepsilon : \Omega \to [0,1], \quad \tilde{S}_m^\varepsilon = \sup_{a\in\mathbb{R}} Y_m^{\varepsilon,a}.$$

Since we are taking here a supremum over a uncountable family of intervals, it is not clear whether the resulting function is still measurable. In fact, this depends on how we chose the versions of the conditional expectation (for each of the uncountable many $a \in \mathbb{R}$). We show in Appendix A that a choice of versions, which ensures that $\tilde{S}_m^\varepsilon$ is $\mathcal{F}$-measurable, exists. In this case we denote by $g_m^\varepsilon : \Omega_m^\perp \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ the measurable function which comes up with the factorization lemma and satisfies $\tilde{S}_m^\varepsilon = g_m^\varepsilon \circ \eta_m^\perp$, and define

$$\tilde{S}_\Lambda(\varepsilon) := \sup_{m\in\Lambda} \operatorname*{ess\,sup}_{(\eta_k)_{k\neq m}} g_m^\varepsilon,$$

where the essential supremum again refers to the measure $\mathbb{P}_{\eta_m^\perp}$.

3. First example: A stationary field

In this first example we consider a random field $\eta = (\eta_k)_{k\in\mathbb{Z}}$ given by Eq. (4), where $d = 1$, and $\operatorname{supp} u = \{0,1\}$ with $u(0) = 1$ and $u(1) = t$ for some $|t| \leq 1$. Moreover, we assume that μ_k equals the uniform distribution on $[0,1]$ for all $k \in \mathbb{Z}$. Thus we have

$$\eta_x = \pi_x + t\pi_{x-1}, \quad x \in \mathbb{Z}^d.$$

First we cite a special case of [5, Lemma 3.1].

Lemma 3.1 ([5])**.** *Let $t \in (0,1]$. Then we have for all $\varepsilon > 0$*

$$\mathbb{P}\big(\eta_0 \in [1+t-\varepsilon, 1+t] \mid \eta_{-1}, \eta_1 \in [1+t-\varepsilon t/2, 1+t]\big) = 1. \tag{6}$$

Let $t \in [-1,0)$. Then we have for all $\varepsilon > 0$

$$\mathbb{P}\big(\eta_0 \in [t, t+\varepsilon] \mid \eta_{-1}, \eta_1 \in [1-\varepsilon|t|/2, 1]\big) = 1.$$

The first part of the lemma can be illustrated as follows. The variables η_k take values up to $1+t$. So if we condition on the event that η_{-1} and η_1 are close to $1+t$, then ω_{-1}, ω_0, ω_1 and ω_2 have to be close to 1. As a consequence, $\eta_0 = \omega_0 + t\omega_1$ is close to $1+t$. The second part of the lemma follows a similar reasoning. We will apply formula (6) in the multiplied form

$\mathbb{P}\big(\eta_0 \in [1+t-\varepsilon, 1+t], \eta_{-1}, \eta_1 \in [1+t-\varepsilon t/2, 1+t]\big) = \mathbb{P}\big(\eta_{-1}, \eta_1 \in [1+t-\varepsilon t/2, 1+t]\big)$

for the proof of

Theorem 3.2. *Let $\varepsilon > 0$ and*

$$a = \begin{cases} 1+t-\varepsilon & \text{if } t > 0, \\ t & \text{if } t < 0. \end{cases}$$

Then

$$\operatorname*{ess\,sup}_{\eta_0^\perp \in \Omega_0^\perp} g_0^{\varepsilon,a} = 1.$$

Proof. Assume the converse, i.e. $b := \operatorname{ess\,sup}_{\eta_0^\perp} g_0^{\varepsilon,a} < 1$. By definition of the conditional expectation we have for all $B \in \sigma(\eta_0^\perp)$ that

$$\mathbb{E}\big(\mathbb{1}_B \mathbb{1}_{\{\eta_0 \in [a,a+\varepsilon]\}}\big) = \mathbb{E}\big(\mathbb{1}_B Y_0^{\varepsilon,a}\big). \tag{7}$$

We choose

$$B = \begin{cases} \{\omega \in \Omega \colon \eta_{-1}, \eta_1 \in [1+t-\varepsilon t/2, 1+t]\} & \text{if } t > 0, \\ \{\omega \in \Omega \colon \eta_{-1}, \eta_1 \in [1-\varepsilon|t|/2, 1]\} & \text{if } t < 0, \end{cases}$$

which is $\sigma(\eta_0^\perp)$-measurable. For the left hand side of Eq. (7) we have by Lemma 3.1 $\mathbb{P}(B \cap \{\eta_0 \in [a, a+\varepsilon]\}) = 1 \cdot \mathbb{P}(B)$. For the right hand side of Eq. (7) we use the factorized version (5) of $Y_0^{\varepsilon,a}$ and obtain by substitution

$$\mathbb{E}\big(\mathbb{1}_B Y_0^{\varepsilon,a}\big) = \int_{\Omega_0^\perp} \mathbb{1}_{B'}(\eta_0^\perp) g_0^{\varepsilon,a}(\eta_0^\perp) \mathrm{d}\mathbb{P}_{\eta_0^\perp}(\eta_0^\perp),$$

where

$$B' = \begin{cases} \{\eta_0^\perp \in \Omega_0^\perp \colon \eta_{-1}, \eta_1 \in [1-t-\varepsilon t/2, 1+t]\} & \text{if } t > 0, \\ \{\eta_0^\perp \in \Omega_0^\perp \colon \eta_{-1}, \eta_1 \in [1-\delta|t|/2, 1]\} & \text{if } t < 0. \end{cases}$$

Since $b < 1$ by our assumption we obtain $\mathbb{E}(\mathbb{1}_B Y_0^{\varepsilon,a}) \leq b\mathbb{P}_{\eta_0^\perp}(B') = b\mathbb{P}(B) < \mathbb{P}(B)$. This is a contradiction to Eq. (7). ■

Corollary 3.3. *For any $\Lambda \subset \mathbb{Z}$ and any $\varepsilon > 0$ we have*

$$\hat{S}_\Lambda(\varepsilon) = \sup_{m \in \Lambda} \sup_{a \in \mathbb{R}} \operatorname*{ess\,sup}_{\eta_0^\perp \in \Omega_0^\perp} g_m^{\varepsilon,a}(\eta_0^\perp) = 1.$$

Proof. Follows from translation invariance and Theorem 3.2. ■

Corollary 3.4. *Assume that $\tilde{S}_m^\varepsilon = \sup_{a \in \mathbb{R}} Y_m^{\varepsilon,a}$ is measurable. Then, for any $\Lambda \subset \mathbb{Z}$ and any $\varepsilon > 0$ we have*

$$\tilde{S}_\Lambda(\varepsilon) := \sup_{m \in \Lambda} \operatorname*{ess\,sup}_{\eta_0^\perp \in \Omega_0^\perp} g_m^{\varepsilon}(\eta_0^\perp) = 1.$$

Proof. Pointwise we have $g_0^\varepsilon(\eta_0^\perp) \geq g_0^{\varepsilon,a}(\eta_0^\perp)$. If we take first the essential supremum with respect to $\eta_0^\perp$ and then supremum with respect to a on both sides, we obtain using Theorem 3.2

$$\operatorname*{ess\,sup}_{\eta_0^\perp \in \Omega_0^\perp} g_0^\varepsilon(\eta_0^\perp) \geq 1.$$

The result now follows by translation invariance. ∎

Remark 3.5. As mentioned before Lemma 3.1 is a special case of [5, Lemma 3.1]. The latter lemma applies to discrete alloy type potentials with $d = 1$, $\operatorname{supp} u = \{0, \dots, n-1\}$, and bounded random i.i.d. random variables ω_k, $k \in \mathbb{Z}^d$. Hence, the conclusions of Theorem 3.2, Corollary 3.3 and Corollary 3.4 can be extended to such models as well.

Remark 3.6 (Invertibility properties). If $|t| < 1$ then the Neumann series shows that the matrix $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ with $a_{i,j} = u(i-j)$, in matrix representation

$$A = \begin{pmatrix} \ddots & & & & \\ \ddots & 1 & & & \\ & t & 1 & & \\ & & t & 1 & \\ & & & \ddots & \ddots \end{pmatrix},$$

is invertible with bounded inverse. Also, the Fourier transform $\hat{u}\colon [0, 2\pi[\to \mathbb{C}$, $\hat{u}(\theta) := \sum_{k \in \mathbb{Z}} u(k) \mathrm{e}^{-\mathrm{i} k \cdot \theta}$ does not vanish on $[0, 2\pi)$. Such conditions have been sucessfully used, e.g., in [7, 6].

Remark 3.7 (Implications for [4]). Corollary 3.4 exhibits an example of a discrete alloy type potential satisfying conditions (S), (H), (R), (D), and (I) of [4], but not condition $(\tilde{R})$. Thus the final sentence of §3.1 in [4] is not correct. In particular, Corollary 3.4 provides a counter-example to Lemma 3.1 of [4].

4. Second example: A non-stationary field

Here we consider an even simple example where all relevant calculations deduce to simple two-dimensional integrals. The induced sequence of random variables is not stationary but the relevant phenomenon is seen clearly in this case. It provides a simple counterexample to Lemma 3.1 in [4].

The specific random variables $\eta_k : \Omega \to \mathbb{R}$, $k \in \mathbb{Z}^d$, we want to study in this second example are given by

$$\eta_{e_1} = \pi_0 + \pi_{e_1}, \quad \eta_k = \pi_k \text{ for all } k \in \mathbb{Z}^d \setminus \{e_1\},$$

where $e_1 = (1, 0, \dots, 0)^\mathrm{T} \in \mathbb{Z}^d$ denotes the unit vector with respect to the first coordinate. Thus they are defined as in Eq. (3) with a suitable linear operator $A : \Omega \to \Omega$.

4.1. **Preliminary estimates.** Let $[a,b]$ and $[c,d]$ denote two compact intervals. Since the random variables π_k, $k \in \mathbb{Z}^d$, are independent we have

$$\mathbb{P}(\pi_0 \in [a,b], \pi_{e_1} \in [c,d]) = \int_{[a,b]} \mu_0(\mathrm{d}x) \int_{[c,d]} \mu_{e_1}(\mathrm{d}y)$$

and

$$\begin{aligned}\mathbb{P}(\eta_0 \in [a,b], \eta_{e_1} \in [c,d]) &= \mathbb{P}(\pi_0 \in [a,b], \pi_{e_1} \in [c-\pi_0, d-\pi_0]) \\ &= \int_{[a,b]} \mu_0(\mathrm{d}x) \int_{[c-x,d-x]} \mu_{e_1}(\mathrm{d}y).\end{aligned}$$

Thus certain probabilities in the infinite product space reduce to two-dimensional integrals. If μ_0 and μ_{e_1} are the uniform distribution on $[0,1]$, we have for all $\varepsilon \in [0,1]$

$$\begin{aligned}\mathbb{P}(\eta_0 \in [0,\varepsilon], \eta_{e_1} \in [0,\varepsilon]) &= \int_0^\varepsilon \mathrm{d}x \int_{-x}^{\varepsilon-x} \mathbb{1}_{[0,1]}(y)\mathrm{d}y = \int_0^\varepsilon \mathrm{d}x \int_0^{\varepsilon-x} \mathrm{d}y \\ &= \int_0^\varepsilon \mathrm{d}x(\varepsilon - x) = \varepsilon^2 - \frac{1}{2}\varepsilon^2 = \frac{1}{2}\varepsilon^2. \qquad (8)\end{aligned}$$

For the global modulus of continuity of the measure μ_0 we have

$$S_0 \colon [0,\infty) \to [0,1], \quad S_0(\varepsilon) = \min(\varepsilon, 1).$$

For any $\varepsilon \in [0,1]$ the set $B := \eta_{e_1}^{-1}([0,\varepsilon]) = \{\omega \in \Omega \mid \eta_{e_1}(\omega) \in [0,\varepsilon]\}$ is measurable, and we have by definition of the conditional expectation and Eq. (8)

$$\begin{aligned}\mathbb{E}(\mathbb{1}_B \mathbb{P}(\eta_0 \in [0,\varepsilon] \mid \eta_0^\perp)) = \mathbb{E}(\mathbb{1}_B Y_0^{\varepsilon,0}) &= \mathbb{E}(\mathbb{1}_B \mathbb{1}_{\{\eta_0 \in [0,\varepsilon]\}}) = \mathbb{E}(\mathbb{1}_B\,(\mathbb{1}_{[0,\varepsilon]} \circ \eta_0)) \\ &= \mathbb{P}(\eta_{e_1} \in [0,\varepsilon], \eta_0 \in [0,\varepsilon]) = \frac{1}{2}\varepsilon^2. \qquad (9)\end{aligned}$$

For $\kappa > 0$, $\varepsilon \in [0, 1/\kappa]$ and $B = \eta_{e_1}^{-1}[0,\varepsilon]$ we have

$$\begin{aligned}\mathbb{E}(\mathbb{1}_B\, S_0(\kappa\varepsilon)) &= \kappa\varepsilon\, \mathbb{P}(B) = \kappa\varepsilon\, \mathbb{P}(\eta_{e_1} \in [0,\varepsilon]) \\ &= \kappa\varepsilon\, \mathbb{P}(\pi_0 \in [0,\varepsilon], \pi_{e_1} \in [-\pi_0, \varepsilon - \pi_0]) \\ &= \kappa\varepsilon \int_0^\varepsilon \mathrm{d}x \int_0^{\varepsilon-x} \mathbb{1}_{[0,1]}(y)\mathrm{d}y.\end{aligned}$$

By the calculation of (8) we obtain

$$\mathbb{E}(\mathbb{1}_B\, S_0(\kappa\varepsilon)) = \kappa\varepsilon \cdot \frac{1}{2}\varepsilon^2 = \frac{1}{2}\kappa\, \varepsilon^3. \qquad (10)$$

By (10) we have also shown that $\mathbb{P}(B) = \varepsilon^2/2$ and thus B has positive measure for all $\varepsilon \in (0,1]$.

4.2. **Main inequality.** We fix $\kappa \in (0,\infty)$ and $\varepsilon \in (0, 1/\kappa)$, and compare the two functions

$$S_0\colon [0,\infty) \to [0,1] \quad \text{and} \quad \tilde{S}_0^\varepsilon\colon \Omega \to [0,1].$$

We have already shown that for $\varepsilon \in (0,1)$ the set B has positive measure. By Eq. (9) and (10) we have

$$\begin{aligned}\mathbb{E}(\mathbb{1}_B \tilde{S}_0^\varepsilon) = \mathbb{E}(\mathbb{1}_B \sup_{a\in\mathbb{R}} Y_0^{\varepsilon,a}) \geq \mathbb{E}(\mathbb{1}_B Y_0^{\varepsilon,0}) = \frac{1}{2}\varepsilon^2 \\ > \frac{1}{2}\kappa\varepsilon^3 = \mathbb{E}(\mathbb{1}_B\, S_0(\kappa\varepsilon)).\end{aligned}$$

Thus we have shown that the two above mentioned functions do not coincide (not even almost surely).

Appendix A. Measurability of the concentration function

We will use here results on the regular version of the condition a distribution of a random variable with respect to a sub-σ-algebra. These can be found, e.g., in §44 of [2].

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathcal{C} \subset \mathcal{A}$ a sub-sigma-algebra. Let $X\colon \Omega \to \mathbb{R}$ be a random variable. Let $Q\colon \Omega \times \mathcal{B}(\mathbb{R}) \to [0,1]$ be a regular version of the conditional distribution of X with respect to $\mathcal{C}$.

Then for each $\varepsilon > 0$ and $a \in \mathbb{R}$

$$\Omega \ni \omega \mapsto Q(\omega, [a, a+\varepsilon])$$

is $\mathcal{C}$-measurable. Consequently, for each $\varepsilon > 0$

$$\sup_{b,\delta\in\mathbb{Q},\delta\in[0,\varepsilon]} Q(\omega, [b, b+\delta])$$

is $\mathcal{C}$-measurable as well. We will show now the following claim:

$$\sup_{a\in\mathbb{R}} Q(\omega, [a, a+\varepsilon]) = \sup_{b,\delta\in\mathbb{Q},\delta\in[0,\varepsilon]} Q(\omega, [b, b+\delta]).$$

Proof. Fix $c \in \mathbb{R}$. Since Q is a regular version of the conditional distribution we have for all $\omega \in \Omega$

$$Q(\omega, [c, c+\varepsilon]) = \sup_{b,\delta\in\mathbb{Q},b\geq c,\delta\geq 0,b+\delta\leq c+\varepsilon} Q(\omega, [b, b+\delta]).$$

(For an arbitrary version of the conditional distribution we would have this statement only for almost all ω, with the exceptional set depending on c.) The last quantity equals

$$\sup_{b,\delta\in\mathbb{Q},b\geq c,\delta\geq 0,b+\delta\leq c+\varepsilon,\delta\leq\varepsilon} Q(\omega, [b, b+\delta])$$

and is bounded from above by

$$\sup_{b,\delta\in\mathbb{Q},b\geq c,\delta\geq 0,\delta\leq\varepsilon} Q(\omega,[b,b+\delta]) \leq \sup_{b,\delta\in\mathbb{Q},\delta\geq 0,\delta\leq\varepsilon} Q(\omega,[b,b+\delta]) \leq \sup_{b\in\mathbb{Q}} Q(\omega,[b,b+\varepsilon]) \leq \sup_{b\in\mathbb{R}} Q(\omega,[b,b+\varepsilon]). \qquad \blacksquare$$

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