Vector duality for convex vector optimization problems with respect to quasi-minimality

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Abstract. We define the quasi-minimal elements of a set with respect to a convex cone and characterize them via linear scalarization. Then we attach to a general vector optimization problem a dual vector optimization problem with respect to quasi-efficient solutions and establish new duality results. By considering particular cases of the primal vector optimization problem we derive vector dual problems with respect to quasi-efficient solutions for both constrained and unconstrained vector optimization problems and the corresponding weak, strong and converse duality statements.

Key Words. quasi-interior, quasi-minimal element, quasi-efficient solution, vector duality

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1 Introduction and Preliminaries

In the last years the role played in optimization by different generalizations the interior of a set, among which let us recall the algebraic interior, also known as the core, the intrinsic core, the relative interior, the strong quasi-relative interior, the quasi-relative interior and the quasi interior, became more and more important, due to both theoretical and practical reasons. Their most common usage was in constructing weaker and weaker regularity conditions for guaranteeing strong duality or certain formulae (see, for instance, [4–8, 10, 12–14, 25]), but in recent works like [2, 3, 16, 17, 19, 24] new minimality concepts for sets were defined by using such generalized interiors, leading to new efficiency notions as solutions to vector optimization problems.

In this paper we consider and characterize via a linear scalarization the quasi-minimal elements of a set with respect to a convex cone. The quasi-minimal elements of a set are defined with respect to a convex cone by making use of its quasi interior. This notion is weaker than the classical minimality with respect to a cone and is actually a generalization of the weak minimality, which can be taken into consideration only when the interior of the ordering cone is nonempty. Then, we attach to a general vector optimization problem

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a vector dual problem with respect to quasi-efficient solutions, providing moreover weak, strong and converse duality statements for this primal-dual pair of vector optimization problems. Afterwards, we derive similar duality treatments for both constrained and unconstrained vector optimization problems. Due to the fact that when the interior of a set is nonempty it coincides with its quasi interior, in case the interior of the ordering cone is nonempty we rediscover the duality theory with respect to weakly efficient solutions presented for instance in [10, Section 4.3.4].

The quasi interior of a set was introduced by Schaefer in [22], while the quasi-relative interior was considered first by Borwein and Lewis in [4]. Both these generalized interiority notions were involved in different ways in dealing with optimization problems in works like [2–8, 12–17, 19, 24, 25]. The most common examples of sets with nonempty quasi interiors and quasi-relative interiors, but with empty interiors and other generalized interiors are the positive cones of the spaces ℓ^p and L^p, with $p \geq 1$.

Our investigations are motivated not only by theoretical reasons, but also by the vector optimization problems where the ordering cones of the image space have empty interiors met in the literature, for instance in [18] or [1], where one can find a finance model with m investors trading securities and having identical expectations on the security payoffs which is modeled as a vector optimization problem whose objective function maps from a portfolio vector space to an ordered payoff vector space that is $L^p(\Omega, \Sigma, P)$, with $p \geq 1$, where (Ω, Σ, P) is an underlying probability space.

The structure of the paper is as follows. In the remainder of this section we establish the framework of our work and present some preliminary results and notions needed in our investigations. In the next one we introduce and characterize the quasi-minimal elements of a set. The third section is dedicated to introducing a vector dual problem with respect to quasi-efficient solutions to a general vector optimization problem and establishing the corresponding weak, strong and converse duality results. Then we formulate constrained and unconstrained vector optimization problems as special cases of the general vector optimization problem and derive for them vector dual problems with respect to quasi-efficient solutions, followed by weak, strong and converse duality statements. In the fifth section we deliver some comparisons between the image sets of the different vector duals attached to a constrained vector optimization problem. We close the paper with a small conclusive section, where we also present some ideas for further research.

Let X be a separated locally convex space, X^* be the topological dual space of X endowed with the corresponding weak* topology and $\langle x^*, x\rangle = x^*(x)$ denote the value at $x \in X$ of the linear continuous functional $x^* \in X^*$.

A *cone* $K \subseteq X$ is a nonempty set which fulfills $\lambda K \subseteq K$ for all $\lambda \geq 0$. A *convex cone* is a cone which is a convex set. A cone $K \subseteq X$ is called *nontrivial* if $K \neq \{0\}$ and $K \neq X$ and *pointed* if $K \cap (-K) = \{0\}$. The *dual cone* of K is $K^* = \{x^* \in X^* : \langle x^*, x\rangle \geq 0\ \forall x \in K\}$. The *projection function* $\Pr_X : X \times Y \to X$ is defined by $\Pr_X(x, y) = x$ for all $(x, y) \in X \times Y$.

For a subset U of X, by $\operatorname{int} U$, $\operatorname{core} U$, $\operatorname{cl} U$, $\operatorname{lin} U$, $\dim U$, $\operatorname{cone} U$, $\operatorname{aff} U$, $\operatorname{lin} U$, δ_U, $\operatorname{sqri} U$ and $\operatorname{ri} U$ we denote its *interior, algebraic interior, closure, linear hull, dimension*, *conical hull, affine hull, linear hull, indicator function*, *strong quasi-relative interior* and, in case $X = \mathbb{R}^n$, *relative interior*, respectively. The *normal cone* associated to the U at $x \in U$ is given by $N_U(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0 \text{ for all } y \in U\}$. The *quasi interior* of U is the set

$$\operatorname{qi} U = \{x \in U : \ \operatorname{cl}(\operatorname{cone}(U - x)) = X\}$$

and the *quasi-relative interior* of $U \subseteq X$ is

$$\operatorname{qri} U = \{x \in U : \operatorname{cl}(\operatorname{cone}(U - x)) \text{ is a linear subspace of } X\}.$$

Some properties of the latter generalized interiority notions follow (cf. [4–6, 8]).

Remark 1 *Let $U \subseteq X$ be a convex set.*

(a) For all $x \in X$, it holds $\operatorname{qri}\{x\} = \{x\}$.

(b) One has

$$\operatorname{int} U \subseteq \operatorname{core} U \subseteq \operatorname{qi} U \subseteq \operatorname{qri} U. \tag{1}$$

When one of the sets in (1) is nonempty, it coincides with all its supersets within this chain of inclusions.

(c) If $x \in U$, one has $x \in \operatorname{qri} U$ if and only if $N_U(x)$ is linear subspace of X^* and, respectively, $x \in \operatorname{qi} U$ if and only if $N_U(x) = \{0\}$.

(d) In case $X = \mathbb{R}^n$, we have that $\operatorname{qi} U = \operatorname{core} U = \operatorname{int} U$ and $\operatorname{qri} U = \operatorname{sqri} U = \operatorname{ri} U$.

A situation where the interior of a set and all the generalized interiors but the quasi interior and the quasi-relative interior are empty can be found below.

Example 1 Let $p = 2$ and consider the real Banach space $\ell^2 = \ell^2(\mathbb{N})$ of the real sequences $(x_n)_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} |x_n|^2 < +\infty$ equipped with the norm $|| \cdot || : \ell^2 \to \mathbb{R}$, $||x|| = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}$ for all $x = (x_n)_{n\in\mathbb{N}} \in \ell^2$, where $\ell^2_+ = \{(x_n)_{n\in\mathbb{N}} \in \ell^2 : x_n \geq 0\ \forall n \in \mathbb{N}\}$ is the positive cone of ℓ^2. Then $\operatorname{int}(\ell^2_+) = \operatorname{core}(\ell^2_+) = \operatorname{sqri}(\ell^2_+) = \emptyset$, but $\operatorname{qi}(\ell^2_+) = \operatorname{qri}(\ell^2_+) = \{(x_n)_{n\in\mathbb{N}} \in \ell^2 : x_n > 0\ \forall n \in \mathbb{N}\}$.

In a separable Banach space the quasi interior of any nonempty convex set not contained in a hyperplane is nonempty (cf. [20]) and the quasi-relative interior of any nonempty convex set is nonempty (cf. [4]). These properties are no longer valid in general if the space is not separable.

Now let us give some properties of the quasi interior of a cone.

Remark 2 *Let $K \subseteq X$ be a convex cone.*

(a) If K is also pointed, then $0 \notin \operatorname{qi} K$.

(b) One has $\operatorname{qi} K + K = \operatorname{qi} K$.

(c) The set $\operatorname{qi} K \cup \{0\}$ is a cone, too.

(d) If K is also closed, then $\operatorname{qi} K^* = \{x^* \in K^* : \langle x^*, x\rangle > 0\ \forall x \in K \setminus \{0\}\}$, a set usually denoted by K^{*0} and known as the quasi interior of the dual cone of K.

The following statements present some useful properties of the quasi-relative interior of a convex set (see, for example [8]).

Proposition 1 *Let U and V be convex subsets X. Then the following statements hold*

(a) $\operatorname{qri} U + \operatorname{qri} V \subseteq \operatorname{qri}(U + V)$;

(b) $\operatorname{qri} U \times \operatorname{qri} V = \operatorname{qri}(U \times V)$*;*

(c) $\operatorname{qri}(U - x) = (\operatorname{qri} U) - x$ *for all* $x \in X$*;*

(d) $\operatorname{qri}(\alpha U) = \alpha(\operatorname{qri} U)$ *for all* $\alpha \in \mathbb{R} \setminus \{0\}$*;*

(e) $\lambda \operatorname{qri} U + (1-\lambda)U \subseteq \operatorname{qri} U$ *for all* $\lambda \in (0, 1]$ *whence* $\operatorname{qri} U$ *is a convex set;*

(f) $\operatorname{qri}(\operatorname{qri} U) = \operatorname{qri} U$*;*

(g) *if* U *is affine then* $\operatorname{qri} U = U$*;*

(h) *if* $\operatorname{qri} U \neq \emptyset$ *then* $\operatorname{cl} \operatorname{qri} U = \operatorname{cl} U$ *and* $\operatorname{cl} \operatorname{cone} \operatorname{qri} U = \operatorname{cl} \operatorname{cone} U$*;*

(i) *if* $U \subseteq V$ *then* $\operatorname{qi} U \subseteq \operatorname{qi} V$*; moreover when* $\operatorname{aff} U = \operatorname{aff} V$ *then* $\operatorname{qri} U \subseteq \operatorname{qri} V$*;*

(j) $\operatorname{qri}(\operatorname{qri} U - \operatorname{qri} V) = \operatorname{qri} U - \operatorname{qri} V$.

In the literature there exists some separation theorems for convex sets by mean of quasi-relative interior (see [5]). We will use in our investigations the following one, which is [8, Theorem 2.7].

Theorem 2 *Let* U *be a nonempty convex subset of* X *and* $x \in U$*. If* $x \notin \operatorname{qri}(U)$ *then there exists* $x^* \in X^* \setminus \{0\}$ *such that* $\langle x^*, y \rangle \leq \langle x^*, x \rangle$ *for all* $y \in U$*.*

We also need the following classical separation statement due to Tuckey.

Theorem 3 *Let* U *and* V *be nonempty convex subsets of the locally convex space* X*, one compact and the other closed. Then* $U \cap V = \emptyset$ *if and only if there exists an* $x^* \in X^* \setminus \{0\}$ *such that* $\sup_{x \in U} \langle x^*, x \rangle < \sup_{x \in V} \langle x^*, x \rangle$*.*

For a convex cone $K \subseteq X$, one can introduce the partial ordering relation "$\leqq_K$" defined by $x \leqq_K y$ if $y - x \in K$, where $x, y \in K$. Denote also $x \leq_K y$ if $x \leqq_K y$ and $x \neq y$. When $\operatorname{qi} K \neq \emptyset$ we denote $x <_K y$ if $y - x \in \operatorname{qi} K$, extending the notation usually considered in the literature for the case $\operatorname{int} K \neq \emptyset$.

If $K \neq \{0\}$, a greatest element with respect to "$\leqq_K$" which does not belong to X denoted by ∞_K is attached to X, and let $X^\bullet = X \cup \{\infty_K\}$. Then for any $x \in X^\bullet$ one has $x \leqq_K \infty_K$ and we consider on $X^\bullet$ the operations $x + \infty_K = \infty_K + x = \infty_K$ for all $x \in X$ and $t \cdot \infty_K = \infty_K$ for all $t \geq 0$. Moreover, consider by convention $\langle v^*, \infty_K \rangle = +\infty$ for all $v^* \in K^*$.

In what follows we consider some notions which extend the classical monotonicity to functions defined on partially ordered spaces.

Definition 1 Let the space X be partially ordered by the convex cone K, a nonempty set $U \subseteq X$ and $f : X \to \overline{\mathbb{R}}$ a given function.

(i) If $f(x) \leq f(y)$ for all $x, y \in U$ such that $x \leqq_K y$ the function f is called K-*increasing* on U.

(ii) If $f(x) < f(y)$ for all $x, y \in U$ such that $x \leq_K y$ the function f is called *strongly* K-*increasing* on U.

(iii) If f is K-increasing on U, $\operatorname{qi} K \neq \emptyset$ and for all $x, y \in U$ fulfilling $x <_K y$ follows $f(x) < f(y)$ the function f is called *strictly K-increasing* on U.

(iv) When $U = X$ we call these classes of functions simply *K-increasing*, *strongly K-increasing* and *strictly K-increasing*, respectively.

Remark 3 *In Definition 1(iii) we extend the notion of strictly K-increasing on U functions given so far in the literature for the case* $\operatorname{int} K \neq \emptyset$ *(or* $\operatorname{core} K \neq \emptyset$*).*

Let us illustrate this definition with the following example (see [10]).

Example 2 Let $x^* \in X^*$. If $x^* \in K^*$, then for all $x_1, x_2 \in X$ such that $x_1 \leqq_K x_2$ we have that $\langle x^*, x_2 - x_1 \rangle \geq 0$. Therefore $\langle x^*, x_1 \rangle \leq \langle x^*, x_2 \rangle$ and this means that the elements of K^* are actually K-increasing functions.

If $x^* \in K^{*0}$, then for all $x_1, x_2 \in X$ such that $x_1 \leq_K x_2$ it holds $\langle x^*, x_2 - x_1 \rangle > 0$. This means by definition that the elements of K^{*0} are strongly K-increasing functions on X.

If $K \subseteq X$ is a convex closed cone, $\operatorname{qi} K \neq \emptyset$, then via Remark 2($d$) $\operatorname{qi} K = \{x \in X : \langle x^*, x \rangle > 0 \ \forall x^* \in K^* \setminus \{0\}\}$ and thus every $x^* \in K^* \setminus \{0\}$ is strictly K-increasing on X.

Nevertheless, we present some notions regarding functions, too. In what follows, for a function $f : X \to \overline{\mathbb{R}}$ we use the classical notations for *domain* $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ and *epigraph* $\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$. The *conjugate function* $f^* : X^* \to \overline{\mathbb{R}}$ is defined by $f^*(x^*) = \sup_{x \in X}\{\langle x^*, x \rangle - f(x)\}$ and the *conjugate function with respect to a nonempty set* $U \subseteq X$ $f_U^* : X^* \to \overline{\mathbb{R}}$ is defined by $f_U^*(x^*) = (f + \delta_U)^*(x^*) = \sup_{x \in U}\{\langle x^*, x \rangle - f(x)\}$. Function f is *proper* if $f(x) > -\infty$ for all $x \in X$ and $\operatorname{dom} f \neq \emptyset$. If $f(x) \in \mathbb{R}$ the *(convex) subdifferential* of f at x is $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x \rangle \ \forall y \in X\}$, while if $f(x) \notin \mathbb{R}$ we take by convention $\partial f(x) = \emptyset$. For $U \subseteq X$ we have for all $x \in U$ that $\partial \delta_U(x) = N_U(x)$. Between a function and its conjugate there is the *Young-Fenchel inequality* $f^*(x^*) + f(x) \geq \langle x^*, x \rangle$ for all $x \in X$ and $x^* \in X^*$. This inequality is fulfilled as an equality if and only if $x^* \in \partial f(x)$. For a linear continuous mapping $A : X \to Y$ we have its *adjoint* $A^* : Y^* \to X^*$ given by $\langle A^* y^*, x \rangle = \langle y^*, Ax \rangle$ for any $(x, y^*) \in X \times Y^*$.

A vector function $F : X \to Y^\bullet$ is said to be *proper* if its *domain* $\operatorname{dom} F = \{x \in X : F(x) \in Y\}$ is nonempty. It is *K-convex* if $F(tx + (1-t)y) \leqq_K tF(x) + (1-t)F(y)$ for all $x, y \in X$ and all $t \in [0, 1]$. The vector function F is said to be K-epi-*closed* if K is closed and its *K-epigraph* $\operatorname{epi}_K F = \{(x, y) \in X \times Y : y \in F(x) + K\}$ is closed, and it is called *K-semicontinuous* if for every $x \in X$, each neighborhood W of zero in Y and for any $b \in Y$ satisfying $b \leqq_K F(x)$, there exists a neighborhood U of x in X such that $F(U) \subseteq b + W + Y \cup \{+\infty_K\}$.

For $v^* \in K^*$ the function $(v^* F) : X \to \overline{\mathbb{R}}$ is defined by $(v^* F)(x) = \langle v^*, F(x) \rangle$, $x \in X$. If F is K-lower semicontinuous then $(v^* F)$ is lower semicontinuous whenever $v^* \in K^* \setminus \{0\}$ and if K is closed, then every K- lower semicontinuous vector function is also K-epi-closed, but not all K-epi-closed vector functions are K-lower semicontinuous, as [9, Example 1] shows.

2 Quasi-efficient solutions

Let V be a separated locally convex vector space partially ordered by the pointed convex cone $K \subseteq V$ with a nonempty quasi interior, and $U \subseteq V$ a nonempty convex set.

Definition 2 An element $\overline{x} \in U$ is said to be a *quasi-minimal element* of U (regarding the partial ordering induced by K) if $(\overline{x} - \operatorname{qi} K) \cap U = \emptyset$.

Remark 4 *Quasi-minimal elements were also considered in works like [16, 19, 24], being usually called quasi-weakly minimal elements. However, we opted for the simpler name presented in Definition 2, even if it is used in the literature also for other types of minimal elements (see, for instance, [21]). However, if the conjecture presented below, namely that* $U + \operatorname{qi} K = \operatorname{qi}(U + K)$ *always holds, turns out to be valid, we believe that the quasi-minimal elements should be actually called weakly minimal. Note also that in [2, 3, 19] one can find quasi-relative minimal elements.*

Analogously one can define *quasi-maximal elements* of U (regarding the partial ordering induced by K), which are defined if $(\overline{x} + \operatorname{qi} K) \cap U = \emptyset$.

We denote by $\operatorname{QMin}(U, K)$ and $\operatorname{QMax}(U, K)$ the sets of all quasi-minimal and quasi-maximal elements of the set U (regarding the partial ordering induced by K), respectively. One can prove that $\operatorname{QMin}(U, -K) = -\operatorname{QMin}(-U, K) = \operatorname{QMax}(U, K)$.

Recall that an element $\overline{x} \in U$ is said to be a *minimal element of* U (regarding the partial ordering induced by K) if there is no $x \in U$ satisfying $x \leq_K \overline{x}$. The set of all minimal elements of U is denoted by $\operatorname{Min}(U, K)$ and it is called the *minimal set of* U (regarding the partial ordering defined by K)

The relation $(\overline{x} - \operatorname{qi} K) \cap U = \emptyset$ in Definition 2 can be equivalently rewritten as $(U - \overline{x}) \cap (-\operatorname{qi} K) = \emptyset$. Whenever the cone K is nontrivial we notice that if we consider as ordering cone $\widehat{K} = \operatorname{qi} K \cup \{0\}$, then $\overline{x} \in \operatorname{QMin}(U, K)$ if and only if $(\overline{x} - \widehat{K}) \cap U = \{\overline{x}\}$, or, equivalently, $\overline{x} \in \operatorname{Min}(U, \widehat{K})$.

If $K \neq V$, any minimal element of U is also quasi-minimal since $(\overline{x} - K) \cap U = \{\overline{x}\}$ implies via Remark 2(a) that $(\overline{x} - \operatorname{qi} K) \cap U = \emptyset$. If $K = V$ then $\operatorname{QMin}(U, K) = \emptyset$.

Note that in case $\operatorname{core} K \neq \emptyset$ (or $\operatorname{int} K \neq \emptyset$) the following investigations rediscover results from [10, Section 2.4.2, Section 2.4.4 and Section 4.3.4], thus they can be seen as generalizations of the latter.

Lemma 4 *It holds* $\operatorname{QMin}(U, K) \subseteq \operatorname{QMin}(U + K, K)$.

Proof. Let us consider an $\overline{x} \in \operatorname{QMin}(U, K)$ assumed not to be a quasi-minimal element of the set $U + K$. Then there is an element $x \in (\overline{x} - \operatorname{qi} K) \cap (U + K) \neq \emptyset$ and there is an $u \in U$ with $\overline{x} - x \in \operatorname{qi} K$ and $x - u \in K$. Consequently, by using Remark 2(b) we obtain that $\overline{x} - u \in \operatorname{qi} K + K = \operatorname{qi} K$, or alternatively $u \in (\overline{x} - \operatorname{qi} K) \cap U$. Hence, $\overline{x}$ is not a quasi-minimal element of the set U, and the conclusion follows by contradiction. ∎

Remark 5 *In Definition 2 and Lemma 4 is not necessary to assume that U is convex.*

Proposition 5 *One has that* $\operatorname{qi}(U + \operatorname{qi} K) = U + \operatorname{qi} K \subseteq \operatorname{qi}(U + K)$.

Proof. From Remark 2(a) we have that $\operatorname{qi}(U + \operatorname{qi} K) \subseteq \operatorname{qi}(U + K)$ and obviously $\operatorname{qi}(U + \operatorname{qi} K) \subseteq U + \operatorname{qi} K$. The only implication left to be prove is $U + \operatorname{qi} K \subseteq \operatorname{qi}(U + \operatorname{qi} K)$.

Let us consider an element $a \in U + \operatorname{qi} K$, so there exist $u \in U$ and $k \in \operatorname{qi} K$ such that $a = u + k$. But $\operatorname{qi} K - k \subseteq \operatorname{qi} K - k + (U - u) = U + \operatorname{qi} K - a$. From here follows that $\operatorname{cone}(\operatorname{qi} K - k) \subseteq \operatorname{cone}(U + \operatorname{qi} K - a)$ and moreover $\operatorname{cl} \operatorname{cone}(\operatorname{qi} K - k) \subseteq \operatorname{cl} \operatorname{cone}(U + \operatorname{qi} K - a)$. But $k \in \operatorname{qi} K = \operatorname{qi}(\operatorname{qi} K)$ and so we have that $\operatorname{cl} \operatorname{cone}(\operatorname{qi} K - k) = V$. Consequently, as $a \in U + \operatorname{qi} K$ follows that $a \in \operatorname{qi}(U + \operatorname{qi} K)$. ∎

If A and B are convex subsets of V, recall that $\operatorname{int}(A+B) = A + \operatorname{int} B$ and $\operatorname{core}(A+B) = A + \operatorname{core} B$ (cf. [23]). Moreover, in all the situations known to us where $\operatorname{qi} B \neq \emptyset$ unlike the interior or algebraic interior of B, it holds $A + \operatorname{qi} B = \operatorname{qi}(A+B)$ for all the convex sets $A \subseteq V$. Consequently, we assume further for the sets U and K dealt with in this section that it holds $U + \operatorname{qi} K = \operatorname{qi}(U+K)$ and we maintain this additional hypothesis for their counterparts in the rest of the paper. Moreover, we conjecture that in general when $A, B \subseteq V$ are convex sets with $\operatorname{qi} B \neq \emptyset$, it holds $A + \operatorname{qi} B = \operatorname{qi}(A+B)$.

Next we formulate some necessary and sufficient characterizations via linear scalarizations of the quasi-minimal elements of the set U with respect to K.

Theorem 6 *If $\overline{x} \in \operatorname{QMin}(U,K)$ then there exists $x^* \in K^* \setminus \{0\}$ such that $\langle x^*, \overline{x}\rangle \leq \langle x^*, x\rangle$, for all $x \in U$.*

Proof. From $\overline{x} \in \operatorname{QMin}(U,K)$ it follows that $u \notin \overline{x} - \operatorname{qi} K$ for all $u \in U$. So, $\overline{x} \notin u + \operatorname{qi} K$ for all $u \in U$. Thus $\overline{x} \notin U + \operatorname{qi} K = \operatorname{qi}(U+K)$. From Proposition 5 follows that $\operatorname{qi}(U+K) \neq \emptyset$ and so $\operatorname{qri}(U+K) = \operatorname{qi}(U+K)$. As $\overline{x} \in U + K$ but $\overline{x} \notin \operatorname{qri}(U+K)$ we can apply Theorem 2. Consequently, there exists $\overline{x}^* \in X^* \setminus \{0\}$ such that

$$\langle \overline{x}^*, x + k\rangle \leq \langle \overline{x}^*, \overline{x}\rangle \text{ for all } x \in U \text{ and } k \in K. \tag{2}$$

Let $\overline{k} \in K \setminus \{0\}$ such that $\langle \overline{x}^*, \overline{k}\rangle > 0$ and $k = \alpha\overline{k}$. For $\alpha > 0$ we obtain a contradiction because the left hand side of (2) is unbounded from above for $\alpha \to +\infty$. Consequently, $\langle \overline{x}^*, \overline{k}\rangle \leq 0$ for all $\overline{k} \in K \setminus \{0\}$ which means that $\overline{x}^* \in -K^*$. Taking $k = 0$ and setting $x^* = -\overline{x}^* \in K^*$ we obtain $\langle x^*, \overline{x}\rangle \leq \langle x^*, x\rangle$ for all $x \in U$. ■

Lemma 7 *Let a function $f : V \to \overline{\mathbb{R}}$ which is strictly K- increasing on U. If there is an element $\overline{x} \in U$ fulfilling $f(\overline{x}) \leq f(x)$ for all $x \in U$, then $\overline{x} \in \operatorname{QMin}(U,K)$.*

Proof. If $\overline{x} \notin \operatorname{QMin}(U,K)$, then there exists $x \in (\overline{x} - \operatorname{qi} K) \cap U$. This implies $f(x) < f(\overline{x})$, which contradicts the assumption. ■

Further let K be also closed. The following theorem is a straightforward conclusion of Lemma 7 and Example 2.

Theorem 8 *If there exist $x^* \in K^* \setminus \{0\}$ and $\overline{x} \in U$ such that for all $x \in U$ it holds $\langle x^*, \overline{x}\rangle \leq \langle x^*, x\rangle$, then $\overline{x} \in \operatorname{QMin}(U,K)$.*

From Theorem 6 and Theorem 8 we obtain an equivalent characterization via linear scalarization for the quasi-minimal elements of U with respect to K.

Corollary 9 *Let $x \in U$. Then $\overline{x} \in \operatorname{QMin}(U,K)$ if and only if there exists $x^* \in K^* \setminus \{0\}$ satisfying $\langle x^*, \overline{x}\rangle \leq \langle x^*, x\rangle$ for all $x \in U$.*

3 General duality results

We consider the vector optimization problem

$$(PVG_q) \qquad \operatorname*{QMin}_{x \in X} F(x),$$

where $F : X \to V^\bullet$ is a proper and K-convex function with $\operatorname{dom} F = \{x \in X : \ F(x) \neq \emptyset\}$ and we are interested in determining the quasi-minimal elements of $F(\operatorname{dom} F)$ with respect to K. We also assume that $F(\operatorname{dom} F) + \operatorname{qi} K = \operatorname{qi}(F(\operatorname{dom} F) + K)$ and K is a closed convex cone.

Definition 3 An element $\overline{x} \in X$ is called *quasi-efficient solution* to the vector optimization problem (PVG_q) if $\overline{x} \in \operatorname{dom} F$ and $F(\overline{x}) \in \operatorname{QMin}(F(\operatorname{dom} F), K)$.

As mentioned in the first section, problems where the quasi-efficient solutions of vector optimization problems can play an important role because the ordering cones of the image spaces have empty interiors, but nonempty quasi interiors, can be found for instance in finance mathematics (see [1, 18]).

Using the vector perturbation function $\Phi : X \times Y \to V^\bullet$ which fulfills $0 \in \operatorname{Pr}_Y(\operatorname{dom} \Phi)$ and $\Phi(x, 0) = F(x)$ for all $x \in X$, the primal vector optimization problem introduced above can be reformulated as

$$(PVG_q) \qquad \operatorname*{QMin}_{x \in X} \Phi(x, 0).$$

To (PVG_q) we attach the following vector dual problem with respect to quasi-efficient solutions

$$(DVG_q) \qquad \operatorname*{QMax}_{(v^*, y^*, v) \in \mathcal{B}_q^G} h_q^G(v^*, y^*, v)$$

where

$$\mathcal{B}_q^G \ = \ \{(v^*, y^*, v) \in (K^* \setminus \{0\}) \times Y^* \times V : \langle v^*, v\rangle \leq -(v^*\Phi)^*(0, -y^*)\}$$

and

$$h_q^G(v^*, y^*, v) = v.$$

Definition 4 An element $(\overline{v}^*, \overline{y}^*, \overline{v}) \in \mathcal{B}_q^G$ is called *quasi-efficient solution* to the vector dual optimization problem (DVG_q) if $(\overline{v}^*, \overline{y}^*, \overline{v}) \in \operatorname{dom} h_q^G$ and $h_q^G(\overline{v}^*, \overline{y}^*, \overline{v}) = \overline{v} \in \operatorname{QMax}(h_q^G(\operatorname{dom}\, h_q^G), K)$.

Next we formulate the weak and strong duality theorems.

Theorem 10 *There are no $x \in X$ and $(v^*, y^*, v) \in \mathcal{B}_q^G$ such that $F(x) <_K h_q^G(v^*, y^*, v)$.*

Proof. We assume the contrary, namely that there exist $x \in X$ and $(v^*, y^*, v) \in \mathcal{B}_q^G$ such that $F(x) <_K h_q^G(v^*, y^*, v) = v$. Then it holds $x \in \operatorname{dom} F$ and $\langle v^*, v\rangle > \langle v^*, F(x)\rangle$. On the other hand $\langle v^*, v\rangle \leq -(v^*\Phi)^*(0, -y^*) \leq (v^*F)(x)$, so we obtained the desired contradiction. ■

Remark 6 *F needs not be K-convex and K-closed in order to formulate the vector dual problem and for proving the weak duality statement.*

For the strong duality we consider the following regularity conditions (cf. [10]). First, a classical condition

$$(RCV^1) \quad \Big| \quad \exists x' \in X \text{ such that } (x',0) \in \operatorname{dom}\Phi \text{ and } \Phi(x',\cdot) \text{ is continuous at } 0;$$

then the most general one that works when X and Y are Fréchet spaces

$$(RCV^2) \quad \left| \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces, } \Phi \text{ is } K\text{- lower semicontinuous and} \\ 0 \in \operatorname{sqri}(\Pr_Y(\operatorname{dom}\Phi)); \end{array} \right.$$

followed by the one good in finite dimensional cases

$$(RCV^3) \quad \Big| \quad \dim(\operatorname{lin}(\Pr_Y(\operatorname{dom}\Phi))) < +\infty \text{ and } 0 \in \operatorname{ri}(\Pr_Y(\operatorname{dom}\Phi));$$

and the closedness type regularity condition

$$(RCV^4) \quad \left| \begin{array}{l} \Phi \text{ is } K\text{-lower semicontinuous and } \Pr_{X^*\times\mathbb{R}}(\operatorname{epi}(v^*\Phi)^*) \text{ is closed in} \\ \text{the topology } w(X^*,X)\times\mathbb{R} \text{ for all } v^* \in K^* \setminus \{0\}. \end{array} \right.$$

Theorem 11 *Assume that one of the regularity conditions (RCV^i), $i \in \{1,\dots,4\}$, is fulfilled. If $\overline{x} \in X$ is a quasi-efficient solution to (PVG_q), then there exists $(\overline{v}^*,\overline{y}^*,\overline{v})$ a quasi-efficient solution to (DVG_q) such that $F(\overline{x}) = h_q^G(\overline{v}^*,\overline{y}^*,\overline{v}) = \overline{v}$.*

Proof. Since $\overline{x} \in X$ is a quasi-efficient solution to (PVG_q) then $\overline{x} \in \operatorname{dom} F$ and $F(\overline{x}) \in \operatorname{QMin}(F(\operatorname{dom} F),K)$. By Corollary 9 there exists $\overline{v}^* \in K^* \setminus \{0\}$ which satisfies

$$\langle \overline{v}^*, F(\overline{x})\rangle = \min_{x\in X}\langle \overline{v}^*, F(x)\rangle = \min_{x\in X}(\overline{v}^*\Phi)(x,0).$$

Applying [10, Theorem 3.2.1 or Theorem 3.2.3] one gets that there exists $\overline{y}^* \in Y^*$ such that

$$\langle \overline{v}^*, F(\overline{x})\rangle = \inf_{x\in X}(\overline{v}^*\Phi)(x,0) = \sup_{y^*\in Y^*}\{-(\overline{v}^*\Phi)^*(0,-y^*)\} = -(\overline{v}^*\Phi)^*(0,-\overline{y}^*).$$

For $\overline{v} = F(\overline{x})$ one has $(\overline{v}^*,\overline{y}^*,\overline{v}) \in \mathcal{B}_q^G$. From Theorem 10 one has that $(\overline{v}^*,\overline{y}^*,\overline{v})$ is a quasi-efficient solution to (DVG_q). ■

Remark 7 Instead of the mentioned regularity conditions, for achieving strong duality it is enough to assume that for all $\overline{v}^* \in K^* \setminus \{0\}$ the scalar optimization problem $\inf_{x\in X}(\overline{v}^*\Phi)(x,0)$ is stable.

Next, we give a preliminary result for the converse duality statement, followed by the mentioned assertion itself.

Theorem 12 *Assume that one of the regularity conditions (RCV^i), $i \in \{1,\dots,4\}$, is fulfilled. Then $V \setminus \operatorname{cl}(F(\operatorname{dom} F)+K) \subseteq \operatorname{core}(h_q^G(\mathcal{B}_q^G))$.*

Proof. Consider $\overline{v}$ be an arbitrary element in $V \setminus \operatorname{cl}(F(\operatorname{dom} F)+K)$. Since the set $\operatorname{cl}(F(\operatorname{dom} F)+K) \subseteq V$ is convex and closed, by Theorem 3 there exists $\overline{v}^* \in K^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\langle \overline{v}^*, \overline{v}\rangle < \alpha < \langle \overline{v}^*, v\rangle, \text{ for all } v \in \operatorname{cl}(F(\operatorname{dom} F)+K).$$

Thus $\langle \overline{v}^*, \overline{v}\rangle < \alpha \le \inf_{x\in X}(\overline{v}^*F)(x) = \inf_{x\in X}(\overline{v}^*\Phi)(x,0)$ and there exists $\overline{y}^* \in Y^*$ such that $\inf_{x\in X}(\overline{v}^*\Phi)(x,0) = -(\overline{v}^*\Phi)^*(0,-\overline{y}^*)$, so $\langle \overline{v}^*, \overline{v}\rangle < -(\overline{v}^*\Phi)^*(0,-\overline{y}^*)$. Obviously $\overline{v}^* \in h_q^G(\mathcal{B}_q^G)$. The rest of the proof follows the lines of [10, Theorem 4.3.3]. ■

Theorem 13 *Assume that one of the regularity conditions (RCV^i), $i \in \{1, \ldots, 4\}$, is fulfilled and that the set $F(\operatorname{dom} F) + K$ is closed. Then for every quasi-efficient solution $(\overline{v}^*, \overline{y}^*, \overline{v})$ to (DVG_q) one has that $\overline{v}$ is a quasi-minimal element of the set $F(\operatorname{dom} F) + K$.*

Proof. We assume that $\overline{v} \notin F(\operatorname{dom} F) + K$. From Theorem 12 follows that $\overline{v} \in \operatorname{core}(h_q^G(\mathcal{B}_q^G))$. From here follows that there exists a $\lambda > 0$ such that $v_\lambda = \overline{v} + \lambda k >_K \overline{v}$ and $v_\lambda \in h_q^G(\mathcal{B}_q^G)$. This contradicts the fact that $(\overline{v}^*, \overline{y}^*, \overline{v})$ is a quasi-efficient solution to (DVG_q). Supposing that $\overline{v}$ is not a quasi-minimal element of $F(\operatorname{dom} F) + K$, it follows that there exist $\overline{x} \in \operatorname{dom} F$ and $\overline{k} \in K$ satisfying $\overline{v} >_K F(\overline{x}) + \overline{k} \geqq_K F(\overline{x})$, but this contradicts Theorem 10. ■

Remark 8 In Theorem 12 and Theorem 13, the regularity conditions (RCV^i), $i \in \{1, \ldots, 4\}$, can be replaced with the weaker assumption that for all $\overline{v}^* \in K^* \setminus \{0\}$ the problem $\inf_{x \in X} \langle \overline{v}^*, F(x) \rangle$ is normal (see [10, Theorem 4.3.3]).

4 Duality results for particular classes of problems

4.1 Constrained vector optimization problems

Let us consider the same framework as in the previous section. Let also Y be partially ordered by the nonempty convex cone $C \subseteq Y$. Moreover, we consider the nonempty convex set $S \subseteq X$, the proper K- convex function $f : X \to V^\bullet$ and the proper $C-$ convex function $g : X \to Y^\bullet$ fulfilling $\operatorname{dom} f \cap S \cap g^{-1}(C) \neq \emptyset$. Assume again that $f(\operatorname{dom} f \cap \mathcal{A}) + \operatorname{qi} K = \operatorname{qi}(f(\operatorname{dom} f \cap \mathcal{A}) + K)$ and K is a closed convex cone. The primal vector optimization problem with geometric and cone constraints that we work with is

$$(PV_q^C) \qquad \operatorname*{QMin}_{x \in \mathcal{A}} f(x),$$

where

$$\mathcal{A} = \big\{ x \in S : g(x) \in -C \big\}.$$

This problem can be seen as a special case of (PVG_q). We construct different vector dual problems to (PV_q^C) with respect to quasi-efficient solutions, by considering different vector perturbation functions. Then we formulate weak, strong and converse duality. Later, in Section 5, we will investigate the connections between the image sets of these problems.

First we consider the Lagrange vector type perturbation function $\Phi_{C_L}^V : X \times Y \to V^\bullet$ given by

$$\Phi_{C_L}^V(x, y) = \begin{cases} f(x), & x \in S, g(x) \in y - C, \\ \infty_K, & \text{otherwise.} \end{cases}$$

Let $v^* \in K^* \setminus \{0\}$, $y^* \in Y^*$, $u \in S$ and $v \in V$. For assigning to (PV_q^C) a vector dual problem which is a special case of (DVG_q) for $\Phi_{C_L}^V$ we need to have $\langle v^*, v \rangle \leq -(v^* \Phi_{C_L}^V)^*(0, -y^*)$. This can be equivalently rewritten as $\langle v^*, v \rangle \leq -((v^* f) + (y^* g) + \delta_S)^*(0, -y^*)$ and $y^* \in C^*$. By using the definition of the conjugate function this relation is equivalent with $\langle v^*, v \rangle \leq \inf_{u \in S} \{(v^* f)(u) + (y^* g)(u)\}$ and $y^* \in C^*$. Thus from (DVG_q) we obtain the *Lagrange type vector dual problem to* (PV_q^C) with respect to quasi-efficient solutions

$$(DV_q^{C_L}) \qquad \operatorname*{QMax}_{(v^*,y^*,v)\in\mathcal{B}_q^{C_L}} h_q^{C_L}(v^*,y^*,v),$$

where

$$\mathcal{B}_q^{C_L} = \{(v^*,y^*,v) \in (K^* \setminus \{0\}) \times C^* \times V : \langle v^*, v\rangle \leq \inf_{u\in S}\{(v^*f)(u) + (y^*g)(u)\}\}$$

and

$$h_q^{C_L}(v^*,y^*,v) = v.$$

For the strong duality we consider the following regularity conditions

$(RCV_{C_L}^1)$ | $\exists x' \in \operatorname{dom} f \cap S$ such that $g(x') \in -\operatorname{int} C$;

$(RCV_{C_L}^2)$ | X and Y are Fréchet spaces, S is closed, f is K- lower semicontinuous, g is $C-$epi closed and $0 \in \operatorname{sqri}(g(\operatorname{dom} f \cap S \cap \operatorname{dom} g) + C)$;

$(RCV_{C_L}^3)$ | $\dim\big(\operatorname{lin}(g(\operatorname{dom} f \cap S \cap \operatorname{dom} g) + C)\big) < +\infty$ and $0 \in \operatorname{ri}(g(\operatorname{dom} f \cap S \cap \operatorname{dom} g) + C)$;

and

$(RCV_{C_L}^4)$ | S is closed, f is K-lower semicontinuous, g is $C-$epi closed and $\bigcup_{y^*\in C^*} \operatorname{epi}((v^*f) + (y^*g) + \delta_S)^*$ is closed in the topology $w(X^*,X) \times \mathbb{R}$ for all $v^* \in K^{*0}$.

Then the weak, strong and converse duality results follow.

Theorem 14 *(a) There are no $x \in X$ and $(v^*,y^*,v) \in \mathcal{B}_q^{C_L}$ such that $f(x) <_K h_q^{C_L}(v^*, y^*,v)$.*

(b) Assume that one of the regularity conditions $(RCV_{C_L}^i)$, $i \in \{1,2,3,4\}$, is fulfilled. If $\overline{x} \in X$ is a quasi-efficient solution to (PV_q^C), then there exists $(\overline{v}^*,\overline{y}^*,\overline{v})$ a quasi-efficient solution to $(DV_q^{C_L})$ such that $f(\overline{x}) = h_q^{C_L}(\overline{v}^*,\overline{y}^*,\overline{v}) = \overline{v}$.

(c) Assume that one of the regularity conditions $(RCV_{C_L}^i)$, $i \in \{1,2,3,4\}$, is fulfilled, and the set $f(\operatorname{dom} f \cap \mathcal{A}) + K$ is closed. Then for every quasi-efficient solution $(\overline{v}^*,\overline{y}^*,\overline{v})$ to $(DVG_q^{C_L})$ one has that $\overline{v}$ is a quasi-minimal element of the set $f(\operatorname{dom} f \cap \mathcal{A}) + K$.

Another vector perturbation function we consider is the Fenchel-Lagrange type vector perturbation function $\Phi_{FL}^V : X \times X \times Y \to V^\bullet$ given by

$$\Phi_{C_{FL}}^V(x,t,y) = \begin{cases} f(x+t), & x \in S, g(x) \in y - C \\ \infty_K, & \text{otherwise.} \end{cases}$$

Let $v^* \in K^* \setminus \{0\}$, $y^* \in C^*$, $t^* \in X^*$, $u \in S$ and $v \in V$. For having a new dual which is a special case of (DVG_q) for $\Phi_{C_{FL}}^V$ we need to have $\langle v^*, v\rangle \leq -(v^*\Phi_{C_{FL}}^V)^*(0,t^*)$. This can be equivalently rewritten as $\langle v^*, v\rangle \leq \langle t^*, u\rangle - (v^*f)^*(t^*) - (y^*g)(u)$ and $y^* \in C^*$. By using the definition of the conjugate function this relation is equivalent with $\langle v^*, v\rangle \leq$

$-(v^*f)^*(t^*) - (y^*g)_S^*(-t^*)$ and $y^* \in C^*$. Thus from (DVG_q) we obtain the *Fenchel-Lagrange type vector dual problem to* (PV_q^C) with respect to quasi-efficient solutions

$$(DV_q^{C_{FL}}) \qquad \underset{(v^*,t^*,y^*,v)\in\mathcal{B}_q^{C_{FL}}}{\mathrm{QMax}} h_q^{C_{FL}}(v^*,t^*,y^*,v),$$

where

$$\mathcal{B}_q^{C_{FL}} = \{(v^*,t^*,y^*,v) \in (K^* \setminus \{0\}) \times X^* \times C^* \times V : \langle v^*, v\rangle \leq -(v^*f)^*(t^*) - (y^*g)_S^*(-t^*)\}$$

and

$$h_q^{C_{FL}}(v^*,t^*,y^*,v) = v.$$

Next we consider the following regularity conditions

$$(RCV_{C_{FL}}^1) \;\; \Big| \;\; \exists x' \in \operatorname{dom} f \cap S \text{ such that } f \text{ is continuous at } x' \text{ and } g(x') \in -\operatorname{int} C;$$

$$(RCV_{C_{FL}}^2) \;\; \left| \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces, } S \text{ is closed, } f \text{ is } K\text{-lower semicontinuous,} \\ g \text{ is } C\text{-epi closed and } 0 \in \operatorname{sqri}(\operatorname{dom} f \times C - \operatorname{epi}_{-C}(-g) \cap (S \times Y)); \end{array} \right.$$

$$(RCV_{C_{FL}}^3) \;\; \left| \begin{array}{l} \dim(\operatorname{lin}(\operatorname{dom} f \times C - \operatorname{epi}_{-C}(-g) \cap (S \times Y))) < +\infty \text{ and} \\ 0 \in \operatorname{ri}(\operatorname{dom} f \times C - \operatorname{epi}_{-C}(-g) \cap (S \times Y)); \end{array} \right.$$

and, respectively,

$$(RCV_{C_{FL}}^4) \;\; \left| \begin{array}{l} S \text{ is closed, } f \text{ is } K\text{-lower semicontinuous, } g \text{ is } C\text{-epi closed and} \\ \operatorname{epi}(v^*f)^* + \bigcup\limits_{y^* \in C^*} \operatorname{epi}((y^*g) + \delta_S)^* \text{ is closed in the topology} \\ w(X^*, X) \times \mathbb{R} \text{ for every } v^* \in K^{*0}. \end{array} \right.$$

Then the weak, strong and converse duality results follow from the general case.

Theorem 15 *(a) There are no $x \in X$ and $(v^*,t^*,y^*,v) \in \mathcal{B}_q^{C_{FL}}$ such that $f(x) <_K h_q^{C_{FL}}(v^*,t^*,y^*,v)$.*

(b) Assume that one of the regularity conditions $(RCV_{C_{FL}}^i)$, $i \in \{1,\ldots,4\}$, is fulfilled. If $\overline{x} \in X$ is a quasi-efficient solution to (PV_q^C), then there exists $(\overline{v}^*,\overline{t}^*,\overline{y}^*,\overline{v})$ a quasi-efficient solution to $(DV_q^{C_{FL}})$ such that $f(\overline{x}) = h_q^{C_{FL}}(\overline{v}^*,\overline{t}^*,\overline{y}^*,\overline{v}) = \overline{v}$.

(c) Assume that one of the regularity conditions $(RCV_{C_{FL}}^i)$, $i \in \{1,\ldots,4\}$, is fulfilled and the set $f(\operatorname{dom} f \cap \mathcal{A}) + K$ is closed. Then for every quasi-efficient solution $(\overline{v}^*,\overline{t}^*,\overline{y}^*,\overline{v})$ to $(DVG_q^{C_{FL}})$ one has that $\overline{v}$ is a quasi-minimal element of the set $f(\operatorname{dom} f \cap \mathcal{A}) + K$.

4.2 Unconstrained vector optimization problems

Using the same framework as in Section 3, we consider the proper K- convex vector functions $f : X \to V^\bullet$ and $h : Y \to V^\bullet$ and $A : X \to Y$ a linear continuous mapping such that $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} h) \neq \emptyset$. Assume again that $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} h) + \operatorname{qi} K = \operatorname{qi}(\operatorname{dom} f \cap A^{-1}(\operatorname{dom} h) + K)$ and K is a closed convex cone. The primal unconstrained vector optimization problem

(PV_q^A) $$\underset{x\in X}{\mathrm{Min}}[f(x)+h(Ax)]$$

is a special case of (PVG_q) where $F = f + h \circ A$.

We consider the vector perturbation function $\Phi_q^A : X \times Y \to V^\bullet$ defined by $\Phi_q^A(x,y) = f(x) + h(Ax+y)$. Using this perturbation function we obtain the vector dual to (PV_q^A) given by

(DV_q^A) $$\underset{(v^*,y^*,v)\in\mathcal{B}_q^A}{\mathrm{QMax}} h_q^A(v^*,y^*,v)$$

where

$$\mathcal{B}_q^A = \{(v^*,y^*,v) \in (K^*\setminus\{0\}) \times Y^* \times V : \langle v^*, v\rangle \le -(v^*f)^*(-A^*y^*) + (v^*h)^*(y^*)\}$$

and

$$h_q^A(v^*,y^*,v) = v.$$

For the primal vector (PV_q^A) and the vector dual (DV_q^A) we have the weak, strong and converse duality statements, that follow from the general case. To guarantee strong duality we have the regularity conditions

(RCV_A^1) | $\exists x' \in \operatorname{dom} f \cap A^{-1}(\operatorname{dom} h)$ such that h is continuous at Ax';

(RC_A^2) | X and Y are Fréchet spaces, f and h are $C-$lower semicontinuous and $0 \in \operatorname{sqri}(\operatorname{dom} h - A(\operatorname{dom} f))$;

(RC_A^3) | $\dim(\operatorname{lin}(\operatorname{dom} h - A(\operatorname{dom} f))) < +\infty$ and $\operatorname{ri}(A(\operatorname{dom} f)) \cap \operatorname{ri}(\operatorname{dom} h) \neq \emptyset$;

and, respectively,

(RC_A^4) | f and h are $C-$lower semicontinuous and $\operatorname{epi}(v^*f)^* + (A^* \times \mathrm{id}_\mathbb{R})(\operatorname{epi}(v^*h)^*)$ is closed in the topology $w(X^*,X)\times\mathbb{R}$, for all $v^* \in K^{*0}$;

where $(A^*\times\mathrm{id}_\mathbb{R})(\operatorname{epi}(v^*h)^*) = \{(x^*,r) \in X^*\times\mathbb{R} : \exists y^* \in Y^* \text{ such that } A^*y^* = x^* \text{ and } (y^*,r) \in \operatorname{epi}(v^*h)^*\}$.

Theorem 16 *(a) There are no $x \in X$ and $(v^*,y^*,v) \in \mathcal{B}_q^A$ such that $f(x) + h(Ax) <_K h_q^A(v^*,y^*,v)$.*

(b) Assume that one of the regularity conditions (RCV_A^i), $i \in \{1,\dots,4\}$, is fulfilled. If $\overline{x} \in X$ is a quasi-efficient solution to (PV_q^A), then there exists $(\overline{v}^*,\overline{y}^*,\overline{v})$ a quasi-efficient solution to (DV_q^A) such that $f(\overline{x}) + h(A\overline{x}) = h_q^A(\overline{v}^*,\overline{y}^*,\overline{v}) = \overline{v}$.

(c) Assume that one of the regularity conditions (RCV_A^i), $i \in \{1,\dots,4\}$, is fulfilled and the set $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} h) + K$ is closed. Then for every quasi-efficient solution $(\overline{v}^*,\overline{y}^*,\overline{v})$ to (DVG_q^A) one has that $\overline{v}$ is a quasi-minimal element of the set $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} h) + K$.

Back to (PV_q^C), seeing it as an unconstrained vector optimization problem, we can attach to it a vector dual problem generated by (DVG_q) by considering the Fenchel type vector perturbation function

$$\Phi_{C_F}^V : X \times Y \to V^\bullet,\ \Phi_{C_F}^V(x,y) = \begin{cases} f(x+y), & x \in \mathcal{A}, \\ \infty_K, & \text{otherwise.} \end{cases}$$

Assume again that $f(\operatorname{dom} f \cap \mathcal{A}) + \operatorname{qi} K = \operatorname{qi}(f(\operatorname{dom} f \cap \mathcal{A}) + K)$ and K is a closed convex cone.

Let $v^* \in K^* \setminus \{0\}$, $t^* \in X^*$, $u \in S$ and $v \in V$. For having a new dual which is a special case of (DVG_q) for $\Phi^V_{C_F}$ we have that $\langle v^*, v\rangle \leq -(v^*\Phi^V_{C_F})^*(0, t^*)$. This can be equivalently rewritten as $\langle v^*, v\rangle \leq \langle t^*, u\rangle - (v^*f)^*(t^*)$. By using the definition of the support function this relation is equivalent with $\langle v^*, v\rangle \leq -(v^*f)^*(t^*) - \sigma_{\mathcal{A}}(-t^*)$. Thus from (DVG_q) we obtain the *Fenchel type vector dual problem to* (PV^C_q) with respect to quasi-efficient solutions

$$(DV^{C_F}_q) \qquad \underset{(v^*,t^*,v)\in\mathcal{B}^{C_F}_q}{\operatorname{QMax}} \; h^{C_F}_q(v^*, t^*, v),$$

where

$$\mathcal{B}^{C_F}_q = \{(v^*, t^*, v) \in (K^* \setminus \{0\}) \times X^* \times V : \; \langle v^*, v\rangle \leq -(v^*f)^*(t^*) - \sigma_{\mathcal{A}}(-t^*)\}$$

and

$$h^{C_F}_q(v^*, t^*, v) = v.$$

From Theorem 16 one can quickly obtain the weak, strong and converse duality statements for (PV^C_q) and $(DV^{C_F}_q)$, too.

5 Comparisons between duals

In this section we compare the image sets of some of the vector duals attached to (PV^C_q) via the Lagrange, Fenchel and Fenchel-Lagrange type vector perturbation functions, respectively.

Proposition 17 *One has that* $h^{C_{FL}}_q(\mathcal{B}^{C_{FL}}_q) \subseteq h^{C_L}_q(\mathcal{B}^{C_L}_q)$.

Proof. Let $(v^*, t^*, y^*, v) \in \mathcal{B}^{C_{FL}}_q$ be an arbitrary element. Using [10, Proposition 3.1.5] we obtain that $\langle v^*, v\rangle \leq -(v^*f)^*(t^*) - (y^*g)^*_S(-t^*) \leq \inf_{u\in S}\{(v^*f)(u) + ((y^*g) + \delta_S)(u)\} = \inf_{u\in S}\{(v^*f)(u) + (y^*g)(u)\}$ and consequently, $v = h^{C_{FL}}_q(v^*, y^*, v) \in h^{C_L}_q(\mathcal{B}^{C_L}_q)$. ■

Remark 9 *A situation when the inclusion in Proposition 17 is not fulfilled as equality can be found in [11, Example 2.2].*

Proposition 18 *One has that* $h^{C_{FL}}_q(\mathcal{B}^{C_{FL}}_q) \subseteq h^{C_F}_q(\mathcal{B}^{C_F}_q)$.

Proof. Let $(v^*, t^*, y^*, v) \in \mathcal{B}^{C_{FL}}_q$ be an arbitrary element. Using [10, Proposition 3.1.6] we obtain that $\langle v^*, v\rangle \leq -(v^*f)^*(t^*) - (y^*g)^*_S(-t^*) \leq -(v^*f)^*(t^*) - \sigma_{\mathcal{A}}(-t^*)$ and consequently, $v = h^{C_{FL}}_q(v^*, y^*, v) \in h^{C_F}_q(\mathcal{B}^{C_F}_q)$. ■

Remark 10 *A situation when the inclusion in Proposition 18 is not fulfilled as equality can be found in [11, Example 2.1].*

Under certain hypotheses, the image sets of the vector duals attached to (PV^C_q) in the precious section coincide.

Theorem 19 *If one of the following conditions* *(a) there exists* $x' \in \operatorname{dom} f \cap S \cap \operatorname{dom} g$ *such that* f *is continuous at* x'*;*

(b) for X *and* Z *Fréchet spaces,* S *closed and* g $C-$*epi closed one has* $0 \in \operatorname{sqri}((\operatorname{dom} g \cap S) - \operatorname{dom} f)$*;*

(c) if $\operatorname{lin}((\operatorname{dom} g \cap S) - \operatorname{dom} f) < +\infty$ *one has* $0 \in \operatorname{ri}((\operatorname{dom} g \cap S) - \operatorname{dom} f)$*;*

is fulfilled, then $h_q^{C_{FL}}(\mathcal{B}_q^{C_{FL}}) = h_q^{C_L}(\mathcal{B}_q^{C_L})$.

Proof. Knowing Proposition 17, we have to prove only that for $v \in h_q^{C_L}(\mathcal{B}_q^{C_L})$ we have that $v \in h_q^{C_{FL}}(\mathcal{B}_q^{C_{FL}})$. Let $v \in h_q^{C_L}(\mathcal{B}_q^{C_L})$, $v^* \in K^* \setminus \{0\}$ and $y^* \in C^*$ such that $(v^*, y^*, v) \in \mathcal{B}_q^{C_L}$. This is equivalent with $\langle v^*, v\rangle \leq \inf_{u \in S}\{(v^*f)(u) + (y^*g)(u)\} = \inf_{u\in S}\{(v^*f)(u) + ((y^*g)+\delta_S)(u)\}$. But $\operatorname{dom}((y^*g)+\delta_S) = S \cap \operatorname{dom} g$ and from [10, Theorem 3.2.6] follows that there exists $\overline{t}^* \in X^*$ fulfilling $\inf_{u\in S}\{(v^*f)(u)+((y^*g)+\delta_S)(u)\} = \sup_{t^*\in X^*}\{-(v^*f)^*(t^*) - (y^*g)_S^*(-t^*)\} = -(v^*f)^*(\overline{t}^*) - (y^*g)_S^*(-\overline{t}^*)$. Consequently, $(v^*, \overline{t}^*, y^*, v) \in \mathcal{B}_q^{C_{FL}}$ and $v \in h_q^{C_{FL}}(\mathcal{B}_q^{C_{FL}})$. ■

Theorem 20 *If one of the following conditions*

(a) there exists $x' \in \operatorname{dom} f \cap S \cap \operatorname{dom} g$ *such that* $g(x') \in -\operatorname{int} C$*;*

(b) for X *and* Z *Fréchet spaces,* S *closed and* g $C-$*epi closed one has* $0 \in \operatorname{sqri}(g(\operatorname{dom} g \cap S) + C)$*;*

(c) if $\operatorname{lin}(g(\operatorname{dom} g \cap S) + C) < +\infty$ *one has* $0 \in \operatorname{ri}(g(\operatorname{dom} g \cap S) + C)$*;*

is fulfilled, then $h_q^{C_{FL}}(\mathcal{B}_q^{C_{FL}}) = h_q^{C_F}(\mathcal{B}_q^{C_F})$.

Proof. From Proposition 18 we have to prove only that for $v \in h_q^{C_F}(\mathcal{B}_q^{C_F})$ we have that $v \in h_q^{C_{FL}}(\mathcal{B}_q^{C_{FL}})$. Let $v \in h_q^{C_F}(\mathcal{B}_q^{C_F})$, $v^* \in K^* \setminus \{0\}$ and $t^* \in X^*$ such that $(v^*, t^*, v) \in \mathcal{B}_q^{C_F}$. This is equivalent with $\langle v^*, v\rangle \leq -(v^*f)^*(t^*) - \sigma_{\mathcal{A}}(-t^*)$. But from [10, Theorem 3.2.9] we have that there exists $\overline{y}^* \in C^*$ such that $\sigma_{\mathcal{A}}(-t^*) = -\inf_{u\in\mathcal{A}}\langle t^*, u\rangle = -\sup_{y^*\in C^*}\inf_{u\in\mathcal{A}}\{\langle t^*, u\rangle + (y^*g)(u)\} = -\inf_{u\in S}\{\langle t^*, u\rangle + (y^*g)(u)\} = (\overline{y}^*g)_S^*(-t^*)$ and then we have $\langle v^*, v\rangle \leq -(v^*f)^*(t^*) - (\overline{y}^*g)_S^*(-t^*)$. Consequently, $(v^*, t^*, \overline{y}^*, v) \in \mathcal{B}_q^{C_{FL}}$ and $v \in h_q^{C_{FL}}(\mathcal{B}_q^{C_{FL}})$. ■

To guarantee the coincidence of the image sets of the vector duals with respect to quasi-efficient solutions we attached to (PV_q^C) one can combine the last two theorems, or, taking advantage of Proposition 17, Proposition 18 and Theorem 15, can formulate the following conclusion.

Corollary 21 *If one of the regularity conditions* $(RCV_{C_{FL}}^i)$, $i \in \{1, \ldots, 4\}$, *is fulfilled, then*

$$h_q^{C_{FL}}(\mathcal{B}_q^{C_{FL}}) = h_q^{C_F}(\mathcal{B}_q^{C_F}) = h_q^{C_L}(\mathcal{B}_q^{C_L}).$$

If additionally, $f(\operatorname{dom}(f \cap \mathcal{A})) + K$ *is closed, then one has*

$$\begin{aligned}\operatorname{QMin}(f(\operatorname{dom} f \cap \mathcal{A}), K) &\subseteq \operatorname{QMax}(h_q^{C_{FL}}(\mathcal{B}_q^{C_{FL}}), K) = \operatorname{QMax}(h_q^{C_L}(\mathcal{B}_q^{C_L}), K) \\ &= \operatorname{QMax}(h_q^{C_F}(\mathcal{B}_q^{C_F}), K) \subseteq \operatorname{QMin}(f(\operatorname{dom} f \cap \mathcal{A}) + K, K).\end{aligned}$$

6 Conclusions and further challenges

We have considered and characterized via linear scalarization the quasi-minimal elements of a set with respect to a convex cone with nonempty quasi interior and possibly empty interior. The notion of quasi-minimality is weaker than the classical minimality with respect to a cone and is actually a generalization of the weak minimality, which can be taken into consideration only when the interior of the ordering cone is nonempty. In order to characterize the quasi-minimal elements of a convex set U with respect to the convex cone K via a linear scalarization we assumed that the quasi interior of $U+K$ coincides with the Minkowski sum of U and the quasi interior of K. We could only show that the first mentioned set always contains the second one, but the mentioned equality is valid when the interior of K is nonempty and also for all the examples we checked, so even if it does not hold in general, we have still provided a generalization of the similar investigations concerning weakly minimal elements.

Then, we attached to a general vector optimization problem a vector dual problem with respect to quasi-efficient solutions, providing moreover the corresponding weak, strong and converse duality statements. Similar duality treatments were derived then for both general unconstrained and constrained vector optimization problems. To the latter we attached three different vector duals with respect to quasi-efficient solutions, by making use of three different perturbation functions and we have also compared their image sets, providing sufficient conditions for their coincidence.

For future research, we plan to investigate further whether the assumption we considered does hold in general or not. Moreover, we intend to extend other properties of weakly minimal elements to quasi-minimal elements and to investigate other possible similar extensions in this direction, for instance of other types of vector duality. Nevertheless, we intend to see how similar duality results can be obtained for other types of generalized minimality notions.

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