Characterizations of ε-duality gap statements for constrained optimization problems

Horaţiu-Vasile Boncea* Sorin-Mihai Grad†

Abstract. In this paper we present different regularity conditions that equivalently characterize various ε-duality gap statements (with $\varepsilon \geq 0$) for constrained optimization problems and their Lagrange and Fenchel-Lagrange duals in separated locally convex spaces, respectively. These regularity conditions are formulated by using epigraphs and ε-subdifferentials. When $\varepsilon = 0$ we rediscover recent results on stable strong and total duality and zero duality gap from the literature.

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1 Preliminaries

Motivated by recent results on stable strong and total duality for constrained convex optimization problems in [3, 6, 8, 9, 12, 13, 17] and the ones on zero duality gap in [15, 16] we introduce in this paper several regularity conditions which characterize ε-duality gap statements (with $\varepsilon \geq 0$) for a constrained optimization problem and its Lagrange and Fenchel-Lagrange dual problems, respectively. The regularity conditions we provide in Section 2 are based on epigraphs, while the ones in Section 3 on ε-subdifferentials. In this way we extend many of the results in the mentioned papers, which are recovered as special cases when $\varepsilon = 0$, delivering thus generalizations of the classical *Farkas-Minkowski* and *basic* constraint qualifications. Moreover some statements in [5, 8, 9, 15, 16], which arise from our results in the special case $\varepsilon = 0$, are extended by removing convexity and topological hypotheses, while various assertions from [15, 16] are improved by working in locally convex spaces instead of Banach spaces and removing the continuity and nonempty domain interior assumptions of the involved functions.

Consider two separated locally convex vector spaces X and Y and their continuous dual spaces X^* and Y^*, endowed with the weak* topologies $w(X^*, X)$ and $w(Y^*, Y)$ respectively. Some of the following notions and results, as well as the statements we prove within in this paper, can be given in the more general framework of linear spaces, but in order to avoid juggling with spaces we decided to consider only locally convex spaces. Let

*Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 1 M. Kogălniceanu Street, 400084 Cluj-Napoca, Romania, e-mail: horatiuboncea@yahoo.com.

†Department of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: grad@mathematik.tu-chemnitz.de.

the nonempty closed convex cone $C \subseteq Y$ and its dual cone $C^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \ \forall y \in C\}$ be given, where we denote by $\langle y^*, y\rangle = y^*(y)$ the value at y of the continuous linear functional y^*. On Y we consider the partial ordering induced by C, "$\leqq_C$", defined by $z \leqq_C y \Leftrightarrow y - z \in C$, $z, y \in Y$. To Y we attach a greatest element with respect to "$\leqq_C$" denoted by ∞_C which does not belong to Y and let $Y^\bullet = Y \cup \{\infty_C\}$. Then for any $y \in Y^\bullet$ one has $y \leqq_C \infty_C$ and we consider on $Y^\bullet$ the following operations: $y + \infty_C = \infty_C + y = \infty_C$ and $t\infty_C = \infty_C$ for all $y \in Y$ and all $t \geq 0$. Moreover, for $y^* \in C^*$ we set $\langle y^*, \infty_C\rangle = +\infty$.

Given a subset U of X, by $\operatorname{cl} U$ we denote its *closure* in the corresponding topology, while its *indicator* function $\delta_U : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and respectively *support* function $\sigma_U : X^* \to \overline{\mathbb{R}}$ are defined as follows

$$\delta_U(x) = \begin{cases} 0, & \text{if } x \in U, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{and } \sigma_U(x^*) = \sup_{x \in U} \langle x^*, x\rangle.$$

We define in the following a notion that extends the one of a closed set, needed for being able to provide generalized closedness type characterizations via epigraphs for ε-duality gap statements. Note that the notion of a *ε-closed set* was considered in the literature in different instances that have nothing in common with our research, see for instance [1,14], while in [18, Definition 3.2] one can find the definition of a *vertically closed set*.

Definition 1 Let $\varepsilon \geq 0$. A set $U \subseteq X \times \mathbb{R}$ is said to be *ε-vertically closed* if $\operatorname{cl} U \subseteq U - (0, \varepsilon)$.

For a function $f : X \to \overline{\mathbb{R}}$ we have its *domain* and *epigraph* defined by $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ and $\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$, respectively. We say that f is *proper* if $f(x) > -\infty$ for all $x \in X$ and $\operatorname{dom} f \neq \emptyset$. The *conjugate of f regarding the set* $U \subseteq X$ is $f_U^* : X^* \to \overline{\mathbb{R}}$, given by $f_U^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in U\}$.

When $U = X$ the conjugate regarding the set U is the classical *(Fenchel-Moreau) conjugate* function of f denoted by f^*. One can easily notice that $\delta_U^* = \sigma_U$. Between a function $f : X \to \overline{\mathbb{R}}$ and its conjugate regarding some set $U \subseteq X$, the *Young-Fenchel's inequality* holds, namely

$$f_U^*(x^*) + f(x) \geq \langle x^*, x\rangle \ \forall x \in U \ \forall x^* \in X^*.$$

Let $f : X \to \overline{\mathbb{R}}$, $x \in X$ with $f(x) \in \mathbb{R}$ and $\varepsilon \geq 0$. The set $\partial_\varepsilon f(x) = \{x^* \in X^* : f(u) - f(x) + \varepsilon \geq \langle x^*, u - x\rangle \ \forall u \in X\}$ is called the *ε-subdifferential* of f at x. When $f(x) \notin \mathbb{R}$ or $\varepsilon < 0$ we take by convention $\partial_\varepsilon f(x) = \emptyset$.

Given a proper function $f : X \to \overline{\mathbb{R}}$, for all $\varepsilon \geq 0$, $x \in X$ and $x^* \in X^*$ one has

$$x^* \in \partial_\varepsilon f(x) \Leftrightarrow f^*(x^*) + f(x) \leq \langle x^*, x\rangle + \varepsilon.$$

For $\varepsilon = 0$, the ε-subdifferential turns out to be the classical (convex) subdifferential. For $U = X$, the set $N_U^\varepsilon(x) = \partial_\varepsilon \delta_U(x)$ is called the *ε-normal set* of U at $x \in X$. When $\varepsilon = 0$, $N_U(x)$ is actually the *(convex) normal cone* of U at x.

Given two proper functions $f, h : X \to \overline{\mathbb{R}}$, their *infimal convolution* is

$$f \square h : X \to \overline{\mathbb{R}}, f \square h(a) = \inf\{f(x) + h(a - x) : x \in X\},$$

and it is called *exact* at some $a \in X$ when there is an $x \in X$ such that $f \square h(a) = f(x) + h(a - x)$.

There are notions given for functions with extended real values that can be formulated also for vector functions as follows.

For a function $g : X \to Y^\bullet$ one has its *domain* $\operatorname{dom} g = \{x \in X : g(x) \in Y\}$. We say that g is *proper* if $\operatorname{dom} g \neq \emptyset$ and C-*convex* if $g(tx + (1-t)y) \leqq_C tg(x) + (1-t)g(y)$ for all $x, y \in X$ and all $t \in [0,1]$. For $\lambda \in C^*$, we define $(\lambda g) : X \to \overline{\mathbb{R}}$, $(\lambda g)(x) = \langle \lambda, g(x) \rangle$ for all $x \in X$. The C-*epigraph* of g is defined by $\operatorname{epi}_C g = \{(x,y) \in X \times Y : y \in g(x) + C\}$. We say that g is C-*epi-closed* if $\operatorname{epi}_C g$ is closed. If (λg) is lower semicontinuous for all $\lambda \in C^*$ we say that g is positively C-*lower semicontinuous* (also known as *star* C-*lower semicontinuous* in the literature).

Let U be a nonempty subset of X and $g : X \to Y^\bullet$ a proper vector function. Denote $\mathcal{A} = \{x \in U : g(x) \in -C\}$ and assume this set non-empty. For a proper function $f : X \to \overline{\mathbb{R}}$ fulfilling $\mathcal{A} \cap \operatorname{dom} f \neq \emptyset$ consider the optimization problem

$$(P) \qquad \inf_{x \in \mathcal{A}} f(x).$$

We denote by $v(P)$ the optimal objective value of the optimization problem (P). In the following we will write min(max) instead of inf(sup) when the corresponding infimum (supremum) is attained. Let us recall some results, needed later.

Lemma 1 *(cf. [2]) Let $f, h : X \longrightarrow \overline{\mathbb{R}}$ be proper convex lower semicontinuous functions, with the intersection of their domains nonempty. Then*

$$\operatorname{epi}(f+h)^* = \operatorname{cl} \operatorname{epi}(f^* \square h^*) = \operatorname{cl}(\operatorname{epi} f^* + \operatorname{epi} h^*).$$

Lemma 2 *(cf. [8, 9]) If U is closed convex and g C-convex and C-epi-closed, one has*

$$\operatorname{epi} \sigma_{\mathcal{A}} = \operatorname{cl}\big(\operatorname{epi} \sigma_U + \bigcup_{\lambda \in C^*} \operatorname{epi}(\lambda g)^*\big) = \operatorname{cl} \bigcup_{\lambda \in C^*} \operatorname{epi}(\lambda g)^*_U.$$

The hypotheses of Lemma 2 are sufficient to guarantee that $\mathcal{A}$ is a closed convex set. When f and $\delta_{\mathcal{A}}$ are proper convex lower semicontinuous functions, one has by Lemma 1 $\operatorname{epi}(f + \delta_{\mathcal{A}})^* = \operatorname{cl}(\operatorname{epi} f^* + \operatorname{epi} \sigma_{\mathcal{A}})$. By Lemma 2, this is further equal to $\operatorname{cl}(\operatorname{epi} f^* + \operatorname{cl} \cup_{\lambda \in C^*} \operatorname{epi}(\lambda g)^*_U)$, which is actually $\operatorname{cl}(\operatorname{epi} f^* + \cup_{\lambda \in C^*} \operatorname{epi}(\lambda g)^*_U)$. Analogously, one can show that under these hypotheses there holds $\operatorname{epi}(f + \delta_{\mathcal{A}})^* = \operatorname{cl}(\operatorname{epi} f^* + \cup_{\lambda \in C^*} \operatorname{epi}(\lambda g)^* + \operatorname{epi} \sigma_U)$.

For $x^* \in X^*$ we also consider the linearly perturbed optimization problem

$$(P_{x^*}) \qquad \inf_{x \in \mathcal{A}} [f(x) + \langle x^*, x \rangle].$$

To (P_{x^*}) one can attach the Lagrange dual problem

$$(D^L_{x^*}) \qquad \sup_{\lambda \in C^*} \inf_{x \in U} [f(x) + \langle x^*, x \rangle + (\lambda g)(x)],$$

which can be equivalently written as

$$(D^L_{x^*}) \qquad \sup_{\lambda \in C^*} -\big(f + (\lambda g)\big)^*_U(-x^*).$$

For a $\lambda \in C^*$, the inner minimization problem that appears in the first formulation of $(D^L_{x^*})$ can be rewritten as

$$\inf_{x \in X} [f(x) + \langle x^*, x \rangle + \delta_U(x) + (\lambda g)(x)].$$

To this problem one can attach different Fenchel type dual problems, obtaining via $(D^L_{x^*})$ some Fenchel-Lagrange type dual problems to (P_{x^*}). The name Fenchel-Lagrange is given to the folowing dual problems because they are thus "combinations" of the classical Fenchel and Lagrange dual problems. Keeping together δ_U and (λg) one gets the following Fenchel-Lagrange type dual problem to (P_{x^*})

$$(\overline{D}_{x^*}) \qquad \sup_{\substack{\lambda \in C^*, \\ \beta \in X^*}} \{-f^*(\beta) - (\lambda g)^*_U(-x^* - \beta)\}.$$

When f, (λg) and δ_U are separated, the following Fenchel-Lagrange type dual problem to (P_{x^*}) is obtained

$$(D_{x^*}) \qquad \sup_{\substack{\lambda \in C^*, \\ \beta, \alpha \in X^*}} \{-f^*(\beta) - (\lambda g)^*(\alpha) - \sigma_U(-x^* - \alpha - \beta)\}.$$

When $x^* = 0$ these duals to (P) are denoted simply by (D^L), $(\overline{D})$ and (D), respectively. Note that when f and δ_U are put together, one can obtain a third Fenchel-Lagrange dual to (P_{x^*}), namely

$$(\widetilde{D}_{x^*}) \qquad \sup_{\substack{\lambda \in C^*, \\ \beta \in X^*}} \{-f^*_U(\beta) - (\lambda g)^*(-x^* - \beta)\}.$$

We will not use it further, but the results given in this paper can be easily adapted for it, too.

Between (P) and (D) one always has *weak duality*, i.e. $v(P) \geq v(D)$. When $v(P) = v(D)$ we say that there is *zero duality gap* between (P) and (D) and if (D) has moreover an optimal solution, the situation is called *strong duality.* If $v(P) - v(D) \leq \varepsilon$, with $\varepsilon \geq 0$, we have ε*-duality gap* for (P) and (D). If one of these situations holds for (P_{x^*}) and (D_{x^*}) for all $x^* \in X^*$, it will be called *stable.*

2 ε-duality gap statements involving epigraphs for Lagrange and Fenchel-Lagrange duality

Motivated by the characterizations of the stable strong duality from [8, 9] we begin this section with several equivalent representations of different instances of ε-duality gap for (P) and its duals by means of epigraphs.

Theorem 3 *Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $\mathcal{A} \cap \operatorname{dom} f \neq \emptyset$ and $\varepsilon \geq 0$. Then the condition*

$$(RCE) \qquad \operatorname{epi}(f + \delta_{\mathcal{A}})^* \subseteq \bigcup_{\lambda \in C^*} (\operatorname{epi} f^* + \operatorname{epi}(\lambda g)^* + \operatorname{epi} \sigma_U) - (0, \varepsilon)$$

holds if and only if for all $x^* \in X^*$ *there exist* $\overline{\lambda} \in C^*$ *and* $\overline{\alpha}, \overline{\beta} \in X^*$ *such that*

$$v(P_{x^*}) \leq -f^*(\overline{\beta}) - (\overline{\lambda}g)^*(\overline{\alpha}) - \sigma_U(-x^* - \overline{\beta} - \overline{\alpha}) + \varepsilon. \tag{1}$$

Proof. "$\Rightarrow$" Let $x^* \in X^*$. We have $v(P_{x^*}) = -\left(f + \delta_{\mathcal{A}}\right)^* (-x^*)$, thus $(-x^*, -v(P_{x^*})) \in \operatorname{epi}\left(f + \delta_{\mathcal{A}}\right)^*$. From (RCE) there exist $\overline{\lambda} \in C^*$ and $\overline{\alpha}, \overline{\beta} \in X^*$ such that $(\overline{\beta}, f^*(\overline{\beta})) \in \operatorname{epi} f^*$, $(\overline{\alpha}, (\overline{\lambda}g)^*(\overline{\alpha})) \in \operatorname{epi}(\overline{\lambda}g)^*$, $(-x^* - \overline{\alpha} - \overline{\beta}, \sigma_U(-x^* - \overline{\alpha} - \overline{\beta})) \in \operatorname{epi} \sigma_U$ fulfilling

$$f^*(\overline{\beta}) + (\overline{\lambda}g)^*(\overline{\alpha}) + \sigma_U(-x^* - \overline{\alpha} - \overline{\beta}) - \varepsilon \leq -v(P_{x^*}),$$

which yields (1).

"$\Leftarrow$" Take now $(-x^*, u) \in \operatorname{epi}(f + \delta_{\mathcal{A}})^*$. This is equivalent to $(f + \delta_{\mathcal{A}})^*(-x^*) \leq u$ and further to $-u \leq v(P_{x^*})$. Because of (1), there are some $\overline{\lambda} \in C^*$ and $\overline{\alpha}, \overline{\beta} \in X^*$ such that

$$-u \leq -f^*(\overline{\beta}) - (\overline{\lambda}g)^*(\overline{\alpha}) - \sigma_U(-x^* - \overline{\alpha} - \overline{\beta}) + \varepsilon,$$

which is equivalent to

$$f^*(\overline{\beta}) + (\overline{\lambda}g)^*(\overline{\alpha}) + \sigma_U(-x^* - \overline{\alpha} - \overline{\beta}) \leq u + \varepsilon.$$

But

$$(\overline{\beta}, f^*(\overline{\beta})) + (\overline{\alpha}, (\overline{\lambda}g)^*(\overline{\alpha})) + (-x^* - \overline{\alpha} - \overline{\beta}, \sigma_U(-x^* - \overline{\alpha} - \overline{\beta})) =$$
$$(-x^*, f^*(\overline{\beta}) + (\overline{\lambda}g)^*(\overline{\alpha}) + \sigma_U(-x^* - \overline{\alpha} - \overline{\beta})).$$

So,

$$(-x^*, u) \in \operatorname{epi} f^* + \operatorname{epi}(\overline{\lambda}g)^* + \operatorname{epi} \sigma_U - (0, \varepsilon).$$

As $(-x^*, u) \in \operatorname{epi}\left(f + \delta_{\mathcal{A}}\right)^*$ was arbitrarily chosen, the validity of (RCE) follows. ■

Analogously, one can prove the following similar statement concerning the dual $(\overline{D}_{x^*})$.

Corollary 4 *Let* $f : X \to \overline{\mathbb{R}}$ *be proper and fulfilling* $\mathcal{A} \cap \operatorname{dom} f \neq \emptyset$ *and* $\varepsilon \geq 0$. *The condition*

$(\overline{RCE})$
$$\operatorname{epi}\left(f + \delta_{\mathcal{A}}\right)^* \subseteq \bigcup_{\lambda \in C^*} (\operatorname{epi} f^* + \operatorname{epi}(\lambda g)^*_U) - (0, \varepsilon)$$

holds if and only if for all $x^* \in X^*$ *there exist* $\overline{\lambda} \in C^*$ *and* $\overline{\beta} \in X^*$ *such that*

$$v(P_{x^*}) \leq -f^*(\overline{\beta}) - (\overline{\lambda}g)^*_U(-x^* - \overline{\beta}) + \varepsilon.$$

Moreover, for the Lagrange dual one has the following characterization.

Corollary 5 *Let* $f : X \to \overline{\mathbb{R}}$ *be proper and fulfilling* $\mathcal{A} \cap \operatorname{dom} f \neq \emptyset$ *and* $\varepsilon \geq 0$. *The condition*

(RCE^L)
$$\operatorname{epi}\left(f + \delta_{\mathcal{A}}\right)^* \subseteq \bigcup_{\lambda \in C^*} \operatorname{epi}(f + (\lambda g))^*_U - (0, \varepsilon)$$

holds if and only if for all $x^* \in X^*$ *there exists* $\overline{\lambda} \in C^*$ *such that*

$$v(P_{x^*}) \leq -\left(f + (\overline{\lambda}g)\right)^*_U (-x^*) + \varepsilon.$$

Remark 1 The quantity in the right-hand side of (1) is not necessarily $v(D_{x^*})+\varepsilon$, as the suprema in (D_{x^*}) are not shown to be attained at $\overline{\lambda}, \overline{\alpha}$ and $\overline{\beta}$, respectively. Though, (1) implies $v(P_{x^*}) \leq v(D_{x^*}) + \varepsilon$ and $(\overline{\lambda}, \overline{\alpha}, \overline{\beta})$ is an ε-optimal solution to (D_{x^*}). This applies to Corollary 4 and Corollary 5, with the corresponding modifications.

If we take $f(x) = 0$ for all $x \in X$, (RCE) becomes

$$(RCE_0) \qquad \operatorname{epi} \sigma_{\mathcal{A}} \subseteq \bigcup_{\lambda \in C^*} \operatorname{epi}(\lambda g)^* + \operatorname{epi}(\sigma_U) - (0, \varepsilon).$$

From Theorem 3 we obtain the following result.

Corollary 6 *The condition (RCE_0) holds if and only if for each $x^* \in X^*$ there exist $\overline{\lambda} \in C^*$ and $\overline{\alpha} \in X^*$ such that*

$$-\sigma_{\mathcal{A}}(x^*) \leq -\sigma_U(-x^* - \overline{\alpha}) - (\overline{\lambda} g)^*(\overline{\alpha}) + \varepsilon.$$

Using it, one can show the following statement.

Corollary 7 *The condition (RCE_0) holds if and only if when the constraint set $\mathcal{A}$ is closed convex for each proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ which satisfies $\mathcal{A} \cap \operatorname{dom} f \neq \emptyset$ and*

$$(CC) \qquad \operatorname{epi} f^* + \operatorname{epi} \sigma_{\mathcal{A}} \text{ is closed,}$$

for each $x^* \in X^*$ there exist $\overline{\lambda} \in C^*$ and $\overline{\alpha}, \overline{\beta} \in X^*$ such that

$$v(P_{x^*}) \leq -f^*(\overline{\beta}) - (\overline{\lambda} g)^*(\overline{\alpha}) - \sigma_U(-x^* - \overline{\alpha} - \overline{\beta}) + \varepsilon.$$

Proof. The sufficiency follows from the previous corollary by taking f linear and continuous.

To prove the necessity, note that since $-v(P_{x^*}) = (f + \delta_{\mathcal{A}})^*(-x^*)$, we have $(-x^*, -v(P_{x^*})) \in \operatorname{epi}(f + \delta_{\mathcal{A}})^*$. From Lemma 1 and (CC) there is a $\overline{\beta} \in X^*$ such that $(\overline{\beta}, f^*(\overline{\beta})) \in \operatorname{epi} f^*$, $(-x^* - \overline{\beta}, \sigma_U(-x^* - \overline{\beta})) \in \operatorname{epi} \sigma_{\mathcal{A}}$ fulfilling $f^*(\overline{\beta}) + \sigma_{\mathcal{A}}(-x^* - \overline{\beta}) = -v(P_{x^*})$. By Corrolary 6, relation (RCE_0) yields that there are some $\overline{\lambda} \in C^*$ and $\overline{\alpha} \in X^*$ such that $-\sigma_{\mathcal{A}}(-x^* - \overline{\beta}) \leq -\sigma_U(-x^* - \overline{\alpha}) - (\overline{\lambda} g)^*(\overline{\alpha} - \overline{\beta}) + \varepsilon$. Consequently, $v(P_{x^*}) \leq -f^*(\overline{\beta}) - \sigma_U(-x^* - \overline{\alpha}) - (\overline{\lambda} g)^*(\overline{\alpha} - \overline{\beta}) + \varepsilon$. ■

Remark 2 When $f(x) = 0$ for all $x \in X$ both $(\overline{RCE})$ and (RCE^L) collapse into the same condition, for which statements similar to Corollary 6 and Corollary 7 can be analogously proven.

Adding convexity and topological hypotheses to the functions and sets considered in Theorem 3, one obtains the following statement. Analogously, one can derive similar statements from its consequences and analogous assertions for the other duals presented above.

Theorem 8 *Let $f : X \to \overline{\mathbb{R}}$ be a proper convex lower semicontinuous function, $g : X \to Y^\bullet$ a proper C-convex and C-epi-closed vector function and $U \subseteq X$ a closed convex set fulfilling $\mathcal{A} \cap \operatorname{dom} f \neq \emptyset$, and $\varepsilon \geq 0$. Then the set $\cup_{\lambda \in C^*}(\operatorname{epi} f^* + \operatorname{epi}(\lambda g)^* + \operatorname{epi} \sigma_U)$ is ε-vertically closed if and only if for all $x^* \in X^*$ there exist $\overline{\lambda} \in C^*$ and $\overline{\alpha}, \overline{\beta} \in X^*$ such that* (1) *holds.*

Remark 3 If $\varepsilon = 0$, Theorem 8 turns out to be [8, Theorem 1]. Adding the necessary topological and convexity hypotheses to Corollary 4, one rediscovers in case $\varepsilon = 0$ [8, Theorem 2], while after the same treatment Corollary 5 collapses into [9, Theorem 1]. Moreover, Corollary 7 rediscovers and extends towards stable strong duality [8, Theorem 4] and analogously one can rediscover and sometimes generalize the similar statements from [8, 9] for the other considered duals to (P).

Inspired by [15,16] we give other regularity conditions which characterize ε-duality gap statements for (P) and its duals. For this, let us define the functions $h^\diamond, h^\diamond_U : X^* \to \overline{\mathbb{R}}$ by $h^\diamond(x^*) = \inf_{\lambda \in C^*}(\lambda g)^*(x^*)$ and $h^\diamond_U(x^*) = \inf_{\lambda \in C^*}(\lambda g)^*_U(x^*)$, for $x^* \in X^*$. From the definitions it follows that $\cup_{\lambda \in C^*} \operatorname{epi}(\lambda g)^* \subseteq \operatorname{epi} h^\diamond$, respectively $\cup_{\lambda \in C^*} \operatorname{epi}(\lambda g)^*_U \subseteq \operatorname{epi} h^\diamond_U$.

Lemma 9 *If g is C-convex and U is convex, the function $h^\diamond_U$ is proper convex. Moreover, if g is also C-epi closed and U additionally closed, it holds* $\operatorname{epi} \sigma_{\mathcal{A}} = \operatorname{cl} \operatorname{epi} h^\diamond_U$.

Proof. The properness and convexity of $h^\diamond_U$ follow analogously to the properties corresponding of $h^\diamond$ shown in [15, Theorem 3.1]. As $\cup_{\lambda \in C^*} \operatorname{epi}(\lambda g)^*_U \subseteq \operatorname{epi} h^\diamond_U$, by Lemma 2 yields $\operatorname{epi} \sigma_{\mathcal{A}} \subseteq \operatorname{cl} \operatorname{epi} h^\diamond_U$.

On the other hand, for any $\lambda \in C^*$ we have $\delta_{\mathcal{A}}(x) \geq (\lambda g)(x) + \delta_U(x)$ for all $x \in X$, which implies $\sigma_{\mathcal{A}}(x^*) \leq (\lambda g)^*_U(x^*)$ for all $x^* \in X^*$, followed by $\sigma_{\mathcal{A}}(x^*) \leq \inf_{\lambda \in C^*}(\lambda g)^*_U(x^*) = h^\diamond_U(x^*)$. This implies that $\operatorname{epi} \sigma_{\mathcal{A}} \supseteq \operatorname{cl} \operatorname{epi} h^\diamond_U$. ■

Using Lemma 9, for f proper convex lower semicontinuous, g moreover C-convex and C-epi closed and U also closed convex, we get

$$\operatorname{epi}(f + \delta_{\mathcal{A}})^* = \operatorname{cl}(\operatorname{epi} f^* + \operatorname{epi} h^\diamond_U) = \operatorname{cl} \operatorname{epi}(f^* \square h^\diamond_U) = \operatorname{cl} \operatorname{epi}(f^* \square h^\diamond \square \sigma_U). \tag{2}$$

Now let us give some other ε-duality gap characterizations by means of $h^\diamond$ and $h^\diamond_U$.

Theorem 10 *Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $\mathcal{A} \cap \operatorname{dom} f \neq \emptyset$ and $\varepsilon \geq 0$. The condition*

$$(RCI) \qquad \operatorname{epi}(f + \delta_{\mathcal{A}})^* \subseteq \operatorname{epi}(f^* \square h^\diamond \square \sigma_U) - (0, \varepsilon)$$

holds if and only if there is stable $\varepsilon-$duality gap for the problems (P) and (D), i.e. one has $\varepsilon-$duality gap for the pair of problems (P_{x^*}) and (D_{x^*}) for all $x^* \in X^*$.

Proof. "$\Rightarrow$" Let $x^* \in X^*$. We have $v(P_{x^*}) = -\left(f + \delta_{\mathcal{A}}\right)^*(-x^*)$, thus $(-x^*, -v(P_{x^*})) \in \operatorname{epi}\left(f + \delta_{\mathcal{A}}\right)^*$. From (RCI) we have

$$(f^* \square h^\diamond \square \sigma_U)(-x^*) = \inf_{\alpha, \beta \in X^*} \{f^*(\beta) + h^\diamond(\alpha) + \sigma_U(-x^* - \alpha - \beta)\} - \varepsilon \leq -v(P_{x^*}) \Longleftrightarrow$$

$$\inf_{\alpha, \beta \in X^*} \{\inf_{\lambda \in C^*}(f^*(\beta) + (\lambda g)^*(\alpha) + \sigma_U(-x^* - \alpha - \beta))\} - \varepsilon \leq -v(P_{x^*}) \Longleftrightarrow$$

$$-\sup_{\substack{\lambda \in C^*,\\ \alpha, \beta \in X^*}} \{-f^*(\beta) - (\lambda g)^*(\alpha) - \sigma_U(-x^* - \alpha - \beta)\} \leq -v(P_{x^*}) + \varepsilon \Longleftrightarrow$$

$$-v(D_{x^*}) \leq -v(P_{x^*}) + \varepsilon \Longleftrightarrow v(P_{x^*}) \leq v(D_{x^*}) + \varepsilon.$$

So, we get ε-duality gap for the pair of problems (P_{x^*}) and (D_{x^*}).

"$\Leftarrow$" Let $x^* \in X^*$. The $\varepsilon-$duality gap inequality for (P_{x^*}) and (D_{x^*}) is equivalent to $(f^* \square h^\diamond \square\, \sigma_U)(-x^*) - \varepsilon \leq -v(P_{x^*}) = \left(f + \delta_{\mathcal{A}}\right)^*(-x^*)$. As $x^* \in X^*$ was arbitrarily taken one gets $\operatorname{epi}(f + \delta_{\mathcal{A}})^* \subseteq \operatorname{epi}(f^* \square h^\diamond \square \sigma_U) - (0, \varepsilon)$. ■

One can give a similar statement for $(\overline{D})$.

Corollary 11 *Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $\mathcal{A} \cap \operatorname{dom} f \neq \emptyset$ and $\varepsilon \geq 0$. The condition*

$$(\overline{RCI}) \qquad \operatorname{epi}(f + \delta_{\mathcal{A}})^* \subseteq \operatorname{epi}(f^* \Box h_U^{\diamond}) - (0, \varepsilon)$$

holds if and only if there is stable $\varepsilon-$duality gap for the problems (P) and $(\overline{D})$, i.e. one has $\varepsilon-$duality gap for the pair of problems (P_{x^*}) and $(\overline{D}_{x^*})$ for all $x^* \in X^*$.

Remark 4 If X and Y are Fréchet spaces, f is proper convex lower semicontinuous, g is C-convex and positively C-lower semicontinuous, U closed convex and $0 \in \operatorname{sqri}(\operatorname{dom} f - \operatorname{dom} g \cap U)$, where by sqri one denotes the *strong quasi relative interior* of the corresponding set, then the condition $(\overline{RCI})$ is satisfied if and only if there is stable ε-duality gap for the problems (P) and (D^L).

A characterization of the Lagrange $\varepsilon-$duality gap without the additional hypotheses from Remark 4 follows.

Corollary 12 *Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $\mathcal{A} \cap \operatorname{dom} f \neq \emptyset$ and $\varepsilon \geq 0$. The condition*

$$(RCI^L) \qquad \operatorname{epi}(f + \delta_{\mathcal{A}})^* \subseteq \operatorname{epi} \inf_{\lambda \in C^*} (f + \lambda g)_U^* - (0, \varepsilon)$$

holds if and only if there is stable $\varepsilon-$duality gap for the problems (P) and (D^L), i.e. one has ε-duality gap for the pair of problems (P_{x^*}) and $(D^L_{x^*})$ for all $x^* \in X^*$.

If we take $f(x) = 0$ for all $x \in X$, (RCI) becomes

$$(RCI_0) \qquad \operatorname{epi}(\sigma_{\mathcal{A}}) \subseteq \operatorname{epi}(h^{\diamond} \Box \sigma_U) - (0, \varepsilon)$$

and we obtain the following results.

Corollary 13 *The condition (RCI_0) holds if and only if for each $x^* \in X^*$*

$$-\sigma_{\mathcal{A}}(x^*) \leq \sup_{\alpha \in C^*} \{-\sigma_U(-x^* - \alpha) - h^{\diamond}(\alpha)\} + \varepsilon.$$

Corollary 14 *The condition (RCI_0) holds if and only if when $\mathcal{A}$ is closed convex for each proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ which satisfies $\mathcal{A} \cap \operatorname{dom} f \neq \emptyset$ and*

$$f^* \Box \delta_{\mathcal{A}}^* \textit{ is a lower semicontinuous function,}$$

for all $x^* \in X^*$ one has

$$v(P_{x^*}) \leq \sup_{\substack{\lambda \in C^*, \\ \alpha, \beta \in X^*}} \{-f^*(\beta) - (\lambda g)^*(\alpha) - \sigma_U(-x^* - \alpha - \beta)\} + \varepsilon.$$

Adding convexity and topological hypotheses to the functions and sets considered in Theorem 10, one obtains the following statement. Analogously, one can derive similar statements from its consequences and analogous assertions for the other duals presented above.

Theorem 15 *Let $f : X \to \overline{\mathbb{R}}$ be a proper convex lower semicontinuous function, $g : X \to Y^\bullet$ a proper C-convex and C-epi-closed vector function and $U \subseteq X$ a closed convex set fulfilling $\mathcal{A} \cap \operatorname{dom} f \neq \emptyset$, and $\varepsilon \geq 0$. Then the set $\operatorname{epi}(f^* \Box h^\diamond \Box \sigma_U)$ is ε-vertically closed if and only if there is stable ε-duality gap for the problems (P) and (D).*

Remark 5 In case $\varepsilon = 0$ and $U = X$, Theorem 15 and the similar statement obtained when adding convexity and topological hypotheses to Corollary 11 extend the Fenchel-Lagrange analogous of [16, Theorem 3.1], while Remark 4 and Corollary 12 (with the corresponding convexity and topological hypotheses) show that [16, Theorem 3.1] actually holds in a more general framework, i.e. without taking the function g continuous. Analogously, Corollary 14 extends and improves, respectively, by dropping the strong nonempty domain interior and continuity hypotheses on the involved functions, statements like [15, Theorem 4.1] and [16, Corollary 3.1].

Now, let us give other stable ε-duality gap statements for (P) and its duals, by making use of other regularity conditions inspired by (2).

Theorem 16 *Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $\mathcal{A} \cap \operatorname{dom} f \neq \emptyset$ and $\varepsilon \geq 0$. The condition*

$$(RCP) \qquad \operatorname{epi}(f + \delta_{\mathcal{A}})^* \subseteq \operatorname{epi} f^* + \operatorname{epi} h^\diamond + \operatorname{epi} \sigma_U - (0, \varepsilon)$$

holds if and only if for all $x^* \in X^*$ there exist $\overline{\alpha}$, $\overline{\beta} \in X^*$ such that

$$v(P_{x^*}) \leq \sup_{\lambda \in C^*} \left\{ -f^*(\overline{\beta}) - (\lambda g)^*(\overline{\alpha}) - \sigma_U(-x^* - \overline{\beta} - \overline{\alpha}) \right\} + \varepsilon. \tag{3}$$

Proof. Let $x^* \in X^*$. We have $v(P_{x^*}) = -(f + \delta_{\mathcal{A}})^*(-x^*)$, thus $(-x^*, -v(P_{x^*})) \in \operatorname{epi}(f + \delta_{\mathcal{A}})^*$. From (RCP) there exist $\overline{\alpha}, \overline{\beta} \in X^*$ such that $(\overline{\beta}, f^*(\overline{\beta})) \in \operatorname{epi} f^*$, $(\overline{\alpha}, h^\diamond(\overline{\alpha})) \in \operatorname{epi} h^\diamond$, $(-x^* - \overline{\alpha} - \overline{\beta}, \sigma_U(-x^* - \overline{\alpha} - \overline{\beta})) \in \operatorname{epi} \sigma_U$ fulfilling

$$f^*(\overline{\beta}) + h^\diamond(\overline{\alpha}) + \sigma_U(-x^* - \overline{\alpha} - \overline{\beta}) - \varepsilon \leq -v(P_{x^*}) \tag{4}$$

and we obtain

$$v(P_{x^*}) \leq -f^*(\overline{\beta}) - h^\diamond(\overline{\alpha}) - \sigma_U(-x^* - \overline{\beta} - \overline{\alpha}) + \varepsilon \iff$$

$$v(P_{x^*}) \leq -f^*(\overline{\beta}) + \sup_{\lambda \in C^*} \{-(\lambda g)^*(\overline{\alpha})\} - \sigma_U(-x^* - \overline{\beta} - \overline{\alpha}) + \varepsilon \iff$$

$$v(P_{x^*}) \leq \sup_{\lambda \in C^*} \left\{ -f^*(\overline{\beta}) - (\lambda g)^*(\overline{\alpha}) - \sigma_U(-x^* - \overline{\beta} - \overline{\alpha}) \right\} + \varepsilon.$$

Viceversa, as the hypothesis means that (4) holds for all $x^* \in X^*$, the validity of (RCP) follows. ■

A similar statement can be given for $(\overline{D})$, too.

Corollary 17 *Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $\mathcal{A} \cap \operatorname{dom} f \neq \emptyset$ and $\varepsilon \geq 0$. The condition*

$$(\overline{RCP}) \qquad \operatorname{epi}(f + \delta_{\mathcal{A}})^* \subseteq \operatorname{epi} f^* + \operatorname{epi} h_U^{\diamond} - (0, \varepsilon)$$

holds if and only if for all $x^* \in X^*$ there exists $\overline{\beta} \in X^*$ such that

$$v(P_{x^*}) \leq \sup_{\lambda \in C^*} \left\{ -f^*(\overline{\beta}) - (\lambda g)_U^*(-x^* - \overline{\beta}) \right\} + \varepsilon.$$

Note that the analogous result for (D^L) collapses into Corollary 12 and when $\varepsilon = 0$ it and both Theorem 16 and Corollary 17 become "pure" stable zero duality gap statements.

Adding convexity and topological hypotheses to the functions and sets considered in Theorem 16, one obtains the following statement. Analogously, one can derive similar statements from its consequences and analogous assertions for the other duals.

Theorem 18 *Let $f : X \to \overline{\mathbb{R}}$ be a proper convex lower semicontinuous function, $g : X \to Y^\bullet$ a proper C-convex and C-epi-closed vector function and $U \subseteq X$ a closed convex set fulfilling $\mathcal{A} \cap \operatorname{dom} f \neq \emptyset$, and $\varepsilon \geq 0$. Then the set $\operatorname{epi} f^* + \operatorname{epi} h^{\diamond} + \operatorname{epi} \sigma_U$ is ε-vertically closed if and only if for all $x^* \in X^*$ there exist $\overline{\alpha}$, $\overline{\beta} \in X^*$ such that* (3) *holds.*

3 ε-duality gap statements involving $\varepsilon-$subdifferentials for Lagrange and Fenchel-Lagrange duality

We introduce regularity conditions to characterize ε-duality gap statements, using ε-subdifferentials, too, when the existence of an ε-optimal solution to the primal problem is assumed. Recall that, for $x^* \in X^*$, $\overline{x} \in \mathcal{A} \cap \operatorname{dom} f$ is an *ε-optimal solution* to (P_{x^*}) if and only if $0 \in \partial_\varepsilon(f + x^* + \delta_{\mathcal{A}})(\overline{x})$, which is equivalent to $-x^* \in \partial_\varepsilon(f + \delta_{\mathcal{A}})(\overline{x})$.

Theorem 19 *Let the proper function $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in \mathcal{A} \cap \operatorname{dom} f$ and $\varepsilon \geq 0$. Then*

$$(RCL) \qquad \partial_\varepsilon(f + \delta_{\mathcal{A}})(\overline{x}) = \bigcup_{\substack{\lambda \in C^*, \\ \varepsilon_i \geq 0,\ i=1,2,3, \\ \varepsilon_1+\varepsilon_2+\varepsilon_3 = \varepsilon + (\lambda g)(\overline{x})}} \left(\partial_{\varepsilon_1} f(\overline{x}) + N_U^{\varepsilon_2}(\overline{x}) + \partial_{\varepsilon_3}(\lambda g)(\overline{x}) \right)$$

holds if and only if for all $x^* \in X^*$ for which $\overline{x}$ is an ε-optimal solution to (P_{x^*}) there exist $\overline{\lambda} \in C^*$ and $\overline{\alpha}, \overline{\beta} \in X^*$ such that

$$f(\overline{x}) + \langle x^*, \overline{x} \rangle \leq -f^*(\overline{\beta}) - (\overline{\lambda} g)^*(\overline{\alpha}) - \sigma_U(-x^* - \overline{\beta} - \overline{\alpha}) + \varepsilon. \tag{5}$$

Proof. "$\Rightarrow$" Let $x^* \in X^*$ such that $-x^* \in \partial_\varepsilon(f + \delta_{\mathcal{A}})(\overline{x})$. Because the condition (RCL) is satisfied at $\overline{x}$, there are some $\overline{\lambda} \in C^*$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$ such that $-x^* \in \partial_{\varepsilon_1} f(\overline{x}) + N_U^{\varepsilon_2}(\overline{x}) + \partial_{\varepsilon_3}(\overline{\lambda} g)(\overline{x})$ and $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon + (\overline{\lambda} g)(\overline{x})$.

Thus there exist $\overline{\beta} \in \partial_{\varepsilon_1} f(\overline{x})$ and $\overline{\alpha} \in \partial_{\varepsilon_3}(\overline{\lambda} g)(\overline{x})$ such that $-x^* - \overline{\alpha} - \overline{\beta} \in N_U^{\varepsilon_2}(\overline{x})$, i.e.

$$f(\overline{x}) + f^*(\overline{\beta}) \leq \langle \overline{\beta}, \overline{x} \rangle + \varepsilon_1,\ (\overline{\lambda} g)^*(\overline{\alpha}) + (\overline{\lambda} g)(\overline{x}) \leq \langle \overline{\alpha}, \overline{x} \rangle + \varepsilon_3 \tag{6}$$

and

$$\sigma_U(-x^* - \overline{\beta} - \overline{\alpha}) + \delta_U(\overline{x}) \leq \langle -x^* - \overline{\beta} - \overline{\alpha}, \overline{x} \rangle + \varepsilon_2. \tag{7}$$

Summing up these inequalities we get

$$f(\overline{x}) + f^*(\overline{\beta}) + (\overline{\lambda}g)^*(\overline{\alpha}) + (\overline{\lambda}g)(\overline{x}) + \sigma_U(-x^* - \overline{\beta} - \overline{\alpha}) + \delta_U(\overline{x}) \leq$$
$$-\langle x^*, \overline{x}\rangle + \varepsilon_1 + \varepsilon_2 + \varepsilon_3,$$

followed, as $\overline{x} \in U$, by (5).

"$\Leftarrow$" Take now $-x^* \in \partial_\varepsilon(f + \delta_{\mathcal{A}})(\overline{x})$. From (5) there are some $\overline{\lambda} \in C^*$ and $\overline{\alpha}$, $\overline{\beta} \in X^*$ such that

$$f(\overline{x}) + \langle x^*, \overline{x}\rangle \leq -f^*(\overline{\beta}) - \sigma_U(-x^* - \overline{\beta} - \overline{\alpha}) - (\overline{\lambda}g)^*(\overline{\alpha}) + \varepsilon.$$

This can be rewritten as

$$f(\overline{x}) + f^*(\overline{\beta}) - \langle \overline{\beta}, \overline{x}\rangle + \delta_U(\overline{x}) + \sigma_U(-x^* - \overline{\beta} - \overline{\alpha}) - \langle -x^* - \overline{\beta} - \overline{\alpha}, \overline{x}\rangle +$$
$$(\overline{\lambda}g)(\overline{x}) + (\overline{\lambda}g)^*(\overline{\alpha}) - \langle \overline{\alpha}, \overline{x}\rangle \leq \varepsilon + (\overline{\lambda}g)(\overline{x}).$$

Using also the Young-Fenchel inequality, it follows that there exist ε_1, ε_2, $\varepsilon_3 \geq 0$, with $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon + (\overline{\lambda}g)(\overline{x})$, such that (6) and (7) hold. Consequently, $\overline{\beta} \in \partial_{\varepsilon_1} f(\overline{x})$, $\overline{\alpha} \in \partial_{\varepsilon_3}(\overline{\lambda}g)(\overline{x})$, $-x^* - \overline{\beta} - \overline{\alpha} \in N_U^{\varepsilon_2}(\overline{x})$. Thus, the inclusion "$\subseteq$" in (RCL) holds and, since, "$\supseteq$" is always true, we obtain the validity of (RCL). ■

Similar statements can be given for $(\overline{D})$ and (D^L), too.

Corollary 20 *Let the proper function $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in \mathcal{A} \cap \operatorname{dom} f$ and $\varepsilon \geq 0$. Then*

$$(\overline{RCL}) \qquad \partial_\varepsilon(f + \delta_{\mathcal{A}})(\overline{x}) = \bigcup_{\substack{\lambda \in C^*, \\ \varepsilon_i \geq 0,\ i=1,2, \\ \varepsilon_1+\varepsilon_2 = \varepsilon + (\lambda g)(\overline{x})}} \left(\partial_{\varepsilon_1} f(\overline{x}) + \partial_{\varepsilon_2}(\delta_U + (\lambda g))(\overline{x})\right)$$

holds if and only if for each $x^* \in X^*$ for which $\overline{x}$ is an ε-optimal solution to (P_{x^*}) there exist $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ such that

$$f(\overline{x}) + \langle x^*, \overline{x}\rangle \leq -f^*(\overline{\beta}) - (\overline{\lambda}g)^*_U(-x^* - \overline{\beta}) + \varepsilon.$$

Corollary 21 *Let the proper function $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in \mathcal{A} \cap \operatorname{dom} f$ and $\varepsilon \geq 0$. Then*

$$(RCL^L) \qquad \partial_\varepsilon(f + \delta_{\mathcal{A}})(\overline{x}) = \bigcup_{\lambda \in C^*} \partial_{\varepsilon + (\lambda g)(\overline{x})}(f + \delta_U + (\lambda g))(\overline{x})$$

holds if and only if for each $x^* \in X^*$ for which $\overline{x}$ is an ε-optimal solution to (P_{x^*}) there exists $\overline{\lambda} \in C^*$ such that

$$f(\overline{x}) + \langle x^*, \overline{x}\rangle \leq \big(f + (\overline{\lambda}g)\big)^*_U(-x^*) + \varepsilon.$$

Remark 6 The quantity in the left-hand side of (5) is not necessarily $v(P)$, while in the right-hand side one has something smaller than $v(D_{x^*}) + \varepsilon$. However, (5) implies $v(P_{x^*}) \leq v(D_{x^*}) + \varepsilon$ and $(\overline{\lambda}, \overline{\alpha}, \overline{\beta})$ is an ε-optimal solution to (D_{x^*}). This applies to Corollary 20 and Corollary 21, with the corresponding modifications.

Remark 7 Taking $\varepsilon = 0$, Theorem 19 becomes [8, Theorem 5] without the topological and convexity assumptions on the involved functions, Corollary 20 is [8, Theorem 6], while Corollary 21 turns into [9, Theorem 3]. Note also that by considering (RCL) for $f(x) = 0$ whenever $x \in X$ one can extend (analogously to Theorem 26, see also Remark 10 and Remark 11) [8, Theorem 9] towards ε-duality gap and the same can be done for their counterparts corresponding to the other considered duals.

As mentioned in Remark 6, the statements given above are not "pure" characterizations of the stable zero duality gap for (P) and its corresponding dual problems. In the following we provide subdifferential formulae that characterize ε-duality gap statements which have in the right-hand side the optimal objective value of the duals plus ε.

Theorem 22 *Let the proper function $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in \mathcal{A} \cap \operatorname{dom} f$ and $\varepsilon \geq 0$. Then*

$$(RCS) \qquad \partial_\varepsilon (f + \delta_\mathcal{A})(\overline{x}) = \bigcap_{\eta>0} \bigcup_{\substack{\lambda \in C^*, \\ \varepsilon_i \geq 0,\ i=1,2,3, \\ \varepsilon_1+\varepsilon_2+\varepsilon_3=\varepsilon+\eta+(\lambda g)(\overline{x})}} \left(\partial_{\varepsilon_1} f(\overline{x}) + N_U^{\varepsilon_2}(\overline{x}) + \partial_{\varepsilon_3}(\lambda g)(\overline{x})\right)$$

holds if and only if for each $x^* \in X^*$ for which $\overline{x}$ is an ε-optimal solution to (P_{x^*}) it holds

$$f(\overline{x}) + \langle x^*, \overline{x}\rangle \leq \sup_{\substack{\lambda \in C^*, \\ \alpha,\beta \in X^*}} \{-f^*(\beta) - \sigma_U(-x^* - \beta - \alpha) - (\lambda g)^*(\alpha)\} + \varepsilon. \tag{8}$$

Proof. "$\Rightarrow$" Let $x^* \in X^*$ such that $-x^* \in \partial_\varepsilon (f + \delta_\mathcal{A})(\overline{x})$. Because the condition (RCS) is satisfied at $\overline{x}$, for each $\eta > 0$ there are some $\overline{\lambda}_\eta \in C^*$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$ such that $-x^* \in \partial_{\varepsilon_1} f(\overline{x}) + N_U^{\varepsilon_2}(\overline{x}) + \partial_{\varepsilon_3}(\overline{\lambda}_\eta g)(\overline{x})$ and $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon + \eta + (\overline{\lambda}_\eta g)(\overline{x})$.

Thus there are some $\overline{\beta}_\eta \in \partial_{\varepsilon_1} f(\overline{x})$ and $\overline{\alpha}_\eta \in \partial_{\varepsilon_3}(\overline{\lambda}_\eta g)(\overline{x})$ such that $-x^* - \overline{\alpha}_\eta - \overline{\beta}_\eta \in N_U^{\varepsilon_2}(\overline{x})$, i.e.

$$f(\overline{x}) + f^*(\overline{\beta}_\eta) \leq \langle \overline{\beta}_\eta, \overline{x}\rangle + \varepsilon_1, (\overline{\lambda}_\eta g)^*(\overline{\alpha}_\eta) + (\overline{\lambda}_\eta g)(\overline{x}) \leq \langle \overline{\alpha}_\eta, \overline{x}\rangle + \varepsilon_3$$

and

$$\sigma_U(-x^* - \overline{\beta}_\eta - \overline{\alpha}_\eta) + \delta_U(\overline{x}) \leq \langle -x^* - \overline{\beta}_\eta - \overline{\alpha}_\eta, \overline{x}\rangle + \varepsilon_2. \tag{9}$$

Summing up these inequalities we get

$$f(\overline{x}) + f^*(\overline{\beta}_\eta) + (\overline{\lambda}_\eta g)^*(\overline{\alpha}_\eta) + (\overline{\lambda}_\eta g)(\overline{x}) + \sigma_U(-x^* - \overline{\beta}_\eta - \overline{\alpha}_\eta) + \delta_U(\overline{x}) \leq -\langle x^*, \overline{x}\rangle + \varepsilon_1 + \varepsilon_2 + \varepsilon_3,$$

followed, as $\overline{x} \in U$, by

$$f(\overline{x}) + \langle x^*, \overline{x}\rangle \leq -f^*(\overline{\beta}_\eta) - (\overline{\lambda}_\eta g)^*(\overline{\alpha}_\eta) - \sigma_U(-x^* - \overline{\beta}_\eta - \overline{\alpha}_\eta) + \varepsilon + \eta,$$

which implies

$$f(\overline{x}) + \langle x^*, \overline{x}\rangle \leq \sup_{\substack{\lambda \in C^*, \\ \alpha,\beta \in X^*}} \{-f^*(\beta) - (\lambda g)^*(\alpha) - \sigma_U(-x^* - \beta - \alpha)\} + \varepsilon + \eta.$$

Letting η converge towards 0, (8) follows.

"$\Leftarrow$" Take now $-x^* \in \partial_\varepsilon (f + \delta_\mathcal{A})(\overline{x})$. Let $\eta > 0$. From (8) there are some $\overline{\lambda}_\eta \in C^*$ and $\overline{\alpha}_\eta$, $\overline{\beta}_\eta \in X^*$ such that

$$f(\overline{x}) + \langle x^*, \overline{x}\rangle \leq -f^*(\overline{\beta}_\eta) - \sigma_U(-x^* - \overline{\beta}_\eta - \overline{\alpha}_\eta) - (\overline{\lambda}_\eta g)^*(\overline{\alpha}_\eta) + \eta + \varepsilon.$$

As $\overline{x} \in U$, this can be rewritten as

$$f(\overline{x}) + f^*(\overline{\beta}_\eta) + \delta_U(\overline{x}) + \sigma_U(-x^* - \overline{\beta}_\eta - \overline{\alpha}_\eta) + (\overline{\lambda}_\eta g)(\overline{x}) +$$

$$(\overline{\lambda}_\eta g)^*(\overline{\alpha}_\eta) + \langle x^*, \overline{x}\rangle \leq \eta + \varepsilon + (\overline{\lambda}_\eta g)(\overline{x}),$$

which implies that

$$f(\overline{x}) + f^*(\overline{\beta}_\eta) - \langle \overline{\beta}_\eta, \overline{x} \rangle + \delta_U(\overline{x}) + \sigma_U(-x^* - \overline{\beta}_\eta - \overline{\alpha}_\eta) - \langle -x^* - \overline{\beta}_\eta - \overline{\alpha}_\eta, \overline{x} \rangle +$$

$$(\overline{\lambda}_\eta g)(\overline{x}) + (\overline{\lambda}_\eta g)^*(\overline{\alpha}_\eta) - \langle \overline{\alpha}_\eta, \overline{x} \rangle \leq \eta + \varepsilon + (\overline{\lambda}_\eta g)(\overline{x}).$$

Using also the Young-Fenchel inequality, it follows that there exist ε_1, ε_2, $\varepsilon_3 \geq 0$, with $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \eta + \varepsilon + (\overline{\lambda}_\eta g)(\overline{x})$, such that (9) holds. So, we get that $\overline{\beta}_\eta \in \partial_{\varepsilon_1} f(\overline{x})$, $\overline{\alpha}_\eta \in \partial_{\varepsilon_3}(\overline{\lambda}_\eta g)(\overline{x})$, $-x^* - \overline{\beta}_\eta - \overline{\alpha}_\eta \in N_U^{\varepsilon_2}(\overline{x})$. Thus, the inclusion "$\subseteq$" in (RCS) holds and, since, "$\supseteq$" is always true, (RCS) is valid. ■

Remark 8 Relation (8) implies $v(P_{x^*}) \leq v(D_{x^*}) + \varepsilon$, without being a consequence of it in general.

Similar statements can be given for $(\overline{D})$ and (D^L), too.

Corollary 23 *Let the proper function $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in \mathcal{A} \cap \operatorname{dom} f$ and $\varepsilon \geq 0$. Then*

$$(\overline{RCS}) \qquad \partial_\varepsilon(f + \delta_\mathcal{A})(\overline{x}) = \bigcap_{\eta > 0} \bigcup_{\substack{\lambda \in C^*, \\ \varepsilon_i \geq 0,\ i=1,2, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon + \eta + (\lambda g)(\overline{x})}} \left(\partial_{\varepsilon_1} f(\overline{x}) + \partial_{\varepsilon_2}\left(\delta_U + (\lambda g)\right)(\overline{x})\right)$$

holds if and only if for each $x^* \in X^*$ for which $\overline{x}$ is an ε-optimal solution to (P_{x^*}) it holds

$$f(\overline{x}) + \langle x^*, \overline{x} \rangle \leq \sup_{\substack{\lambda \in C^*, \\ \beta \in X^*}} \{-f^*(\beta) - (\lambda g)_U^*(-x^* - \beta)\} + \varepsilon.$$

Corollary 24 *Let the proper function $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in \mathcal{A} \cap \operatorname{dom} f$ and $\varepsilon \geq 0$. Then*

$$(RCS^L) \qquad \partial_\varepsilon(f + \delta_\mathcal{A})(\overline{x}) = \bigcap_{\eta > 0} \bigcup_{\lambda \in C^*} \partial_{\varepsilon + \eta + (\lambda g)(\overline{x})}(f + \delta_U + (\lambda g))(\overline{x})$$

holds if and only if for each $x^* \in X^*$ for which $\overline{x}$ is an ε-optimal solution to (P_{x^*}) it holds

$$f(\overline{x}) + \langle x^*, \overline{x} \rangle \leq \sup_{\lambda \in C^*} \inf_{x \in U} [f(x) + \langle x^*, x \rangle + (\lambda g)(x)] + \varepsilon.$$

Remark 9 When $f(x) = 0$ for all $x \in X$, for $\overline{x} \in \mathcal{A}$ the condition (RCS) becomes

$$(RCS_0) \qquad N_\mathcal{A}^\varepsilon(\overline{x}) = \bigcap_{\eta > 0} \bigcup_{\substack{\lambda \in C^*, \\ \varepsilon_i \geq 0,\ i=1,2, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon + \eta + (\lambda g)(\overline{x})}} \left(N_U^{\varepsilon_1}(\overline{x}) + \partial_{\varepsilon_2}(\lambda g)(\overline{x})\right).$$

We also consider the condition

$$(RCS_0') \qquad N_\mathcal{A}^{\varepsilon'}(\overline{x}) = \bigcap_{\eta > 0} \bigcup_{\substack{\lambda \in C^*, \\ \varepsilon_i \geq 0,\ i=1,2, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon' + \eta + (\lambda g)(\overline{x})}} \left(N_U^{\varepsilon_1}(\overline{x}) + \partial_{\varepsilon_2}(\lambda g)(\overline{x})\right) \ \forall \varepsilon' \in [0, \varepsilon].$$

We have the next result, as a direct consequence of Theorem 22, which characterizes the ε-Fenchel-Lagrange duality gap for optimization problems consisting in minimizing linear functionals that have an ε-minimum over $\mathcal{A}$ at $\overline{x}$.

Corollary 25 *For $\overline{x} \in \mathcal{A}$, the condition (RCS_0) holds if and only if for each $x^* \in X^*$ which has an ε-minimum over $\mathcal{A}$ at $\overline{x}$ one has*

$$\langle x^*, \overline{x} \rangle \leq \sup_{\substack{\lambda \in C^*, \\ \alpha \in X^*}} \{-\sigma_U(-x^* - \alpha) - (\lambda g)^*(\alpha)\} + \varepsilon.$$

The next theorem gives via (RCS_0) at some $\overline{x} \in \mathcal{A}$ an ε-duality gap statement for convex optimization problems consisting in minimizing over the set $\mathcal{A}$ of proper convex lower semicontinuous functions $f : X \to \overline{\mathbb{R}}$ which attain an ε-minimum over $\mathcal{A}$ at $\overline{x}$ and fulfill the condition

$$(FRC) \qquad f^* \square \delta^*_{\mathcal{A}} \text{ is lower semicontinuous and exact at } 0,$$

and their Fenchel-Lagrange type dual problems.

Theorem 26 *Let $f : X \to \overline{\mathbb{R}}$ be a proper convex lower semicontinuous function, $\mathcal{A}$ be closed convex and $\varepsilon \geq 0$. When (FRC) is fulfilled, $\overline{x} \in \mathcal{A} \cap \operatorname{dom} f$ is an ε-optimal solution to (P) and the condition (RCS'_0) holds, then*

$$f(\overline{x}) \leq \sup_{\substack{\lambda \in C^*, \\ \alpha, \beta \in X^*}} \{-f^*(\beta) - (\lambda g)^*(\alpha) - \sigma_U(-\alpha - \beta)\} + \varepsilon. \tag{10}$$

Proof. Take a function f as requested in the hypothesis. We have

$$f(\overline{x}) \leq -(f + \delta_{\mathcal{A}})^*(0) + \varepsilon$$

and (FRC) guarantees that there is some $\overline{\beta} \in X^*$ such that $(f+\delta_{\mathcal{A}})^*(0) = f^*(\overline{\beta}) + \sigma_{\mathcal{A}}(-\overline{\beta})$. Further we get

$$f(\overline{x}) + f^*(\overline{\beta}) + \sigma_{\mathcal{A}}(-\overline{\beta}) + \delta_{\mathcal{A}}(\overline{x}) \leq \langle \overline{\beta}, \overline{x} \rangle + \langle -\overline{\beta}, \overline{x} \rangle + \varepsilon,$$

therefore there exist $\varepsilon_1, \varepsilon_2 \geq 0$ with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ such that $\overline{\beta} \in \partial_{\varepsilon_1} f(\overline{x})$ and $-\overline{\beta} \in N^{\varepsilon_2}_{\mathcal{A}}(\overline{x})$. By (RCS'_0) for each $\eta > 0$ there are some $\overline{\lambda}_\eta \in C^*$ and $\varepsilon_3, \varepsilon_4 \geq 0$ with $\varepsilon_3 + \varepsilon_4 = \varepsilon_2 + (\overline{\lambda}_\eta g)(\overline{x}) + \eta$ such that $-\overline{\beta} \in \partial_{\varepsilon_3}(\overline{\lambda}_\eta g)(\overline{x}) + N^{\varepsilon_4}_U(\overline{x})$. This implies that there exists $\overline{\alpha} \in X^*$ such that $\overline{\alpha} \in \partial_{\varepsilon_3}(\overline{\lambda}_\eta g)(\overline{x})$ and $-\overline{\alpha} - \overline{\beta} \in N^{\varepsilon_4}_U(\overline{x})$, so we have

$$\begin{aligned} & f(\overline{x}) + f^*(\overline{\beta}) + \delta_U(\overline{x}) + \sigma_U(-\overline{\alpha} - \overline{\beta}) + (\overline{\lambda}_\eta g)(\overline{x}) + (\overline{\lambda}_\eta g)^*(\overline{\alpha}) \\ \leq {} & \langle \overline{\beta}, \overline{x} \rangle - \langle \overline{\alpha} + \overline{\beta}, \overline{x} \rangle + \langle \overline{\alpha}, \overline{x} \rangle + \varepsilon_1 + \varepsilon_3 + \varepsilon_4 + \eta = \varepsilon_1 + \varepsilon_2 + (\overline{\lambda}_\eta g)(\overline{x}) + \eta. \end{aligned}$$

Further

$$f(\overline{x}) \leq -f^*(\overline{\beta}) - (\overline{\lambda}_\eta g)^*(\overline{\alpha}) - \sigma_U(-\overline{\alpha} - \overline{\beta}) + \varepsilon + \eta,$$

which yields

$$f(\overline{x}) \leq \sup_{\substack{\lambda \in C^*, \\ \alpha, \beta \in X^*}} \{-f^*(\beta) - (\lambda g)^*(\alpha) - \sigma_U(-\alpha - \beta)\} + \varepsilon + \eta.$$

Letting η converge towards 0, the proof is complete. ■

Remark 10 In the preceding theorem, the relation (10) yields the condition (RCS_0), without being implied by it in general. For the other duals one can obtain similar results with Theorem 26, too.

Remark 11 In case $\varepsilon = 0$ the assertion from Theorem 26 turns, via Remark 10, into an equivalence, improving [15, Theorem 4.2] by removing the continuity assumptions on the functions involved.

4 Conclusions and further research

We provided new characterizations for ε-duality gap statements (with $\varepsilon \geq 0$) for a constrained optimization problem and its Lagrange and Fenchel-Lagrange dual problems, respectively, by means of epigraphs and ε-subdifferentials, respectively. After formulating the results in the most general frameworks, with the functions taken only proper, we added to them convexity and topological hypotheses, which led to extending several recent results on stable strong and total duality for constrained optimization problems from [5, 8, 9, 15, 16] and in some cases to improving them by removing the the continuity and nonempty domain interior assumptions of the involved functions.

We intend to extend our investigations to other classes of optimization problems, for instance the ones containing compositions of functions as considered in [4]. Moreover, an extension of the notion of a set closed regarding another one (cf. [4, 7, 10]) in the direction of ε-vertical closedness can be considered, too. Note that one can formulate also ε-optimality conditions statements for (P) and its considered dual problems, extending thus the corresponding optimality conditions statements from [8, 9]. Moreover, ε-Farkas type results can be given for the considered problem by combining the statements from this paper with ideas from [11]. Nevertheless, by defining an (η, ε)-*saddle point* of a function $L : X \times Y \to \overline{\mathbb{R}}$, where $\eta, \varepsilon \geq 0$, to be some $(\overline{x}, \overline{y}) \in X \times Y$ for which $L(\overline{x}, y) - \eta \leq L(\overline{x}, \overline{y}) \leq L(x, \overline{y}) + \varepsilon$ for all $(x, y) \in X \times Y$, the connections between such points of the Lagrangian functions attached to the considered dual problems to (P_{x^*}) and the ε-duality gap for the corresponding primal-dual pair of problems can be investigated, too. An interesting question posed to us by Dr. R.I. Boţ refers to a possible convergence of the ε-optimal solutions of the dual problems towards optimal solutions of theirs when ε tends to 0. This, together with the ideas mentioned above, remains subject to future research.

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References

[1] Y. Anbo, *Nonstandard arguments and the characterization of independence in generic structures*, RIMS Kôkyûroku 1646, 4–17, 2009.

[2] R.I. Boţ, G. Wanka: *A weaker regularity condition for subdifferential calculus and Fenchel duality in infinite dimensional spaces*, Nonlinear Analysis: Theory, Methods & Applications 64(12), 2787–2804, 2006.

[3] R.I. Boţ, S.-M. Grad, *Lower semicontinuous type regularity conditions for subdifferential calculus*, Optimization Methods and Software 25(1), 37–48, 2010.

[4] R.I. Boţ, S.-M. Grad, G. Wanka, *A new constraint qualification for the formula of the subdifferential of composed convex functions infinite dimensional spaces*, Mathematische Nachrichten 281(8), 1088–1107, 2008.

[5] R.I. Boţ, S.-M. Grad, G. Wanka: *Duality in Vector Optimization*, Springer-Verlag Berlin Heidelberg, 2009.

[6] R.I. Boţ, S.-M. Grad, G. Wanka: *Generalized Moreau-Rockafellar results for composed convex functions*, Optimization 58(7), 917–933, 2009.

[7] R.I. Boţ, S.-M. Grad, G. Wanka: *Maximal monotonicity for the precomposition with a linear operator*, SIAM Journal on Optimization 17(4), 1239–1252, 2006.

[8] R.I. Boţ, S.-M. Grad, G. Wanka: *New regularity conditions for strong and total Fenchel-Lagrange duality in infinite dimensional spaces*, Nonlinear Analysis: Theory, Methods & Applications 69(1), 323–336, 2008.

[9] R.I. Boţ, S.-M. Grad, G. Wanka: *On strong and total Lagrange duality for convex optimization problems*, Journal of Mathematical Analysis and Applications 337(2), 1315–1325, 2008.

[10] R.I. Boţ, S.-M. Grad, G. Wanka: *Weaker constraint qualifications in maximal monotonicity*, Numerical Functional Analysis and Optimization 28(1–2), 27–41, 2007.

[11] R.I. Boţ, G. Wanka, *Farkas-type results with conjugate functions*, SIAM Journal on Optimization 15(2), 540–554, 2005.

[12] D.H. Fang, C. Li, K.F. Ng: *Constraint qualifications for optimality conditions and total Lagrange dualities in convex infinite programming*, Nonlinear Analysis: Theory, Methods & Applications 73(5), 1143-1159, 2010.

[13] D.H. Fang, C. Li, X.Q Yang: *Stable and total Fenchel duality for DC optimization problems in locally convex spaces*, SIAM Journal on Optimization 21(3), 730-760, 2011.

[14] H.M. Friedman, *A way out*, in: G. Link (Ed.), "One Hundred Years of Russell's Paradox", de Gruyter, 49–86, 2004.

[15] V. Jeyakumar, G.Y. Li: *New dual constraint qualifications characterizing zero duality gaps of convex programs and semidefinite programs*, Nonlinear Analysis: Theory, Methods & Applications 71(12), 2239–2249, 2009.

[16] V. Jeyakumar, G.Y. Li: *Stable zero duality gaps in convex programming: Complete dual characterizations with applications to semidefinite programs*, Journal of Mathematical Analysis and Applications 360, 156–167, 2009.

[17] C. Li, D.H. Fang, G. López, M.A. López: *Stable and total Fenchel duality for convex optimization problems in locally convex spaces*, SIAM Journal on Optimization 20(2), 1032-1051, 2009.

[18] A.M. Rubinov, B.M. Glover: *Quasiconvexity via two step functions*, in: J.-P. Crouzeix, J.E. Martínez Legaz, M. Volle (Eds.), "Generalized Convexity, Generalized Monotonicity: Recent Results", Nonconvex Optimization and Its Applications Vol. 27 , Kluwer Acad. Publ., Dordrecht, 159-183, 1998.