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An extended approach for lifting clique tree inequalities

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Abstract. We present a new lifting approach for strengthening arbitrary clique tree inequalities that are known to be facet defining for the symmetric traveling salesman problem in order to get stronger valid inequalities for the symmetric quadratic traveling salesman problem (SQTSP). Applying this new approach to the subtour elimination constraints (SEC) leads to two new classes of facet defining inequalities of SQTSP. For the special case of the SEC with two nodes we derive all known conflicting edges inequalities for SQTSP. Furthermore we extend the presented approach to the asymmetric quadratic traveling salesman problem (AQTSP).

Keywords: traveling salesman problem, quadratic traveling salesman problem, polyhedral combinatorics
MSC: 90C57, 90C27

1 Introduction

The symmetric traveling salesman problem (STSP) asks for a cost-minimal tour in a complete edge-weighted undirected graph. In contrast to this, in the symmetric *quadratic* traveling salesman problem (SQTSP) the costs are not associated to the edges but to each three nodes that are traversed in succession. This leads to a minimization problem with a quadratic objective function. The problem was introduced by Jäger and Molitor [13] in connection with an application in biology and its polyhedral structure was studied in [6]. Special cases are the angular-metric traveling salesman problem [1] used in the design of robot paths and the traveling salesman problem with reload costs [2] used in the planning of transport and telecommunication systems. SQTSP can be stated as follows.

We consider complete undirected 2-graphs $G = (V, E)$ with node set V, $|V| = n$, and set of 2-edges $E = V^{\langle 3\rangle} := \{\langle u, v, w\rangle = \langle w, v, u\rangle : u, v, w \in V, |\{u, v, w\}| = 3\}$ with associated set of edges $V^{\{2\}} := \{\{u, v\} : u, v \in V, u \neq v\}$. We often simply write ijk instead of $\langle i, j, k\rangle$ and ij instead of $\{i, j\}$. A 2-cycle K of length $k > 2$ in a 2-graph G is a set of k 2-edges $K = \{v_1v_2v_3, v_2v_3v_4, \dots, v_{k-1}v_kv_1, v_kv_1v_2\}$ with pairwise distinct v_i. The 2-edges $ijk \in K$ are associated with a set of edges $K^{\{2\}} := \{ij \in V^{\{2\}} : \exists\, ijk \in K\}$. A 2-cycle of length n is called a *tour* and the set of all tours is denoted by $\mathcal{K}_n = \{K : K \text{ 2-cycle in } G, |K| = n\}$.

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The STSP is often modeled as

$$\sum_{ij\in V^{\{2\}}} x_{ij} = 2, \qquad i \in V, \tag{1}$$

$$\sum_{ij\in S^{\{2\}}} x_{ij} \le |S| - 1, \qquad S \subset V, 2 \le |S| \le n-2, \tag{2}$$

$$x_{ij} \in \{0,1\}, \qquad ij \in V^{\{2\}}, \tag{3}$$

see *e. g.* [3], and the considered polytope is

$$P_{\mathbf{STSP}_n} := \operatorname{conv}\left\{x \in \{0,1\}^{V^{\{2\}}} : (1), (2)\right\}.$$

For the SQTSP we additionally introduce the binary variables $y_{ijk} \in \{0,1\}, ijk \in V^{\langle 3\rangle}$, and have the equalities

$$x_{ij} = \sum_{k\colon ijk\in V^{\langle 3\rangle}} y_{ijk} = \sum_{k\colon kij\in V^{\langle 3\rangle}} y_{kij}, \qquad ij \in V^{\{2\}}. \tag{4}$$

The corresponding polytope reads

$$P_{\mathbf{SQTSP}_n} := \operatorname{conv}\left\{(x,y) \in \{0,1\}^{V^{\{2\}}\cup V^{\langle 3\rangle}} : (1), (2), (4)\right\}.$$

There are two canonical types of 2-edges for strengthening inequalities of STSP in order to get stronger inequalities for SQTSP. The types of 2-edges are described in detail below. While the approach presented in [6] only adds 2-edges of one type we develop a lifting strategy that allows to use both types.

We shortly repeat the approach in [6]. It is based on the observation that a 2-edge ikj almost acts as the edge ij in the sense that the two nodes i, j are close in a tour. The approach reads as follows. Let $\sum_{ij\in V^{\{2\}}} a_{ij}x_{ij} \le b$ be a valid inequality of STSP with coefficients $a_{ij} \ge 0, ij \in V^{\{2\}}$. Let $V_a = \{i \in V\colon \exists ij \in V^{\{2\}} \text{ with } a_{ij} > 0\}$. Then in the case $|V_a| < \frac{n}{2}$

$$\sum_{ij\in V^{\{2\}}} a_{ij}x_{ij} + \sum_{\substack{ikj\in V^{\langle 3\rangle}:\\ a_{ik}=a_{kj}=0}} a_{ij}y_{ikj} \le b \tag{5}$$

and if $\frac{n}{2} \le |V_a| < n$ and $\bar{t} \in V \setminus V_a$

$$\sum_{ij\in V^{\{2\}}} a_{ij}x_{ij} + \sum_{\substack{ikj\in V^{\langle 3\rangle}:\\ a_{ik}=a_{kj}=0, k\neq \bar{t}}} a_{ij}y_{ikj} \le b \tag{6}$$

are valid inequalities of $P_{\mathbf{SQTSP}_n}$. Applying the strengthening to the simplest subtour elimination constraints on two nodes, *i. e.* the upper bound constraints for the x-variables, $x_{ij} \le 1, ij \in V^{\{2\}}$, leads to the simplest of the so called *conflicting edges inequalities* $x_{ij} + \sum_{ikj\in V^{\langle 3\rangle}, k\in S_1} y_{ikj} \le 1, ij \in V^{\{2\}}, S_1 = V \setminus \{i,j\}$, that are valid for $P_{\mathbf{SQTSP}_n}, n \ge 5$, by (5) and that define facets of $P_{\mathbf{SQTSP}_n}, n \ge 6$ [6]. Besides these constraints there are further conflicting edges inequalities [6] that also represent a strengthening of $x_{ij} \le 1, ij \in V^{\{2\}}$, and are facet defining for $P_{\mathbf{SQTSP}_n}, n \ge 6$, that cannot be obtained by this strengthening approach. These read as follows:

$$x_{ij} + \sum_{ikj\in V^{\langle 3\rangle}\colon k\in S_1} y_{ikj} + \sum_{kil\in V^{\langle 3\rangle}\colon k,l\in S_2} y_{kil} \le 1, \tag{7}$$

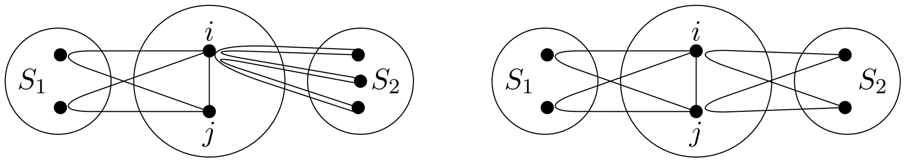


Figure 1: Visualization of inequalities (7) on the left and of (8) on the right side. At most one of the edges (straight line) and 2-edges (curved lines) can be contained in a tour. For (7), left figure, the set S_2 may contain further nodes that lead to further 2-edges that can be added to the inequality. But for (8) $|S_2| = 2$ is essential for preserving feasibility. In both cases S_1 might be enlarged.

for $i, j \in V, i \neq j, S_1, S_2 \in V \setminus \{i, j\}, V = \{i, j\} \cup S_1 \cup S_2, S_1 \cap S_2 = \emptyset, S_1 \neq \emptyset, |S_2| \geq 3$ and

$$x_{ij} + \sum_{ikj \in V^{\langle 3 \rangle}:\, k \in S_1} y_{ikj} + y_{s_1 i s_2} + y_{s_1 j s_2} \leq 1, \tag{8}$$

for $ij \in V^{\{2\}}, S_2 = \{s_1, s_2\}, |\{i, j, s_1, s_2\}| = 4, S_1 = V \setminus \{i, j, s_1, s_2\}$. Apart from 2-edges ikj that act as the edge ij inequalities (7) and (8) contain variables of 2-edges with middle node i and partially j, for example 2-edges $kil \in V^{\langle 3 \rangle}, k, l \in S_2$ in (7). Because $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = V \setminus \{i, j\}$ the inequalities remain valid [6]. Note, since $|S_2| = 2$ in (8) the presence of both 2-edges $s_1 i s_2, s_1 j s_2$ would imply a subtour if $n \geq 5$. If S_2 contains more than two nodes, for example $S_2 = \{t_1, t_2, t_3\}$ the 2-edges $t_1 i t_2, t_2 j t_3$ may be contained in a tour at the same time. Figure 1 shows a visualization of (7) and (8).

The aim of this paper is to improve the understanding of complex inequality classes of $P_{\mathbf{SQTSP}_n}$ and of linearizations of combinatorial optimization problems with quadratic objective function in general. For this we develop a strengthening approach for the large class of clique tree inequalities of $P_{\mathbf{STSP}_n}$ [11] that extends the approach in [6]. The lifted variant of $\sum_{ij \in V^{\{2\}}} a_{ij} x_{ij} \leq b, a \geq 0$, may contain 2-edges $ikj, i, j \in V_a, k \in \tilde{S}_1$, as well as $kil, i \in \tilde{V}_a \subset V_a, k, l \in \tilde{S}_2$ with $S_1 \subset \tilde{S}_1, S_2 \subset \tilde{S}_2, V_a \dot{\cup} S_1 \dot{\cup} S_2 = V$, *i. e.* the lifted inequality has the form

$$\sum_{ij \in V^{\{2\}}} a_{ij} x_{ij} + \sum_{\substack{ikj \in V^{\langle 3 \rangle}: \\ k \in \tilde{S}_1, a_{ik} = a_{kj} = 0}} a_{ij} y_{ikj} + \sum_{\substack{kil \in V^{\langle 3 \rangle}: \\ i \in \tilde{V}_a, k, l \in \tilde{S}_2, \\ a_{ki} = a_{il} = 0}} y_{kil} \leq b. \tag{9}$$

In order to obtain a valid inequality the sets $\tilde{S}_1, \tilde{S}_2, \tilde{V}_a$ have to be chosen appropriately.

Indeed, the new approach allows to lift arbitrary clique tree inequalities [11] that are a very general class and include many other well known inequalities like, *e. g.*, subtour elimination constraints [3], 2-matching inequalities [4] and comb inequalities [9, 10]. The proof of the conditions for feasibility is somewhat tricky and highly depends on the structure of the clique tree inequalities (see Section 2), but the new lifting automatically yields important facet-defining inequalities. In the special case of the bound constraints $x_{\hat{\imath}\hat{\jmath}} \leq 1, \hat{\imath}\hat{\jmath} \in V^{\{2\}}$, we exactly obtain the conflicting edges inequalities (7) and (8) setting $V_a = \{\hat{\imath}, \hat{\jmath}\}, \tilde{V}_a = \{\hat{\imath}\}, S_1 = \tilde{S}_1, S_2 = \tilde{S}_2$ resp. $V_a = \tilde{V}_a = \{\hat{\imath}, \hat{\jmath}\}, S_1 = \tilde{S}_1, S_2 = \tilde{S}_2$ in (9).

Finally in Section 4, we show how to extend the presented ideas to the asymmetric equivalents of the clique tree inequalities [7] for the asymmetric traveling salesman problem [8]. These inequalities are known to be facet defining for the ATSP, see [7]. We will extend our lifting approach to inequalities that do not have a symmetric counterpart for the SQTSP [6].

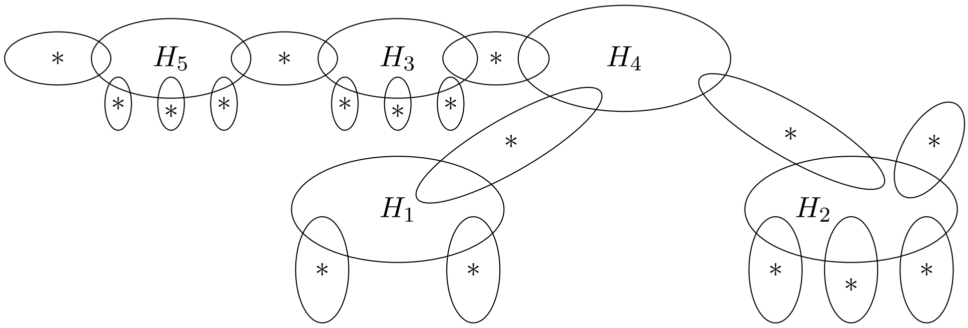


Figure 2: Visualization of the structure of a clique tree C with handles $H_i, i = 1, \dots, 5$, and 17 teeth. Each clique is indicated by an ellipse and a star $*$ symbolizes that each tooth has to contain at least one node not contained in any handle by Definition 1.

2 Clique tree inequalities

A clique tree inequality corresponding to a SQTSP on n nodes is defined as follows.

Definition 1 (see Definition 2.16 in [11]) *A clique tree $C = (V_C, E_C)$ with node set V_C and set of edges E_C is a connected graph whose maximal cliques fulfill the following. The cliques (a clique D is identified by its nodes, its edges are $D^{\{2\}}$) can be partitioned into families of handles $\mathcal{H}$ and teeth $\mathcal{T}$ with $\forall\, X \in \mathcal{H} \cup \mathcal{T}\colon X \subseteq V_C$, and $\forall\, H, H' \in \mathcal{H}, H \neq H'\colon H \cap H' = \emptyset$ and $\forall\, T, T' \in \mathcal{T}, T \neq T'\colon T \cap T' = \emptyset$. So there holds $H^{\{2\}} \subset E_C, H \in \mathcal{H}$, and $T^{\{2\}} \subset E_C, T \in \mathcal{T}$. Each of the teeth fulfills $2 \leq |T| \leq n-2, |T \setminus (\bigcup_{H \in \mathcal{H}} H)| \geq 1, T \in \mathcal{T}$, and each handle H intersects with an odd number greater one of teeth,* i.e., *$|\{T \in \mathcal{T}\colon H \cap T \neq \emptyset\}|$ is greater than or equal to three and odd for $H \in \mathcal{H}$. If a handle $H \in \mathcal{H}$ and a tooth $T \in \mathcal{T}$ fulfill $H \cap T \neq \emptyset$ then deleting all nodes in $H \cap T$ and all incident edges enlarges the number of components of graph C.*

Figure 2 shows the structure of an example clique tree. For a clique tree C with handles $\mathcal{H}$ and teeth $\mathcal{T}$ the corresponding clique tree inequality [11] reads

$$\sum_{Z \in \mathcal{H} \cup \mathcal{T}} \sum_{kl \in Z^{\{2\}}} x_{kl} \leq \sum_{H \in \mathcal{H}} |H| + \sum_{T \in \mathcal{T}} (|T| - t(T)) - \tfrac{|\mathcal{T}|+1}{2} =: s(C), \tag{10}$$

where $t(T) = |\{H \in \mathcal{H}\colon H \cap T \neq \emptyset\}|, T \in \mathcal{T}$, and is facet defining for $P_{\mathbf{STSP}_n}$ [11]. The right-hand side of (10) is often called *size* of C and denoted by $s(C)$. Note, the coefficients of all edges kl with $k, l \in H \cap T, k \neq l$, for some $H \in \mathcal{H}, T \in \mathcal{T}$ are two because such an edge is counted once for the handle H and once for the tooth T.

3 The new lifting approach

In this section we will extend the lifting approach (6) for the clique tree inequalities. Let $(a^1)^T x + (a^2)^T y \leq b, a^1 \geq 0, a^2 \geq 0$, be an appropriately lifted clique tree inequality. It is called a^1-a^2-*dominated* if for any $K \in \mathcal{K}_n$ there exists a *dominating* tour $\bar{K} \in \mathcal{K}_n$ with $(a^1)^T x_{K^{\{2\}}} + (a^2)^T y_K \leq (a^1)^T x_{\bar{K}^{\{2\}}} + (a^2)^T y_{\bar{K}} = (a^1)^T x_{\bar{K}^{\{2\}}} \leq b$, *i. e.* the coefficients of all 2-edges in $\bar{K}$ have to be zero. Here $x_{\bar{K}^{\{2\}}}, y_{\bar{K}}$ denote the incidence vectors of all edges $V^{\{2\}}$ resp. all 2-edges $V^{\langle 3 \rangle}$ of the 2-cycle $\bar{K}$. For proving that a lifted clique tree inequality $(a^1)^T x + (a^2)^T y \leq b$ is a^1-a^2-dominated we study the structure of the intersection of an arbitrary tour in G and a clique tree C. Figure 3 shows an example for the intersection of

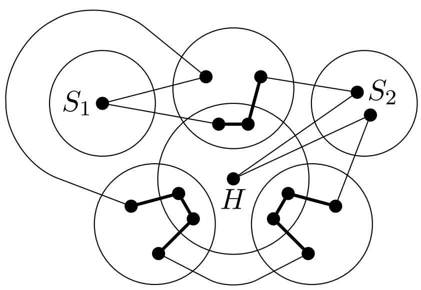


Figure 3: Intersection of the edges $K^{\{2\}}$ of a tour $K \in \mathcal{K}_n$ and the edges E_C of a clique tree $C = (V_C, E_C)$ with $|\mathcal{H}| = 1, |\mathcal{T}| = 3$ and $|V_C| = n-3$ and $V = V_C \cup S_1 \cup S_2, |S_1| = 1, |S_2| = 2$. All edges in the intersection are highlighted in bold.

the edges $K^{\{2\}}$ of a tour $K \in \mathcal{K}_n$ and of the edges E_C of a clique tree $C = (V_C, E_C)$ with $|\mathcal{H}| = 1, |\mathcal{T}| = 3$ and $|V_C| = n - 3$.

In constructing dominating tours we will use the following notations. A path $P = u_1 \dots u_k$ is a sequence of pairwise different nodes $u_1, \dots, u_k$, $k \in \mathbb{N}$, so that $u_i u_{i+1}$, $i = 1, \dots, k-1$, are edges. The subpath $u_i \dots u_j$, $1 \le i \le j \le k$, of P is denoted by $u_i P u_j$. Note that the subpath may contain only one node if $i = j$. If we want to emphasize that u_1 is an end node of a path P we simply write $P = Pu_1$ resp. $P = u_1 P$. For two paths $P = Pu$ and $Q = uQ$ their concatenation is the path $PQ = PuQ$. The set of nodes of a path P is denoted by $V(P)$, the set of edges by $E(P)$.

Let $G = (V, E)$ be a complete undirected 2-graph with $|V| = n$. If $|V_C| < n$ holds for a clique tree $C = (V_C, E_C) \subset G$ it is easy to see that the intersection of the edges $K^{\{2\}}$ of a tour $K \in \mathcal{K}_n$ and E_C leads to a set of edges that form paths. Furthermore we consider all isolated nodes $v \in V_C \setminus \{w \in V_C \colon \exists\, wz \in K^{\{2\}} \cap E_C\}$ as paths.

For proving that a lifted inequality $(a^1)^T x + (a^2)^T y \le b, a^1 \ge 0, a^2 \ge 0$, is a^1-a^2-dominated we have to show that for each tour $K \in \mathcal{K}_n$ there exists a dominating tour $\bar{K} \in \mathcal{K}_n$ such that the left-hand side calculated for $\bar{K}$ is at least as high as the left-hand side for K and $a^2_{ijk} = 0$ for all $ijk \in \bar{K}$. We do this by considering not only the edges $K^{\{2\}} \cap E_C$ but also all 2-edges $ijk \in K$ with nonzero coefficients. In our approach such a coefficient might be nonzero because $j \in \tilde{S}_1, ik \in E_C$ or $j \in \tilde{V}_C \subset V_C, i, k \in \tilde{S}_2$, see (9). For constructing $\bar{K}$ we start with a set of paths, more precisely a set of edges of paths. Let $K^{\{2\}}_E$ denote the set of all edges corresponding to a tour K. Then we have $(K^{\{2\}} \cap E_C) \subset K^{\{2\}}_E$. If a tour K contains a 2-edge $ijk \in K$ with coefficient $a^2_{ijk} > 0$ and $j \in \tilde{S}_1, ik \in E_C$ according to (9) we add the edge ik to $K^{\{2\}}_E$. If set S_2 is nonempty the set $K^{\{2\}}_E$ corresponds to a set of paths, see the proof of the lifting approach in [6]. Indeed, $K^{\{2\}}_E$ cannot correspond to a tour because it does not visit the nodes in S_2 and its edges cannot form subtours because $K \in \mathcal{K}_n$. The situation is a bit more complicated for 2-edges $ijk \in K$ with $a^2_{ijk} > 0$ and $i, k \in \tilde{S}_2, j \in \tilde{V}_a \subset V_C$ according to (9). Then we want to preserve the information that node j has this property and we will call such a node a *blocked node*. This leads to the definition of *path systems* containing *blocked nodes*.

Definition 2 *A* path system *in $C = (V_C, E_C)$ is a pair $(\mathcal{P}, B)$ of a set of paths $\mathcal{P}$ and a set of* blocked nodes B *so that*

(i) $\forall\, P, Q \in \mathcal{P}, P \ne Q \colon V(P) \cap V(Q) = \emptyset,$

(ii) $\forall\, P \in \mathcal{P} \colon V(P) \cap B = \emptyset,$

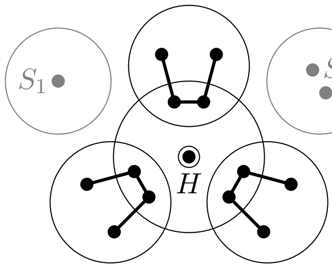


Figure 4: Visualization of a possible path system $(\mathcal{P}, B)$ for the example in Figure 3. The set of paths $\mathcal{P}$ comprises three paths of length three (straight lines) and B contains exactly one blocked node in handle H that is highlighted.

(iii) $\bigcup_{P\in\mathcal{P}} V(P) \cup B = V_C$.

Note that $\mathcal{P}$ may contain paths that contain only one node. The weight *of a path system is*

$$\omega(\mathcal{P}, B) = |B| + \sum_{P\in\mathcal{P}} (|E(P) \cap E(\mathcal{H})| + |E(P) \cap E(\mathcal{T})|),$$

where $E(\mathcal{H}) = \bigcup_{H\in\mathcal{H}} E(H)$, $E(\mathcal{T}) = \bigcup_{T\in\mathcal{T}} E(T)$. We denote with $\mathcal{E}(\mathcal{P}) := \{v \in V_C \setminus B \colon \exists\, P = vP \in \mathcal{P}\}$ the set of all end nodes of paths in $\mathcal{P}$.

Figure 4 shows the path system for the example in Figure 3 with one blocked node in H. The weight of a path system $(\mathcal{P}, B)$ is defined so that the left-hand side of a lifted clique tree inequality (9) computed for a tour $K \in \mathcal{K}_n$ equals $\omega(\mathcal{P}, B)$ with $\mathcal{P}, B$ as follows. The edges of the paths are obtained by $(K^{\{2\}} \cap E_C) \cup \{ik \in E_C \colon j \in \tilde{S}_1, a^2_{ijk} > 0, ijk \in K\}$ and $B = \{j \in V_C \colon i, k \in \tilde{S}_2, a^2_{ijk} > 0, ijk \in K\}$. By an appropriate choice of $\tilde{S}_1, \tilde{S}_2, \tilde{V}_a$ we will ensure that this construction leads indeed to a path system associated with K.

Instead of working directly with the inequalities or the tours, we look at the corresponding path systems and study their properties. We show that there exists a path system $(\mathcal{P}', B')$, $B' = \emptyset$ for each path system $(\mathcal{P}, B)$ with $(T \setminus \bigcup_{H\in\mathcal{H}} H) \not\subseteq B$ for all $T \in \mathcal{T}$ and the path systems fulfill $\omega(\mathcal{P}, B) \le \omega(\mathcal{P}', B')$. This is achieved in two main steps. First we simplify the path system and get $(\hat{\mathcal{P}}, \hat{B}), \omega(\mathcal{P}, B) \le \omega(\hat{\mathcal{P}}, \hat{B})$. For this we modify $(\mathcal{P}, B)$ such that no tooth contains a blocked node, *i. e.* $\hat{B} \cap \bigcup_{T\in\mathcal{T}} T = \emptyset$. Since the nonempty intersections of handles $H \in \mathcal{H}$ and teeth $T \in \mathcal{T}$ are very important for the next step we force that the paths $\hat{\mathcal{P}}$ have the following simple structure. For each handle $H \in \mathcal{H}$ and tooth $T \in \mathcal{T}$ with $H \cap T \neq \emptyset$ there is exactly one path $P \in \hat{\mathcal{P}}$ that has a nonempty intersection with $H \cap T$ and this intersection comprises exactly one (connected) subpath of P. If after this simplification there exists a handle $H \in \mathcal{H}$ with $\hat{B} \cap H \neq \emptyset$ we reorder the paths in such a way that after reordering one of the paths has an end node in H and we can enlarge this path by connecting it to a path of all nodes $\hat{B} \cap H \neq \emptyset$ and delete these nodes from B resp. $\hat{B}$. If we end up in a path system $(\mathcal{P}', B'), B' = \emptyset, \omega(\mathcal{P}, B) \le \omega(\mathcal{P}', B')$ we show that we indeed can construct a dominating tour $\bar{K}$ with the desired properties. More precisely, $\bar{K}$ contains only 2-edges whose coefficients are zero in the lifted inequality. So the left-hand side of the lifted inequality as well as of the original inequality calculated for $\bar{K}$ is at least as big as the left-hand side calculated for K.

First we prove that a blocked node $v \in \bigcup_{T\in\mathcal{T}} T$ can easily be eliminated without reducing the weight.

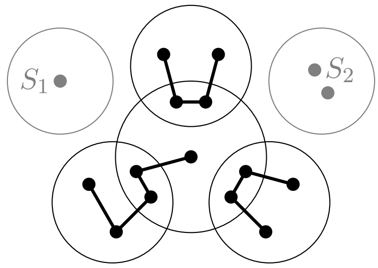


Figure 5: Visualization of a path system $(\mathcal{P}', B'), B' = \emptyset$ that fulfills $\omega(\mathcal{P}', B') = \omega(\mathcal{P}, B)$ for $(\mathcal{P}, B)$ the path system in Figure 4. One of the paths in Figure 4 was reordered and we connected the blocked node to it.

Lemma 3 *Let $(\mathcal{P}, B)$ be a path system on a clique tree $C = (V_C, E_C)$ with handles $\mathcal{H}$ and teeth $\mathcal{T}$. If for each tooth $T \in \mathcal{T}$ it holds $(T \setminus \bigcup_{H \in \mathcal{H}} H) \not\subset B$ then there exists a path system $(\mathcal{P}', B')$ with $\omega(\mathcal{P}', B') \geq \omega(\mathcal{P}, B)$ and*

$$B' \cap (\bigcup_{T \in \mathcal{T}} T) = \emptyset. \tag{11}$$

Proof. Let $(\mathcal{P}, B)$ be a path system with minimal number of blocked nodes $|B \cap (\bigcup_{T \in \mathcal{T}} T)| > 0$ fulfilling $(T \setminus \bigcup_{H \in \mathcal{H}} H) \not\subset B$ for each tooth $T \in \mathcal{T}$. Assume there exists a tooth $T \in \mathcal{T}$ with $B \cap T = \{v_1, \ldots, v_m\} \neq \emptyset$. By assumption on the path system there exists a node $w \in (T \setminus \bigcup_{H \in \mathcal{H}} H)$ with $w \notin B$. Let $P \in \mathcal{P}$ be the path with $w \in V(P)$. If $P = Pw$, *i. e.* w is an end node, we enlarge P to a path $P' := Pwv_1v_2 \ldots v_m$ and set $\mathcal{P}' := (\mathcal{P} \setminus \{P\}) \cup \{P'\}, B' := B \setminus \{v_1, \ldots, v_m\}$. Otherwise, if $P = pPwzPq$, $w \neq z$, we enlarge P to a path $P' := pPwv_1v_2 \ldots v_m zPq$ and set $\mathcal{P}' := (\mathcal{P} \setminus \{P\}) \cup \{P'\}, B' := B \setminus \{v_1, \ldots, v_m\}$. In both cases $(\mathcal{P}', B')$ fulfills $\omega(\mathcal{P}', B') \geq \omega(\mathcal{P}, B)$ because the number of edges in T that are counted in ω is enlarged by at least m and the number of blocked nodes is reduced by exactly m. $\square$

As can be seen in Figure 4 it might sometimes be impossible to directly eliminate a blocked node $v \in (H \setminus \bigcup_{T \in \mathcal{T}} T) \cap B, H \in \mathcal{H}$, by connecting it to a path with an end node in H. Here a reordering of the paths of other handles and teeth might be needed, see Figure 5. If each handle $H \in \mathcal{H}$ intersects with a $T \in \mathcal{T}, T \cap (\bigcup_{H' \in \mathcal{H}, H' \neq H} H') = \emptyset$ this reordering can be found relatively simple, see Figure 5. But if no such T exists for $H \in \mathcal{H}$, see H_4 in Figure 2, it is not obvious how to restructure the path system. In order to reduce the number of cases that have to be considered for such path systems resp. blocked nodes we simplify the structure of the path system.

Lemma 4 *Let $(\mathcal{P}, B)$ be a path system on a clique tree $C = (V_C, E_C)$ with handles $\mathcal{H}$, teeth $\mathcal{T}$ and $\bigcup_{T \in \mathcal{T}} T \cap B = \emptyset$. Then there exists a path system $(\mathcal{P}', B)$ with $\omega(\mathcal{P}', B) \geq \omega(\mathcal{P}, B)$ that fulfills the property*

a) $\mathcal{H}$-$\mathcal{T}$-loop-free*: for all $H \in \mathcal{H}, T \in \mathcal{T}, H \cap T \neq \emptyset$ there exists no $P = Pxx'Py'yP \in \mathcal{P}'$ with $x, y \in H \cap T$ and $V(x'Py') \cap (H \cap T) = \emptyset$,*

and $\mathcal{E}(\mathcal{P}) \subseteq \mathcal{E}(\mathcal{P}')$.

Proof. Let $(\mathcal{P}, B)$ be a path system with minimal number of $\mathcal{H}$-$\mathcal{T}$-loops. In this proof the set of blocked nodes remains unchanged. Let $P \in \mathcal{P}$ be a path with $P = pPxx'Py'yPq \in \mathcal{P}$ such that there exists $H \in \mathcal{H}, T \in \mathcal{T}$ with $x, y \in H \cap T, x', y' \in (H \cup T) \setminus (H \cap T)$

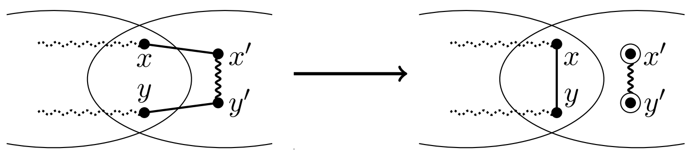


Figure 6: Visualization of an $\mathcal{H}$-$\mathcal{T}$-loop and of the reordered path system according to the proof of Lemma 4. In all figures straight lines correspond to edges. We use solid waved lines if there is a (possibly empty) path between the nodes and dotted waved lines for unspecified continuations of paths. All end nodes are highlighted.

and $V(x'Py') \cap H \cap T = \emptyset$ (see Fig. 6). Then we can replace P by two new paths $P' := x'Py', P'' := pPxyPq$ setting $\mathcal{P}' := (\mathcal{P} \setminus \{P\}) \cup \{P', P''\}$. Because the edges xx', yy' are counted once in ω and xy is counted twice it holds $\omega(\mathcal{P}', B) = \omega(\mathcal{P}, B)$ and the considered system was not minimal because $\mathcal{P}'$ contains one $\mathcal{H}$-$\mathcal{T}$-loop less than $\mathcal{P}$. Furthermore it holds $\mathcal{E}(\mathcal{P}) \subsetneq \mathcal{E}(\mathcal{P}')$ because all end nodes in $\mathcal{P}$ remain end nodes of $\mathcal{P}'$ and we get at least one additional end node. $\square$

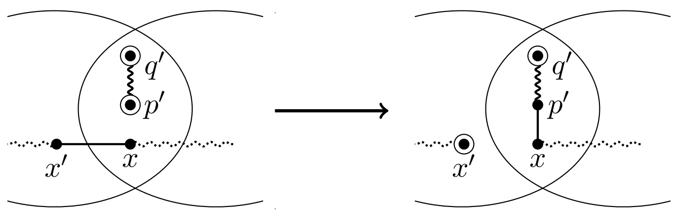
Lemma 5 *Let $(\mathcal{P}, B)$ be a path system on a clique tree $C = (V_C, E_C)$ with handles $\mathcal{H}$, teeth $\mathcal{T}$ and $\bigcup_{T \in \mathcal{T}} T \cap B = \emptyset$ that fulfills property a). Then there exists a path system $(\mathcal{P}', B)$ with $\omega(\mathcal{P}', B) \geq \omega(\mathcal{P}, B)$ that fulfills the properties a) and*

b) $\mathcal{H}$-$\mathcal{T}$-one-path: for all $H \in \mathcal{H}, T \in \mathcal{T}, H \cap T \neq \emptyset$ there exists exactly one path $P \in \mathcal{P}'$ with $V(P) \cap (H \cap T) \neq \emptyset$.

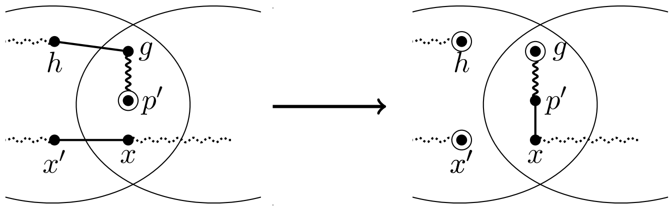
Furthermore it holds $\{Z \in \mathcal{H} \cup \mathcal{T} \colon Z \cap \mathcal{E}(\mathcal{P}) \neq \emptyset\} \subset \{Z \in \mathcal{H} \cup \mathcal{T} \colon Z \cap \mathcal{E}(\mathcal{P}') \neq \emptyset\}$, i. e., all $Z \in \mathcal{H} \cup \mathcal{T}$ that contain an end node in $(\mathcal{P}, B)$ also contain an end node in $(\mathcal{P}', B)$ but there might be further $Z' \in \mathcal{H} \cup \mathcal{T}$ containing end nodes for $(\mathcal{P}', B)$.

Proof. Let $(\mathcal{P}, B)$ be an $\mathcal{H}$-$\mathcal{T}$-loop-free path system with minimal number of paths $P \in \mathcal{P}$ with $V(P) \cap (H \cap T) \neq \emptyset$ for all $H \in \mathcal{H}, T \in \mathcal{T}, H \cap T \neq \emptyset$. In this proof the set of blocked nodes remains unchanged. Let $H \in \mathcal{H}, T \in \mathcal{T}, H \cap T \neq \emptyset$ and assume there exist two paths $P, P' \in \mathcal{P}, P \neq P'$, with $V(P) \cap (H \cap T) \neq \emptyset$, $V(P') \cap (H \cap T) \neq \emptyset$. We distinguish two main cases. First, if $V(P) \subset (H \cap T)$ and $V(P') \subset (H \cap T)$ we can simply join the two paths $P = Px, P' = yP'$ to one path $P'' := PxyP'$, *i. e.*, $\mathcal{P}' := (\mathcal{P} \setminus \{P, P'\}) \cup \{P''\}$, $\omega(\mathcal{P}', B) = \omega(\mathcal{P}, B) + 2$. Both end nodes of P'' lie in $H \cap T$. If w. l. o. g. $V(P) \not\subset (H \cap T)$ with $P = pPxx'Pq, x \in H \cap T, x' \in (H \cup T) \setminus (H \cap T)$ and

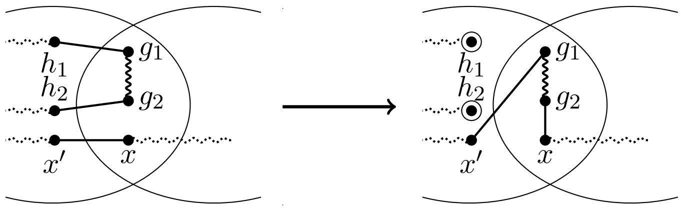
1) $V(P') \subset (H \cap T), P' = p'P'q'$: then $P'' := pPxp'P'q', \mathcal{P}' := (\mathcal{P} \setminus \{P, P'\}) \cup \{P'', x'Pq\}$ fulfills condition a), $\omega(\mathcal{P}', B) = \omega(\mathcal{P}, B) + 1$ and q' is an end node in H and T.



2) $\{gh\} = \{z_1z_2 \in P' \colon z_1 \in H \cap T, z_2 \in (H \cup T) \setminus (H \cap T)\}, g \in H \cap T, P' = p'P'ghP'q'$: then $P'' := pPxp'P'g, \mathcal{P}' := (\mathcal{P} \setminus \{P, P'\}) \cup \{P'', hP'q', x'Pq\}$ fulfills condition a), $\omega(\mathcal{P}', B) = \omega(\mathcal{P}, B)$ and g is an end node in H and T.



3) $\{g_1h_1, g_2h_2\} = \{z_1z_2 \in P' : z_1 \in H \cap T, z_2 \in (H \cup T) \setminus (H \cap T)\}, g_1h_1 \neq g_2h_2, g_1, g_2 \in H \cap T, P' = p'P'h_1g_1P'g_2h_2P'q'$: we set $P'' := pPxg_2P'g_1x'Pq$. Then $(\mathcal{P}', B)$ with $\mathcal{P}' = (\mathcal{P} \setminus \{P, P'\}) \cup \{P'', p'P'h_1, h_2P'q'\}$ fulfills condition a) and $\omega(\mathcal{P}', B) = \omega(\mathcal{P}, B)$ but the number of paths $P \in \mathcal{P}'$ with $V(P) \cap (H \cap T)$ in the considered T, H has been reduced be one. Because we only enlarged path P by putting a path between nodes x, x' and some edges are deleted it holds $\{Z \in \{H, T\} : \mathcal{E}(\mathcal{P}) \cap Z \neq \emptyset\} \subset \{Z \in \{H, T\} : \mathcal{E}(\mathcal{P}') \cap Z \neq \emptyset\}$.



Note that because of $\bigcup_{T \in \mathcal{T}} T \cap B = \emptyset$ and a) these are all cases to consider. In all cases the considered system was not minimal and so the statement follows. $\square$

Definition 6 *A path system $(\mathcal{P}, B)$ that satisfies $\bigcup_{T \in \mathcal{T}} T \cap B = \emptyset$ and conditions a)–b) is called* simple *path system.*

Now we show how to delete blocked nodes in $v \in B \cap (\bigcup_{H \in \mathcal{H}} H \setminus (\bigcup_{T \in \mathcal{T}} T))$. Therefore we transform simple path systems so that we can move end nodes of paths to other teeth or handles.

Lemma 7 *Let $C = (V_C, E_C)$ be a clique tree with handles $\mathcal{H}$, teeth $\mathcal{T}$ and $(\mathcal{P}, B)$ be a simple path system in C. Let $X, Y \in \mathcal{H} \cup \mathcal{T}$, $X \neq Y$, $X \cap Y \neq \emptyset$ so that $X \cap \mathcal{E}(\mathcal{P}) = \emptyset$ and $Y \cap \mathcal{E}(\mathcal{P}) \neq \emptyset$. Then there exists a path system $(\mathcal{P}', B)$ with $X \cap \mathcal{E}(\mathcal{P}') \neq \emptyset$ and $\omega(\mathcal{P}', B) = \omega(\mathcal{P}, B)$.*

Before we prove Lemma 7 we state the following corollary.

Corollary 8 *Let $C = (V_C, E_C)$ be a clique tree with handles $\mathcal{H}$ and teeth $\mathcal{T}$ and let $(\mathcal{P}, B)$ be a simple path system in C. Fix some $X \in \mathcal{H} \cup \mathcal{T}$. Then there exists a simple path system $(\mathcal{P}', B)$ with $\mathcal{E}(\mathcal{P}') \cap X \neq \emptyset$ and $\omega(\mathcal{P}', B) = \omega(\mathcal{P}, B)$.*

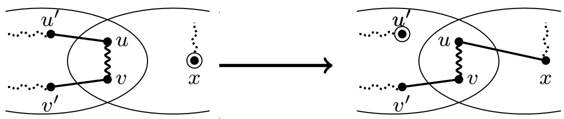
Proof. Because $\mathcal{P}$ is a nonempty set of paths there must be a sequence $X_1, \ldots, X_k$, $k \in \mathbb{N}$, $X_i \in \mathcal{H} \cup \mathcal{T}$, $i = 1, \ldots, k$, with $X_1 = X$, $\mathcal{E}(\mathcal{P}) \cap X_i = \emptyset$, $i = 1, \ldots, k-1$, and $\mathcal{E}(\mathcal{P}) \cap X_k \neq \emptyset$. Applying Lemma 7 repeatedly on $X_{k-1}, X_{k-2}, \ldots, X_1$ we end up with a path system $(\mathcal{P}', B)$ with the desired properties. $\square$

Note that Corollary 8 allows to transform any simple path system so that some specific tooth or handle contains at least one end node.

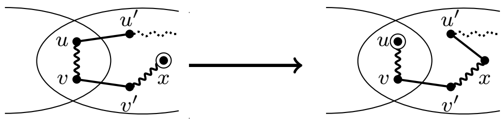
Proof (of Lemma 7). We prove the assertion by induction on $l := |\mathcal{H} \cup \mathcal{T}|$. Obviously the claim holds for $l = 1$, so we may assume it holds for all clique trees with $|\mathcal{H} \cup \mathcal{T}| < l$.

Let $S := X \cap Y \neq \emptyset$. $S \subset X$ contains no end nodes by assumption on X, thus there must be exactly one path $P \in \mathcal{P}$ (since $\mathcal{P}$ is simple) with $P = pPu'uPvv'Pq$ and $V(uPv) \subseteq S$ and $u', v' \notin S$. We distinguish w. l. o. g. five cases depending on the positions of u', v' and the end node $x \in Y \setminus S$.

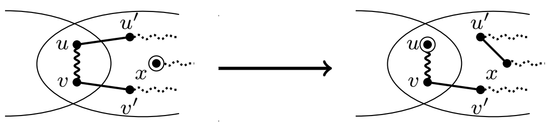
1) $u', v' \in X$: The end node $x \in Y \setminus S$ does not belong to P. Let x be the end node of path $Q = Qx \in \mathcal{P}$ (possibly $V(Q) = \{x\}$) and set $P' := pPu'$ and $Q' := QxuPq$. Because Q and P are disjoint paths Q' does not contain a cycle and hence is a path. The path system $\mathcal{P}' := (\mathcal{P} \setminus \{P, Q\}) \cup \{P', Q'\}$ fulfills the requirements: It holds $E(\mathcal{P}') = (E(\mathcal{P}) \setminus \{u'u\}) \cup \{xu\}$ and $u'u, xu \notin S^{\{2\}}$, thus $\omega(\mathcal{P}', B) = \omega(\mathcal{P}, B)$ and $(\mathcal{P}', B)$ remains simple (no $\mathcal{H}$-$\mathcal{T}$-loop can arise) and contains the end node $u' \in \mathcal{E}(\mathcal{P}) \cap X$.



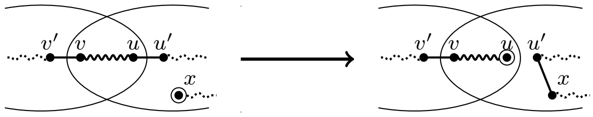
2) $u', v' \in Y$ and $P = pPu'Pv'Px$, *i. e.* $(x = q)$: The end node $x \in Y \setminus S$ is an end node of P. We set $P' := pPu'xPu$ and $\mathcal{P}' := (\mathcal{P} \setminus \{P\}) \cup \{P'\}$. As above it is easy to see that $\mathcal{P}'$ remains simple and contains the end node $u \in \mathcal{E}(\mathcal{P}') \cap X$. Because of $E(\mathcal{P}') = (E(\mathcal{P}) \setminus \{u'u\}) \cup \{u'x\}$ we have $\omega(\mathcal{P}', B) = \omega(\mathcal{P}, B)$, thus $\mathcal{P}'$ fulfills the requirements.



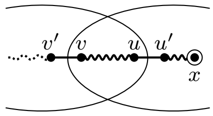
3) $u', v' \in Y$ and $x \notin V(P)$: x is end node of a path $Q = Qx \in \mathcal{P}$, $P \neq Q$. We set $P' := uPq$ and $Q' := Qxu'Pp$ (both are paths because $P \neq Q$) and $\bar{\mathcal{P}} := (\mathcal{P} \setminus \{P, Q\}) \cup \{P', Q'\}$. $\bar{\mathcal{P}}$ contains end node $u \in \mathcal{E}(\bar{\mathcal{P}}) \cap X$ and by $E(\bar{\mathcal{P}}) = (E(\mathcal{P}) \setminus \{u'u\}) \cap \{xu'\}$ we have $\omega(\bar{\mathcal{P}}, B) = \omega(\mathcal{P}, B)$. The path $Q' \in \bar{\mathcal{P}}$ may contain an $\mathcal{H}$-$\mathcal{T}$-loop, but applying Lemma 4 to $(\bar{\mathcal{P}}, B)$ leads to a simple path system $(\mathcal{P}', B)$ with $u \in \mathcal{E}(\bar{\mathcal{P}}) \subset \mathcal{E}(\mathcal{P}')$ and $\omega(\mathcal{P}', B) = \omega(\bar{\mathcal{P}}, B) = \omega(\mathcal{P}, B)$.



4) $v' \in X$, $u' \in Y$ and $x \notin V(P)$: Analogous to the previous case we set $P' := uPq$ and $Q' := Qxu'Pp$ leading to a system $\bar{\mathcal{P}}$ that may contain an $\mathcal{H}$-$\mathcal{T}$-loop. As before applying Lemma 4 leads to a simple system $(\mathcal{P}', B)$ fulfilling the requirements.



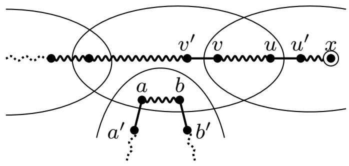
5) $v' \in X$, $u' \in Y$ and $x \in V(P)$: This is the most difficult case. We have to distinguish further cases depending on $X \in \mathcal{H}$ or $X \in \mathcal{T}$ and the intersections with further sets. Note that we may assume $\mathcal{E}(\mathcal{P}) \cap Y = \{x\}$ because otherwise we could apply case 4).



5.1) $X \in \mathcal{H}$.

5.1.1) $\exists\, Z \in \mathcal{T} \setminus \{Y\}$ and $Q = dQa'aQbb'Qd' \in \mathcal{P}$ with $V(aQb) \subseteq S' := X \cap Z$, $a', b' \in Z \setminus S'$.

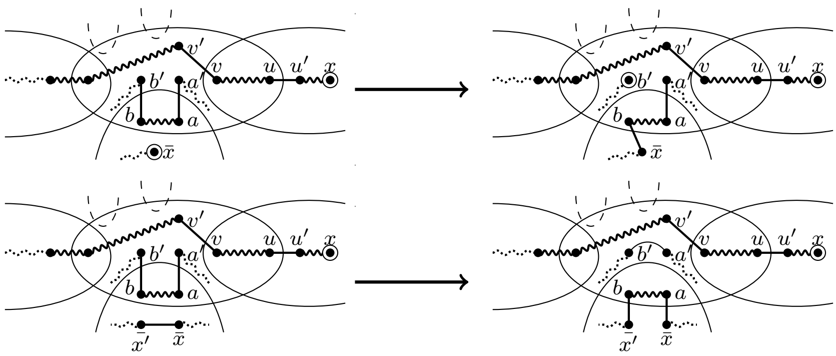
We use induction. Let $C' = (V'_C, E'_C)$ be the subcliquetree that arises by deleting X from $\mathcal{H}$, *i. e.* C' is the component of $C \setminus (X \setminus Z)$ that contains Z. Define the path system $(\bar{\mathcal{P}}, \bar{B})$ with $\bar{\mathcal{P}} := \{P \in \mathcal{P} \colon V(P) \subseteq V'_C\}$, $\bar{B} := B \cap V'_C$. Note that by assumption there is no path $P \in \mathcal{P}$ with $V(P) \cap V'_C \neq \emptyset$ and $V(P) \setminus V'_C \neq \emptyset$. Now we apply Corollary 8 to C' with path system $(\bar{\mathcal{P}}, \bar{B})$. This gives us a simple path system $(\bar{\mathcal{P}}', \bar{B})$ in C' with an end node $x' \in Z \cap \mathcal{E}(\bar{\mathcal{P}}')$. We combine $\bar{\mathcal{P}}'$ and $\mathcal{P} \setminus \bar{\mathcal{P}}$ to a path system $(\tilde{\mathcal{P}}, B)$ in C with an end node in Z. Note that this path system is not necessarily simple because $X \cap Z$ is not an intersection of a tooth and a handle in C'. But we can apply first Lemma 4 and then Lemma 5 to $(\tilde{\mathcal{P}}, B)$. This leads to a simple path system $(\tilde{\mathcal{P}}', B)$ preserving at least one end node in Z. Now by construction $\tilde{\mathcal{P}}'$ has either an end node in $S' \subset Z$ or it contains a path $\tilde{Q} = \tilde{Q}c'c\tilde{Q}dd'\tilde{Q}$ with $V(c\tilde{Q}d) \subseteq S'$ and $c', d' \in Z \setminus S'$. Thus we are now either in case 2) or 3) using Z instead of Y.



5.1.2) $\exists\, Z \in \mathcal{T} \setminus \{Y\}$ and $Q = dQa'aQbb'Qd' \in \mathcal{P}$ with $V(aQb) \subseteq S' := X \cap Z$, $a', b' \in X \setminus S'$ (possibly $Q = P$, but we only visualize the case $Q \neq P$, the other case looks quite similar).

Because $Z \in \mathcal{T}$ there exists an $\bar{x} \in Z \setminus \bigcup_{H \in \mathcal{H}} H$. If this $\bar{x}$ is an end node, *i. e.* $\exists\, R \in \mathcal{P}$ with $R = \bar{x}R$, then we set $\tilde{\mathcal{P}} := (\mathcal{P} \setminus \{P, R\}) \cup \{d'Qb', dQb\bar{x}R\}$ and get $\omega(\mathcal{P}, B) = \omega(\tilde{\mathcal{P}}, B)$ as well as an end node $b' \in X$.

If this $\bar{x}$ is not an end node, *i. e.* $\exists\, R \in \mathcal{P}$ with $R = rR\bar{x}\bar{x}'Rr'$, then we set $\tilde{\mathcal{P}} := (\mathcal{P} \setminus \{P, R\}) \cup \{d'Qb'a'Qd, rR\bar{x}aQb\bar{x}'Rr'\}$. This fulfills $\omega(\mathcal{P}, B) = \omega(\tilde{\mathcal{P}}, B)$ and we can apply case 5.1.1).

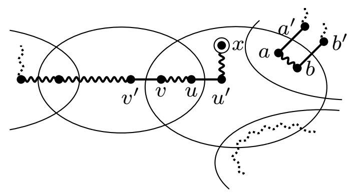


Because by assumption X contains no end node and is a handle $X \in \mathcal{H}$, which intersects with an odd number of teeth greater than one, we know that those two cases above are exhaustive.

5.2) $X \in \mathcal{T}$. In this case $Y \in \mathcal{H}$. We consider three cases, the first two are analogous to the previous two cases.

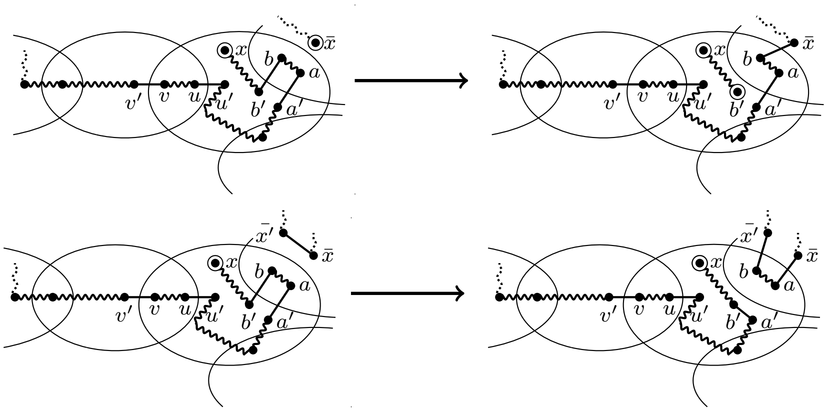
5.2.1) $\exists\, Z \in \mathcal{T} \setminus \{X\}$ and $Q = dQa'aQbb'Qd' \in \mathcal{P}$ with $V(aQb) \subseteq S' := Y \cap Z$, $a', b' \in Z \setminus S'$.

As in 5.1.1) we get by induction hypothesis a simple path system $(\tilde{\mathcal{P}}, B)$ with an end node in Z. Analogous to either case 2) or 3) we get a path system $(\tilde{\mathcal{P}}', B)$ with *an additional* end node $\tilde{x} \neq x$ in $Y \cap Z$, so we may apply case 4) with end node $\tilde{x}$.

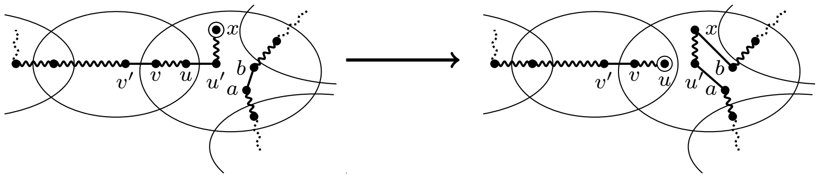


5.2.2) $\exists\, Z \in \mathcal{T} \setminus \{X\}$ and $P = pPuPa'aPbb'Px \in \mathcal{P}$ with $V(aPb) \subseteq S' := Y \cap Z$, $a', b' \in Y \setminus S'$.

Because $Z \in \mathcal{T}$ there exists an $\bar{x} \in Z \setminus \bigcup_{H \in \mathcal{H}} H$. If this $\bar{x}$ is an end node, *i. e.* $\exists\, R \in \mathcal{P}$ with $R = \bar{x}R$, then we set $\tilde{\mathcal{P}} := (\mathcal{P} \setminus \{P, R\}) \cup \{pPuPb\bar{x}R, b'Px\}$. If this $\bar{x}$ is not an end node, *i. e.* $\exists\, R \in \mathcal{P}$ with $R = rR\bar{x}\bar{x}'Rr'$, then we set $\tilde{\mathcal{P}} := (\mathcal{P} \setminus \{P, R\}) \cup \{pPuPa'b'Px, rR\bar{x}aPb\bar{x}'Rr'\}$. In both cases $\omega(\mathcal{P}, B) = \omega(\tilde{\mathcal{P}}, B)$. In the first case it holds $b' \in \mathcal{E}(\tilde{\mathcal{P}})$ that is not contained in the path containing the subpath $v'vPuu'$ and in the second case we can apply case 4) respectively 5.2.1).



5.2.3) If neither of the previous two cases occurs we know by $Y \in \mathcal{H}$ (and thus Y intersects with an odd number of sets of $\mathcal{T}$) that $\mathcal{P}$ must contain a path $Q = \tilde{p}QabQ\tilde{q}$ with $ab \in E(Y) \setminus E(\mathcal{T})$. We set $P' := uPq$ and $Q' := \tilde{p}Qau'PxbQ\tilde{q}$ and set $\mathcal{P}' := (\mathcal{P} \setminus \{P, Q\}) \cup \{P', Q'\}$. It is easy to check that $\mathcal{P}'$ is simple, contains the end node $u \in X \cap \mathcal{E}(\mathcal{P}')$ and it holds $\omega(\mathcal{P}', B) = \omega(\mathcal{P}, B)$.



$\square$

If there is a blocked node $v \in B \cap H$ in a handle $H \in \mathcal{H}$ and H contains an end node w, *i. e.* there exists $P = Pw \in \mathcal{P}$, we can force an edge vw and delete v from B.

Corollary 9 *Let $(\mathcal{P}, B)$ be a simple path system. Then there exists a path system $(\mathcal{P}', \emptyset)$ with $\omega(\mathcal{P}, B) \leq \omega(\mathcal{P}', \emptyset)$.*

Proof. Let $X \in \mathcal{H}$ with $B \cap X \neq \emptyset$. Corollary 8 implies there is a path system $(\tilde{\mathcal{P}}, B)$ with $\omega(\tilde{\mathcal{P}}, B) = \omega(\mathcal{P}, B)$ and $x \in X \cap \mathcal{E}(\tilde{\mathcal{P}})$ and $B \cap X = \{b_1, \dots, b_k\} \neq \emptyset$. Let $P \in \tilde{\mathcal{P}}$ be the path with $P = Px$. Set $\tilde{\mathcal{P}}' := (\tilde{\mathcal{P}} \setminus \{P\}) \cup \{Pxb_1 \dots b_k\}$ and $\tilde{B}' := B \setminus \{b_1, \dots, b_k\}$. Then $\omega(\tilde{\mathcal{P}}', \tilde{B}') = \omega(P, B)$. Continuing like this removing all blocked nodes we get a path system $(\mathcal{P}', \emptyset)$ as desired. $\square$

Our main result is the following theorem that includes two variants for lifting clique tree inequalities. In order to use the previous results for path systems by transforming a given tour K the lifted inequalities have to fulfill that a 2-edge $ijk, i, k \in S_2$ and the edges ij, jk are not counted simultaneously for the tour (blocked nodes are not contained in paths). Furthermore we have to achieve that for each $T \in \mathcal{T}$ there exists a node $v \in T \setminus \bigcup_{H \in \mathcal{H}} H$ that does not become a blocked node.

Theorem 10 *Let $C = (V_C, E_C)$ be a clique tree with handles $\mathcal{H}$ and teeth $\mathcal{T}$ according to Definition 1. Let $S_1, S_2 \subset V \setminus V_C$ be two sets with $S_1 \cap S_2 = \emptyset, V = V_C \cup S_1 \cup S_2, S_1 \neq \emptyset, S_2 \neq \emptyset$. Let $W_1, W_2 \subseteq V_C$ be two arbitrary subsets satisfying*

$$
\begin{aligned}
&\bigcup_{H \in \mathcal{H}} H \subseteq W_i, i = 1, 2, \\
&\forall T \in \mathcal{T}, |T \setminus \bigcup_{H \in \mathcal{H}} H| < |S_2|: \quad |T \setminus W_1| = 1 \\
&\forall T \in \mathcal{T}: \qquad\qquad\qquad\qquad\quad |T \setminus W_2| = 1
\end{aligned}
$$

Then the inequalities

$$\sum_{Z \in \mathcal{H} \cup \mathcal{T}} \sum_{kl \in Z^{\{2\}}} x_{kl} + \sum_{Z \in \mathcal{H} \cup \mathcal{T}} \sum_{\substack{kl \in Z^{\{2\}}, m \in S_1 \cup V_C: \\ km, lm \notin E_C}} y_{kml} + \sum_{\substack{kml \in V^{\langle 3 \rangle}: \\ k,l \in S_2, m \in W_1}} y_{kml} \leq s(C), \tag{12}$$

$$\sum_{Z \in \mathcal{H} \cup \mathcal{T}} \sum_{kl \in Z^{\{2\}}} x_{kl} + \sum_{Z \in \mathcal{H} \cup \mathcal{T}} \sum_{\substack{kl \in Z^{\{2\}}, \\ m \in S_1}} y_{kml} + \sum_{\substack{kml \in V^{\langle 3 \rangle}: \, k,l \in S_2 \cup V_C \\ m \in W_2, km, ml \notin E_C}} y_{kml} \leq s(C) \tag{13}$$

are valid for $P_{\mathbf{SQTSP}_n}$ for each choice of W_1, W_2.

The choice of sets W_1, W_2 ensures that the corresponding path system that is used in the proof fulfills $(T \setminus \bigcup_{H \in \mathcal{H}} H) \nsubseteq B$ either by definition or as otherwise a subtour would be implied.

Remark 11 *For $S_2 = \{\bar{t}\}$ and $|V_C| \geq \frac{|V|}{2}$ inequalities (12) are equivalent to clique tree inequalities strengthened by (6).*

Using the previous results we are able to prove our main result Theorem 10.

Proof (of Theorem 10). Let $K \in \mathcal{K}_n$ be an arbitrary tour. We have to show that K fulfills (12) and (13) for appropriate choices of a clique tree $C = (V_C, E_C)$ and sets S_1, S_2, W_1, W_2. We will show this by constructing dominating tours $\bar{K}_1, \bar{K}_2 \in \mathcal{K}_n$ that fulfill the following properties

- they do not contain 2-edges with coefficients greater zero in (12) resp. (13),

- calculating the left-hand side of (12) resp. (13) for K these values are not greater than the computed left-hand side of (10) for $\bar{K}_1$ resp. $\bar{K}_2$.

Then the validity of (12) resp. (13) follows from the validity of the clique tree inequalities (10) for $P_{\mathbf{STSP}_n}$.

We start with the validity of (12) and build up a path system $(\mathcal{P}_1, B_1)$ that is empty at the beginning. The blocked nodes B_1 correspond to the 2-edges counted in the last sum,

$$B_1 := \{v \in V_C \colon uvw \in K \text{ with } u, w \in S_2, v \in W_1\}.$$

Note that by construction we have $|\{uvw \in K \colon u, w \in S_2\}| = 1$ for all $v \in B_1$, hence each blocked node corresponds to exactly one counted 2-edge.

Next, we specify all edges of the path system and collect them in the set P_1.

$$P_1 := (K^{\{2\}} \cap E_C) \cup \{uw \in E_C \colon \exists\, v \in S_1 \cup V_C, uvw \in K, uv, vw \notin E_C\}.$$

Because K is a tour, and thus does not contain a subtour, and $S_2 \neq \emptyset$ we know by the same arguments as in the proofs of the standard lifting approach [6] that for each node v it holds $|\{vw \in P_1\}| \leq 2$. Furthermore the edges in P_1 do not form a cycle because this cycle would not visit all nodes (in particular, it would not visit S_2) and would be therefore a subtour. By construction it holds for each $v \in B_1$ that $\{uv \in P_1\} = \emptyset$. This implies that the edges of P_1 together with the isolated nodes in $V_C \setminus (V(P_1) \cup B_1)$ correspond to a set of paths $\mathcal{P}_1$ so that $(\mathcal{P}_1, B_1)$ is a path system in C. We further know that each tooth $T \in \mathcal{T}$ fulfills $(T \setminus \bigcup_{H \in \mathcal{H}} H) \not\subset B_1$, either by the definition of W_1 or because otherwise K would contain a subtour using only nodes in T and S_2.

We denote by $(\mathcal{P}_1^{(k)}, B_1^{(k)}), k \in \mathbb{N}$, a family of path systems. Using Lemma 3, Lemma 4 and Lemma 5 we get a simple path system $(\bar{\mathcal{P}}_1, \bar{B}_1)$ and by applying Corollary 9 we get $(\mathcal{P}_1^{(1)}, B_1^{(1)}), B_1^{(1)} = \emptyset$. This path system can then be extended to a cycle $\bar{K}_1$ in the following way. As long as there exists a $Z \in \mathcal{H} \cup \mathcal{T}$ such that $\mathcal{P}_1^{(k)}, k \in \mathbb{N}$, contains two paths $P, Q \in \mathcal{P}_1^{(k)}, P \neq Q, P = Px, x \in Z$, and $Q = Qy, y \in Z$, we set $\mathcal{P}_1^{(k+1)} = (\mathcal{P}_1^{(k)} \setminus \{P, Q\}) \cup \{PxyQ\}$ that enlarges the weight, *i. e.*, $\omega(\mathcal{P}_1^{(k)}, \emptyset) + 1 \leq \omega(\mathcal{P}_1^{(k+1)}, \emptyset)$. Let $(\mathcal{P}^{(\hat{k})}, \emptyset)$ denote the path system after the path connection steps. Then we get a tour $\bar{K}_1$ be simply hanging the paths in $\mathcal{P}^{\hat{k}}$ at the end nodes together until there remains only one path. At its end nodes this path is connected to the path $s_1^1 \dots s_{|S_1|}^1 s_1^2 \dots s_{|S_2|}^2$ of the nodes $\{s_1^1, \dots, s_{|S_1|}^1\} \in S_1$ and $\{s_1^2, \dots, s_{|S_2|}^2\} \in S_2$. By construction the 2-edges $ijk \in \bar{K}_1$ have coefficient zero in (12). Because $\omega(\mathcal{P}_1, B_1) \leq \omega(\bar{\mathcal{P}}_1, \bar{B}_1) \leq \omega(\mathcal{P}_1^{(\hat{k})}, \emptyset)$ the left-hand side of (10) (or (12)) calculated for $\bar{K}_1$ is at least as big as the left-hand side of (12) for K_1. Thus, (12) is valid for $P_{\mathbf{SQTSP}_n}$ under the given assumptions.

For the construction of $\bar{K}_2$ we start with defining the set of blocked nodes B_2 corresponding to the 2-edges counted in the last sum,

$$B_2 := \{v \in V_C \colon uvw \in K \text{ with } u, w \in S_2 \cup V_C, km, ml \notin E_C, v \in W_2\}.$$

The edge set P_2 is defined as

$$P_2 := (K^{\{2\}} \cap E_C) \cup \{uw \in E_C \colon \exists\, v \in S_1, uvw \in K\}.$$

Because even less edges are inserted we know by the same arguments as above that for each node v it holds $|\{vw \in P_2\}| \leq 2$ and that the edges in P_2 do not form a cycle. Again by construction it holds for each $v \in B_2$ that $\{uv \in P_2\} = \emptyset$. So analogously

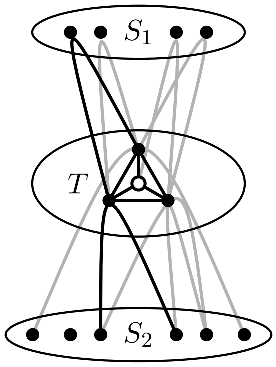


Figure 7: Visualization of a subtour elimination constraint for one tooth with four nodes lifted according to (13) with W_2 the three corner nodes of the triangle. At most three of the edges and 2-edges can be contained in a tour.

to above we get the path system $(\mathcal{P}_2, B_2)$ in C. Again we know that each tooth $T \in \mathcal{T}$ fulfills $(T \setminus \bigcup_{H\in\mathcal{H}} H) \not\subset B_2$ by the definition of W_2. As above we get a new path system $(\mathcal{P}'_2, B'_2), B'_2 = \emptyset$. This path system can then be extended to a cycle $\bar{K}_2$ by simply hanging the paths in $\mathcal{P}'_2$ at the end nodes together to one long path. These is connected to the path $s^1_1 \dots s^1_{|S_1|} s^2_1 \dots s^2_{|S_2|}$ of the nodes $s^1_1, \dots, s^1_{|S_1|} \in S_1$ and $s^2_1, \dots, s^2_{|S_2|} \in S_2$ on both ends. By construction the 2-edges $ijk \in \bar{K}_2$ have coefficient zero in (13) and the left-hand side of (10) is at least as big as the left-hand side of (13) for K_2. This proves Theorem 10. $\square$

The result in Theorem 10 can be seen as general lifting approach for improving clique tree inequalities for $\mathbf{SQTSP}_n$. Applying it to the subtour elimination constraints, which are a special case of the clique tree constraints with exactly one tooth and no handle, leads to the following result.

Remark 12 *Let $I, S_1, S_2 \subset V, V = I \dot\cup S_1 \dot\cup S_2, I \cap S_1 = \emptyset, I \cap S_2 = \emptyset, S_1 \cap S_2 = \emptyset, S_1 \neq \emptyset$. The inequalities*

$$\sum_{ij\in I^{\{2\}}} x_{ij} + \sum_{ikj\in V^{\langle 3\rangle}\,:\; i,j\in I, k\in S_1} y_{ikj} + \sum_{kil\in V^{\langle 3\rangle}\,:\; i\in I, k,l\in S_2} y_{kil} \leq |I| - 1 \tag{14}$$

for $2 \leq |S_2| \leq |I|$, and for some $\bar{\imath} \in I$

$$\sum_{ij\in I^{\{2\}}} x_{ij} + \sum_{ikj\in V^{\langle 3\rangle}\,:\; i,j\in I, k\in S_1} y_{ikj} + \sum_{kil\in V^{\langle 3\rangle}\,:\; i\in I\setminus\{\bar{\imath}\}, k,l\in S_2} y_{kil} \leq |I| - 1 \tag{15}$$

for $|S_2| > |I| \geq 2$ are valid for $P_{\mathbf{SQTSP}_n}$, because they are clique-tree constraints (12) *resp.* (13) *with $V_C = I$, $\mathcal{H} = \emptyset$ and $\mathcal{T} = \{I\}$. In the case $|I| = 2$ inequalities* (14) *are equivalent to the special* conflicting edges inequalities (8) *and* (15) *are equivalent to the standard* conflicting edges inequalities (7). *Furthermore one can prove for $|I| \geq 3$ with methods similar to the ones used in [6] for proving the facetness of several inequality classes that inequalities* (14) *and* (15) *as above define facets of $P_{\mathbf{SQTSP}_n}$ if $|S_1 \cup S_2| \geq 5$ or $|S_2| = 3$ (see Appendix).*

Unfortunately one cannot expect that applying this approach leads to facets in general. Here future research is needed to improve the understanding of complex facet classes of $P_{\mathbf{SQTSP}_n}$.

4 The asymmetric case

The asymmetric quadratic traveling salesman problem is the problem of finding a directed Hamiltonian 2-cycle (a tour) in a weighted directed 2-graph $G = (V, A)$ with node set $V, |V| = n$, set of directed 2-arcs $A = V^{(3)} = \{(i,j,k) \colon i,j,k \in V, |\{i,j,k\}| = 3\}$, set of associated directed arcs $V^{(2)} = \{(i,j) \colon i,j \in V, i \neq j\}$ and 2-arc weights $c_{(i,j,k)}, (i,j,k) \in A$. The corresponding polytope reads

$$
\begin{aligned}
P_{\mathbf{AQTSP}_n} = \operatorname{conv}\{(\bar{x},\bar{y}) \in \{0,1\}^{n(n-1)+n(n-1)(n-2)} \colon & \\
& \sum_{(i,j)\in V^{(2)}} \bar{x}_{(i,j)} = \sum_{(j,i)\in V^{(2)}} \bar{x}_{(j,i)} = 1 \text{ for all } i \in V; \\
& \bar{x}_{(i,j)} = \sum_{(i,j,k)\in V^{(3)}} \bar{y}_{(i,j,k)} = \sum_{(k,i,j)\in V^{(3)}} \bar{y}_{(k,i,j)} \text{ for all } (i,j) \in V^{(2)}; \\
& \sum_{(i,j)\in S^{(2)}} \bar{x}_{(i,j)} \leq |S| - 1 \text{ for all } S \subset V, 2 \leq |S| \leq n-2\},
\end{aligned}
$$

see [5]. It equals the convex hull over all incidence vectors of directed Hamiltonian 2-cycles.

Definition 13 *A valid inequality* $(a^1)^T\bar{x} + (a^2)^T\bar{y} \leq b$ *of* $P_{\mathbf{AQTSP}_n}$ *is called* coefficient-symmetric *if* $(a^1)_{(i,j)} = (a^1)_{(j,i)}$ *for all* $(i,j) \in V^{(2)}$ *and* $(a^2)_{(i,j,k)} = (a^2)_{(k,j,i)}$ *for all* $(i,j,k) \in V^{(3)}$.

Like for the STSP and ATSP [12], a valid inequality $(a^1)^T x + (a^2)^T y \leq b$ for $P_{\mathbf{SQTSP}_n}$ leads to a coefficient-symmetric valid inequality $(\hat{a}^1)^T\bar{x} + (\hat{a}^2)^T\bar{y} \leq b$ of $P_{\mathbf{AQTSP}_n}$ with coefficients $(\hat{a}^1)_{(i,j)} = (\hat{a}^1)_{(j,i)} = (a^1)_{ij}$ for $ij \in V^{\{2\}}$ and $(\hat{a}^2)_{(i,j,k)} = (\hat{a}^2)_{(k,j,i)} = (a^2)_{ijk}$ for $ijk \in V^{\langle 3\rangle}$ and vice versa. In [7] Fischetti proved that the coefficient-symmetric variant of the clique tree inequalities (10) define facets for ATSP. He also remarked that coefficient-symmetric facets of $P_{\mathbf{ATSP}_n}$ lead to facets of $P_{\mathbf{STSP}_n}$. With these arguments one can show that the coefficient-symmetric variants of (12) and (13) are valid inequalities for $P_{\mathbf{AQTSP}_n}$. For the simple case of the bound constraints $x_{ij} \leq 1, ij \in V^{\{2\}}$ resp. the subtour elimination constraints $\bar{x}_{(i,j)} + \bar{x}_{(j,i)} \leq 1, i,j \in V, i \neq j$, we get all coefficient-symmetric conflicting arcs inequalities presented in [5]. These define facets of $P_{\mathbf{AQTSP}_n}$ for appropriately chosen n and sets S_1, S_2. But [5] contains a class of facet-defining inequalities that are strengthenings of $\bar{x}_{(i,j)} + \bar{x}_{(j,i)} \leq 1, i,j \in V, i \neq j$, that are not coefficient-symmetric. Before repeating these we introduce the following notation: Given sets $L_1, L_2, L_3 \subset V$, we simply write

$$
\bar{y}_{(L_1,L_2,L_3)} := \sum_{\substack{(i,j,k)\in V^{(3)}: \\ i\in L_1, j\in L_2, k\in L_3}} \bar{y}_{(i,j,k)}.
$$

Let $C = (V_C, E_C)$ be a clique tree. Then we write

$$
\bar{y}_{(L_1,L_2,L_3)_{E_C}} := \sum_{\substack{(i,j,k)\in V^{(3)}: \\ i\in L_1, j\in L_2, k\in L_3 \\ ik\in E_C}} \bar{y}_{(i,j,k)} \text{ and } \bar{y}_{(L_1,L_2,L_3)^2_{E_C}} := \sum_{\substack{(i,j,k)\in V^{(3)}: \\ i\in L_1, j\in L_2, k\in L_3 \\ \exists\, H\in\mathcal{H}, T\in\mathcal{T}, i,k\in H\cap T}} \bar{y}_{(i,j,k)}.
$$

With this notation the inequalities in [5] read as follows:

$$
\bar{x}_{(i,j)} + \bar{x}_{(j,i)} + \bar{y}_{(i,S_3,j)} + \bar{y}_{(j,S_4,i)} + \bar{y}_{(S_3,i,S_4)} \leq 1 \tag{16}
$$

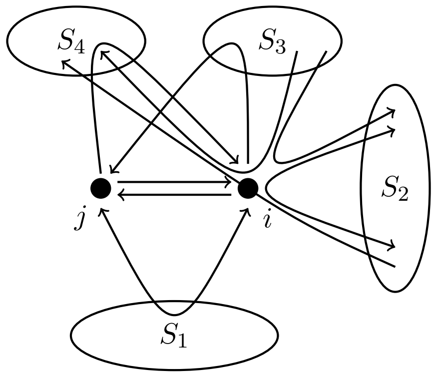


Figure 8: Visualization of inequalities (17): at most one of these arcs (straight lines) and 2-arcs (curved lines) can be contained in a directed Hamiltonian 2-cycle.

for $i, j \in V, i \neq j, S_3, S_4 \subset V \setminus \{i, j\}, S_3 \cap S_4 = \emptyset, V = \{i, j\} \dot\cup S_3 \dot\cup S_4, |S_3| \geq 2, |S_4| \geq 2$. One can further combine the two approaches for getting conflicting arcs inequalities. It is easy to see that

$$\bar{x}_{(i,j)} + \bar{x}_{(j,i)} + \bar{y}_{(i,S_1,j)} + \bar{y}_{(j,S_1,i)} + \bar{y}_{(S_2 \cup S_3, i, S_2 \cup S_4)} + \bar{y}_{(i,S_3,j)} + \bar{y}_{(j,S_4,i)} \leq 1 \tag{17}$$

are valid for $P_{\mathbf{AQTSP}_n}, n \geq 5$, for $i, j \in V, i \neq j, S_o \subset V \setminus \{i, j\}, o = 1, \ldots, 4, S_o \cap S_p = \emptyset, o, p = 1, \ldots, 4, o \neq p$. A visualization of these can be found in Figure 8.

Inequalities (16) and (17) motivate the question if such a strengthening can also be applied to the coefficient-symmetric clique tree inequalities

$$\sum_{Z \in \mathcal{H} \cup \mathcal{T}} \sum_{(k,l) \in Z^{(2)}} \bar{x}_{kl} \leq s(C)$$

for a given clique tree according to Definition 1. The following theorem answers this question.

Theorem 14 *Let $C = (V_C, E_C)$ be a clique tree with handles $\mathcal{H}$ and teeth $\mathcal{T}$ according to Definition 1. Let $S_1, S_2, S_3, S_4 \subset V \setminus V_C$ be sets with $S_o \cap S_p = \emptyset, o, p = 1, \ldots, 4, o \neq p$, $V = V_C \dot\cup S_1 \dot\cup S_2 \dot\cup S_3 \dot\cup S_4$ and with $(S_2 \neq \emptyset \vee |V_C| < |S_1 \cup S_3 \cup S_4|)$. Let W be an arbitrary subset satisfying*

$$\bigcup_{H \in \mathcal{H}} H \subseteq W, \text{ and } \forall T \in \mathcal{T}\colon |T \setminus W| = 1$$

and $W_1, W_2 \subset W, W_1 \cap W_2 = \emptyset, W_1 \cup W_2 = W$. Then the inequalities

$$\sum_{Z \in \mathcal{H} \cup \mathcal{T}} \sum_{(k,l) \in Z^{(2)}} \bar{x}_{(k,l)} + \sum_{Z \in \mathcal{H} \cup \mathcal{T}} \sum_{\substack{(k,l) \in Z^{(2)}, \\ m \in S_1 \cup V_C, \\ km, ml \notin E_C}} \bar{y}_{(k,m,l)} + \bar{y}_{(S_2 \cup S_3, W_1, S_2 \cup S_4)} + \bar{y}_{(S_2 \cup S_4, W_2, S_2 \cup S_3)}$$
$$+ \bar{y}_{(W_1, S_3, V_C \setminus W_1)_{E_C}} + \bar{y}_{(W_2, S_4, V_C \setminus W_2)_{E_C}} + \bar{y}_{(W_1, S_3, W_2)^2_{E_C}} + \bar{y}_{(W_2, S_4, W_1)^2_{E_C}} \leq s(C) \tag{18}$$

are valid for $P_{\mathbf{AQTSP}_n}, n \geq 5$, for each choice of W_1, W_2, W and $S_i, i = 1, \ldots, 4$.

Proof. We prove this by showing that for each directed Hamiltonian 2-cycle $\vec{K}$ there exists a directed Hamiltonian 2-cycle $\vec{K}'$ such that $\vec{K}'$ does not use a 2-arc with nonzero coefficient and the left-hand side of (18) calculated for $\vec{K}'$ is at least as big as the left-hand side of (18) with $\vec{K}$. First we construct an arc set $\vec{L}$ via

$$\begin{aligned}\vec{L} :=&\{(i,j) \in V^{(2)}\colon ij \in E_C, (i,j) \in \vec{K}^{(2)}\}\\ &\cup \{(i,j) \in V^{(2)}\colon (i,k,j) \in \vec{K}, k \in S_1 \cup V_C, ik, kj \notin E_C\}\\ &\cup \{(i,j) \in V^{(2)}\colon i \in W_1, k \in S_3, j \in V_C \setminus W_1, ij \in E_C, (i,k,j) \in \vec{K}\}\\ &\cup \{(i,j) \in V^{(2)}\colon i \in W_2, k \in S_4, j \in V_C \setminus W_2, ij \in E_C, (i,k,j) \in \vec{K}\}.\end{aligned}$$

It holds $|\{(i,j) \in \vec{L}\}| \leq 1$ and $|\{(j,i) \in \vec{L}\}| \leq 1$ for all $i \in V_C$ because an in- or out-degree of a node $i \in V_C$ larger than one would imply that $\vec{K}$ is not a directed Hamiltonian 2-cycle (exactly one arc enters resp. leaves a node). Furthermore the arcs in $\vec{L}$ do not contain a directed cycle because this would imply a directed 2-cycle in $\vec{K}$ not visiting the nodes in S_2 if $S_2 \neq \emptyset$ or a directed 2-cycle in $\vec{K}$ of length at most $2|V_C| < n$ if $|V_C| < |S_1 \cup S_3 \cup S_4|$. Let L be a set of edges with $L := \{ij \in V^{\{2\}}\colon (i,j) \in \vec{L} \vee (j,i) \in \vec{L}\}$. Then L is the edge set of a set of paths. In the next step we build a set of nodes B (later the blocked nodes). We set

$$\begin{aligned}B :=&\{j \in W_1\colon i \in S_2 \cup S_3, k \in S_2 \cup S_4, (i,j,k) \in \vec{K}\}\\ &\cup \{j \in W_2\colon i \in S_2 \cup S_4, k \in S_2 \cup S_3, (i,j,k) \in \vec{K}\}.\end{aligned}$$

Because $\vec{K}$ is a (directed) tour we know that a node $j \in B$ can either lie on one 2-arc from $S_2 \cup S_3$ to $S_2 \cup S_4$ or on one 2-arc from $S_2 \cup S_4$ to $S_2 \cup S_3$. Furthermore it holds $\{j \in V_C\colon ij \in L\} = \emptyset$ for all $i \in B$ by construction of L and B. So we can set up a path system $(\mathcal{P}, B)$ with $\mathcal{P}$ formed by all paths corresponding to L and all isolated nodes $v \in V_C \setminus (B \cup \{i \in V\colon \exists\, ij \in L\})$. The weight $\omega(\mathcal{P}, B)$ equals the left-hand side of (18) computed for $\vec{K}$. Now we apply the results of the previous section, more precisely Lemma 3, Lemma 4, Lemma 5 and Lemma 7, to $(\mathcal{P}, B)$ and get a simple path system $(\bar{\mathcal{P}}, \emptyset)$ with $\omega(\bar{\mathcal{P}}, \emptyset) \geq \omega(\mathcal{P}, B)$. Let $(\mathcal{P}^{(k)}, \emptyset), k \in \mathbb{N}$, be a family of path systems with $(\mathcal{P}^{(1)}, \emptyset) = (\bar{\mathcal{P}}, \emptyset)$. As long as there exist a $Z \in \mathcal{H} \cup \mathcal{T}$ with $P, Q \in \mathcal{P}^{(k)}, P \neq Q, xP = P, yQ = Q, x, y \in Z$ we join the two path to one path $P' = PxyQ$ and set $\mathcal{P}^{(k+1)} = (\mathcal{P}^{(k)} \setminus \{P, Q\}) \cup \{P'\}$. Let $(\mathcal{P}^{(\hat{k})}, \emptyset)$ be the path system after the joining operations. Then we join all paths to one large path and connect this to the path $s^1_1 \dots s^1_{|S_1|} s^2_1 \dots s^2_{|S_2|} s^3_1 \dots s^3_{|S_3|} s^4_1 \dots s^4_{|S_4|}$ of all nodes in S_1, S_2, S_3, S_4. Closing this path to an undirected tour and orienting it with one of the two orientations gives a directed Hamiltonian 2-cycle $\vec{K}'$ that fulfills the desired properties. □

If the clique tree C consists of only one tooth T with $T = \{i, j\}, i \neq j$ we receive (17) with $W = \{i\}, W_1 = \{i\}, W_2 = \emptyset$. Figure 9 visualizes a strengthened subtour elimination constraint with one tooth of size $|T| = 3$.

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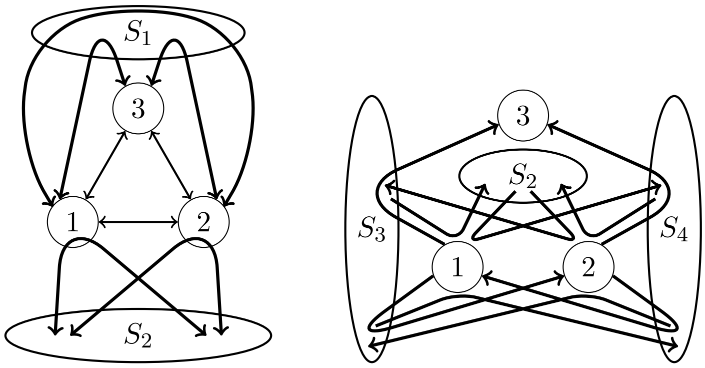

Figure 9: Visualization of a strengthened subtour elimination constraint as a special case of a clique tree inequality ($\mathcal{H} = \emptyset, \mathcal{T} = \{T\}$) with $W = \{1, 2\}, W_1 = \{1\}, W_2 = \{2\}, T = \{1, 2, 3\}, n \geq 5$ in Theorem 14. At most two of the arcs (straight lines) and 2-arcs (curved lines) can be contained in a directed tour. In order to simplify the presentation we have drawn the arcs and 2-arcs in two pictures.

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Appendix

Theorem 15 *Inequalities* (14) *define facets of* $P_{\mathbf{SQTSP}_n}$ *for* $I, S_1, S_2 \subset V, V = I\dot{\cup}S_1\dot{\cup}S_2$, $I \cap S_1 = \emptyset$*,* $I \cap S_2 = \emptyset$*,* $S_1 \cap S_2 = \emptyset$*,* $S_1 \neq \emptyset$*,* $|I| \geq 3$*,* $2 \leq |S_2| \leq |I|$ *and (*$|S_1 \cup S_2| \geq 5$ *or* $|S_2| \geq 3$*).*

Proof. We used the proof-framework of the proof of the dimension of $P_{\mathbf{SQTSP}_n}$ in [6]. To keep the proof self-contained we will repeat the notations used there. We prove this result by constructing $f(n) := 3 \cdot \binom{n}{3} + \binom{n}{2} - n^2$ affinely (linearly) independent tours in three main steps. In the first step we determine the rank of some specially structured tours $\bar{C}^{\bar{n},1}_{dim}$ by means of a computer algebra system and take the largest affinely independent subset $C^{\bar{n},1}_{dim} \subset \bar{C}^{\bar{n},1}_{dim}$. In the second and the third step we build tours so that each tour contains at least one 2-edge that is not contained in any tour constructed before. So, considering a matrix formed by the incidence vectors of these tours, we get a block with full row rank and a lower triangular matrix with ones on the main diagonal and zeros in the block of the first step for those variables that form the main diagonal in the second and third step. It is easy to see that the constructed matrix has full row rank.

We set, w. l. o. g., $I = \{i_1 = n - |I| + 1, \ldots, i_{|I|} = n\}$, $S_1 = \{1, \ldots, |S_1|\}$ and denote by $\bar{I}$ all nodes of I that are not explicitly mentioned in the tours, in arbitrary order.

($\textsf{\textbf{Step}}_{(14)}\mathbf{1}$) If $|S_1 \cup S_2| \geq 5$ we know $\{1, \ldots, 5\} \cap I = \emptyset$. We set $\bar{n} = 5$ and can use the same construction as in [6] building tours $\bar{C}^{\bar{n},1}_{dim} = \{K \in \mathcal{K}_n \colon \{\langle \bar{n}+1, \bar{n}+2, \bar{n}+3\rangle, \langle \bar{n}+2, \bar{n}+3, \bar{n}+4\rangle, \ldots, \langle n-2, n-1, n\rangle\} \subset K, \{n-1, n\} \in K^{\{2\}}\}$. A largest affinely independent subset of $\bar{C}^{\bar{n},1}_{dim}$ contains 54 tours that are collect in set $C^{\bar{n},1}_{dim}$. In the case $|S_1| = 1, |S_2| = 3$ it holds $5 \in I$ and so setting $\bar{n} = 5$ we have to restrict to tours $\tilde{C}^{\bar{n},1}_{dim,(14)} = \{K \in \mathcal{K}_n \colon \{\langle \bar{n}+1, \bar{n}+2, \bar{n}+3\rangle, \langle \bar{n}+2, \bar{n}+3, \bar{n}+4\rangle, \ldots, \langle n-2, n-1, n\rangle\} \subset K, \{n-1, n\} \in K^{\{2\}}, (\{5, n\} \in K^{\{2\}} \vee \{5, 6\} \in K^{\{2\}} \vee \langle n, 1, 5\rangle \in K \vee \langle 5, 1, 6\rangle \in K \vee \langle 2, 5, 3\rangle \in K \vee \langle 2, 5, 4\rangle \in K \vee \langle 3, 5, 4\rangle \in K)\}$. The rank reduces by one to 53 in this case.

($\textsf{\textbf{Step}}_{(14)}\mathbf{2}$) The set $C^{\bar{n},2}_{dim} = \bigcup_{\bar{n}<k<n-1} T_k$ is formed iteratively. For each $k \in \{\bar{n}+1, \ldots, n-2\}$ we build a set of tours T_k that contains n_k tours $t^1_k, \ldots, t^{n_k}_k$. The tour construction uses five substeps. During each substep the order of the tours is arbitrary. In each substep we append new rows of incidence vectors of tours to a large matrix built by the affinely independent tours. At the end we have to check that the tours indeed fulfill the described matrix structure (a lower triangular matrix with ones on the main diagonal).

Let k be fixed with $\bar{n} < k < n-1$. All tours presented next are represented by the order of the nodes, *i. e.*, a tour $t = \{v_1v_2v_3, v_2v_3v_4, \ldots, v_{n-1}v_nv_1, v_nv_1v_2\}$ is represented by $v_1\, v_2\, v_3 \ldots v_{n-1}\, v_n$. Only the relevant parts of the tours are specified. The node sequence $(k+2)\,(k+3) \ldots (n-2)\,(n-1)$ is subsumed and denoted by the symbol ϖ_k. If some nodes are not explicitly mentioned and the completion of the tour is arbitrary we denote this by "$\ldots$". We underline the decisive 2-edge (the three corresponding nodes) $e^i_k, i = 1, \ldots, n_k$, that is used for forming the triangular structure. It belongs to one of the four types

(**Type-I1**) $\langle a,k,b\rangle, a,b \in \{1,\dots,k-1\}, a<b$,

(**Type-I2**) $\langle k,a,k+1\rangle, a \in \{2,\dots,k-1\}$,

(**Type-I3**) $\langle a,b,k+1\rangle, a,b \in \{1,\dots,k-1\}, a \neq b$.

(**Type-I4**) $\langle n,a,k\rangle, \langle n,k,a\rangle, a \in \{1,\dots,k-1\}$.

In [6] the standard construction for fixed k is

(**I1**) $\dots \underline{a\,k\,1}\,(k+1)\,\varpi_k\,n \dots$, for $a \in \{2,\dots,k-1\}$
(the 2-edge $\langle k,1,k+1\rangle$ is not used as an e_k^i),

(**I2**) $\dots 1\,\underline{k\,a\,(k+1)}\,\varpi_k\,n \dots$, for $a \in \{2,\dots,k-1\}$,

(**I3**) $\dots \underline{a\,k\,b}\,(k+1)\,\varpi_k\,n \dots$, for $a,b \in \{2,\dots,k-1\}, a<b$,

(**I4**) $\dots k\,\underline{a\,b\,(k+1)}\,\varpi_k\,n \dots$, for $a,b \in \{1,\dots,k-1\}$, $a \neq b$,

(**I5**) $\dots (k+1)\,\varpi_k\,\underline{n\,a\,b} \dots$, for $a,b \in \{1,\dots,k\}, a \neq b, k \in \{a,b\}$.

These substeps fulfill the desired triangular structure (proof of Claim 1 in the proof of Theorem 2.3 in [6]).

As long as $k \in S_1 \cup S_2$ the nodes in I lie next to each other and so the corresponding tours define roots of (14). Adaptations of (**I1**)–(**I5**) are needed for the case $k \in I$. We start with a specific ordering for $k = i_1$ for the case $|S_1 \cup S_2| \geq 5$.

(**I**$^{i_1}_{(14)}$**1**) $\dots \underline{a\,i_1\,1}\,i_2\,\varpi_k\,n \dots$, for $a \in (S_1 \cup S_2) \setminus \{1\}$
(the 2-edge $\langle i_1,1,i_2\rangle$ is not used as an $e_k^{\hat{\imath}}$; the same 2-edge is not used in (**I1**), too),

(**I**$^{i_1}_{(14)}$**2a**) $\dots 1\,\underline{i_1\,a\,i_2}\,\varpi_k\,n \dots$, for $a \in S_1 \setminus \{1\}$,

(**I**$^{i_1}_{(14)}$**3a**) $\dots \underline{a\,i_1\,b}\,i_2\,\varpi_k\,n \dots$, for $a \in (S_1 \cup S_2) \setminus \{1\}, b \in S_1 \setminus \{1\}, b<a$,

(**I**$^{i_1}_{(14)}$**4a**) $\dots \underline{a\,b\,i_2}\,\varpi_k\,n\,1\,i_1 \dots$, for $a,b \in (S_1 \cup S_2) \setminus \{1\}$
(the 2-edge $\langle n,1,i_1\rangle$ is not used as an $e_k^{\hat{\imath}}$; it is the one specific tour that is lost in comparison to the dimension proof in [6]),

(**I**$^{i_1}_{(14)}$**5a**) $\begin{cases} \dots m\,o\,i_2\,\varpi_k\,\underline{n\,i_1\,a} \dots, & \text{for } a \in S_1 \cup S_2, \\ \dots m\,o\,i_2\,\varpi_k\,\underline{n\,a\,i_1} \dots, & \text{for } a \in S_1 \setminus \{1\}, \end{cases}$
with $m,o \in (S_1 \cup S_2) \setminus \{1\}, |\{a,m,o\}| = 3$,

(**I**$^{i_1}_{(14)}$**4b**) $\dots \underline{a\,b\,i_2}\,\varpi_k\,n\,i_1 \dots$, for $a,b \in S_1 \cup S_2, 1 \in \{a,b\}, a \neq b$,

(**I**$^{i_1}_{(14)}$**3b**) $\dots i_2\,\varpi_k\,n\,1\,\underline{a\,i_1\,b} \dots$, for $a,b \in S_2, a<b$,

(**I**$^{i_1}_{(14)}$**5b**) $\dots i_2\,\varpi_k\,\underline{n\,a\,i_1}\,m \dots$, for $a \in S_2$ with $m \in S_2, m \neq a$,

(**I**$^{i_1}_{(14)}$**2b**) $\dots m\,\underline{i_1\,a\,i_2}\,\varpi_k\,n \dots$, for $a \in S_2$ with $m \in S_2, m \neq a$.

All tours in (**I**$^{i_1}_{(14)}$**1**)–(**I**$^{i_1}_{(14)}$**2b**) define roots of (14) because the nodes i_2 to n lie next to each other and for i_1 it holds that either i_1 lies next to node n, or there is exactly one node between i_1 and i_2 resp. n and this node belongs to S_1 or i_1 lies between two nodes in S_2. Furthermore we have to show that all underlined 2-edges are not used in a tour of a previous substep. It suffices to look only at previous substeps for the same $k = i_1$.

- Tours in (**I**$^{i_1}_{(14)}$**2a**): all tours in (**I**$^{i_1}_{(14)}$**1**) contain the 2-edge $\langle i_1,1,i_2\rangle$.
- Tours in (**I**$^{i_1}_{(14)}$**3a**): all tours in (**I**$^{i_1}_{(14)}$**1**)–(**I**$^{i_1}_{(14)}$**2a**) contain the edge $\{i_1,1\}$.

- Tours in ($\mathbf{I}^{i_1}_{(14)}$**4a**): all tours in ($\mathbf{I}^{i_1}_{(14)}$**1**)–($\mathbf{I}^{i_1}_{(14)}$**3a**) contain a 2-edge $\langle i_1, \tilde{a}, i_2\rangle \in V^{\langle 3\rangle}$.
- Tours in ($\mathbf{I}^{i_1}_{(14)}$**5a**): all tours in ($\mathbf{I}^{i_1}_{(14)}$**1**)–($\mathbf{I}^{i_1}_{(14)}$**3a**) contain a 2-edge $\langle n, \tilde{a}, \tilde{b}\rangle \in V^{\langle 3\rangle}$ with $\tilde{a}, \tilde{b} \in S_1 \cup S_2$ and the tours in ($\mathbf{I}^{i_1}_{(14)}$**4a**) contain the 2-edge $\langle n, 1, i_1\rangle$.
- Tours in ($\mathbf{I}^{i_1}_{(14)}$**4b**): all tours in ($\mathbf{I}^{i_1}_{(14)}$**1**)–($\mathbf{I}^{i_1}_{(14)}$**3a**) contain a 2-edge $\langle i_1, \tilde{a}, i_2\rangle \in V^{\langle 3\rangle}$ and in ($\mathbf{I}^{i_1}_{(14)}$**4a**) the nodes a, b and in ($\mathbf{I}^{i_1}_{(14)}$**5a**) the nodes m, o are not allowed to be 1.
- Tours in ($\mathbf{I}^{i_1}_{(14)}$**3b**): all tours ($\mathbf{I}^{i_1}_{(14)}$**1**)–($\mathbf{I}^{i_1}_{(14)}$**4b**) contain an edge $\{n, i_1\}$ or an edge $\{i_1, \tilde{a}\}, \tilde{a} \in S_1$.
- Tours in ($\mathbf{I}^{i_1}_{(14)}$**5b**): all tours in ($\mathbf{I}^{i_1}_{(14)}$**1**)–($\mathbf{I}^{i_1}_{(14)}$**3a**) contain a 2-edge $\langle n, \tilde{a}, \tilde{b}\rangle \in V^{\langle 3\rangle}$ with $\tilde{a}, \tilde{b} \in S_1 \cup S_2$ and the tours in ($\mathbf{I}^{i_1}_{(14)}$**4a**)–($\mathbf{I}^{i_1}_{(14)}$**3b**) contain the edge $\{n, i_1\}$ or an edge $\{n, \tilde{a}\}, \tilde{a} \in S_1$.
- Tours in ($\mathbf{I}^{i_1}_{(14)}$**2b**): all tours in ($\mathbf{I}^{i_1}_{(14)}$**1**)–($\mathbf{I}^{i_1}_{(14)}$**3a**) contain a 2-edge $\langle i_1, \tilde{a}, i_2\rangle \in V^{\langle 3\rangle}$ with $\tilde{a} \in S_1$ and there are at least two nodes between i_1 and i_2 in the tours in ($\mathbf{I}^{i_1}_{(14)}$**4a**)–($\mathbf{I}^{i_1}_{(14)}$**5b**) because $i_1 \geq 6$.

All in all, we constructed exactly one tour less than described in (**Type-I1**)–(**Type-I4**) for this k.

For $k \in I, k \neq i_1, k \leq n-2$ the substeps presented next provide tours having the desired root structure.

($\mathbf{I}_{(14)}$**1**) $\ldots \underline{a\,k\,1}\,(k+1)\,\varpi_k\,n\,\bar{I}\ldots$, for $a \in \{2, \ldots, k-1\}$ (the 2-edge $\langle k, 1, k+1\rangle$ is not used as an $e^{\hat{\imath}}_{k}$, see (**I1**)),

($\mathbf{I}_{(14)}$**2a**) $\ldots 1\,\underline{k\,a\,(k+1)}\,\varpi_k\,n\,\bar{I}\ldots$, for $a \in \{2, \ldots, k-1\} \setminus S_2$,

($\mathbf{I}_{(14)}$**3a**) $\ldots \underline{a\,k\,b}\,(k+1)\,\varpi_k\,n\,\bar{I}\ldots$, for $a \in \{2, \ldots, k-1\}, b \in S_1 \setminus \{1\}, b < a$,

($\mathbf{I}_{(14)}$**3b**) $\ldots \underline{a\,k\,b}\,(k+1)\,\varpi_k\,n\,\bar{I}\ldots$, for $a \in \{1, \ldots, k-1\} \setminus S_1, b \in \{1, \ldots, k-1\} \cap I, a < b$,

($\mathbf{I}_{(14)}$**4a**) $\ldots \underline{a\,b\,(k+1)}\,\varpi_k\,n\,1\,\bar{I}\,k\ldots$, for $a, b \in S_2, a \neq b$,

($\mathbf{I}_{(14)}$**5a**) $\ldots m\,o\,(k+1)\,\varpi_k\,\underline{n\,a\,b}\,\bar{I}\ldots$, for $a, b \in \{1, \ldots, k\} \cap (I \cup S_1), k \in \{a, b\}$, with $m, o \in S_2, |\{a, b, m, o\}| = 4$,

($\mathbf{I}_{(14)}$**4b**) $\ldots \underline{a\,b\,(k+1)}\,\varpi_k\,n\,k\,\bar{I}\ldots$, for $a, b \in S_1 \cup S_2, \{a, b\} \cap S_1 \neq \emptyset, a \neq b$,

($\mathbf{I}_{(14)}$**4c**) $\ldots \bar{I}\,k\,\underline{a\,b\,(k+1)}\,\varpi_k\,n\ldots$, for $a, b \in \{1, \ldots, k-1\} \setminus S_2, \{a, b\} \cap I \neq \emptyset, a \neq b$,

($\mathbf{I}_{(14)}$**4d**) $\ldots \underline{a\,b\,(k+1)}\,\varpi_k\,n\,k\,\bar{I}\,1\ldots$, for $a \in S_2, b \in \{1, \ldots, k-1\} \cap I$,

($\mathbf{I}_{(14)}$**4e**) $\ldots m\,\underline{a\,b\,(k+1)}\,\varpi_k\,n\,k\,\bar{I}\,1\ldots$, for $a \in \{1, \ldots, k-1\} \cap I, b \in S_2$ with $m \in S_2, m \neq b$,

($\mathbf{I}_{(14)}$**5b**) $\ldots \bar{I}\,(k+1)\,\varpi_k\,\underline{n\,k\,a}\ldots$, for $a \in S_2$,

($\mathbf{I}_{(14)}$**3c**) $\ldots (k+1)\,\varpi_k\,n\,\bar{I}\,\underline{a\,k\,b}\ldots$, for $a, b \in S_2, a < b$,

($\mathbf{I}_{(14)}$**5c**) $\ldots \bar{I}\,(k+1)\,\varpi_k\,\underline{n\,a\,k}\,m\ldots$, for $a \in S_2$ with $m \in S_2, m \neq a$,

($\mathbf{I}_{(14)}$**2b**) $\ldots m\,\underline{k\,a\,(k+1)}\,\varpi_k\,n\,\bar{I}\ldots$, for $a \in S_2$ with $m \in S_2, m \neq a$

The tours in ($\mathsf{I}_{(14)}\mathbf{1}$)–($\mathsf{I}_{(14)}\mathbf{2b}$) define roots of (14) because in ($\mathsf{I}_{(14)}\mathbf{1}$)–($\mathsf{I}_{(14)}\mathbf{5b}$) all nodes in I lie next to each other, partially with exactly one node from S_1 between them. In ($\mathsf{I}_{(14)}\mathbf{3c}$)–($\mathsf{I}_{(14)}\mathbf{2b}$) the nodes $I \setminus \{k\}$ lie next to each other and node k lies between two nodes of S_2. It remains to prove that all underlined 2-edges are not used in a tour of a previous substep.

- Tours in ($\mathsf{I}_{(14)}\mathbf{2a}$): all tours in ($\mathsf{I}_{(14)}\mathbf{1}$) contain the 2-edge $\langle k, 1, k+1\rangle$.
- Tours in ($\mathsf{I}_{(14)}\mathbf{3a}$), ($\mathsf{I}_{(14)}\mathbf{3b}$): the two substeps use different 2-edges of type (**Type-I2**). So we can treat them together. All tours in ($\mathsf{I}_{(14)}\mathbf{1}$)–($\mathsf{I}_{(14)}\mathbf{2a}$) contain the edge $\{k, 1\}$.
- Tours in ($\mathsf{I}_{(14)}\mathbf{4a}$): all tours in ($\mathsf{I}_{(14)}\mathbf{1}$)–($\mathsf{I}_{(14)}\mathbf{3b}$) contain a 2-edge $\langle k, \tilde{a}, k+1\rangle \in V^{\langle 3\rangle}$.
- Tours in ($\mathsf{I}_{(14)}\mathbf{5a}$): in all tours in ($\mathsf{I}_{(14)}\mathbf{1}$)–($\mathsf{I}_{(14)}\mathbf{4a}$) there are at least two nodes between node n and node k on both sides. Note, $\bar{I}$ represents at least one node in ($\mathsf{I}_{(14)}\mathbf{4a}$).
- Tours in ($\mathsf{I}_{(14)}\mathbf{4b}$)–($\mathsf{I}_{(14)}\mathbf{4e}$): the four substeps use different 2-edges of type (**Type-I3**). So we can treat them together. All tours in ($\mathsf{I}_{(14)}\mathbf{1}$)–($\mathsf{I}_{(14)}\mathbf{3b}$) contain a 2-edge $\langle k, \tilde{a}, k+1\rangle \in V^{\langle 3\rangle}$. The tours in ($\mathsf{I}_{(14)}\mathbf{4a}$)–($\mathsf{I}_{(14)}\mathbf{5a}$) contain a 2-edge $\langle \tilde{a}, \tilde{b}, k+1\rangle \in V^{\langle 3\rangle}, \tilde{a}, \tilde{b} \in S_2, \tilde{a} \neq \tilde{b}$.
- Tours in ($\mathsf{I}_{(14)}\mathbf{5b}$): in all tours in ($\mathsf{I}_{(14)}\mathbf{1}$)–($\mathsf{I}_{(14)}\mathbf{4a}$), ($\mathsf{I}_{(14)}\mathbf{4c}$) there are at least two nodes between node n and node k on both sides. All tours in ($\mathsf{I}_{(14)}\mathbf{5a}$)–($\mathsf{I}_{(14)}\mathbf{4b}$), ($\mathsf{I}_{(14)}\mathbf{4d}$)–($\mathsf{I}_{(14)}\mathbf{4e}$) contain a 2-edge $\langle n, \tilde{a}, \tilde{b}\rangle, \tilde{a}, \tilde{b} \in \{k\} \cup I \cup S_1$.
- Tours in ($\mathsf{I}_{(14)}\mathbf{3c}$): all tours in ($\mathsf{I}_{(14)}\mathbf{1}$)–($\mathsf{I}_{(14)}\mathbf{3a}$) contain an edge $\{k, \tilde{a}\}$, $\tilde{a} \in S_1$ and all tours in ($\mathsf{I}_{(14)}\mathbf{3b}$)–($\mathsf{I}_{(14)}\mathbf{5b}$) contain an edge $\{k, \tilde{b}\}, \tilde{b} \in I \cup S_1$ (note $n \in I$).
- Tours in ($\mathsf{I}_{(14)}\mathbf{5c}$): in all tours in ($\mathsf{I}_{(14)}\mathbf{1}$)–($\mathsf{I}_{(14)}\mathbf{4a}$), ($\mathsf{I}_{(14)}\mathbf{4c}$) there are at least two nodes between node n and node k on both sides. All tours in ($\mathsf{I}_{(14)}\mathbf{5a}$)–($\mathsf{I}_{(14)}\mathbf{4b}$), ($\mathsf{I}_{(14)}\mathbf{4d}$)–($\mathsf{I}_{(14)}\mathbf{4e}$) contain a 2-edge $\langle n, \tilde{a}, \tilde{b}\rangle \in V^{\langle 3\rangle}, \tilde{a}, \tilde{b} \in \{k\} \cup I \cup S_1$. The tours in ($\mathsf{I}_{(14)}\mathbf{5b}$) contain the edge $\{n, k\}$ and each tour in ($\mathsf{I}_{(14)}\mathbf{3c}$) contains an edge $\{n, \tilde{a}\}, \tilde{a} \in I$.
- Tours in ($\mathsf{I}_{(14)}\mathbf{2b}$): all tours in ($\mathsf{I}_{(14)}\mathbf{1}$)–($\mathsf{I}_{(14)}\mathbf{3b}$) contain a 2-edge $\langle k, \tilde{a}, k+1\rangle \in V^{\langle 3\rangle}, \tilde{a} \in S_1 \cup I$, and in the tours in ($\mathsf{I}_{(14)}\mathbf{4a}$)–($\mathsf{I}_{(14)}\mathbf{5c}$) there are at least two nodes between nodes $k+1$ and k on both sides.

Because, in total, the same 2-edges are underlined and used for building the triangular structure we get exactly $\frac{3}{2}k^2 - \frac{3}{2}k - 1$ tours for $k \in \{\bar{n}+1, \ldots, n-2\} \setminus \{i_1\}$ ($\frac{1}{2}(k-1)(k-2)$ with (**Type-I1**), $k-2$ with (**Type-I2**), $(k-1)(k-2)$ with (**Type-I3**) and $2(k-1)$ with (**Type-I4**)), see proof of Claim 3 in the proof of Theorem 2.3 in [6].

($\mathsf{Step}_{(14)}\mathbf{3}$) In all tours in ($\mathsf{Step}_{(14)}\mathbf{1}$) and ($\mathsf{Step}_{(14)}\mathbf{2}$) the nodes $n-1$ and n are adjacent. Now we construct tours $C^{\bar{n},3}_{dim,(14)} = \{t^1_L, \ldots, t^{n_L}_L\}$ in which $n-1$ and n do not lie next to each other. Each tour will contain a 2-edge $e^i_L, i = 1, \ldots, n_L$, of one of the types

(**Type-L1**) $\langle a, n-1, b\rangle, a, b \in \{1, \ldots, n-2\}, a < b$,

(**Type-L2**) $\langle a, n, b\rangle, a, b \in \{1, \ldots, n-2\}, a < b$, (**Type-L3**) $\langle n-1, a, n\rangle, a \in \{1, \ldots, n-2\}$.

Except for one all of these 2-edges are used as e_L^i. During the following substeps we will ensure that each underlined e_L^i is not used in a previous substep (the order in each substep is arbitrary) and not in (**Step**$_{(14)}$**1**) and (**Step**$_{(14)}$**2**). Indeed, the substeps are only slightly modified in comparison to the ones used in the third step in the original dimension proof in [6]. We specify the position of $\bar{I}$ here and split up some of the original substeps in several successive ones in order to simplify the presentation. We set $w_1 = 1, w_2, w_3 \in S_2, w_2 \neq w_3$ with $W = \{w_1, w_2, w_3\}$.

(**L**$_{(14)}$**1a**) $\ldots \underline{a\,(n-1)\,b}\,\bar{I}\,w_1\,n\,w_2 \ldots$, for $a \in \{1, \ldots, n-2\} \setminus \{w_1, w_2\}, b \in I \setminus \{n-1, n\}, a < b$,
(the 2-edge $\langle w_1, n, w_2\rangle$ is not used as an $e_L^{\hat{i}}$),

(**L**$_{(14)}$**1b**) $\ldots \underline{a\,(n-1)\,b}\,\bar{I}\,w_1\,n\,w_2 \ldots$, for $a \in (S_1 \cup S_2) \setminus \{w_1, w_2\}, b \in S_1 \setminus \{w_1\}$, $a > b$,

(**L**$_{(14)}$**1c**) $\ldots \underline{a\,(n-1)\,b}\,\bar{I}\,w_1\,n\,w_2 \ldots$, for $a, b \in S_2 \setminus \{w_2\}, a < b$,

(**L**$_{(14)}$**2**) $\begin{cases} \ldots m\,(n-1)\,\bar{I}\,\underline{w_1\,n\,w_3} \ldots, & \text{with } m \in (S_1 \cup S_2) \setminus W, \\ \ldots m\,(n-1)\,\bar{I}\,\underline{w_2\,n\,w_3} \ldots, & \text{with } m \in (S_1 \cup S_2) \setminus W, \end{cases}$

(**L**$_{(14)}$**3**) $\ldots \underline{a\,(n-1)\,w_1}\,\bar{I}\,w_2\,n\,w_3 \ldots$, for $a \in \{1, \ldots, n-2\} \setminus W$,

(**L**$_{(14)}$**4**) $\ldots \underline{w_2\,(n-1)\,a}\,\bar{I}\,w_1\,n\,w_3 \ldots$, for $a \in \{1, \ldots, n-2\} \setminus W$,

(**L**$_{(14)}$**5a**) $\ldots \underline{a\,n\,w_1}\,\bar{I}\,(n-1)\,w_2 \ldots$, for $a \in (S_1 \cup S_2) \setminus W$,

(**L**$_{(14)}$**5b**) $\ldots \underline{a\,n\,b}\,\bar{I}\,(n-1)\,w_1 \ldots$, for $a \in \{w_2, w_3\}, b \in (S_1 \cup S_2) \setminus W$,

(**L**$_{(14)}$**5c**) $\ldots \underline{w_1\,n\,a}\,\bar{I}\,m\,(n-1)\,o \ldots$, for $a \in I \setminus \{n-1, n\}$ with $m, o \in \{1, \ldots, n-2\}, |\{a, m, o\}| = 3$, and $((m, o \in S_2, \{m, o\} \not\subset W) \vee (m \in I \cup S_1))$

(**L**$_{(14)}$**5d**) $\ldots \underline{a\,n\,b}\,w_1\,(n-1)\,\bar{I}\,m \ldots$, for $a \in \{w_2, w_3\}, b \in I \setminus \{n-1, n\}$ with $m \in \{1, \ldots, n-2\} \setminus W, |\{a, b, m\}| = 3$,

(**L**$_{(14)}$**6**) $\begin{cases} \ldots \underline{w_2\,(n-1)\,w_1}\,\bar{I}\,n\,w_3 \ldots, \\ \ldots \underline{w_3\,(n-1)\,w_1}\,\bar{I}\,n\,w_2 \ldots, \\ \ldots \underline{w_2\,(n-1)\,w_3}\,\bar{I}\,n\,w_1 \ldots, \end{cases}$

(**L**$_{(14)}$**7a**) $\ldots \underline{a\,n\,b}\,\bar{I}\,w_1\,(n-1) \ldots$, for $a \in \{1, \ldots, n-2\} \setminus W, b \in I \setminus \{n-1, n\}, a < b$,

(**L**$_{(14)}$**7b**) $\ldots \underline{a\,n\,b}\,\bar{I}\,(n-1) \ldots$, for $a \in \{1, \ldots, n-2\} \setminus (I \cup W), b \in S_1 \setminus W, a > b$,

(**L**$_{(14)}$**7c**) $\ldots \underline{a\,n\,b}\,\bar{I}\,(n-1) \ldots$, for $a, b \in S_2 \setminus W, a < b$,

(**L**$_{(14)}$**8a**) $\ldots \underline{(n-1)\,a\,n}\,\bar{I} \ldots$, for $a \in (S_1 \cup I) \setminus \{n-1, n\}$,

(**L**$_{(14)}$**8b**) $\ldots \bar{I}\,\underline{(n-1)\,a\,n}\,m \ldots$, for $a \in S_2$ with $m \in S_2, m \neq a$.

It follows from the proof of Claim 2 in the proof of Theorem 2.3 in [6] (and is indeed easy to check) that all underlined 2-edges are not used in a previous substep and that we build exactly $n^2 - 4n + 3$ tours in (**Step**$_{(14)}$**3**) ($\frac{1}{2}(n-2)(n-3)$ of type (**Type-L1**), $\frac{1}{2}(n-2)(n-3) - 1$ of type (**Type-L2**) and $(n-2)$ of type (**Type-L3**)).

- All 2-edges underlined in substeps with the same number belong to the same type and are in pairwise conflict. So we subsume all substeps with the same number to one in the following investigations.
- Tours in (**L**$_{(14)}$**2**): all tours created in (**L**$_{(14)}$**1a**)–(**L**$_{(14)}$**1c**) contain the 2-edge $\langle w_1, n, w_2\rangle$.

- Tours in ($\mathsf{L}_{(14)}$**3**), ($\mathsf{L}_{(14)}$**4**): all tours created in ($\mathsf{L}_{(14)}$**1a**)–($\mathsf{L}_{(14)}$**2**) contain a 2-edge $\langle a, n-1, b\rangle \in V^{\langle 3\rangle}, a, b \in \{1, \ldots, n-2\} \setminus \{w_1, w_2\}$.
- Tours in ($\mathsf{L}_{(14)}$**5a**)–($\mathsf{L}_{(14)}$**5d**): all tours created in ($\mathsf{L}_{(14)}$**1a**)–($\mathsf{L}_{(14)}$**4**) contain a 2-edge $c \in \{\langle w_1, n, w_2\rangle, \langle w_1, n, w_3\rangle, \langle w_2, n, w_3\rangle\}$.
- Tours in ($\mathsf{L}_{(14)}$**6**): all tours created in ($\mathsf{L}_{(14)}$**1a**)–($\mathsf{L}_{(14)}$**5d**) contain none of the three 2-edges $\langle w_1, n-1, w_2\rangle, \langle w_1, n-1, w_3\rangle, \langle w_2, n-1, w_3\rangle$.
- Tours in ($\mathsf{L}_{(14)}$**7a**)–($\mathsf{L}_{(14)}$**7c**): all tours created in ($\mathsf{L}_{(14)}$**1a**)–($\mathsf{L}_{(14)}$**4**) contain a 2-edge $c \in \{\langle w_1, n, w_2\rangle, \langle w_1, n, w_3\rangle, \langle w_2, n, w_3\rangle\}$. In ($\mathsf{L}_{(14)}$**5a**)–($\mathsf{L}_{(14)}$**6**) node n is adjacent to one of the nodes $w_1, w_2, w_3 \in W$.
- Tours in ($\mathsf{L}_{(14)}$**8a**), ($\mathsf{L}_{(14)}$**8b**): in all tours created in ($\mathsf{L}_{(14)}$**1a**)–($\mathsf{L}_{(14)}$**7c**) there are at least two nodes between nodes $n-1$ and n on both sides.

It remains to check the root property of all tours constructed. There is one large block of nodes in I, partially with one node of S_1 between two of these nodes, in all substeps except for ($\mathsf{L}_{(14)}$**1c**), one tour in ($\mathsf{L}_{(14)}$**2**), ($\mathsf{L}_{(14)}$**3**), some tours in ($\mathsf{L}_{(14)}$**5b**)–($\mathsf{L}_{(14)}$**5c**), one tours in ($\mathsf{L}_{(14)}$**6**), ($\mathsf{L}_{(14)}$**7c**) and ($\mathsf{L}_{(14)}$**8b**). In these node $n-1$ or node n does not belong to that block but lies between two nodes in S_2. □

All in all we created exactly $f(n)$ tours and so one tour less than in the proof of the dimension in [6]. If $|S_1 \cup S_2| = 4$ we get one tour less in ($\mathsf{Step}_{(14)}$**1**) by the special structure of the tours and for $|S_1 \cup S_2| \geq 5$ we lost one tour in ($\mathsf{Step}_{(14)}$**2**) for $k = i_1$. Thus, inequalities (14) define facets of $P_{\mathbf{SQTSP}_n}, n \geq 7$.

Theorem 16 *Inequalities* (15) *define facets of* $P_{\mathbf{SQTSP}_n}$ *if* $I, S_1, S_2 \subset V, V = I \dot\cup S_1 \dot\cup S_2$, $I \cap S_1 = \emptyset$, $I \cap S_2 = \emptyset$, $S_1 \cap S_2 = \emptyset$, $S_1 \neq \emptyset$, $|S_2| > |I| \geq 3$.

Proof. We set, w. l. o. g., $I = \{i_1 = n - |I| + 1, \ldots, i_{|I|} = n\}$, $\bar{\imath} = n-1$, $S_1 = \{1, \ldots, |S_1|\}$. Again we use the proof-framework of Theorem 2.3 in [6], similar to the proof of Theorem 15, with its notation and explain the differences only. Additionally, we denote by $\bar{I}$ all nodes of I and by $\bar{S}_1$ all nodes of S_1 that are not explicitly mentioned, in arbitrary order.

($\mathsf{Step}_{(15)}$**1**) By $|S_1| \geq 1$ and $|S_2| \geq 4$ we know $\{1, \ldots, 5\} \cap I = \emptyset$. So setting $\bar{n} = 5$ we can use the same construction as in ($\mathsf{Step}_{(14)}$**1**) taking a largest affinely independent subset $C_{dim}^{\bar{n},1}$ of set $\bar{C}_{dim}^{\bar{n},1}$ containing 54 tours.

($\mathsf{Step}_{(15)}$**2**) As long as $k \in S_1 \cup S_2$ the nodes in I lie next to each other and so the corresponding tours define roots of (14). Adaptations are needed for the case $k \in I$. We start with a specific ordering for $k = i_1$. Here we can use ($\mathsf{I}_{(14)}^{i_1}$**1**)–($\mathsf{I}_{(14)}^{i_1}$**2b**) because all corresponding tours define also roots of (15) by the same arguments as in the proof of Theorem 15. Similarly for $n-2 \geq k > i_1$, constructing substeps ($\mathsf{I}_{(14)}$**1**)–($\mathsf{I}_{(14)}$**2b**) in the proof of Theorem 15 provide roots of (15) and can be applied here.

($\mathsf{Step}_{(15)}$**3**) Some adaptations of the construction in step three are needed specifying the position of $\bar{I}$ and splitting up some of the substeps in several successive ones. We set $W = \{w_1, w_2, w_3\} \subset S_2, |W| = 3$.

($\mathsf{L}_{(15)}$**1a**) $\ldots \underline{a\,(n-1)\,b}\,\bar{I}\,w_1\,n\,w_2 \ldots$, for $a \in \{1, \ldots, n-2\} \setminus \{w_1, w_2\}, b \in I \setminus \{n-1, n\}, a < b$,
(the 2-edge $\langle w_1, n, w_2\rangle$ is not used as an $e_L^{\hat{\imath}}$),

($\mathsf{L}_{(15)}$1b) $\ldots \underline{a\,(n-1)\,b}\,\bar{I}\,w_1\,n\,w_2 \ldots$, for $a \in (S_1 \cup S_2) \setminus \{w_1, w_2\}, b \in S_1,\ a > b$,

($\mathsf{L}_{(15)}$1c) $\ldots \underline{a\,(n-1)\,b}\,\eta_{\bar{I},\bar{S}_2}\,w_1\,n\,w_2\,\bar{S}_1 \ldots$, for $a, b \in S_2 \setminus \{w_1, w_2\}, a < b$, with $\eta_{\bar{I},\bar{S}_2}$ denoting a path of all nodes in $\bar{I} = I \setminus \{n-1, n\}$ and in $\bar{S}_2 = S_2 \setminus \{a, b, w_1, w_2\}$. This path starts with a node $v \in \bar{I}$, then an alternating sequence of the remaining nodes in $\bar{S}_2$ and in $\bar{I} \setminus \{v\}$, (so that each node $w \in I$ lies between two nodes of S_2) and depending of the size of S_2 in comparison to I a block of nodes in S_2.

($\mathsf{L}_{(15)}$2) $\begin{cases} \ldots 1\,(n-1)\,\bar{I}\,\underline{w_1\,n\,w_3} \ldots, \\ \ldots 1\,(n-1)\,\bar{I}\,\underline{w_2\,n\,w_3} \ldots, \end{cases}$

($\mathsf{L}_{(15)}$3a) $\ldots \underline{w_1\,(n-1)\,a}\,\bar{I}\,w_2\,n\,w_3 \ldots$, for $a \in \{1, \ldots, n-2\} \cap (I \cup S_1)$,

($\mathsf{L}_{(15)}$3b) $\ldots \underline{w_1\,(n-1)\,a}\,\eta_{\bar{I},\bar{S}_2}\,w_2\,n\,w_3\,\bar{S}_1 \ldots$, for $a \in \{1, \ldots, n-2\} \cap (S_2 \setminus W)$ with $\eta_{\bar{I},\bar{S}_2}$ as above with $\bar{I} = I \setminus \{n-1, n\}, \bar{S}_2 = S_2 \setminus (\{a\} \cup W)$,

($\mathsf{L}_{(15)}$4a) $\ldots \underline{w_2\,(n-1)\,a}\,\bar{I}\,w_1\,n\,w_3 \ldots$, for $a \in \{1, \ldots, n-2\} \cap (I \cup S_1)$,

($\mathsf{L}_{(15)}$4b) $\ldots \underline{w_2\,(n-1)\,a}\,\eta_{\bar{I},\bar{S}_2}\,w_1\,n\,w_3\,\bar{S}_1 \ldots$, for $a \in \{1, \ldots, n-2\} \cap (S_2 \setminus W)$ with $\eta_{\bar{I},\bar{S}_2}$ as above with $\bar{I} = I \setminus \{n-1, n\}, \bar{S}_2 = S_2 \setminus (\{a\} \cup W)$,

($\mathsf{L}_{(15)}$5a) $\ldots \underline{a\,n\,b}\,\bar{I}\,1\,(n-1) \ldots$, for $a \in W, b \in I \setminus \{n-1, n\}$,

($\mathsf{L}_{(15)}$5b) $\ldots \underline{a\,n\,b}\,\bar{I}\,(n-1) \ldots$, for $a \in W, b \in S_1$,

($\mathsf{L}_{(15)}$5c) $\ldots \underline{a\,n\,b}\,\bar{I}\,1\,(n-1) \ldots$, for $a \in W, b \in S_2 \setminus W$

($\mathsf{L}_{(15)}$6) $\begin{cases} \ldots \underline{w_1\,(n-1)\,w_2}\,\eta_{\bar{I},\bar{S}_2}\,n\,w_3\,\bar{S}_1 \ldots, \\ \ldots \underline{w_1\,(n-1)\,w_3}\,\eta_{\bar{I},\bar{S}_2}\,n\,w_2, \bar{S}_1 \ldots, \\ \ldots \underline{w_2\,(n-1)\,w_3}\,\eta_{\bar{I},\bar{S}_2}\,n\,w_1, \bar{S}_1 \ldots, \end{cases}$
with $\eta_{\bar{I},\bar{S}_2}$ as above with $\bar{I} = I \setminus \{n-1, n\}, \bar{S}_2 = S_2 \setminus W$,

($\mathsf{L}_{(15)}$7a) $\ldots \underline{a\,n\,b}\,\bar{I}\,1\,(n-1) \ldots$, for $a \in \{1, \ldots, n-2\} \setminus (S_1 \cup W), b \in I \setminus \{n-1, n\}, a < b$,

($\mathsf{L}_{(15)}$7b) $\ldots \underline{a\,n\,b}\,\bar{I}\,(n-1) \ldots$, for $a \in \{1, \ldots, n-2\} \setminus (I \cup W), b \in S_1, a > b$,

($\mathsf{L}_{(15)}$7c) $\ldots \underline{a\,n\,b}\,\bar{I}\,(n-1) \ldots$, for $a, b \in S_2 \setminus W, a < b$,

($\mathsf{L}_{(15)}$8a) $\ldots \bar{S}_1\,\bar{I}\,\underline{(n-1)\,a\,n} \ldots$, for $a \in (S_1 \cup I) \setminus \{n-1, n\}$,

($\mathsf{L}_{(15)}$8b) $\ldots \bar{I}\,\underline{(n-1)\,a\,n}\,m \ldots$, for $a \in S_2$ with $m \in S_2, m \neq a$,

($\mathsf{L}_{(15)}$7d) $\ldots \underline{a\,n\,b}\,\bar{I}\,(n-1) \ldots$, for $a \in \{1, \ldots, n-2\} \cap S_1, b \in I \setminus \{n-1, n\}$.

It follows from the proof of Claim 2 in the proof of Theorem 2.3 in [6] and from the proof of Theorem 15 (or is easy to check) in combination with the fact that no 2-edge $\langle \tilde{a}, n, \tilde{b} \rangle, \tilde{a} \in S_1, \tilde{b} \in I$, is contained in the tours in (**$\mathsf{L}_{(15)}$8a**)–(**$\mathsf{L}_{(15)}$8b**) that all underlined 2-edges are not used in a previous substep and that we build exactly $n^2 - 4n + 3$ tours in (**$\mathsf{Step}_{(15)}$3**). It remains to check the root property of all tours constructed.

- Tours in (**$\mathsf{L}_{(15)}$1a**), (**$\mathsf{L}_{(15)}$1b**), (**$\mathsf{L}_{(15)}$2**), (**$\mathsf{L}_{(15)}$3a**), (**$\mathsf{L}_{(15)}$4a**), (**$\mathsf{L}_{(15)}$5c**), (**$\mathsf{L}_{(15)}$7c**), (**$\mathsf{L}_{(15)}$8b**), (**$\mathsf{L}_{(15)}$7d**): the tours contain one large block of nodes in $I \setminus \{n\}$, partially with one node of S_1 between two of these nodes, and n lies between two nodes that belong to S_2.
- Tours in (**$\mathsf{L}_{(15)}$1c**), (**$\mathsf{L}_{(15)}$3b**), (**$\mathsf{L}_{(15)}$4b**), (**$\mathsf{L}_{(15)}$6**): each node in I lies between two nodes that belong to S_2.

- Tours in ($\mathsf{L}_{(15)}$**5a**), ($\mathsf{L}_{(15)}$**5b**), ($\mathsf{L}_{(15)}$**7a**), ($\mathsf{L}_{(15)}$**7b**), ($\mathsf{L}_{(15)}$**8a**): the tours contain one large block of nodes in I, partially with one node of S_1 between two of these nodes.

All in all we created the same number of tour as in the proof of Theorem 15, more precisely $f(n)$ tours. So inequalities (15) define facets of $P_{\mathbf{SQTSP}_n}, n \geq 8$. $\square$