Sparse polynomial interpolation in Chebyshev bases

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We study the problem of reconstructing a sparse polynomial in a basis of Chebyshev polynomials (Chebyshev basis in short) from given samples on a Chebyshev grid of $[-1, 1]$. A polynomial is called M-sparse in a Chebyshev basis, if it can be represented by a linear combination of M Chebyshev polynomials. We show that an M-sparse polynomial of maximum degree $2N-1$ can be theoretically recovered from $2M$ samples on a Chebyshev grid. As efficient recovery methods, Prony–like methods are used. The reconstruction results are mainly presented for bases of Chebyshev polynomials of first and second kind, respectively. But similar issues can be obtained for bases of Chebyshev polynomials of third and fourth kind, respectively.

Key words and phrases: Sparse interpolation, Chebyshev basis, Chebyshev polynomial, sparse polynomial, Prony–like method, ESPRIT, matrix pencil factorization, companion matrix, Prony polynomial, eigenvalue problem, rectangular Toeplitz-plus-Hankel matrix.

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1 Introduction

The central issue of compressive sensing is the recovery of sparse signals from a rather small set of measurements, where a sparse signal can be represented in some basis by a linear combination with few nonzero coefficients. For example, a 1-periodic trigonometric polynomial of maximum degree $N-1$ with only M nonzero exponential terms can be recovered by $\mathcal{O}(M \log^4(N))$ sampling points that are randomly chosen from the equidistant grid $\{\frac{j}{N};\, j = 0, \ldots, N-1\}$, where $M \ll N$ (see [20]). Recently, Rauhut and Ward [18] have presented a recovery method of a polynomial of maximum degree $N-1$

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given in Legendre expansion with M nonzero terms, where $\mathcal{O}(M \log^4(N))$ random samples are taken independently according to the Chebyshev probability measure of $[-1, 1]$. The recovery algorithms in compressive sensing are often based on ℓ_1–minimization. Exact recovery of sparse signals or functions can be ensured only with a certain probability. The method of [18] can extended to sparse polynomial interpolation in a basis of Chebyshev polynomials too.

In contrast to these random recovery methods, there exist also deterministic methods for the reconstruction of an exponential sum

$$H(t) := \sum_{j=1}^{M} c_j \, \mathrm{e}^{\mathrm{i} f_j t} \quad (t \in \mathbb{R})$$

with distinct frequencies $f_j \in [-\pi, \pi)$ and complex coefficients. Such methods are the Prony–like methods [16], such as the classical Prony method, annihilating filter method [5], ESPRIT (Estimation of Signal Parameters via Rotational Invariance Techniques) [19], matrix pencil method [8, 7], and approximate Prony method [3, 15]. This approach allows the recovery of all parameters of H, i.e. M, f_j and c_j for $j = 1, \ldots, M$, from equidistant samples $H(k)$ $(k = 0, \ldots, 2N-1)$, where $N \geq M$. Prony–like methods can be applied also for the reconstruction of sparse trigonometric polynomials [16, Example 4.2]. Note that the classical Prony method is equivalent to the annihilating filter method. Unfortunately, the classical Prony method is very sensitive to noise in the sampled data. Hence numerous modifications have been proposed in order to improve the numerical behavior of the Prony method. Efficient Prony–like methods are ESPRIT, matrix pencil methods, and approximate Prony methods. This procedures are important within many disciplines in sciences and engineering (see [13]). For a survey of the most successful methods for the data fitting problem with linear combinations of complex exponentials, we refer to [12]. Note that a variety of papers compare the statistical properties of the different algorithms, see e.g. [8, 1, 2, 6]. Similar results for our new suggested algorithms are of great interest, but are behind the scope of this paper.

In this paper, we present a new deterministic approach to sparse polynomial interpolation in a basis of Chebyshev polynomials, if relatively few samples of a Chebyshev grid of $[-1, 1]$ are given. Note that Chebyshev grids are much better suited for the recovery of polynomials than uniform grids (see [4]). For $n \in \mathbb{N}_0$, the *nth Chebyshev polynomial of first kind* can be defined by

$$T_n(x) := \cos(n \arccos x) \quad (x \in [-1, 1])$$

(see for example [11, p. 2]). These polynomials are orthogonal with respect to the weight $(1-x^2)^{-1/2}$ on $(-1, 1)$ (see [11, p. 73]) and form the *Chebyshev*-1 *basis*.
Let M and N be integers with $1 \leq M < N$. A polynomial of maximum degree $2N-1$

$$h(x) = \sum_{k=0}^{2N-1} b_k \, T_k(x)$$

is called to be *M-sparse in the Chebyshev-1 basis*, if M coefficients b_k are nonzero and if the other $2N - M$ coefficients vanish. Then such a M-sparse polynomial h can be represented in the form

$$h(x) = \sum_{j=1}^{M} c_j \, T_{n_j}(x) \tag{1.1}$$

with $c_j := b_{n_j} \neq 0$ and $0 \leq n_1 < n_2 < \ldots < n_M < 2N$. The integer M is called the *Chebyshev-1 sparsity* of the polynomial (1.1).

Recently the authors have presented a unified approach to Prony–like methods for the parameter estimation of an exponential sum [16], namely the classical Prony method, the matrix pencil method [7], and the ESPRIT method [19]. The main idea is based on the evaluation of the eigenvalues of a matrix which is similar to the companion matrix of the Prony polynomial. To this end we have computed the singular value decomposition (SVD) or the QR decomposition of a special Toeplitz-plus-Hankel matrix (T+H matrix). The aim of this paper is to generalize this unified approach in order to obtain stable algorithms for an interpolation problem of a sparse polynomial (1.1) in the Chebyshev-1 basis. A similar spare interpolation problem on a grid of $[1, \infty]$, which is solved by the Prony method, was explored in [10, 9]. However we use a deterministic sampling set of $[-1, 1]$, and need only, at least theoretically, $2M$ samples for the reconstruction of the $2M$ parameters. Theorem 2.6 shows that an M-sparse polynomial (1.1) in a Chebyshev basis can be reconstructed from only $2M$ samples. A Prony–like method for sparse Legendre reconstruction was suggested by Peter, Plonka and Rosça in [14]. This method can be also generalized to other polynomial systems, but one needs there high order derivatives of the sparse polynomial.

The outline of this paper is as follows. In Section 2, we collect some useful properties of T+H matrices and Vandermonde–like matrices. Further we formulate the algorithms, if the order M is known and if only $2M$ sampled data (1.1) are given. We find a factorization of the T+H matrix in Lemma 2.2 and prove an interesting relation between the Prony polynomial and its companion matrix in Lemma 2.5. Thereby we are able to present the algorithms. In Section 3, we obtain corresponding results on sparse polynomial interpolation for unknown Chebyshev-1 sparsity M. Furthermore one can improve the numerical stability of the algorithms by using more sampling values. In Section 4, we discuss the sparse interpolation in the basis of Chebyshev polynomials of second kind. Finally we present some numerical experiments in Section 5, where we apply our methods to sparse polynomial interpolation.

In the following we use standard notations. By $\mathbb{N}_0$, we denote the set of all nonnegative integers. The Kronecker symbol is δ_k. The linear space of all column vectors with N real components is denoted by $\mathbb{R}^N$, where $\boldsymbol{o}$ is the corresponding zero vector. The linear space of all real M-by-N matrices is denoted by $\mathbb{R}^{M \times N}$, where $\boldsymbol{O}_{M,N}$ is the corresponding zero matrix. For a matrix $\boldsymbol{A}_{M,N} \in \mathbb{R}^{M \times N}$, its transpose is denoted by $\boldsymbol{A}_{M,N}^{\mathrm{T}}$, and its Moore–Penrose pseudoinverse by $\boldsymbol{A}_{M,N}^{\dagger}$. A square matrix $\boldsymbol{A}_{M,M}$ is abbreviated to $\boldsymbol{A}_M$. By $\boldsymbol{I}_M$ we denote the M-by-M identity matrix. By $\operatorname{null} \boldsymbol{A}_{M,N}$ we denote the null space of a matrix $\boldsymbol{A}_{M,N}$. Further we use the known submatrix notation. Thus $\boldsymbol{A}_{M,M+1}(1:M,\,2:M+1)$ is the submatrix of $\boldsymbol{A}_{M,M+1}$ obtained by extracting rows 1 through M and columns 2 through $M+1$, and $\boldsymbol{A}_{M,M+1}(1:M,\,M+1)$ means the last column vector of $\boldsymbol{A}_{M,M+1}$. Definitions are indicated by the symbol :=. Other notations are introduced when needed.

2 Interpolation for known Chebyshev–1 sparsity

For $u_N := \cos\frac{\pi}{2N-1}$ we form the nonequidistant Chebyshev grid $\{u_{N,k} := T_k(u_N) = \cos\frac{k\pi}{2N-1};\, k = 0,\ldots,2N-1\}$ of the interval $[-1,\,1]$. Note that $T_{2N-1}(u_{N,k}) = (-1)^k$ $(k=0,\ldots,2N-1)$. We consider the following problem of *sparse polynomial interpolation in the Chebyshev*-1 *basis*: For given sampled data $h_k := h(u_{N,k}) = h(\cos\frac{k\pi}{2N-1})$ $(k = 0,\ldots,2M-1)$ determine all parameters n_j and c_j $(j=1,\ldots,M)$ of the sparse polynomial (1.1). If we substitute $x=\cos t$ $(t\in[0,\,\pi])$, then we see that the above interpolation problem is closely related to the *interpolation problem of the sparse, even trigonometric polynomial*

$$g(t) := h(\cos t) = \sum_{j=1}^{M} c_j\,\cos(n_j t) \quad (t\in[0,\,\pi])\,, \tag{2.1}$$

where the sampled values $g(\frac{k\pi}{2N-1}) = h_k$ $(k=0,\ldots,2M-1)$ are given (see [17]).
We introduce the *Prony polynomial* P of degree M with the leading coefficient 2^{M-1}, whose roots are $x_j := T_{n_j}(u_N) = \cos\frac{n_j\pi}{2N-1}$ $(j=1,\ldots,M)$, i.e.

$$P(x) = 2^{M-1}\prod_{j=1}^{M}\big(x-\cos\frac{n_j\pi}{2N-1}\big)\,. \tag{2.2}$$

Then the Prony polynomial P can be represented in the Chebyshev-1 basis by

$$P(x) = \sum_{l=0}^{M} p_l\,T_l(x) \quad (p_M := 1)\,. \tag{2.3}$$

The coefficients p_j of the Prony polynomial (2.3) can be characterized as follows:

Lemma 2.1 *For $k=0,\,1,\,\ldots$, the sampled data h_k and the coefficients p_l of the Prony polynomial* (2.3) *satisfy the equations*

$$\sum_{j=0}^{M-1}(h_{j+k}+h_{|j-k|})\,p_j = -(h_{k+M}+h_{|M-k|})\,. \tag{2.4}$$

Proof. Using $\cos(\alpha+\beta)+\cos(\alpha-\beta) = 2\,\cos\alpha\,\cos\beta$, we obtain by (2.1) that

$$\begin{aligned} h_{j+k}+h_{|j-k|} &= 2\sum_{l=1}^{M} c_l\left(\cos\frac{n_l(j+k)\pi}{2N-1}+\cos\frac{n_l(j-k)\pi}{2N-1}\right)\\ &= 2\sum_{l=1}^{M} c_l\,\cos\frac{n_l j\pi}{2N-1}\,\cos\frac{n_l k\pi}{2N-1}\,. \end{aligned} \tag{2.5}$$

Thus we conclude that

$$\begin{aligned}\sum_{j=0}^{M} \big(h_{j+k} + h_{|j-k|}\big)\, p_j &= 2 \sum_{l=1}^{M} c_l \, \cos \frac{n_l k \pi}{2N-1} \, \sum_{j=0}^{M} p_j \, \cos \frac{n_l j \pi}{2N-1} \\ &= 2 \sum_{l=1}^{M} c_l \, \cos \frac{n_l k \pi}{2N-1} \, P\big(\cos \frac{n_l \pi}{2N-1}\big) = 0\,.\end{aligned}$$

By $p_M = 1$, this implies the assertion (2.4). ∎

Introducing the vectors $\boldsymbol{h}(k) := (h_{j+k} + h_{|j-k|})_{j=0}^{M-1}$ $(k = 0, \ldots, M)$ and the *square* T+H *matrix*

$$\begin{aligned}\boldsymbol{H}_M(0) &:= (h_{j+k} + h_{|j-k|})_{j,k=0}^{M-1} = \big(\ \boldsymbol{h}(0) \quad \boldsymbol{h}(1) \quad \ldots \quad \boldsymbol{h}(M-1)\ \big) \\ &= \begin{pmatrix} 2\,h_0 & 2\,h_1 & \ldots & 2\,h_{M-1} \\ 2\,h_1 & h_2 + h_0 & \ldots & h_M + h_{M-2} \\ \vdots & \vdots & & \vdots \\ 2\,h_{M-1} & h_M + h_{M-2} & \ldots & h_{2M-2} + h_0 \end{pmatrix},\end{aligned}$$

then by (2.4) the vector $\boldsymbol{p} := (p_k)_{k=0}^{M-1}$ is a solution of the linear system

$$\boldsymbol{H}_M(0)\, \boldsymbol{p} = -\boldsymbol{h}(M)\,. \tag{2.6}$$

Lemma 2.2 *Let M and N be integers with $1 \le M \le N$. Further let h be an M-sparse polynomial of degree at most $2N-1$ in the Chebyshev-1 basis.*
If $h(u_{N,j}) = 0$ for $j = 0, \ldots, M-1$, then h is identically zero. Further the Vandermonde-like matrix

$$\boldsymbol{V}_M(\boldsymbol{x}) := \big(T_{n_j}(u_{N,k})\big)_{k=0,j=1}^{M-1,M} = \big(T_k(x_j)\big)_{k=0,j=1}^{M-1,M} = \Big(\cos \frac{n_j k \pi}{2N-1}\Big)_{k=0,j=1}^{M-1,M}$$

with $\boldsymbol{x} := (x_j)_{j=1}^{M}$ is nonsingular and the T+H *matrix $\boldsymbol{H}_M(0)$ can be factorized in the following form*

$$\boldsymbol{H}_M(0) = 2\, \boldsymbol{V}_M(\boldsymbol{x})\, (\operatorname{diag} \boldsymbol{c})\, \boldsymbol{V}_M(\boldsymbol{x})^{\mathrm{T}} \tag{2.7}$$

and is nonsingular.

Proof. 1. Assume that the Vandermonde-like matrix $\boldsymbol{V}_M(\boldsymbol{x})$ is singular. Then there exists a vector $\boldsymbol{d} = (d_l)_{l=0}^{M-1} \ne \boldsymbol{o}$ such that $\boldsymbol{d}^{\mathrm{T}}\, \boldsymbol{V}_M(\boldsymbol{x}) = \boldsymbol{o}^{\mathrm{T}}$. We consider the even trigonometric polynomial D of order at most $M-1$ given by

$$D(t) = \sum_{l=0}^{M-1} d_l \, \cos(lt) \quad (t \in \mathbb{R})\,.$$

Hence $\boldsymbol{d}^{\mathrm{T}}\, \boldsymbol{V}_M(\boldsymbol{x}) = \boldsymbol{o}^{\mathrm{T}}$ implies that $t_j = \frac{n_j \pi}{2N-1} \in [0,\, \pi]$ $(j = 1, \ldots, M)$ are roots of D. These M roots are distinct, because $0 \le n_1 < \ldots < n_M < 2N$. But this is impossible,

since the even trigonometric polynomial $D \neq 0$ of degree at most $M-1$ cannot have M distinct roots in $[0, \pi]$. Therefore, $\boldsymbol{V}_M(\boldsymbol{x})$ is nonsingular.
If $h(u_{N,j}) = 0$ for $j = 0, \ldots, M-1$, then $\boldsymbol{V}_M(\boldsymbol{x})\, \boldsymbol{c} = \boldsymbol{o}$. Since $\boldsymbol{V}_M(\boldsymbol{x})$ is nonsingular, $\boldsymbol{c}$ is equal to $\boldsymbol{o}$, such that h is identically zero.
2. The factorization (2.7) of the T+H matrix $\boldsymbol{H}_M(0)$ follows immediately from (2.5). Since $c_j \neq 0$ $(j = 1, \ldots, M)$, $\mathrm{diag}\, \boldsymbol{c}$ is nonsingular. Further the Vandermonde–like matrix $\boldsymbol{V}_M(\boldsymbol{x})$ is nonsingular, such that $\boldsymbol{H}_M(0)$ is nonsingular too. ■

Introducing the matrix

$$\boldsymbol{P}_M := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 & -p_0 \\ 1 & 0 & 1 & \ldots & 0 & 0 & -p_1 \\ 0 & 1 & 0 & \ldots & 0 & 0 & -p_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & -p_{M-3} \\ 0 & 0 & 0 & \ldots & 1 & 0 & 1-p_{M-2} \\ 0 & 0 & 0 & \ldots & 0 & 1 & -p_{M-1} \end{pmatrix} \in \mathbb{R}^{M \times M}$$

and using the linear system (2.6), we see that

$$\boldsymbol{H}_M(0)\, \boldsymbol{P}_M = \boldsymbol{H}_M(1) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{h}(0) & \ldots & \boldsymbol{h}(M-2) \end{pmatrix}$$

with the T+H matrix

$$\boldsymbol{H}_M(1) := \begin{pmatrix} \boldsymbol{h}(1) & \boldsymbol{h}(2) & \ldots & \boldsymbol{h}(M) \end{pmatrix} = \big(h_{j+k+1} + h_{|j-k-1|}\big)_{j,k=0}^{M-1} \in \mathbb{R}^{M \times M}\,.$$

This T+H matrix has the following properties:

Lemma 2.3 *The* T+H *matrix* $\boldsymbol{H}_M(1)$ *can be factorized in the following form*

$$\boldsymbol{H}_M(1) = 2\, \boldsymbol{V}_M(\boldsymbol{x})\, (\mathrm{diag}\, \boldsymbol{c})\, \boldsymbol{V}'_M(\boldsymbol{x})^{\mathrm{T}} \tag{2.8}$$

with the Vandermonde-like matrix $\boldsymbol{V}'_M(\boldsymbol{x}) := \big(T_k(x_j)\big)_{k,j=1}^{M}$. *Further the matrices* $\boldsymbol{H}_M(1)$ *and* $\boldsymbol{V}'_M(\boldsymbol{x})$ *are nonsingular.*

Proof. 1. By Lemma 2.1 we know that

$$\sum_{k=0}^{M} (h_{j+k} + h_{|j-k|})\, p_k = 0 \quad (j = 0, \ldots, 2N-M-1)\,.$$

Consequently we obtain

$$\boldsymbol{H}_M(0)\, (p_k)_{k=0}^{M-1} = -\boldsymbol{h}(M)\,, \quad \boldsymbol{H}_M(1)\, (p_{k+1})_{k=0}^{M-1} = -p_0\, \boldsymbol{h}(0)\,,$$

where

$$p_0 = 2^{M-1}\, (-1)^M \prod_{j=1}^{M} \cos \frac{n_j \pi}{2N-1}$$

does not vanish. This implies that

$$\boldsymbol{h}(M) \in \operatorname{span}\{\boldsymbol{h}(0), \ldots, \boldsymbol{h}(M-1)\}, \quad \boldsymbol{h}(0) \in \operatorname{span}\{\boldsymbol{h}(1), \ldots, \boldsymbol{h}(M)\}.$$

Thus we obtain that $\operatorname{rank} \boldsymbol{H}_M(0) = \operatorname{rank} \boldsymbol{H}_M(1) = M$.
2. The (j,k)th element of the matrix product $2\,\boldsymbol{V}_M(\boldsymbol{x})\,(\operatorname{diag}\boldsymbol{c})\,\boldsymbol{V}'_M(\boldsymbol{x})^{\mathrm{T}}$ can be analogously computed as (2.5) such that

$$2\sum_{l=1}^{M} c_l\, T_{n_l}(u_{N,j})\, T_{n_l}(u_{N,k}) = h_{j+k+1} + h_{|j-k-1|}\,.$$

Since $\boldsymbol{H}_M(1)$, $\boldsymbol{V}_M(\boldsymbol{x})$, and $\operatorname{diag}\boldsymbol{c}$ are nonsingular, it follows from (2.8) that the Vandermonde-like matrix $\boldsymbol{V}'_M(\boldsymbol{x})$ is nonsingular too. ■

In the following Lemmas 2.4 and 2.5 we show that the zeros of (2.3) can be computed via an eigenvalue problem. To this end, we represent the Chebyshev polynomial T_M in the form of a determinant.

Lemma 2.4 *Let M be a positive integer. Further let $\boldsymbol{E}_M := \operatorname{diag}(\frac{1}{2}, 1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^M$ and the modified shift matrix*

$$\boldsymbol{S}_M := \left(\delta_{j-k-1} + \delta_{j-k+1}\right)_{j,k=0}^{M-1} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{M\times M}.$$

Then

$$\det(2\,\boldsymbol{E}_M\,x - \boldsymbol{S}_M) = T_M(x) \quad (x \in \mathbb{R}).$$

Proof. We show this by induction. For $M = 1$ and $M = 2$ it follows immediately the assertion. For $M \ge 3$ we compute the determinant

$$\begin{vmatrix} x & -1 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2x & -1 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 2x & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2x & -1 & 0 \\ 0 & 0 & 0 & \ldots & -1 & 2x & -1 \\ 0 & 0 & 0 & \ldots & 0 & -1 & 2x \end{vmatrix}$$

using cofactors of the last row (cf. [11, p. 18]). Then we obtain the known recursion of the Chebyshev polynomials $T_M(x) = 2x\,T_{M-1}(x) - T_{M-2}(x)$ (see [11, p. 2]). This completes the proof. ■

Now we show that $\frac{1}{2}\,\boldsymbol{E}_M^{-1}\,\boldsymbol{P}_M$ is the companion matrix of the Prony polynomial (2.3) in the Chebyshev-1 basis.

Lemma 2.5 *Let M be a positive integer. Then $\frac{1}{2}\,\boldsymbol{E}_M^{-1}\,\boldsymbol{P}_M$ is the companion matrix of the Prony polynomial* (2.3) *in the Chebyshev-*1 *basis, i.e.*

$$\det\big(2\,x\,\boldsymbol{E}_M-\boldsymbol{P}_M\big)=2^{M-1}\det\big(x\,\boldsymbol{I}_M-\frac{1}{2}\,\boldsymbol{E}_M^{-1}\,\boldsymbol{P}_M\big)=P(x)\quad(x\in\mathbb{R})\,.$$

Proof. Applying Lemma 2.4, we compute $\det(2\,x\,\boldsymbol{E}_M-\boldsymbol{P}_M)$ using cofactors of the last column. Then we obtain

$$\det\big(2\,x\,\boldsymbol{E}_M-\boldsymbol{P}_M\big)=T_M(x)+\sum_{l=0}^{M-1}p_l\,T_l(x)=P(x)\quad(x\in\mathbb{R})\,.$$

Otherwise it follows that

$$\det\big(2\,x\,\boldsymbol{E}_M-\boldsymbol{P}_M\big)=\det(2\,\boldsymbol{E}_M)\det\big(x\,\boldsymbol{I}_M-\frac{1}{2}\,\boldsymbol{E}_M^{-1}\,\boldsymbol{P}_M\big)$$

with $\det(2\,\boldsymbol{E}_M)=2^{M-1}$. This completes the proof. ■

Theorem 2.6 *Let M and N be integers with $1\le M<N$. Let h be a M-sparse polynomial of maximum degree $2N-1$ in the Chebyshev-*1 *basis.*
Then the M coefficients $c_j\in\mathbb{R}$ $(j=1,\ldots M)$ and the M nonnegative integers n_j $(j=1,\ldots M)$ of (1.1) *can be reconstructed from the $2M$ samples $h_k=h(\cos\frac{k\pi}{2N-1})$ $(k=0,\ldots,2M-1)$.*

Proof. We form the equation (2.6). The matrix $\boldsymbol{H}_M(0)$ is nonsingular by Lemma 2.2. By Lemma 2.5, the eigenvalues of the companion matrix $\frac{1}{2}\,\boldsymbol{E}_M^{-1}\,\boldsymbol{P}_M$ of the Prony polynomial (2.3) in the Chebyshev-1 basis coincide with the zeros of (2.3). Note that

$$\boldsymbol{P}_M=\boldsymbol{S}_M-\begin{pmatrix}\boldsymbol{o} & \ldots & \boldsymbol{o} & \boldsymbol{p}\end{pmatrix}$$

i.e., we compute the zeros of the Prony polynomial (2.2) as an eigenvalue problem such that we obtain the nonnegative integers n_j $(j=1,\ldots M)$. We form the Vandermonde-like matrix $\boldsymbol{V}_M(\boldsymbol{x})$ with $x_j=T_{n_j}(u_N)$ $(j=1,\ldots,M)$, which is nonsingular by Lemma 2.2, and obtain finally the coefficients $c_j\in\mathbb{R}$ $(j=1,\ldots,M)$. ■

Thus we can summarize:

Algorithm 2.7 *(Prony method for sparse Chebyshev-*1 *interpolation)*
Input: $h_k=h(u_{N,k})\in\mathbb{R}$ $(k=0,\ldots,2M-1)$, $M\in\mathbb{N}$ *Chebyshev-*1 *sparsity of* (1.1).

1. *Solve the square system*

$$\boldsymbol{H}_M(0)\,(p_j)_{j=0}^{M-1}=-\boldsymbol{h}(M)\,.$$

2. *Determine the simple roots x_j $(j=1,\ldots M)$ of the Prony polynomial* (2.3), *where $1\ge x_1>x_2>\ldots>x_M\ge-1$, and compute then $n_j:=[\frac{2N-1}{\pi}\,\arccos x_j]$ $(j=1,\ldots,M)$, where $[x]:=\lfloor x+0.5\rfloor$ means rounding of $x\in\mathbb{R}$ to the nearest integer.*

3. *Compute* $c_j \in \mathbb{R}$ $(j = 1, \ldots, M)$ *as solution of the square Vandermonde-like system*

$$\boldsymbol{V}_M(\boldsymbol{x})\,\boldsymbol{c} = (h_k)_{k=0}^{M-1}$$

with $\boldsymbol{c} := (c_j)_{j=1}^{M}$.

Output: $n_j \in \mathbb{N}_0$ $(0 \le n_1 < n_2 < \ldots < n_M < 2N)$, $c_j \in \mathbb{R}$ $(j = 1, \ldots, M)$.

Now we show that the *matrix pencil method* follows directly from the Prony method. In other words, the matrix pencil method is a simplified Prony method. First we observe that

$$\boldsymbol{H}_M(0) = 2\,\boldsymbol{V}_M(\boldsymbol{x})\,(\operatorname{diag}\boldsymbol{c})\,\boldsymbol{V}_M(\boldsymbol{x})^{\mathrm{T}}\,.$$

Since $c_j \neq 0$ $(j = 1, \ldots, M)$, the matrix $\boldsymbol{H}_M(0)$ has the rank M and is invertible. Note that the Chebyshev-1 sparsity of the polynomial (1.1) coincides with the rank of $\boldsymbol{H}_M(0)$. Hence we conclude that

$$\begin{aligned}
\det\left(2\,x\,\boldsymbol{H}_M(0)\,\boldsymbol{E}_M - \boldsymbol{H}_M(0)\,\boldsymbol{P}_M\right) &= \det\left(\boldsymbol{H}_M(0)\right)\det\left(2\,x\,\boldsymbol{E}_M - \boldsymbol{P}_M\right)\\
&= \det\left(\boldsymbol{H}_M(0)\right)P(x)
\end{aligned}$$

such that the eigenvalues of the *square matrix pencil*

$$2\,x\,\boldsymbol{H}_M(0)\,\boldsymbol{E}_M - \boldsymbol{H}_M(0)\,\boldsymbol{P}_M \quad (x \in \mathbb{R}^M) \tag{2.9}$$

are exactly $x_j = \cos\frac{n_j\pi}{2N-1} \in [-1,\,1]$ $(j = 1, \ldots, M)$. Each eigenvalue x_j of the matrix pencil (2.9) is simple and has a right eigenvector $\boldsymbol{v} = (v_k)_{k=0}^{M-1}$ with

$$v_{M-1} = T_M(x_j) = -\sum_{l=0}^{M-1} p_l\,T_l(x_j)\,.$$

The other components $v_{M-2}, \ldots, v_0$ can be computed recursively from the linear system

$$\boldsymbol{P}_M\,\boldsymbol{v} = 2\,x_j\,\boldsymbol{E}_M\,\boldsymbol{v}\,.$$

Hence we obtain $\boldsymbol{H}_M(0)\,\boldsymbol{P}_M\,\boldsymbol{v} = 2\,x_j\,\boldsymbol{H}_M(0)\,\boldsymbol{E}_M\,\boldsymbol{v}$, where the matrices can be represented in the following form

$$\begin{aligned}
\boldsymbol{H}_M(0)\,\boldsymbol{P}_M &= \boldsymbol{H}_M(1) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{h}(0) & \ldots & \boldsymbol{h}(M-2) \end{pmatrix},\\
2\,\boldsymbol{H}_M(0)\,\boldsymbol{E}_M &= \boldsymbol{H}_M(0) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{h}(1) & \ldots & \boldsymbol{h}(M-1) \end{pmatrix}.
\end{aligned}$$

Example 2.8 In the case $M = 3$ we have to solve the linear system

$$\begin{pmatrix} 0 & 1 & -p_0 \\ 1 & 0 & 1-p_1 \\ 0 & 1 & -p_2 \end{pmatrix}\begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x_j\,v_0 \\ 2x_j\,v_1 \\ 2x_j\,v_2 \end{pmatrix}$$

with

$$v_2 = T_3(x_j) = -\sum_{l=0}^{2} p_l\,T_l(x_j)\,.$$

Then we determine the other components of the eigenvector $\boldsymbol{v} = (v_l)_{l=0}^2$ as

$$\begin{aligned} v_1 &= -p_1\, T_0(x_j) - (2p_0 + p_2)\, T_1(x_j) - p_1\, T_2(x_j)\,, \\ v_0 &= -(p_0 + p_2)\, T_0(x_j) - 2p_1\, T_1(x_j) - 2p_0\, T_2(x_j)\,. \end{aligned}$$

□

In the following, we factorize the square T+H matrices $\boldsymbol{H}_M(s)$ $(s = 0,\, 1)$ simultaneously. Therefore we introduce the rectangular T+H matrix

$$\boldsymbol{H}_{M,M+1} := \begin{pmatrix} \boldsymbol{H}_M(0) & \boldsymbol{H}_M(1)(1:M,\, M) \end{pmatrix} = \begin{pmatrix} \boldsymbol{h}(0) & \boldsymbol{h}(1) & \ldots & \boldsymbol{h}(M) \end{pmatrix} \tag{2.10}$$

such that conversely

$$\boldsymbol{H}_M(s) = \boldsymbol{H}_{M,M+1}(1:M,\, 1+s:M+s) \quad (s = 0, 1)\,. \tag{2.11}$$

Then we compute the QR factorization of $\boldsymbol{H}_{M,M+1}$ with column pivoting and obtain

$$\boldsymbol{H}_{M,M+1}\, \boldsymbol{\Pi}_{M+1} = \boldsymbol{Q}_M\, \boldsymbol{R}_{M,M+1}$$

with an orthogonal matrix $\boldsymbol{Q}_M$, a permutation matrix $\boldsymbol{\Pi}_{M+1}$, and a trapezoidal matrix $\boldsymbol{R}_{M,M+1}$, where $\boldsymbol{R}_{M,M+1}(1:M,\, 1:M)$ is a nonsingular upper triangular matrix. Note that the permutation matrix $\boldsymbol{\Pi}_{M+1}$ is chosen such that the diagonal entries of $\boldsymbol{R}_{M,M+1}(1:M,\, 1:M)$ have nonincreasing absolute values. Using the definition

$$\boldsymbol{S}_{M,M+1} := \boldsymbol{R}_{M,M+1}\, \boldsymbol{\Pi}_{M+1}^{\mathrm{T}}\,,$$

we infer that by (2.11)

$$\boldsymbol{H}_M(s) = \boldsymbol{Q}_M\, \boldsymbol{S}_M(s) \quad (s = 0,\, 1)\,,$$

where

$$\boldsymbol{S}_M(s) := \boldsymbol{S}_{M,M+1}(1:M,\, 1+s:M+s) \quad (s = 0, 1)\,.$$

Hence we can factorize the matrices $2\, \boldsymbol{H}_M(0)\, \boldsymbol{E}_M$ and $\boldsymbol{H}_M(0)\, \boldsymbol{P}_M$ in the following form

$$\begin{aligned} 2\, \boldsymbol{H}_M(0)\, \boldsymbol{E}_M &= \boldsymbol{H}_M(0) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{h}(1) & \ldots & \boldsymbol{h}(M-1) \end{pmatrix} = \boldsymbol{Q}_M\, \boldsymbol{S}'_M(0)\,, \\ \boldsymbol{H}_M(0)\, \boldsymbol{P}_M &= \boldsymbol{H}_M(1) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{h}(0) & \ldots & \boldsymbol{h}(M-2) \end{pmatrix} = \boldsymbol{Q}_M\, \boldsymbol{S}'_M(1)\,, \end{aligned}$$

where

$$\boldsymbol{S}'_M(0) := \boldsymbol{S}_M(0) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{S}_M(1)(1:M,\, 1:M-1) \end{pmatrix}\,, \tag{2.12}$$

$$\boldsymbol{S}'_M(1) := \boldsymbol{S}_M(1) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{S}_M(0)(1:M,\, 1:M-1) \end{pmatrix}\,. \tag{2.13}$$

Since $\boldsymbol{Q}_M$ is orthogonal, the generalized eigenvalue problem of the matrix pencil (2.9) is equivalent to the generalized eigenvalue problem of the matrix pencil

$$x\, \boldsymbol{S}'_M(0) - \boldsymbol{S}'_M(1) = \boldsymbol{S}'_M(0) \left(x\, \boldsymbol{I}_M - \left(\boldsymbol{S}'_M(0)\right)^{-1} \boldsymbol{S}'_M(1) \right) \quad (x \in \mathbb{R})\,.$$

Since $\boldsymbol{H}_M(0)$ is nonsingular by Lemma 2.2, the matrix $2\, \boldsymbol{H}_M(0)\, \boldsymbol{E}_M$ is nonsingular too. Hence $\boldsymbol{S}'_M(0) = 2\, \boldsymbol{Q}_M^*\, \boldsymbol{H}_M(0)\, \boldsymbol{E}_M$ is invertible.

We summarize this method:

Algorithm 2.9 *(Matrix pencil factorization based on* QR *decomposition for sparse Chebyshev-1 interpolation)*

Input: $h_k = h(u_{N,k}) \in \mathbb{R}$ $(k = 0, \ldots, 2M-1)$, $M \in \mathbb{N}$ *Chebyshev-1 sparsity of* (1.1).

1. *Compute the* QR *factorization with column pivoting of the rectangular* T+H *matrix* (2.10) *and form the matrices* (2.12) *and* (2.13).
2. *Determine the eigenvalues* $x_j \in [-1,\,1]$ $(j = 1, \ldots, M)$ *of the square matrix*

$$\big(\boldsymbol{S}'_M(0)\big)^{-1}\,\boldsymbol{S}'_M(1)\,,$$

where x_j *are ordered in the following way* $1 \ge x_1 > x_2 > \ldots > x_M \ge -1$. *Form* $n_j := [\frac{2N-1}{\pi}\,\arccos x_j]$ $(j = 1, \ldots, M)$.
3. *Compute* $c_j \in \mathbb{R}$ $(j = 1, \ldots, M)$ *as solution of the square Vandermonde-like system*

$$\boldsymbol{V}_M(\boldsymbol{x})\,\boldsymbol{c} = (h_k)_{k=0}^{M-1}$$

with $\boldsymbol{x} := (x_j)_{j=1}^M$ *and* $\boldsymbol{c} := (c_j)_{j=1}^M$.

Output: $n_j \in \mathbb{N}_0$ $(0 \le n_1 < n_2 < \ldots < n_M < 2N)$, $c_j \in \mathbb{R}$ $(j = 1, \ldots, M)$.

In contrast to Algorithm 2.9, we use now the singular value decomposition (SVD) of the rectangular Hankel matrix (2.10) and obtain a method which is known as the ESPRIT method. Applying the SVD to $\boldsymbol{H}_{M,M+1}$, we obtain

$$\boldsymbol{H}_{M,M+1} = \boldsymbol{U}_M\,\boldsymbol{D}_{M,M+1}\,\boldsymbol{W}_{M+1}$$

with orthogonal matrices $\boldsymbol{U}_M$, $\boldsymbol{W}_{M+1}$ and a diagonal matrix $\boldsymbol{D}_{M,M+1}$, whose diagonal entries are the ordered singular values $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_M > 0$ of $\boldsymbol{H}_{M,M+1}$. Introducing

$$\boldsymbol{D}_M := \boldsymbol{D}_{M,M+1}(1:M,\,1:M)\,, \quad \boldsymbol{W}_{M,M+1} := \boldsymbol{W}_{M+1}(1:M,\,1:M+1)\,,$$

we can simplify the SVD of (2.10) by

$$\boldsymbol{H}_{M,M+1} = \boldsymbol{U}_M\,\boldsymbol{D}_M\,\boldsymbol{W}_{M,M+1}\,.$$

Note that $\boldsymbol{W}_{M,M+1}\,\boldsymbol{W}_{M,M+1}^{\mathrm{T}} = \boldsymbol{I}_M$. Setting

$$\boldsymbol{W}_M(s) := \boldsymbol{W}_{M,M+1}(1:M,\,1+s:M+s) \quad (s = 0,\,1)\,,$$

it follows from (2.11) that $\boldsymbol{H}_M(s) = \boldsymbol{U}_M\,\boldsymbol{D}_M\,\boldsymbol{W}_M(s)$ $(s = 0,\,1)$. Hence we can factorize the matrices $2\,\boldsymbol{H}_M(0)\,\boldsymbol{E}_M$ and $\boldsymbol{H}_M(0)\,\boldsymbol{P}_M$ in the following form

$$\begin{aligned} 2\,\boldsymbol{H}_M(0)\,\boldsymbol{E}_M &= \boldsymbol{H}_M(0) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{h}(1) & \ldots & \boldsymbol{h}(M-1) \end{pmatrix} = \boldsymbol{U}_M\,\boldsymbol{D}_M\,\boldsymbol{W}'_M(0)\,, \\ \boldsymbol{H}_M(0)\,\boldsymbol{P}_M &= \boldsymbol{H}_M(1) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{h}(0) & \ldots & \boldsymbol{h}(M-2) \end{pmatrix} = \boldsymbol{U}_M\,\boldsymbol{D}_M\,\boldsymbol{W}'_M(1)\,, \end{aligned}$$

where

$$\boldsymbol{W}'_M(0) := \boldsymbol{W}_M(0) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{W}_M(1)(1:M,\,1:M-1) \end{pmatrix}\,, \tag{2.14}$$

$$\boldsymbol{W}'_M(1) := \boldsymbol{W}_M(1) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{W}_M(0)(1:M,\,1:M-1) \end{pmatrix}\,. \tag{2.15}$$

Clearly, $\boldsymbol{W'}_M(0) = 2\,\boldsymbol{D}_M^{-1}\,\boldsymbol{U}_M^{\mathrm{T}}\,\boldsymbol{H}_M(0)\,\boldsymbol{E}_M$ is a nonsingular matrix by construction. Then we infer that the generalized eigenvalue problem of the matrix pencil (2.9) is equivalent to the generalized eigenvalue problem of the matrix pencil

$$x\,\boldsymbol{W'}_M(0) - \boldsymbol{W'}_M(1) = \boldsymbol{W'}_M(0)\,\Big(x\,\boldsymbol{I}_M - \big(\boldsymbol{W'}_M(0)\big)^{-1}\,\boldsymbol{W'}_M(1)\Big)\,,$$

since $\boldsymbol{U}_M$ is orthogonal and $\boldsymbol{D}_M$ is invertible. Therefore we obtain that

$$\boldsymbol{P}_M = \big(\boldsymbol{H}_M(0)\big)^{-1}\,\boldsymbol{U}_M\,\boldsymbol{D}_M\,\boldsymbol{W'}_M(1)\,.$$

Algorithm 2.10 *(*ESPRIT *method for sparse Chebyshev*-1 *interpolation)*

Input: $h_k \in \mathbb{R}$ $(k = 0, \ldots, 2M-1)$, $M \in \mathbb{N}$ *Chebyshev*-1 *sparsity of* (1.1).

1. *Compute the* SVD *of the Hankel matrix* (2.10) *and form the matrices* (2.14) *and* (2.15).
2. *Determine the eigenvalues* $x_j \in [-1,\,1]$ $(j = 1, \ldots M)$ *of* $\big(\boldsymbol{W'}_M(0)\big)^{-1}\,\boldsymbol{W'}_M(1)$, *where* x_j *are ordered in the following form* $1 \ge x_1 > x_2 > \ldots > x_M \ge -1$. *Form* $n_j := [\frac{2N-1}{\pi}\,\arccos x_j]$ $(j = 1, \ldots, M)$.
3. *Compute the coefficients* $c_j \in \mathbb{R}$ $(j = 1, \ldots, M)$ *as solution of the square Vandermonde-like system*

$$\boldsymbol{V}_M(\boldsymbol{x})\,\boldsymbol{c} = (h_k)_{k=0}^{M-1}$$

with $\boldsymbol{x} := (x_j)_{j=1}^M$ *and* $\boldsymbol{c} := (c_j)_{j=1}^M$.

Output: $n_j \in \mathbb{N}_0$ $(0 \le n_1 < n_2 < \ldots < n_M < 2N)$, $c_j \in \mathbb{R}$ $(j = 1, \ldots, M)$.

Remark 2.11 The last step of the Algorithms 2.7 – 2.10 can be replaced by the computation of the real coefficients c_j $(j = 1, \ldots, M)$ as least squares solution of the overdetermined Vandermonde-like $\boldsymbol{V}_{2M,M}(\boldsymbol{x})\,\boldsymbol{c} = (h_k)_{k=0}^{2M-1}$ with the rectangular Vandermonde-like matrix

$$\boldsymbol{V}_{2M,M}(\boldsymbol{x}) := \big(T_k(x_j)\big)_{k=0,j=1}^{2M-1,M} = \big(\cos\frac{n_j k \pi}{2N-1}\big)_{k=0,j=1}^{2M-1,M}\,.$$

□

In the case of sparse Chebyshev-1 interpolation of (1.1) with known Chebyshev-1 sparsity M, we have seen that each method determines the eigenvalues x_j $(j = 1, \ldots, M)$ of the matrix pencil $2\,x\,\boldsymbol{E}_M - \boldsymbol{P}_M$, where $\frac{1}{2}\,\boldsymbol{E}_M^{-1}\,\boldsymbol{P}_M$ is the companion matrix of the Prony polynomial (2.3) in the Chebyshev-1 basis.

3 Interpolation for unknown Chebyshev-1 sparsity

Now we consider the more general case of interpolation of the sparse polynomial (1.1) with unknown Chebyshev-1 sparsity M. Let $L \in \mathbb{N}$ be convenient upper bound of the sparsity M with $M \le L \le N$. In order to improve the stability, we allow to choose more sampling points. Therefore we introduce an additional parameter K with $L \le K \le N$ such that we use $K + L$ sampling points of (1.1), more precisely we assume that noiseless sampled data $h_k = h(u_{N,k})$ $(k = 0, \ldots, L+K-1)$ are given. With the $L+K$ sampled data $h_k \in \mathbb{R}$ $(k = 0, \ldots, L+K-1)$ we form the *rectangular* T+H *matrices*

$$\begin{aligned} \boldsymbol{H}_{K,L+1} &:= \big(h_{l+m} + h_{|l-m|}\big)_{l,m=0}^{K-1,L}, &(3.1)\\ \boldsymbol{H}_{K,L}(s) &:= \big(h_{l+m+s} + h_{|l-m-s|}\big)_{l,m=0}^{K-1,L-1} \quad (s = 0,\, 1)\,. &(3.2)\end{aligned}$$

Then $\boldsymbol{H}_{K,L}(1)$ is a shifted version of the T+H matrix $\boldsymbol{H}_{K,L}(0)$ and

$$\begin{aligned} \boldsymbol{H}_{K,L+1} &= \big(\ \boldsymbol{H}_{K,L}(0) \quad \boldsymbol{H}_{K,L}(1)(1:K,\, L)\ \big)\,,\\ \boldsymbol{H}_{K,L}(s) &= \boldsymbol{H}_{K,L+1}(1:K,\, 1+s:L+s) \quad (s = 0,\, 1)\,. &(3.3)\end{aligned}$$

Note that in the special case $M = L = K$ we obtain again the matrices (2.10) and (2.11). Using the coefficients p_k $(k = 0, \ldots, M-1)$ of the Prony polynomial (2.3), we form the vector $\boldsymbol{p}_L := (p_k)_{k=0}^{L-1}$ with $p_M := 1$, $p_{M+1} = \ldots = p_{L-1} := 0$. By $\boldsymbol{S}_L := \big(\delta_{k-l-1} + \delta_{k-l+1}\big)_{k,l=0}^{L-1}$ we denote the sum of forward and backward shift matrix, where δ_k is the Kronecker symbol. Analogously, we introduce $\boldsymbol{p}_{L+1} := (p_k)_{k=0}^{L}$ with $p_L := 0$, if $L > M$, and $\boldsymbol{S}_{L+1} := \big(\delta_{k-l-1} + \delta_{k-l+1}\big)_{k,l=0}^{L}$.

Lemma 3.1 *Let $L,\, K,\, M,\, N \in \mathbb{N}$ with $M \le L \le K \le N$ be given. Furthermore, let $h_k = h(u_{N,k})$ $(k = 0, \ldots, L+K-1)$ be noiseless sampled data of the sparse polynomial* (1.1) *with $c_j \in \mathbb{R} \setminus \{0\}$ $(j = 1, \ldots, M)$. Then*

$$\operatorname{rank} \boldsymbol{H}_{K,L+1} = \operatorname{rank} \boldsymbol{H}_{K,L}(s) = M \quad (s = 0,\, 1)\,. \qquad (3.4)$$

If $L = M$, then $\operatorname{null} \boldsymbol{H}_{K,M+1} = \operatorname{span}\{\boldsymbol{p}_{M+1}\}$ *and* $\operatorname{null} \boldsymbol{H}_{K,M}(s) = \{\boldsymbol{o}\}$ *for $s = 0,\, 1$. If $L > M$, then*

$$\begin{aligned} \operatorname{null} \boldsymbol{H}_{K,L+1} &= \operatorname{span}\{\boldsymbol{p}_{L+1},\, \boldsymbol{S}_{L+1}\boldsymbol{p}_{L+1},\, \ldots,\, \boldsymbol{S}_{L+1}^{L-M}\boldsymbol{p}_{L+1}\}\,,\\ \operatorname{null} \boldsymbol{H}_{K,L}(s) &= \operatorname{span}\{\boldsymbol{p}_L,\, \boldsymbol{S}_L\boldsymbol{p}_L,\, \ldots,\, \boldsymbol{S}_L^{L-M-1}\boldsymbol{p}_L\} \quad (s = 0,\, 1)\end{aligned}$$

and

$$\begin{aligned} \dim(\operatorname{null} \boldsymbol{H}_{K,L+1}) &= L - M + 1\,,\\ \dim(\operatorname{null} \boldsymbol{H}_{K,L}(s)) &= L - M \quad (s = 0,\, 1)\,.\end{aligned}$$

Proof. 1. For $x_j = T_{n_j}(u_N)$ $(j = 1, \ldots, M)$, we introduce the *rectangular Vandermonde-like matrices*

$$\begin{aligned} \boldsymbol{V}_{K,M}(\boldsymbol{x}) &:= \big(T_{k-1}(x_j)\big)_{k,j=1}^{K,M} = \big(\cos\frac{n_j(k-1)\pi}{2N-1}\big)_{k,j=1}^{K,M}, &(3.5)\\ \boldsymbol{V}'_{K,M}(\boldsymbol{x}) &:= \big(T_k(x_j)\big)_{k,j=1}^{K,M} = \big(\cos\frac{n_j k\pi}{2N-1}\big)_{k,j=1}^{K,M},\end{aligned}$$

which have the rank M, since $\boldsymbol{V}_M(\boldsymbol{x})$ and $\boldsymbol{V}'_M(\boldsymbol{x})$ are nonsingular by Lemmas 2.2 and 2.3. Then the rectangular T+H matrices (3.1) and (3.2) can be factorized in the following form

$$\begin{aligned}
\boldsymbol{H}_{K,L+1} &= 2\,\boldsymbol{V}_{K,M}(\boldsymbol{x})\,(\mathrm{diag}\,\boldsymbol{c})\,\boldsymbol{V}_{L+1,M}(\boldsymbol{x})^{\mathrm{T}}\,,\\
\boldsymbol{H}_{K,L}(0) &= 2\,\boldsymbol{V}_{K,M}(\boldsymbol{x})\,(\mathrm{diag}\,\boldsymbol{c})\,\boldsymbol{V}_{L,M}(\boldsymbol{x})^{\mathrm{T}}\,,\\
\boldsymbol{H}_{K,L}(1) &= 2\,\boldsymbol{V}_{K,M}(\boldsymbol{x})\,(\mathrm{diag}\,\boldsymbol{c})\,\boldsymbol{V}'_{L,M}(\boldsymbol{x})^{\mathrm{T}}
\end{aligned}$$

with $\boldsymbol{x} = (x_j)_{j=1}^M$ and $\boldsymbol{c} = (c_j)_{j=1}^M$. This can be shown in similar way as in the proof of Lemma 2.2. Since $c_j \neq 0$ and since $x_j \in [-1,\,1]$ are distinct, we obtain (3.4). Using rank estimation, we can determine the rank and thus the Chebyshev-1 sparsity of the sparse polynomial (2.3). By (3.4) and $\boldsymbol{H}_{K,L+1}\,\boldsymbol{p}_{M+1} = \boldsymbol{o}$ (see (2.4)), the 1–dimensional null space of $\boldsymbol{H}_{K,L+1}$ is spanned by $\boldsymbol{p}_{M+1}$. Furthermore, the null spaces of $\boldsymbol{H}_{K,L}(s)$ are trivial for $s = 0,\,1$.

2. Assume that $L > M$. From

$$\sum_{m=0}^{M} p_m\,\big(h_{l+m+s} + h_{|l-m-s|}\big) = 0 \quad (l = 0,\,\ldots,\,2N-M-s-1;\; s = 0,\,1)$$

it follows that

$$\boldsymbol{H}_{K,L+1}\,(\boldsymbol{S}_{L+1}^{j}\,\boldsymbol{p}_{L+1}) = \boldsymbol{o} \quad (j = 0,\,\ldots,\,L-M)$$

and analogously

$$\boldsymbol{H}_{K,L}(s)\,(\boldsymbol{S}_{L}^{j}\,\boldsymbol{p}_{L}) = \boldsymbol{o} \quad (j = 0,\,\ldots,\,L-M-1;\; s = 0,\,1)\,,$$

where $\boldsymbol{o}$ denotes the corresponding zero vector. By $p_M = 1$, we see that the vectors $\boldsymbol{S}_{L+1}^{j}\,\boldsymbol{p}_{L+1}$ $(j = 0,\,\ldots,\,L-M)$ and $\boldsymbol{S}_{L}^{j}\,\boldsymbol{p}_{L}$ $(j = 0,\,\ldots,\,L-M-1)$ are linearly independent and located in $\mathrm{null}\,\boldsymbol{H}_{K,L+1}$, and $\mathrm{null}\,\boldsymbol{H}_{K,L}(s)$, respectively.

3. Let again $L > M$. Now we prove that $\mathrm{null}\,\boldsymbol{H}_{K,L+1}$ is contained in the linear span of the vectors $\boldsymbol{S}_{L+1}^{j}\boldsymbol{p}_{L+1}$ $(j = 0, \ldots, L-M)$. Let $\boldsymbol{u} = (u_l)_{l=0}^{L} \in \mathbb{R}^{L+1}$ be an arbitrary right eigenvector of $\boldsymbol{H}_{K,L+1}$ related to the eigenvalue 0 and let U be the corresponding polynomial

$$U(x) = \sum_{l=0}^{L} u_l\,T_l(x) \quad (x \in \mathbb{R})\,.$$

Using the noiseless sampled data $h_k = h(u_{N,k})$ $(k = 0, \ldots, 2N-1)$, we obtain

$$0 = \sum_{m=0}^{L} (h_{l+m} + h_{|l-m|})\,u_m = \sum_{m=0}^{L} u_m\,\Big(\sum_{j=1}^{M} c_j\,\big[T_{n_j}(u_{N,l+m}) + T_{n_j}(u_{N,|l-m|})\big]\Big)\,.$$

Thus by $T_{n_j}(u_{N,l+m}) + T_{n_j}(u_{N,|l-m|}) = T_{l+m}(x_j) + T_{|l-m|}(x_j) = 2\,T_l(x_j)\,T_m(x_j)$ it follows that

$$0 = 2\sum_{j=1}^{M} c_j\,T_l(x_j)\,U(x_j) \quad (l = 0, \ldots, 2N-L-1)$$

and hence by (3.5)

$$\boldsymbol{V}_{K,M}(\boldsymbol{x})\,\big(c_j\,U(x_j)\big)_{j=1}^{M} = \boldsymbol{o}\,.$$

Since $x_j \in [-1, 1]$ $(j = 1, \ldots, M)$ are distinct by assumption, the square Vandermonde-like matrix $\boldsymbol{V}_M(\boldsymbol{x})$ is nonsingular by Lemma 2.2. Hence we obtain $U(x_j) = 0$ $(j = 1, \ldots, M)$ by $c_j \neq 0$. Thus it follows that $U(x) = P(x)\,R(x)$ with certain polynomial

$$R(x) = \sum_{k=0}^{L-M} r_k\,T_k(x) \quad (x \in \mathbb{R};\, r_k \in \mathbb{R})\,.$$

But this means for the coefficients of the polynomials P, R, and U that

$$\boldsymbol{u} = r_0\,\boldsymbol{p}_{L+1} + \frac{1}{2}\,r_1\,\boldsymbol{S}_{L+1}\,\boldsymbol{p}_{L+1} + \ldots + \frac{1}{2}\,r_{L-M}\,\boldsymbol{S}_{L+1}^{L-M}\,\boldsymbol{p}_{L+1}\,.$$

Hence the vectors $\boldsymbol{S}_{L+1}^{j}\,\boldsymbol{p}_{L+1}$ $(j = 0, \ldots, L-M)$ form a basis of null $\boldsymbol{H}_{K,L+1}$ such that $\dim(\text{null}\,\boldsymbol{H}_{K,L+1}) = L - M + 1$. Similarly, one can show the results for the other Hankel matrices (3.2). This completes the proof. ∎

The Prony method for sparse Chebyshev-1 interpolation (with unknown Chebyshev-1 sparsity M) is based on the following result.

Lemma 3.2 *Let L, K, M, $N \in \mathbb{N}$ with $M \leq L \leq K \leq N$ be given. Let $h_k = h(u_{N,k})$ $(k = 0, \ldots, L+K-1)$ be noiseless sampled data of the sparse polynomial* (1.1) *with $c_j \in \mathbb{R} \setminus \{0\}$. Then following assertions are equivalent:*
(i) *The polynomial*

$$Q(x) := \sum_{k=0}^{L} q_k\,T_k(x) \quad (x \in \mathbb{R};\, q_L := 1) \tag{3.6}$$

with real coefficients q_k has M distinct zeros $x_j \in [-1, 1]$ $(j = 1, \ldots, M)$.
(ii) *The vector $\boldsymbol{q} = (q_k)_{k=0}^{L-1}$ is a solution of the linear system*

$$\boldsymbol{H}_{K,L}(0)\,\boldsymbol{q} = -\boldsymbol{h}(L) \quad \big(\boldsymbol{h}(L) := \big(h_{L+m} + h_{|L-m|}\big)_{m=0}^{K-1}\big). \tag{3.7}$$

(iii) *The matrix $\boldsymbol{Q}_L := \boldsymbol{S}_L - \begin{pmatrix} \boldsymbol{o} & \ldots & \boldsymbol{o} & \boldsymbol{q} \end{pmatrix} \in \mathbb{R}^{L \times L}$ has the property*

$$\boldsymbol{H}_{K,L}(0)\,\boldsymbol{Q}_L = \boldsymbol{H}_{K,L}(1) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{h}(0) & \ldots & \boldsymbol{h}(L-2) \end{pmatrix}\,. \tag{3.8}$$

Further the eigenvalues of $\frac{1}{2}\,\boldsymbol{E}_L^{-1}\,\boldsymbol{Q}_L$ coincide with the zeros of the polynomial (3.6).

Proof. 1. From (i) it follows (ii): Assume that $Q(x_j) = 0$ $(j = 1, \ldots, M)$. For $m = 0, \ldots, K$, we compute the sums

$$s_m := \sum_{k=0}^{L} (h_{k+m} + h_{|k-m|})\,q_k\,.$$

Using $h_k = h(u_{N,k})$ $(k = 0, \ldots, 2N-1)$, (1.1), and the known identities (see e.g. [11, p. 17 and p. 31])

$$2\,T_j(x)\,T_k(x) = T_{j+k}(x) + T_{|j-k|}(x)\,, \quad T_j\big(T_k(x)\big) = T_{j+k}(x) \quad (j,\, k \in \mathbb{N}_0)\,,$$

we obtain

$$\begin{aligned} s_m &= \sum_{k=0}^{L} q_k \left[h\big(T_{k+m}(u_N)\big) + h\big(T_{|k-m|}(u_N)\big)\right] \\ &= \sum_{l=1}^{M} c_l \sum_{k=0}^{L} q_k \left[T_{k+m}(x_l) + T_{|k-m|}(x_l)\right] = 2 \sum_{l=1}^{M} c_l\, T_m(x_l)\, Q(x_l) = 0\,. \end{aligned}$$

By $q_L = 1$ this implies that

$$\sum_{k=0}^{L-1} (h_{k+m} + h_{|k-m|})\, q_k = -1\, (h_{L+m} + h_{|L-m|}) \quad (m = 0, \ldots, K)\,.$$

Hence we get (3.7).
2. From (ii) it follows (iii): Assume that $\boldsymbol{q} = (q_l)_{l=0}^{L-1}$ is a solution of the linear system (3.7). Then by

$$\begin{aligned} \boldsymbol{H}_{K,L}(0)\, (\delta_{k-j})_{k=0}^{L-1} &= \boldsymbol{h}(j) = \big(h_{k+j} + h_{|k-j|}\big)_{k=0}^{K-1} \quad (j = 1, \ldots, L-1)\,, \\ -\,\boldsymbol{H}_{K,L}(0)\, \boldsymbol{q} &= \boldsymbol{h}(L) = \big(h_{k+L} + h_{|k-L|}\big)_{k=0}^{K-1}\,, \end{aligned}$$

we obtain (3.8) column by column.
3. From (iii) it follows (i): By (3.8) we obtain (3.7), since the last column of $\boldsymbol{Q}_L$ reads $(\delta_{L-2-j})_{j=0}^{L-1} - \boldsymbol{q}$ and since the last column of

$$\boldsymbol{H}_{K,L}(1) + \big(\ \boldsymbol{o} \quad \boldsymbol{h}(0) \quad \ldots \quad \boldsymbol{h}(L-2)\ \big)$$

is equal to $\boldsymbol{h}(L) + \boldsymbol{h}(L-2)$. Then (3.7) implies

$$\sum_{k=0}^{L} (h_{k+m} + h_{|k-m|})\, q_k = 0 \quad (m = 0, \ldots, K).$$

As shown in the first step, we obtain

$$\sum_{l=1}^{M} c_l\, T_m(x_l)\, Q(x_l) = 0 \quad (m = 0, \ldots, K)\,,$$

i.e. by (3.5) finally $\boldsymbol{V}_{K,M}(\boldsymbol{x})\, \big(c_l\, Q(x_l)\big)_{l=1}^{M} = \boldsymbol{o}$. Especially we conclude that

$$\boldsymbol{V}_M(\boldsymbol{x})\, \big(c_l\, Q(x_l)\big)_{l=1}^{M} = \boldsymbol{o}\,.$$

Since $x_j \in [-1, 1]$ $(j = 1, \ldots, M)$ are distinct, the square Vandermonde-like matrix $\boldsymbol{V}_M(\boldsymbol{x})$ is nonsingular by Lemma 2.2 such that $Q(x_j) = 0$ $(j = 1, \ldots, M)$.
4. From Lemma 2.5, it follows that

$$\det\big(2x\, \boldsymbol{E}_L - \boldsymbol{Q}_L\big) = Q(x) \quad (x \in \mathbb{R})\,.$$

Hence the eigenvalues of the square matrix $\frac{1}{2}\,\boldsymbol{E}_L^{-1}\,\boldsymbol{Q}_L$ coincide with the zeros of the polynomial (3.6). This completes the proof. ∎

In the following, we denote a polynomial (3.6) as a *modified Prony polynomial* of degree L $(M \le L \le N)$, if the corresponding coefficient vector $\boldsymbol{q} = (q_k)_{k=0}^{L-1}$ is a solution of the linear system (3.7). Then (3.6) has the same zeros $x_j \in [-1,\,1]$ $(j = 1,\ldots,M)$ as the Prony polynomial (2.3), but (3.6) has $L-M$ additional zeros, if $L > M$. The eigenvalues of $\frac{1}{2}\,\boldsymbol{E}_L^{-1}\,\boldsymbol{Q}_L$ coincide with the zeros of the polynomial Q.

Now we formulate Lemma 3.2 as an algorithm. Since the unknown coefficients c_j $(j = 1,\ldots,M)$ do not vanish, we can assume that $|c_j| > \varepsilon$ for convenient bound ε $(0 < \varepsilon \ll 1)$.

Algorithm 3.3 *(Prony method for sparse Chebyshev-1 interpolation)*

Input: $L,\,K,\,N \in \mathbb{N}$ $(N \gg 1,\ 3 \le L \le K \le N$, *$L$ is upper bound of the Chebyshev-1 sparsity M of* (1.1)), $h_k = h(u_{N,k}) \in \mathbb{R}$ $(k = 0,\ldots,L+K-1)$, $0 < \varepsilon \ll 1$.

1. *Compute the least squares solution* $\boldsymbol{q} = (q_k)_{k=0}^{L-1}$ *of the rectangular linear system* (3.7).
2. *Determine the simple roots* $\tilde{x}_j \in [-1,\,1]$ $(j = 1,\ldots,\tilde{M})$ *of the modified Prony polynomial* (3.6), *i.e., compute all eigenvalues* $\tilde{x}_j \in [-1,\,1]$ $(j = 1,\ldots,\tilde{M})$ *of the companion matrix* $\frac{1}{2}\,\boldsymbol{E}_L^{-1}\,\boldsymbol{Q}_L$. *Assume that $\tilde{x}_j$ are ordered in the following form* $1 \ge \tilde{x}_1 > \tilde{x}_2 > \ldots > \tilde{x}_M \ge -1$. *Note that* $\operatorname{rank}\boldsymbol{H}_{K,L}(0) = M \le \tilde{M}$.
3. *Compute* $\tilde{c}_j \in \mathbb{R}$ $(j = 1,\ldots,\tilde{M})$ *as least squares solution of the overdetermined linear Vandermonde–like system*

$$\boldsymbol{V}_{L+K,\tilde{M}}(\tilde{\boldsymbol{x}})\,(\tilde{c}_j)_{j=1}^{\tilde{M}} = (h_k)_{k=0}^{L+K-1}$$

with $\tilde{\boldsymbol{x}} := (\tilde{x}_j)_{j=1}^{\tilde{M}}$ *and* $\boldsymbol{V}_{L+K,\tilde{M}}(\tilde{\boldsymbol{x}}) := \big(T_k(\tilde{x}_j)\big)_{k=0,j=1}^{L+K-1,\tilde{M}}$.

4. *Delete all the* $\tilde{x}_l$ $(l \in \{1,\ldots,\tilde{M}\}$ *with* $|\tilde{c}_l| \le \varepsilon$ *and denote the remaining values by* x_j $(j = 1,\ldots,M)$ *with* $M \le \tilde{M}$. *Calculate* $n_j := [\frac{2N-1}{\pi}\,\arccos x_j]$ $(j = 1,\ldots,M)$.
5. *Repeat step 3 and compute* $\boldsymbol{c} = (c_j)_{j=1}^{M} \in \mathbb{R}^M$ *as least squares solution of the overdetermined linear Vandermonde–like system*

$$\boldsymbol{V}_{L+K,M}(\boldsymbol{x})\,\boldsymbol{c} = (h_k)_{k=0}^{L+K-1}$$

with $\boldsymbol{x} := (x_j)_{j=1}^{M}$ *and* $\boldsymbol{V}_{L+K,M}(\boldsymbol{x}) := \big(T_k(x_j)\big)_{k=0,j=1}^{L+K-1,M} = \big(\cos\frac{n_j k\pi}{2N-1}\big)_{k=0,j=1}^{L+K-1,M}$.

Output: $M \in \mathbb{N}$, $n_j \in \mathbb{N}_0$ $(0 \le n_1 < n_2 < \ldots < n_M < 2N)$, $c_j \in \mathbb{R}$ $(j = 1,\ldots,M)$.

Now we show that the Prony method for sparse Chebyshev-1 interpolation can be simplified to a matrix pencil method. As known, a rectangular matrix pencil may not have eigenvalues in general. But this is not the case for our *rectangular matrix pencil*

$$2x\,\boldsymbol{H}_{K,L}(0)\,\boldsymbol{E}_L - \boldsymbol{H}_{K,L}(0)\,\boldsymbol{Q}_L\,, \tag{3.9}$$

which has $x_j \in [-1,\,1]$ $(j = 1,\ldots,M)$ as eigenvalues. Note that by (3.8) both matrices $\boldsymbol{H}_{K,L}(0)\,\boldsymbol{E}_L$ and $\boldsymbol{H}_{K,L}(0)\,\boldsymbol{Q}_L$ are known by the given sampled data h_k $(k = 0,\ldots,2N-$

1). The matrix pencil (3.9) has at least the eigenvalues $x_j \in [-1,\,1]$ $(j = 1, \ldots, M)$. If $\boldsymbol{v} \in \mathbb{C}^L$ is a right eigenvector related to x_j, then by

$$\big(2x_j\, \boldsymbol{H}_{K,L}(0)\, \boldsymbol{E}_L - \boldsymbol{H}_{K,L}(0)\, \boldsymbol{Q}_L\big)\, \boldsymbol{v} = \boldsymbol{H}_{K,L}(0)\, \big(2x_j\, \boldsymbol{E}_L - \boldsymbol{Q}_L\big)\, \boldsymbol{v}$$

and

$$\det\big(2x_j\, \boldsymbol{E}_L - \boldsymbol{Q}_L\big) = Q(x_j) = 0$$

we see that $\boldsymbol{v} = (v_k)_{k=0}^{L-1}$ is a right eigenvector of the square eigenvalue problem

$$\frac{1}{2}\, \boldsymbol{E}_L^{-1}\, \boldsymbol{Q}_L\, \boldsymbol{v} = x_j\, \boldsymbol{v}\,.$$

A right eigenvector can be determined by

$$v_{L-1} = T_L(x_j) = -\sum_{l=0}^{L-1} q_l\, T_l(x_j)\,,$$

whereas the other components $v_{L-2}, \ldots, v_0$ can be computed recursively from the linear system

$$\boldsymbol{Q}_L\, \boldsymbol{v} = 2x_j\, \boldsymbol{E}_L\, \boldsymbol{v}\,.$$

Now we factorize the rectangular T+H matrices (3.2) simultaneously. For this reason, we compute the QR decomposition of the rectangular T+H matrix (3.1). By (3.4), the rank of the T+H matrix $\boldsymbol{H}_{K,L+1}$ is equal to M. Hence $\boldsymbol{H}_{K,L+1}$ is rank deficient. Therefore we apply QR factorization with column pivoting and obtain

$$\boldsymbol{H}_{K,L+1}\, \boldsymbol{\Pi}_{L+1} = \boldsymbol{U}_K\, \boldsymbol{R}_{K,L+1}$$

with an orthogonal matrix $\boldsymbol{U}_K$, a permutation matrix $\boldsymbol{\Pi}_{L+1}$, and a trapezoidal matrix

$$\boldsymbol{R}_{K,L+1} = \begin{pmatrix} \boldsymbol{R}_{K,L+1}(1:M,\, 1:L+1) \\ \boldsymbol{O}_{K-M,L+1} \end{pmatrix},$$

where $\boldsymbol{R}_{K,L+1}(1:M,\, 1:M)$ is a nonsingular upper triangular matrix. By the QR decomposition we can determine the rank M of the T+H matrix (3.1) and hence the Chebyshev-1 sparsity of the sparse polynomial (1.1). Note that the permutation matrix $\boldsymbol{\Pi}_{L+1}$ is chosen such that the diagonal entries of $\boldsymbol{R}_{K,L+1}(1:M,\, 1:M)$ have nonincreasing absolute values. We denote the diagonal matrix containing these diagonal entries by $\boldsymbol{D}_M$. With

$$\boldsymbol{S}_{K,L+1} := \boldsymbol{R}_{K,L+1}\, \boldsymbol{\Pi}_{L+1}^{\mathrm{T}} = \begin{pmatrix} \boldsymbol{S}_{K,L+1}(1:M,\, 1:L+1) \\ \boldsymbol{O}_{K-M,L+1} \end{pmatrix}, \tag{3.10}$$

we infer that by (3.3)

$$\boldsymbol{H}_{K,L}(s) = \boldsymbol{U}_K\, \boldsymbol{S}_{K,L}(s) \quad (s = 0,\, 1)$$

with

$$\boldsymbol{S}_{K,L}(s) := \boldsymbol{S}_{K,L+1}(1:K,\, 1+s:L+s) \quad (s=0,\,1)\,.$$

Hence we can factorize the matrices $2\,\boldsymbol{H}_{K,L}(0)\,\boldsymbol{E}_L$ and $\boldsymbol{H}_{K,L}(0)\,\boldsymbol{Q}_L$ in the following form

$$\begin{aligned} 2\,\boldsymbol{H}_{K,L}(0)\,\boldsymbol{E}_L &= \boldsymbol{H}_{K,L}(0) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{h}(1) & \ldots & \boldsymbol{h}(L-1) \end{pmatrix} = \boldsymbol{U}_K\,\boldsymbol{S}'_{K,L}(0)\,, \\ \boldsymbol{H}_{K,L}(0)\,\boldsymbol{Q}_L &= \boldsymbol{H}_{K,L}(1) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{h}(0) & \ldots & \boldsymbol{h}(L-2) \end{pmatrix} = \boldsymbol{U}_K\,\boldsymbol{S}'_{K,L}(1)\,, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{S}'_{K,L}(0) &:= \boldsymbol{S}_{K,L}(0) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{S}_{K,L}(1)(1:K,\,1:L-1) \end{pmatrix}\,, \\ \boldsymbol{S}'_{K,L}(1) &:= \boldsymbol{S}_{K,L}(1) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{S}_{K,L}(0)(1:K,\,1:L-1) \end{pmatrix}\,. \end{aligned}$$

Since $\boldsymbol{U}_K$ is orthogonal, the generalized eigenvalue problem of the matrix pencil (3.9) is equivalent to the generalized eigenvalue problem of the matrix pencil

$$x\,\boldsymbol{S}'_{K,L}(0) - \boldsymbol{S}'_{K,L}(1) \quad (x \in \mathbb{R})\,.$$

Using the special structure of (3.10), we can simplify the matrix pencil

$$x\,\boldsymbol{T}_{M,L}(0) - \boldsymbol{T}_{M,L}(1) \quad (x \in \mathbb{R}) \tag{3.11}$$

with

$$\boldsymbol{T}_{M,L}(s) := \boldsymbol{S}_{K,L}(1:M,\, 1+s:L+s) \quad (s=0,\,1)\,. \tag{3.12}$$

Here one can use the matrix $\boldsymbol{D}_M$ as diagonal preconditioner and proceed with

$$\boldsymbol{T}'_{M,L}(s) := \boldsymbol{D}_M^{-1}\,\boldsymbol{T}_{M,L}(s)\,. \tag{3.13}$$

Then the generalized eigenvalue problem of the transposed matrix pencil

$$x\,\boldsymbol{T}'_{M,L}(0)^{\mathrm{T}} - \boldsymbol{T}'_{M,L}(1)^{\mathrm{T}}$$

has the same eigenvalues as the matrix pencil (3.11) except for the zero eigenvalues and it can be solved as eigenvalue problem of the M-by-M matrix

$$\boldsymbol{F}_M^{QR} := \left(\boldsymbol{T}'_{M,L}(0)^{\mathrm{T}}\right)^{\dagger}\,\boldsymbol{T}'_{M,L}(1)^{\mathrm{T}}\,. \tag{3.14}$$

Finally we obtain the nodes $x_j \in [-1,\,1]$ $(j=1,\ldots,M)$ as the eigenvalues of (3.14).

Algorithm 3.4 *(Matrix pencil factorization based on* QR *decomposition for sparse Chebyshev-*1 *interpolation)*

Input: $L,\,K,\,N \in \mathbb{N}$ $(N \gg 1,\ 3 \le L \le K \le N,\ L$ *is upper bound of the Chebyshev-*1 *sparsity* M *of* (1.1)$)$, $h_k = h(u_{N,k}) \in \mathbb{R}$ $(k=0,\ldots,L+K-1)$.

1. *Compute* QR *factorization of the rectangular* T+H *matrix* (3.1). *Determine the rank of* (3.1) *and form the matrices* (3.12) *and* (3.13).
2. *Determine the eigenvalues* $x_j \in [-1, 1]$ $(j = 1, \ldots, M)$ *of the square matrix* (3.14). *Assume that* x_j *are ordered in the following form* $1 \geq x_1 > x_2 > \ldots > x_M \geq -1$. *Calculate* $n_j := [\frac{2N-1}{\pi} \arccos x_j]$ $(j = 1, \ldots, M)$.
3. *Compute the coefficients* $c_j \in \mathbb{R}$ $(j = 1, \ldots, M)$ *as least squares solution of the overdetermined linear Vandermonde–like system*

$$\boldsymbol{V}_{L+K,M}(\boldsymbol{x})\,(c_j)_{j=1}^{M} = (h_k)_{k=0}^{L+K-1}$$

with $\boldsymbol{x} := (x_j)_{j=1}^{M}$ *and* $\boldsymbol{V}_{L+K,M}(\boldsymbol{x}) := \big(T_k(x_j)\big)_{k=0,j=1}^{L+K-1,M} = \big(\cos\frac{n_j k\pi}{2N-1}\big)_{k=0,j=1}^{L+K-1,M}$.

Output: $M \in \mathbb{N}$, $n_j \in \mathbb{N}_0$ $(0 \leq n_1 < n_2 < \ldots < n_M < 2N)$, $c_j \in \mathbb{R}$ $(j = 1, \ldots, M)$.

In the following we derive the ESPRIT method by similar ideas as above, but now we use the SVD of the T+H matrix (3.1), which is rank deficient by (3.4). Therefore we use the factorization

$$\boldsymbol{H}_{K,L+1} = \boldsymbol{U}_K\,\boldsymbol{D}_{K,L+1}\,\boldsymbol{W}_{L+1}\,,$$

where $\boldsymbol{U}_K$ and $\boldsymbol{W}_{L+1}$ are orthogonal matrices and where $\boldsymbol{D}_{K,L+1}$ is a rectangular diagonal matrix. The diagonal entries of $\boldsymbol{D}_{K,L+1}$ are the singular values of (3.1) arranged in nonincreasing order $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_M > \sigma_{M+1} = \ldots = \sigma_{L+1} = 0$. Thus we can determine the rank M of the Hankel matrix (3.1) which coincides with the Chebyshev-1 sparsity of the polynomial (1.1). Introducing the matrices

$$\begin{aligned} \boldsymbol{D}_{K,M} &:= \boldsymbol{D}_{K,L+1}(1:K,\,1:M) = \begin{pmatrix} \operatorname{diag}(\sigma_j)_{j=1}^{M} \\ \boldsymbol{O}_{K-M,M} \end{pmatrix}, \\ \boldsymbol{W}_{M,L+1} &:= \boldsymbol{W}_{L+1}(1:M,\,1:L+1)\,, \end{aligned}$$

we can simplify the SVD of the Hankel matrix (3.1) as follows

$$\boldsymbol{H}_{K,L+1} = \boldsymbol{U}_K\,\boldsymbol{D}_{K,M}\,\boldsymbol{W}_{M,L+1}\,.$$

Note that $\boldsymbol{W}_{M,L+1}\,\boldsymbol{W}_{M,L+1}^{\mathrm{T}} = \boldsymbol{I}_M$. Setting

$$\boldsymbol{W}_{M,L}(s) = \boldsymbol{W}_{M,L+1}(1:M,\,1+s:L+s) \quad (s = 0,\,1)\,, \tag{3.15}$$

it follows from (3.3) that $\boldsymbol{H}_{K,L}(s) = \boldsymbol{U}_K\,\boldsymbol{D}_{K,M}\,\boldsymbol{W}_{M,L}(s)$ $(s = 0,\,1)$. Hence we can factorize the matrices $2\,\boldsymbol{H}_{K,L}(0)\,\boldsymbol{E}_L$ and $\boldsymbol{H}_{K,L}(0)\,\boldsymbol{Q}_L$ in the following form

$$\begin{aligned} 2\,\boldsymbol{H}_{K,L}(0)\,\boldsymbol{E}_L &= \boldsymbol{H}_{K,L}(0) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{h}(1) & \ldots & \boldsymbol{h}(L-1) \end{pmatrix} = \boldsymbol{U}_K\,\boldsymbol{D}_{K,M}\,\boldsymbol{W}'_{K,L}(0)\,, \\ \boldsymbol{H}_{K,L}(0)\,\boldsymbol{Q}_L &= \boldsymbol{H}_{K,L}(1) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{h}(0) & \ldots & \boldsymbol{h}(L-2) \end{pmatrix} = \boldsymbol{U}_K\,\boldsymbol{D}_{K,M}\,\boldsymbol{W}'_{K,L}(1)\,, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{W}'_{K,L}(0) &:= \boldsymbol{W}_{K,L}(0) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{W}_{K,L}(1)(1:K,\,1:L-1) \end{pmatrix}, \\ \boldsymbol{W}'_{K,L}(1) &:= \boldsymbol{W}_{K,L}(1) + \begin{pmatrix} \boldsymbol{o} & \boldsymbol{W}_{K,L}(0)(1:K,\,1:L-1) \end{pmatrix}. \end{aligned}$$

Since $\boldsymbol{U}_K$ is orthogonal, the generalized eigenvalue problem of the rectangular matrix pencil (3.9) is equivalent to the generalized eigenvalue problem of the matrix pencil

$$x\,\boldsymbol{D}_{K,M}\,\boldsymbol{W}'_{M,L}(0) - \boldsymbol{D}_{K,M}\,\boldsymbol{W}'_{M,L}(1)\,. \tag{3.16}$$

If we multiply the transposed matrix pencil (3.16) from the right side with

$$\begin{pmatrix} \operatorname{diag}(\sigma_j^{-1})_{j=1}^M \\ \boldsymbol{O}_{K-M,M} \end{pmatrix},$$

we obtain the generalized eigenvalue problem of the matrix pencil

$$x\,\boldsymbol{W}'_{M,L}(0)^{\mathrm{T}} - \boldsymbol{W}'_{M,L}(1)^{\mathrm{T}}\,,$$

which has the same eigenvalues as the matrix pencil (3.16) except for the zero eigenvalues. Finally we determine the nodes $x_j \in [-1,\,1]$ $(j = 1, \ldots, M)$ as eigenvalues of the matrix

$$\boldsymbol{F}_M^{SVD} := \big(\boldsymbol{W}'_{M,L}(0)^{\mathrm{T}}\big)^{\dagger}\,\boldsymbol{W}'_{M,L}(1)^{\mathrm{T}}\,. \tag{3.17}$$

Thus the ESPRIT algorithm reads as follows:

Algorithm 3.5 *(*ESPRIT *method for sparse Chebyshev*-1 *interpolation)*
Input: $L,\,K,\,N \in \mathbb{N}$ *(*$N \gg 1,\ 3 \le L \le K \le N$*, L is upper bound of the Chebyshev*-1 *sparsity M of* (1.1)*),* $h_k = h(u_{N,k}) \in \mathbb{R}$ $(k = 0, \ldots, L+K-1)$.

1. *Compute the* SVD *of the rectangular* T+H *matrix* (3.1)*. Determine the rank M of* (3.1) *and form the matrices* (3.15).
2. *Compute all eigenvalues* $x_j \in [-1,\,1]$ $(j = 1, \ldots, M)$ *of the square matrix* (3.17)*. Assume that the eigenvalues are ordered in the following form* $1 \ge x_1 > x_2 > \ldots > x_M \ge -1$*. Calculate* $n_j := [\frac{2N-1}{\pi}\,\arccos x_j]$ $(j = 1, \ldots, M)$.
3. *Compute the coefficients* $c_j \in \mathbb{R}$ $(j = 1, \ldots, M)$ *as least squares solution of the overdetermined linear Vandermonde–like system*

$$\boldsymbol{V}_{L+K,M}(\boldsymbol{x})\,\boldsymbol{c} = (h_k)_{k=0}^{L+K-1}$$

with $\boldsymbol{x} := (x_j)_{j=1}^M$ *and* $\boldsymbol{c} := (c_j)_{j=1}^M$.

Output: $M \in \mathbb{N}$, $n_j \in \mathbb{N}_0$ $(0 \le n_1 < n_2 < \ldots < n_M < 2N)$, $c_j \in \mathbb{R}$ $(j = 1, \ldots, M)$.

4 Sparse polynomial interpolation in Chebyshev-2 basis

In the following, we sketch the sparse interpolation in another Chebyshev basis. For $n \in \mathbb{N}_0$ and $x \in (-1,\,1)$, the *Chebyshev polynomial of second kind* is defined by

$$U_n(x) := (1-x^2)^{-1/2}\,\sin\big((n+1)\,\arccos x\big)$$

(see for example [11, p. 3]). These polynomials are orthogonal with respect to the weight $(1-x^2)^{1/2}$ on $[-1,\,1]$ (see [11, p. 74]) and form the *Chebyshev*-2 *basis*.

For $M,\, N \in \mathbb{N}$ with $M \le N$, we consider a polynomial h of degree at most $N-1$, which is *M-sparse in the Chebyshev-2 basis*, i.e.

$$h(x) = \sum_{j=1}^{M} c_j\, U_{n_j}(x) \tag{4.1}$$

with $0 \le n_1 < n_2 < \ldots < n_M \le N-1$. Note that the sparsity depends on the choice of Chebyshev basis. Using $T_0 = U_0$, $T_1 = U_1/2$ and $T_n = (U_n - U_{n-2})/2$ for $n \ge 2$ (cf. [11, p. 4]), we obtain for $N \gg 2$

$$U_{N-2} + U_{N-1} = T_0 + 2\,(T_1 + \ldots + T_{N-1})\,.$$

Thus the 2-sparse polynomial $U_{N-2} + U_{N-1}$ in the Chebyshev-2 basis is not a sparse polynomial in the Chebyshev-1 basis. For shortness, we restrict us on the discussion of the sparse polynomial interpolation in the Chebyshev-2 basis. But we emphasize that one can extend this approach the Chebyshev polynomials of third and fourth kind (see [11, p. 5]), which are defined for $n \in \mathbb{N}_0$ by

$$V_n(x) := \frac{\cos((n+\frac{1}{2})\,\arccos x)}{\cos(\frac{1}{2}\,\arccos x)}\,, \quad W_n(x) := \frac{\sin((n+\frac{1}{2})\,\arccos x)}{\sin(\frac{1}{2}\,\arccos x)} \quad (x \in (-1,\,1))\,.$$

Substituting $x = \cos t$, we obtain for all $t \in [0,\,\pi]$

$$\sin t\; h(\cos t) = \sum_{j=1}^{M} c_j\, \sin\big((n_j+1)\,t\big)\big)\,.$$

By sampling at $t = \frac{\pi k}{2N-1}$ $(k = 0, \ldots, 2N-1)$, it follows that

$$\tilde{h}_k := \sin\frac{\pi k}{2N-1}\, h\big(\cos\frac{\pi k}{2N-1}\big) = \sum_{j=1}^{M} c_j\, \sin\big((n_j+1)\,\frac{\pi k}{2N-1})\big)\,. \tag{4.2}$$

Further we set $\tilde{h}_{-k} := -\tilde{h}_k$ $(k = 1, \ldots, 2N-1)$. In this case, we introduce the *Prony polynomial* by

$$\tilde{P}(x) := 2^{M-1} \prod_{j=1}^{M} \Big(x - \cos\frac{(n_j+1)\pi}{2N-1}\Big)\,, \tag{4.3}$$

which can be represented again in the Chebyshev-1 basis in the form

$$\tilde{P}(x) = \sum_{l=0}^{M} p_l\, T_l(x) \quad (p_M = 1)\,.$$

The coefficients p_l of the Prony polynomial (4.3) can be characterized as follows:

Lemma 4.1 *For all $k = 0, \ldots, 2N-1$, the scaled sampled values* (4.2) *and the coefficients p_l of the Prony polynomial* (4.3) *fulfil the equations*

$$\sum_{j=0}^{M-1} (\tilde{h}_{j+k} - \tilde{h}_{j-k})\, p_j = -(\tilde{h}_{M+k} - \tilde{h}_{M-k})\,.$$

Proof. Using $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\,\sin\alpha\,\cos\beta$, we obtain for $j, k = 0, \ldots, M$

$$\tilde{h}_{j+k} - \tilde{h}_{j-k} = 2 \sum_{l=1}^{M} c_l\, \sin\frac{(n_l+1)\pi k}{2N-1}\, \cos\frac{(n_l+1)\pi j}{2N-1}\,. \tag{4.4}$$

Note that the equation (4.4) is trivial for $k = 0$ and therefore omitted. From (4.4) it follows that

$$\begin{aligned} \sum_{j=0}^{M} (\tilde{h}_{j+k} - \tilde{h}_{j-k})\, p_j &= 2 \sum_{j=0}^{M} p_j \sum_{l=1}^{M} c_l\, \sin\frac{(n_l+1)\pi k}{2N-1}\, \cos\frac{(n_l+1)\pi j}{2N-1} \\ &= 2 \sum_{l=1}^{M} c_l\, \sin\frac{(n_l+1)\pi k}{2N-1}\, \tilde{P}\big(\cos\frac{(n_l+1)\pi j}{2N-1}\big) = 0\,. \end{aligned}$$

By $p_M = 1$, this implies the assertion. ■

If we introduce the T+H matrix

$$\tilde{\boldsymbol{H}}_M(0) := \big(\tilde{h}_{j+k} - \tilde{h}_{j-k}\big)_{k=1,j=0}^{M,M-1}$$

and the vector $\tilde{\boldsymbol{h}}(M) := (\tilde{h}_{M+k} - \tilde{h}_{M-k})_{k=1}^{M}$, then by Lemma 4.1 the vector $\boldsymbol{p} := (p_j)_{j=0}^{M-1}$ is a solution of the linear system

$$\tilde{\boldsymbol{H}}_M(0)\, \boldsymbol{p} = -\tilde{\boldsymbol{h}}(M)\,. \tag{4.5}$$

By (4.4), the T+H matrix $\tilde{\boldsymbol{H}}_M(0)$ can be factorized in the form

$$\tilde{\boldsymbol{H}}_M(0) = 2\, \boldsymbol{V}_M^s\, (\operatorname{diag}\boldsymbol{c})\, \big(\boldsymbol{V}_M^c\big)^{\mathrm{T}} \tag{4.6}$$

with the Vandermonde-like matrices

$$\boldsymbol{V}_M^c := \Big(\cos\frac{(n_l+1)\pi j}{2N-1}\Big)_{j=0,l=1}^{M-1,M}\,, \quad \boldsymbol{V}_M^s := \Big(\sin\frac{(n_l+1)\pi k}{2N-1}\Big)_{k,l=1}^{M}$$

and the diagonal matrix of $\boldsymbol{c} = (c_l)_{l=1}^{M}$. Both Vandermonde-like matrices are nonsingular. Assume that $\boldsymbol{V}_M^c$ is singular. Then there exists a vector $\boldsymbol{d} = (d_l)_{l=0}^{M-1} \neq \boldsymbol{o}$ with $\boldsymbol{d}^{\mathrm{T}}\, \boldsymbol{V}_M^c = \boldsymbol{o}^{\mathrm{T}}$. Introducing

$$D(x) := \sum_{l=0}^{M-1} d_l\, \cos(lx)\,,$$

this even trigonometric polynomial of order at most $M-1$ has M distinct zeros $\frac{(n_l+1)\pi}{2N-1} \in (0,\,\pi]$ $(j = 1, \ldots, M)$. But this can be only the case, if D vanishes identically. Similarly, one can see that $\boldsymbol{V}_M^s$ is nonsingular too. From (4.6) it follows that $\tilde{\boldsymbol{H}}_M(0)$ is also nonsingular. Thus we obtain:

Algorithm 4.2 *(Prony method for sparse Chebyshev-2 interpolation)*
Input: $\tilde{h}_k \in \mathbb{R}$ $(k = 0, \ldots, 2M-1)$, $M \in \mathbb{N}$ *Chebyshev-2 sparsity of* (4.1).

1. *Solve the square linear system* (4.5).
2. *Determine the simple roots* $\tilde{x}_j$ $(j = 1, \ldots M)$ *of the Prony polynomial* (4.3), *where* $1 \geq \tilde{x}_1 > \tilde{x}_2 > \ldots > \tilde{x}_M \geq -1$, *and compute then* $n_j := [\frac{2N-1}{\pi} \arccos \tilde{x}_j] - 1$ $(j = 1, \ldots, M)$.
3. *Compute* $c_j \in \mathbb{R}$ $(j = 1, \ldots, M)$ *as solution of the square Vandermonde-like system*

$$\boldsymbol{V}_M^s \, \boldsymbol{c} = (\tilde{h}_k)_{k=0}^{M-1}$$

with $\boldsymbol{c} := (c_j)_{j=1}^M$.

Output: $n_j \in \mathbb{N}_0$ $(0 \leq n_1 < n_2 < \ldots < n_M < 2N)$, $c_j \in \mathbb{R}$ $(j = 1, \ldots, M)$.

Immediately we can see that the Algorithms 3.4 and 3.5 can be generalized in a straightforward manner, since the Prony polynomial $\tilde{P}$ is represented in the Chebyshev-1 basis. We will denote these generalizations by Algorithms $\widetilde{3.4}$ and $\widetilde{3.5}$, respectively.

5 Numerical examples

Now we illustrate the behavior and the limits of the suggested algorithms. Using IEEE standard floating point arithmetic with double precision, we have implemented our algorithms in MATLAB. In the Examples 5.1 – 5.3, an M-sparse polynomial is given in the form (1.1) with Chebyshev polynomials T_{n_j} of degree n_j and real coefficients $c_j \neq 0$ $(j = 1, \ldots, M)$. We compute the absolute error of the coefficients by

$$e(\boldsymbol{c}) := \max_{j=1,\ldots,M} |c_j - \tilde{c}_j| \quad (\boldsymbol{c} := (c_j)_{j=1}^M),$$

where $\tilde{c}_j$ are the coefficients computed by our algorithms. In Example 5.4 we generalize the method to a sparse nonpolynomial interpolation. Finally in Example 5.5, we present an example of sparse polynomial interpolation in the Chebyshev-2 basis.

Example 5.1 We start with the following example. We choose $M = 5$, $c_j = j$, $u_N := \cos \frac{\pi}{2N-1}$ and $(n_1, n_2, n_3, n_4, n_5) = (6, 12, 176, 178, 200)$ in (1.1). The symbols + and − in the Table 5.1 mean that all degrees n_j are correctly reconstructed and accordingly the reconstruction fails. Since after a successful reconstruction the last step is the same in the Algorithms 3.3 – 3.5, we present the error $e(\boldsymbol{c})$ in the last column of the Table 5.1. □

Example 5.2 It is difficult to reconstruct a sparse polynomial (1.1) in the case, if some degrees n_j of the Chebyshev polynomials T_{n_j} differ only a little. Therefore we consider the sparse polynomial (1.1) with $(n_1, n_2, n_3, n_4, n_5) = (60, 120, 1760, 1780, 2000)$ and again $c_j = j$ $(j = 1, \ldots, 5)$. The results are shown in Table 5.2. □

N	K	L	Alg. 3.3	Alg. 3.4	Alg. 3.5	$e(\boldsymbol{c})$
101	5	5	+	+	+	4.26e-14
200	5	5	+	+	+	7.11e-15
300	5	5	−	−	−	−
300	6	5	+	+	+	1.38e-14
400	6	5	−	−	−	−
400	7	5	+	+	+	3.82e-14
500	7	5	−	−	−	−
500	8	5	−	−	+	7.28e-14
500	9	5	+	+	+	3.82e-14
1000	70	5	−	−	+	6.22e-15
1000	65	10	+	+	−	6.22e-15
1000	73	5	+	+	−	5.33e-15
1000	90	5	+	+	+	2.66e-15
1000	100	100	−	+	+	4.44e-15

Table 5.1: Results of Example 5.1.

N	K	L	Alg. 3.3	Alg. 3.4	Alg. 3.5	$e(\boldsymbol{c})$
2000	50	50	−	+	+	1.78e-15
4000	50	50	−	+	+	2.66e-15
5000	60	5	+	+	+	8.88e-16

Table 5.2: Results of Example 5.2.

Example 5.3 Similarly as in Example 5.1, we choose $M = 5$, $c_j = j$ $(j = 1, \ldots, 5)$ and $(n_1, n_2, n_3, n_4, n_5) = (6, 12, 176, 178, 200)$. We reconstruct the sparse polynomial (1.1) from samples of a random Chebyshev grid. For this purpose, we choose a random integer $\sigma \in [1, N-1]$ such that its inverse σ^{-1} modulo $2N-1$ exists. Assume that N fulfils the conditions $n_j \le 2N-1$. By

$$\begin{aligned}
T_{n_j}(u_{N,k}) &= \cos(\frac{kn_j\pi}{2N-1}) \\
&= \begin{cases} \cos(\frac{(\sigma k)(\sigma^{-1}n_j \mod (2N-1))\pi}{2N-1}) & \text{if } \sigma^{-1}n_j \mod (2N-1) \le N, \\ \cos(\frac{(\sigma k)\big(2N-1-(\sigma^{-1}n_j \mod (2N-1))\big)\pi}{2N-1}) & \text{if } \sigma^{-1}n_j \mod (2N-1) > N \end{cases} \\
&= \begin{cases} T_{\sigma^{-1}n_j \mod (2N-1)}(u_{N,\sigma k}) & \text{if } \sigma^{-1}n_j \mod (2N-1) < N, \\ T_{2N-1-(\sigma^{-1}n_j \mod (2N-1))}(u_{N,\sigma k}) & \text{if } \sigma^{-1}n_j \mod (2N-1) \ge N \end{cases}
\end{aligned}$$

we are able to recover the degrees n_j from the sampling set $u_{N,\sigma k} = \cos\frac{\sigma k \pi}{2N-1}$ for $k = 0, \ldots, K+L-1$. The main advantage is that the degrees $\sigma^{-1} n_j$ are much better separated than the original degrees n_j. The results are shown in the Table 5.3. Note that the Algorithm 3.3 determines the eigenvalues $\tilde{x}_j$, which give the correct degrees n_j after step 2, but the selection of these correct degrees fails in general in step 4. □

N	K	L	σ	Alg. 3.3	Alg. 3.4	Alg. 3.5	$e(\boldsymbol{c})$
2000	10	10	98	−	+	+	2.17e-10
4000	10	10	3294	−	+	+	1.46e-05
5000	10	10	1586	−	+	+	6.08e-07
8000	10	10	3053	−	+	+	1.60e-04

Table 5.3: Results of Example 5.3.

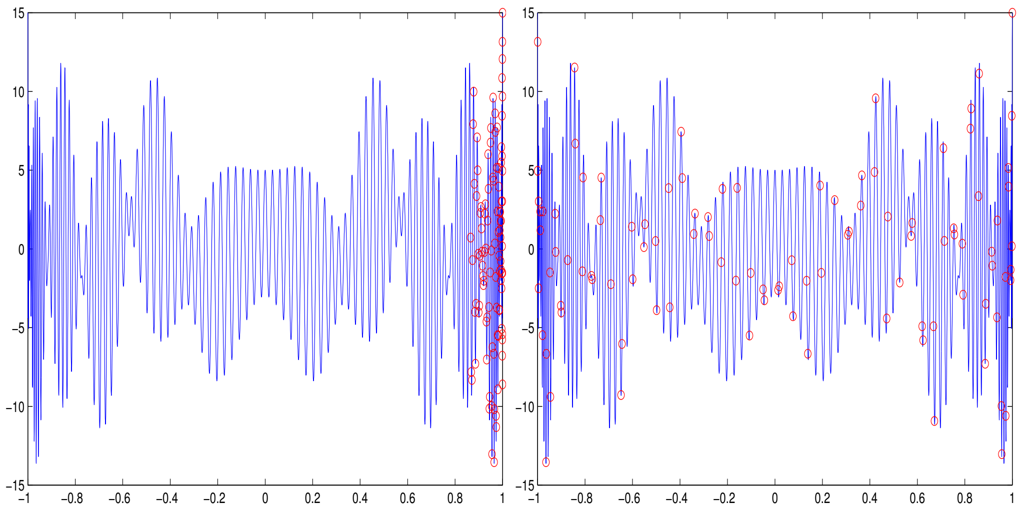

Figure 5.1: The sparse polynomial (1.1) of Example 5.3 for $N = 300$ and 100 samples with $\sigma = 1$ (left) and $\sigma = 251$ (right).

Example 5.4 This example shows a straightforward generalization to a sparse non-polynomial interpolation. We consider special functions the form

$$h(x) := \sum_{j=1}^{M} c_j \cos(\nu_j \arccos(x)) \quad (x \in [-1, 1]),$$

where $\nu_j \in \mathbb{R}$ with $0 \le \nu_1 < \ldots < \nu_M < 2N$ are not necessarily integers. Using

$t = \arccos(x)$, we obtain

$$g(t) = \sum_{j=1}^{M} c_j \, \cos(\nu_j t) \quad (t \in [0,\, \pi])\,.$$

As in Example 5.1 we choose $M = 5$, $c_j = j$, $u_N := \cos \frac{\pi}{2N-1}$ and $(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5) = (6.1, 12.2, 176.3, 178.4, 200.5)$. We compute the error of the values $\nu_j \in \mathbb{R}$ by

$$e(\boldsymbol{\nu}) := \max_{j=1,\ldots,5} |\nu_j - \tilde{\nu}_j| \quad (\boldsymbol{\nu} := (\nu_j)_{j=1}^{5})\,,$$

where $\tilde{\nu}_j$ are the values computed by our algorithms. This corresponding errors $e(\boldsymbol{\nu})$ are shown in the Table 5.4. We sample the function g at the nodes $\frac{k\pi}{2N-1}$ for $k = 0, \ldots, L+K-1$ and present the error $e(\boldsymbol{c})$ in the last column of Table 5.4 based on Algorithm 3.3. The results show that the Algorithms 3.4 and 3.5 can be used to find the entries ν_j and the coefficients c_j. □

N	K	L	Alg. 3.3	Alg. 3.4	Alg. 3.5	$e(\boldsymbol{c})$
120	10	10	1.64e+02	2.46e-09	2.48e-09	5.09e-09
120	20	20	1.23e+02	3.87e-10	3.92e-10	5.89e-10

Table 5.4: Results of Example 5.4.

Example 5.5 Finally, we consider a sparse polynomial (4.1) in Chebyshev-2 basis. To this end, we choose $M = 5$, $c_j = j$ $(j = 1, \ldots, 5)$, $u_N := \cos \frac{\pi}{2N-1}$ and $(n_1, n_2, n_3, n_4, n_5) = (6, 12, 176, 178, 190)$. The symbols $+$ and $-$ in the Table 5.5 mean that all degrees n_j of the Chebyshev polynomials U_{n_j} are correctly reconstructed and accordingly the reconstruction fails. Remember that the generalizations of Algorithms 3.4 and 3.5 for the Chebyshev-2 basis are denoted by Algorithms $\widetilde{3.4}$ and $\widetilde{3.5}$, respectively. Since after a successful reconstruction the last step is the same in our algorithms, we present the error $e(\boldsymbol{c})$ in the last column of the Table 5.5. From Table 5.5 we observe that the algorithms for sparse polynomial interpolation in Chebyshev-2 basis behaves very similar as the algorithms for sparse polynomial interpolation in Chebyshev-1 basis. □

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N	K	L	Alg. 4.2	Alg. $\widetilde{3.4}$	Alg. $\widetilde{3.5}$	$e(\boldsymbol{c})$
100	5	5	+	+	+	2.35e-14
200	5	5	+	+	+	5.86e-14
300	5	5	–	–	–	–
300	6	5	+	+	+	7.84e-02
300	7	5	+	+	+	1.38e-13

Table 5.5: Results of Example 5.5.

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