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The Asymmetric Quadratic Traveling Salesman Problem*

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Abstract. The quadratic traveling salesman problem asks for a tour of minimal costs where the costs are associated with each two arcs that are traversed in succession. This structure arises, *e. g.*, if the succession of two arcs represents the costs of loading processes in transport networks or a switch between different technologies in communication networks. Based on a quadratic integer program we present a linearized integer programming formulation and study the corresponding polyhedral structure of the asymmetric quadratic traveling salesman problem (AQTSP), where the costs may depend on the direction of traversal. The constructive approach that is used to establish the dimension of the underlying polyhedron allows to prove the facetness of several classes of valid inequalities. Some of them are related to the Boolean quadric polytope. Two new classes are presented that exclude conflicting configurations. Among these the first one is separable in polynomial time, the separation problem for the second class is **NP**-complete under certain conditions. We provide a general strengthening approach that allows to lift valid inequalities for the asymmetric traveling salesman problem (ATSP) to stronger valid inequalities for AQTSP. Applying this approach to the subtour elimination constraints gives rise to facet defining inequalities, but finding a maximally violated inequality among these is **NP**-complete. For the (D_3)-, (D_4^-)-, (D_4^+)-inequalities of the ATSP the strengthening approach is not sufficient to obtain a facet. First computational results are presented to illustrate the importance of the new inequalities. In particular, with these inequalities the running times can be improved for some real-world instances from biology by orders of magnitude in comparison to other state of the art methods in the literature.

Keywords: combinatorial optimization, polyhedral combinatorics, quadratic 0-1 programming, reload cost model
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1 Introduction

The Traveling Salesman Problem (TSP) is one of the best studied combinatorial optimization problems [5, 18, 22] and well-known to be **NP**-complete. Given a directed graph the

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task is to find a directed Hamiltonian cycle of minimal cost where costs are attached to the *arcs*, *i. e.*, to each two nodes that are traversed in succession in the tour. The Quadratic Traveling Salesman Problem (QTSP) differs from the TSP in that the costs depend on each three successively traversed nodes, a *2-arc*, of a tour. Because a 2-arc is present in a tour if and only if the two corresponding arcs are contained in the tour we speak of a quadratic TSP.

The QTSP was introduced by Jäger and Molitor [11, 21] in connection with an application in biology. Indeed, for the recognition of transcription factor binding sites one may employ Permuted (Variable Length) Markov models [25]. These models can be solved using an iterative algorithm that requires the solution of a TSP and a QTSP in each step.

A special case of the QTSP is the TSP with reload costs [4, 12, 14, 24] that arises in the planning of telecommunication networks whenever the costs for switching between different technologies are high or in the design of transport networks whenever the costs for loading processes are high in comparison to the transportation costs. In this setting one is given an arc-colored graph and the task is to find a tour over n nodes with minimal weighted sum of color changes along the tour.

In 1999 Aggarwal et. al. [3] introduced the Angular-Metric TSP (Angle-TSP) which is used in robotics. Given n points in the Euclidean space one looks for a tour with minimal total direction change, *i. e.*, the costs depend on the angle of the path from point i over point j to point k. The QTSP includes this problem and allows to handle extensions of it that take not only the angles but also arc- and 2-arc-dependent costs into account, *e. g.* the distances between the points.

In this work we consider the *Asymmetric* QTSP (AQTSP), *i. e.*, the costs of a 2-arc may depend on the direction of traversal. Based on a quadratic integer model we provide an integer linear programming formulation. Following the same line of argumentation used in [10] for the Symmetric QTSP (SQTSP), where the direction of traversal is irrelevant, we investigate the polyhedral structure of the AQTSP. In order to highlight the relationship to the SQTSP we stay close to the notation and structure of [10]. First we determine the dimension of the associated AQTSP polyhedron $P_{\mathbf{AQTSP}_n}$. For $n \geq 8$ an explicit construction is given that allows to generate the affinely independent tours. In a first step we determine the rank of a small number of specially structured tours so that the rank does not depend on n and after that we build tours in such a way that each tour contains at least one 2-arc that is not used in any of the tours constructed before.

The same proof technique allows to establish the facetness of several classes of valid inequalities of $P_{\mathbf{AQTSP}_n}$ (Section 3). In particular, we present facets that are related to the Boolean Quadric Polytope [23] (Section 3.1). Among these there are two classes that ensure that only one element can be chosen from a set of conflicting arcs and 2-arcs (Section 3.2). Both classes contain an exponential family of inequalities. The separation problem for one family can be solved in polynomial time and for the other, for which there exists no equivalent for the SQTSP, it is **NP**-complete under certain conditions. In Section 3.3 a general strengthening approach is presented that allows to lift almost all inequalities with nonnegative coefficients for the Asymmetric TSP (ATSP) to stronger valid inequalities for AQTSP. This approach suffices to lift the subtour elimination constraints to facet defining inequalities for AQTSP. In general, further strengthenings are required to obtain not only valid inequalities but facets for AQTSP. As examples, we present further lifted (D_3)-, (D_4^-)- and (D_4^+)-inequalities [15, 16] that define AQTSP facets. Although the main idea of the dimension proof, and so of the facet proofs, is similar to the proofs in [10] we present the results and the proofs in detail to keep the paper self-contained.

Finally we demonstrate the usefulness of the new inequalities by some computational results in Section 4 comparing the basic integer programming formulation with the formulation improved by the new cutting planes. Using these in a branch-and-cut algorithm enables us to solve real-world instances from biology with sizes up to 100 nodes in less than 605 seconds, all of them without branching. This improves the running times presented in [21] from days to seconds. Without the new cutting planes the running times are much higher and we get relatively large root gaps. Furthermore we tested rather small random TSP instances with reload costs and random Angle-TSP instances with additional 2-arc-dependent costs and compared the corresponding root gaps with and without the additional cutting planes.

A further way to solve non-convex quadratic integer programs is to convexify the objective function by the methods presented in [6, 7] and to use a solver for convex quadratic integer programs for the modified objective function. Because the approaches were not competitive concerning the lower bounds at the root node and the running times for the SQTSP we did not apply them for the AQTSP.

2 The model and the dimension of its associated polyhedron

A directed *2-graph* G is a pair (V, A) that consists of a node set $V = \{1, \dots, n\}, n \geq 3$, and a set of directed *2-arcs* A. Directed 2-arcs are ordered triples of three distinct nodes, *i. e.*, elements of the set $V^{(3)} := \{(i, j, k) \colon i, j, k \in V, |\{i, j, k\}| = 3\}$. A 2-arc $(i, j, k) \in V^{(3)}$ can be interpreted as a directed path on three nodes formed by the two arcs $(i, j), (j, k) \in V^{(2)} := \{(i, j) \colon i, j \in V, i \neq j\}$, $i \neq k$. If there is no danger of confusion we simply write ij instead of (i, j) and ijk instead of (i, j, k). We consider the complete directed 2-graph on V with $A := V^{(3)}$.

A directed 2-cycle C of length $k > 2$ in a directed 2-graph G is a set of k directed 2-arcs $C = \{v_1v_2v_3, v_2v_3v_4, \dots, v_{k-2}v_{k-1}v_k, v_{k-1}v_kv_1, v_kv_1v_2\} \subset A$ with pairwise distinct v_i. It induces a set of arcs $C^{(2)} := \{ij \in V^{(2)} \colon ijk \in C\}$. Our task is to find a directed 2-cycle C in a complete directed 2-graph $G = (V, A)$ with $n = |V|$ nodes, called a *tour*, that minimizes the sum of given weights c_a over all 2-arcs $a \in C$. With $\mathcal{C}_n = \{C \colon C \text{ 2-cycle in } G, |C| = n\}$ denoting the set of all tours on n nodes the optimization problem reads

$$\min \left\{ c(C) := \sum_{a \in C} c_a \colon C \in \mathcal{C}_n \right\}.$$

For a directed 2-cycle C we define the incidence vector $(x^C, y^C) \in \{0,1\}^{V^{(2)} \cup V^{(3)}}$ by

$$\forall\, a \in V^{(2)} \colon x_a^C = \begin{cases} 1 & \text{if } a \in C^{(2)}, \\ 0 & \text{if } a \notin C^{(2)}, \end{cases} \quad \text{and} \quad \forall\, a \in V^{(3)} \colon y_a^C = \begin{cases} 1 & \text{if } a \in C, \\ 0 & \text{if } a \notin C. \end{cases}$$

An integer programming model for all incidence vectors of tours is given by

$$\sum_{j \colon ij \in V^{(2)}} x_{ij} = \sum_{j \colon ji \in V^{(2)}} x_{ji} = 1, \qquad i \in V, \tag{1}$$

$$x_{ij} = \sum_{k \colon ijk \in V^{(3)}} y_{ijk} = \sum_{k \colon kij \in V^{(3)}} y_{kij}, \qquad ij \in V^{(2)}, \tag{2}$$

$$\sum_{\substack{ij \in V^{(2)} \colon \\ i \in S, j \in V \setminus S}} x_{ij} \geq 1, \qquad S \subset V, 2 \leq |S| \leq n-2, \tag{3}$$

$$x_{ij} \in \{0,1\}, y_{ijk} \in [0,1], \qquad ij \in V^{(2)}, ijk \in V^{(3)}. \tag{4}$$

Equalities (1), called *degree constraints*, ensure that each node is entered and left exactly once. The constraints (3) are the well known *subtour elimination constraints* [8]. Coupling constraints (2) may be seen as a kind of flow conservation for each $ij \in V^{(2)}$, because the sum of the in-flow into ij via 2-arcs $kij \in V^{(3)}$ has to be the same as the out-flow out of ij via 2-arcs $ijk \in V^{(3)}$.

The presented model (1)–(4) combines the well known formulation of the *Asymmetric Traveling Salesman Polytope* $P_{\mathbf{ATSP}_n} := \operatorname{conv}\{x^C \in \{0,1\}^{V^{(2)}} : C \in \mathcal{C}_n\} = \operatorname{conv}\{x \in \{0,1\}^{V^{(2)}}, (1), (3)\}$ [8] with the linearization of the quadratic terms in the objective function of the following quadratic integer program

$$\min_{\left\{x \in \{0,1\}^{V^{(2)}} : (1),(3)\right\}} \sum_{ijk \in V^{(3)}} c_{ijk} x_{ij} x_{jk}. \tag{5}$$

For proving that this is indeed a formulation of all incidence vectors of directed 2-cycles it suffices to show that the integrality of $y_{ijk}, ijk \in V^{(3)}$, follows from the integrality of the x-variables. For this we have to check that $x_{ij}x_{jk} = y_{ijk}$ for all $ij, jk \in V^{(2)}$ (with $ijk \in V^{(3)}$) and integral x. If $x_{ij} = 0$ equations (2) imply $y_{ijk} = 0$ for all $ijk \in V^{(3)}$, so consider the case $x_{ij} = x_{jk} = 1$. Assume $y_{ijk} < 1$, but then there exists $ijl \in V^{(3)}, l \neq k$, with $y_{ijl} > 0$ by (2) which implies $x_{jl} = 1$ (again by (2)). That contradicts $\sum_{m \in V :\, jm \in V^{(2)}} x_{jm} = 1$.

This paper is mainly devoted to the study of the polytope arising as the convex hull of all incidence vectors of directed 2-cycles, the *Asymmetric Quadratic Traveling Salesman Polytope*

$$P_{\mathbf{AQTSP}_n} := \operatorname{conv}\left\{(x^C, y^C) : C \in \mathcal{C}_n\right\} = \operatorname{conv}\left\{(x,y) \in \{0,1\}^{V^{(2)} \cup V^{(3)}} : (1), (2), (3)\right\}.$$

We start with determining the dimension of $P_{\mathbf{AQTSP}_n}$. We first calculate the rank of the corresponding constraint matrix.

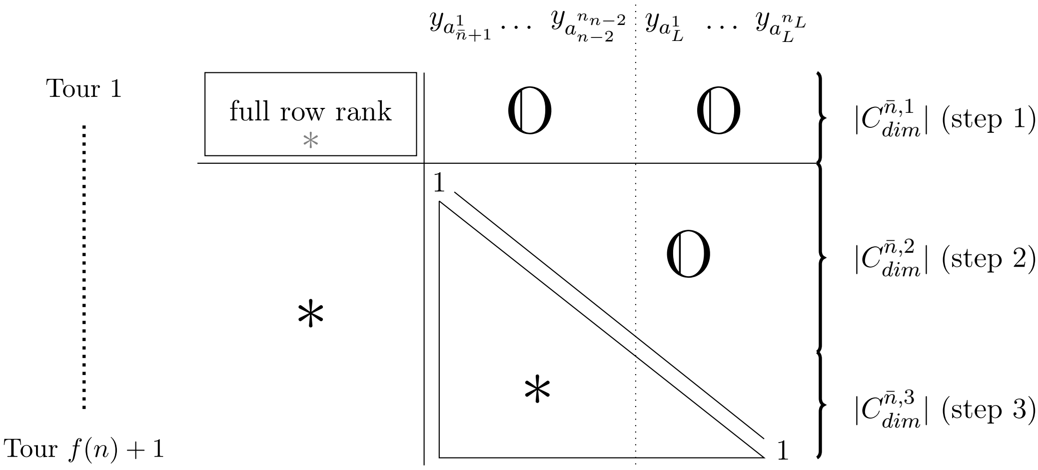
Lemma 2.1 *The constraint matrix corresponding to* (1) *and* (2) *has rank* $2n + 2n(n-1) - 1 - n = 2n^2 - n - 1$ *for* $n \geq 4$.

Proof. It is well known that the constraint matrix of the assignment polytope (1) has rank $2n - 1$. So we concentrate on the second part. We have to show that the rank of (2) equals $2n^2 - 3n$. The rows of the corresponding constraint matrix are denoted $r_{i,j,\bullet}$ for $x_{ij} = \sum_{ijk \in V^{(3)}} y_{ijk}$ and $r_{\bullet,i,j}$ for $x_{ij} = \sum_{kij \in V^{(3)}} y_{kij}$ for all $ij \in V^{(2)}$. We prove the rank formula in two steps. First we show that the matrix that arises after deleting the rows $r_{i,i+1,\bullet}, i = 1, \ldots, n-1$, and $r_{n,1,\bullet}$ has full row rank and then that the deleted rows are dependent on the others. The linear independence of the selected rows is equivalent to $\sum_{ij \in V^{(2)} :\, i+1 \neq j, (i,j) \neq (n,1)} \alpha_{i,j,\bullet} r_{i,j,\bullet} + \sum_{ij \in V^{(2)}} \alpha_{\bullet,i,j} r_{\bullet,i,j} = 0 \;\Rightarrow\; \alpha_{i,j,\bullet} = 0$ for all $ij \in V^{(2)}, i+1 \neq j, (i,j) \neq (n,1)$, and $\alpha_{\bullet,i,j} = 0$ for all $ij \in V^{(2)}$. After deletion of rows $r_{i,i+1,\bullet}, i = 1, \ldots, n-1$, and $r_{n,1,\bullet}$ the variables $y_{i(i+1)k}, i = 1, \ldots, n-1, k \in V \backslash \{i, i+1\}, y_{n1k}, k \in V \backslash \{1, n\}$, have exactly one nonzero entry in the matrix. Therefore the alphas of the corresponding rows, $\alpha_{\bullet,i,k}, k \in V \backslash \{i-1, i\}, k = 2, \ldots, n, \alpha_{\bullet,1,k}, k \in V \backslash \{1, n\}$, have to be zero. This implies that the variables $y_{kil}, k, l \in V \setminus \{i-1, i\}, i = 2, \ldots, n$ and $y_{k1l}, k, l \in V \setminus \{1, n\}$ have exactly one nonzero entry and so $\alpha_{k,i,\bullet}, k \in V \setminus \{i-1, i\}, i = 2, \ldots, n, \alpha_{k,1,\bullet}, k \in V \setminus \{1, n\}$ have to be zero. At this point only $\alpha_{\bullet,i,i-1}, i \in 2, \ldots, n, \alpha_{\bullet,1,n}$ may be nonzero but the variables $y_{(i+1)i(i-1)}, i = 2, \ldots, n-1, y_{21n}, y_{1n(n-1)}$ have only one nonzero entry which completes the first part of the proof. It remains to show that, w. l. o. g., row $r_{1,2,\bullet}$ is a linear combination of rows not deleted. This follows because $\sum_{i2 \in V^{(2)}} r_{i,2,\bullet} = \sum_{2i \in V^{(2)}} r_{\bullet,2,i}$ is equivalent to $r_{1,2,\bullet} = \sum_{2i \in V^{(2)}} (1 \cdot r_{\bullet,2,i}) + \sum_{i2 \in V^{(2)}, i \neq 1} (-1 \cdot r_{i,2,\bullet})$. □

This proves that the dimension of $P_{\mathbf{AQTSP}_n}$ is at most $f(n) := n(n-1)^2-(2n^2-n-1) = n^3-4n^2+2n+1$. That it is exactly this value for $n \geq 8$ is shown next in Theorem 2.2. The structure of the proof and the ideas used are similar to the proof of the dimension of the Symmetric Quadratic Traveling Salesman Polytope in [10]. Nonetheless we present the proof in detail to keep the presentation self-contained. Furthermore the proofs of the facetness of the valid inequalities to be presented in Section 3 are based on it.

Theorem 2.2 *The dimension of $P_{\mathbf{AQTSP}_n}$ equals $f(n)$ for all $n \geq 8$.*

Proof. We prove this by constructing $f(n)+1$ affinely independent tours, in dependence on a fixed small parameter $\bar{n}$, that are collected in the set $C_{dim}^{\bar{n}} = C_{dim}^{\bar{n},1} \dot{\cup} C_{dim}^{\bar{n},2} \dot{\cup} C_{dim}^{\bar{n},3}$ and by showing that $|C_{dim}^{\bar{n}}| = f(n)+1$. This is done in three main steps building the following matrix structure where each row corresponds to an incidence vector of a tour. In the first step we explicitly determine the rank of some specially structured tours $\tilde{C}_{dim}^{\bar{n},1}$ and take a largest affinely independent subset $C_{dim}^{\bar{n},1} \subseteq \tilde{C}_{dim}^{\bar{n},1}$. After that we iteratively add tours, each of these contains at least one 2-arc that is not used in a previous tour. We achieve this by the order in which the tours are added. Finally, tours are constructed with unused 2-arcs containing the nodes $n-1, n$ in step 3.



1. We fix $\bar{n}$ to a small value. In this proof $\bar{n}=6$ is sufficient but for some facets $\bar{n}=7,8$ is needed. In the set $\tilde{C}_{dim}^{\bar{n},1}$ we collect all tours containing a directed path of nodes $\{(\bar{n}+1),\ldots,n\}$ with increasing order of all nodes and an arbitrary permutation of nodes $\{1,\ldots,\bar{n}\}$, $\tilde{C}_{dim}^{\bar{n},1} = \{C \in \mathcal{C}_n \colon (i,i+1) \in C^{(2)}, i = \bar{n}+1,\ldots,n-1\}$. The rank $r_{\bar{n}}$ of the matrix formed by the incidence vectors of these rows is independent of $n \geq \bar{n}+2$ and so we can determine $r_{\bar{n}}$ easily by a computer algebra system. This gives $r_6 = 144$, $r_7 = 245$ and $r_8 = 384$. Now we select $r_{\bar{n}}$ tours of $\tilde{C}_{dim}^{\bar{n},1}$ whose incidence vectors are affinely independent and these form $C_{dim}^{\bar{n},1} \subset \tilde{C}_{dim}^{\bar{n},1}$ with $|C_{dim}^{\bar{n},1}| = r_{\bar{n}}$.

2. In the second step we build $C_{dim}^{\bar{n},2} = \bigcup_{\bar{n}+1 \leq k \leq n-2} T_k$ by iteratively constructing sets of tours $T_k = \{t_k^1,\ldots,t_k^{n_k}\}$ for $k = \bar{n}+1,\ldots,n-2$ so that specific matrix entries, corresponding to 2-arcs not contained in any tour of $C_{dim}^{\bar{n},1}$, of the incidence vectors of these tours form a lower triangular matrix, establishing the affine independence of the tours. This structure is obtained by ordering the tours in such a way that each tour $t_k^i, i = 1,\ldots,n_k$, contains one 2-arc a_k^i that fulfills

$$a_k^i \notin C \text{ for all } C \in \left(C_{dim}^{\bar{n},1} \cup \left(\bigcup_{\bar{n}<h<k} T_h \right) \cup \left(\bigcup_{1 \leq h < i} \{t_k^h\} \right) \right). \tag{6}$$

For each $k \in \{\bar{n}+1, \ldots, n-2\}$ tours are built during five iteration steps ($\mathsf{I}j$), $j = 1, \ldots, 5$, and their corresponding incidence vectors are appended in sequence of increasing j. Ordering the columns appropriately, *i. e.*, $y_{a^1_{\bar{n}+1}}, \ldots, y_{a^{n_{\bar{n}+1}}_{\bar{n}+1}}, \ldots, y_{a^1_{n-2}}, \ldots, y_{a^{n_{n-2}}_{n-2}}$ with arbitrary order within an iteration step ($\mathsf{I}j$), these columns form a lower triangular matrix.

Consider a fixed k with $\bar{n}+1 \leq k \leq n-2$. In order to simplify the presentation we only write down the relevant parts of the tours (tour direction from left to right). For this purpose we replace the nodes of the (possibly empty) common node sequence $(k+2)\,(k+3)\ldots(n-2)\,(n-1)$ by the symbol ϖ_k and nodes that are irrelevant for the decisive structure may appear in any order within the parts denoted by "$\ldots$". The 2-arc a^i_k is marked by underlining its three corresponding nodes. Each 2-arc a^i_k has one of the four types

(**Type-I1**) $(a, k, b), a, b \in \{1, \ldots, k-1\}, a \neq b$,

(**Type-I2**) $(k, a, b), a, b \in \{1, \ldots, k-1\}, a \neq b, (a, b) \neq (1, 2)$,

(**Type-I3**) $(a, b, k+1), a, b \in \{1, \ldots, k-1\}, a \neq b$,

(**Type-I4**) $(n, a, k), (n, k, a), a \in \{1, \ldots, k-1\}$.

Only 2-arc $(k, 1, 2)$ is not used as an a^i_k.

The tours of T_k are built during the following five steps.

(**I1**) $\ldots \underline{a\,k\,b}\,(k+1)\,\varpi_k\,n \ldots$, for $a, b \in \{1, \ldots, k-1\}, a \neq b$,

(**I2**) $\ldots k\,1\,2\,\underline{a\,b\,(k+1)}\,\varpi_k\,n \ldots$, for $a, b \in \{3, \ldots, k-1\}, a \neq b$ (the 2-arc $(k, 1, 2)$ is not used as an a^i_k),

(**I3**) $\ldots \underline{k\,a\,b}\,m\,o\,(k+1)\,\varpi_k\,n \ldots$, for $a, b \in \{1, \ldots, k-1\}, (a, b) \neq (1, 2)$, with $m, o \in \{3, \ldots, k-1\}, |\{a, b, m, o\}| = 4$,

(**I4**) $\ldots k\,\underline{a\,b\,(k+1)}\,\varpi_k\,n \ldots$, for $a, b \in \{1, \ldots, k-1\}, a \neq b, \{a, b\} \cap \{1, 2\} \neq \emptyset$,

(**I5**) $\ldots (k+1)\,\varpi_k\,\underline{n\,a\,b} \ldots$, for $a, b \in \{1, \ldots, k\}, a \neq b, k \in \{a, b\}$.

Claim 1: The 2-arcs underlined above fulfill condition (6).

Proof of Claim 1. By construction all tours $t \in C^{\bar{n},1}_{dim} \cup \bigcup_{\bar{n}+1 \leq j < k} T_j$ contain the arc $(k, k+1)$ and a 2-arc $(n, a, b), a, b \in \{1, \ldots, k-1\}$. Thus the 2-arcs of (**Type-I1**)–(**Type-I4**) have not been used before. The 2-arcs $a^i_k, a^{\hat{i}}_k$ underlined during the same iteration step ($\mathsf{I}j$) are not in conflict because they belong both to the same 2-arc type and at most one 2-arc of a fixed type can be present in a tour. It remains to show that the 2-arcs of iteration step ($\mathsf{I}j$) are not contained in the tours built during iteration steps ($\mathsf{I}l$), $l < j$.

- Tours in step (**I2**): All tours in (**I1**) contain a 2-arc $(k, \tilde{b}, k+1), \tilde{b} \in \{1, \ldots, k-1\}$ and by (1), (2) no 2-arc $(a, b, k+1), a, b \in \{1, \ldots, k-1\}, a \neq b$.
- Tours in step (**I3**): All tours in (**I1**) contain a 2-arc $(k, \tilde{b}, k+1), \tilde{b} \in \{1, \ldots, k-1\}$ and by (1), (2) no 2-arc $(k, a, b), a, b \in \{1, \ldots, k-1\}$. Furthermore all tours in (**I2**) contain the 2-arc $(k, 1, 2)$ that is explicitly forbidden in (**I3**).
- Tours in step (**I4**): All tours in (**I1**) contain a 2-arc $(k, \tilde{b}, k+1), \tilde{b} \in \{1, \ldots, k-1\}$ and by (1), (2) no 2-arc $(a, b, k+1), a, b \in \{1, \ldots, k-1\}, a \neq b$. In (**I2**)–(**I3**) 2-arcs were restricted to $(\tilde{a}, \tilde{b}, k+1), \tilde{a}, \tilde{b} \in \{3, \ldots, k-1\}, \tilde{a} \neq \tilde{b}$; so none contains one of 1 or 2.
- Tours in step (**I5**): In all tours in (**I1**)–(**I4**) there is a 2-arc $(n, \bar{a}, \bar{b}), \bar{a}, \bar{b} \in \{1, \ldots, k-1\}, \bar{a} \neq \bar{b}$, and by (1), (2) no 2-arc of (**Type-I4**). Note, that for $\bar{n} \geq 6$

and so $n-2 \geq k \geq 7$ there are at least two nodes between n and k in tour direction in steps (**I2**), (**I3**).

This proves Claim 1.

3. All tours constructed so far fulfill

$$C_{dim}^{\bar{n},1} \cup C_{dim}^{\bar{n},2} \subset \left\{ C \in \mathcal{C}_n \colon (n-1,n) \in C^{(2)} \right\}, \tag{7}$$

i. e., they contain the arc $(n-1,n)$. In this step we construct tours $t_L^i, i = 1, \ldots, n_L$, with 2-arcs a_L^i of previously unused types. (Only the 2-arcs $(1,2,n)$ and $(1,n-1,2)$ are not used as a_L^i.)

(**Type-L1**) $(a,b,n), a,b \in \{1,\ldots,n-2\}, a \neq b, (a,b) \neq (1,2)$,

(**Type-L2**) $(n-1,a,b), a,b \in \{1,\ldots,n-2\}, a \neq b$,

(**Type-L3**) $(a,n,b), a,b \in \{1,\ldots,n-2\}, a \neq b$,

(**Type-L4**) $(a,n-1,b), a,b \in \{1,\ldots,n-2\}, a \neq b, (a,b) \neq (1,2)$,

(**Type-L5**) $(n-1,a,n), a \in \{1,\ldots,n-2\}$,

(**Type-L6**) $(n,a,n-1), a \in \{1,\ldots,n-2\}$.

Again the order of the tours is chosen in such a way that the underlined 2-arcs fulfill

$$a_L^i \notin C \text{ for all } C \in C_{dim}^{\bar{n},1} \cup C_{dim}^{\bar{n},2} \cup \{t_L^1,\ldots,t_L^{i-1}\} \tag{8}$$

in order to attain a lower triangular matrix structure of the corresponding incidence vectors. All tours of step (**L**j) are created before the start of steps (**L**l), $l > j$, and the order within each step is arbitrary.

(**L1**) $\ldots 1\,2\,n\,\underline{n-1\,a\,b}\ldots$, for $a,b \in \{3,\ldots,n-2\}, a \neq b$
(2-arc $(1,2,n)$ is not used as an a_L^i),

(**L2**) $\ldots \underline{a\,b\,n}\,(n-1)\,m\,o\ldots$, for $a,b \in \{1,\ldots,n-2\}, a \neq b, (a,b) \neq (1,2)$, with $m,o \in \{3,\ldots,n-2\}, |\{a,b,m,o\}| = 4$,

(**L3**) $\ldots n\,\underline{n-1\,a\,b}\ldots$, for $a,b \in \{1,\ldots,n-2\}, a \neq b, \{a,b\} \cap \{1,2\} \neq \emptyset$,

(**L4**) $\ldots 1\,(n-1)\,2\,\underline{a\,n\,b}\ldots$, for $a,b \in \{3,\ldots,n-2\}, a \neq b$
(2-arc $(1,n-1,2)$ is not used as an a_L^i),

(**L5**) $\ldots \underline{a\,(n-1)\,b}\,m\,n\,o\ldots$, for $a,b \in \{1,\ldots,n-2\}, (a,b) \neq (1,2)$, with $m,o \in \{3,\ldots,n-2\}, |\{a,b,m,o\}| = 4$,

(**L6**) $\ldots (n-1)\,m\,\underline{a\,n\,b}\ldots$, for $a,b \in \{1,\ldots,n-2\}, \{a,b\} \cap \{1,2\} \neq \emptyset$, with $m \in \{1,\ldots,n-2\}, |\{a,b,m\}| = 3$,

(**L7**) $\ldots \underline{(n-1)\,a\,n}\ldots$, for $a \in \{1,\ldots,n-2\}$,

(**L8**) $\ldots \underline{n\,a\,(n-1)}\ldots$, for $a \in \{1,\ldots,n-2\}$.

Claim 2: The 2-arcs of tours $t_L^i \in C_{dim}^{\bar{n},3}$ underlined in (**L1**)–(**L8**) fulfill condition (8).

Proof of Claim 2. By condition (7) and (1), (2) we know that all 2-arcs of types (**Type-L1**)–(**Type-L6**) are not contained in tours $t \in C_{dim}^{\bar{n},1} \cup C_{dim}^{\bar{n},2}$. As in the inductive steps, the underlined 2-arcs of a step (**L**j) belong to one type and are in conflict by (1), (2). It remains to show (8) for each underlined 2-arc a_L^i for tours of step (**L**j) with increasing j.

- Tours in step (**L2**): All tours in (**L1**) contain the 2-arc $(1,2,n)$.

- Tours in step (**L3**): In (**L1**)–(**L2**) the underlined 2-arcs are explicitly forbidden, *i. e.*, the 2-arcs are restricted to $(n-1, \tilde{a}, \tilde{b}), \tilde{a}, \tilde{b} \in \{3, \ldots, n-2\}, \tilde{a} \neq \tilde{b}$.
- Tours in step (**L4**): All tours in (**L1**)–(**L3**) contain the arc $(n, n-1)$.
- Tours in step (**L5**): All tours in (**L1**)–(**L3**) contain the arc $(n, n-1)$ and the tours in (**L4**) are restricted to contain $(1, n-1, 2)$.
- Tours in step (**L6**): All tours in (**L1**)–(**L3**) contain the arc $(n, n-1)$. In (**L4**)–(**L5**) the underlined 2-arcs are explicitly forbidden, *i. e.*, 2-arcs are restricted to $(\tilde{a}, n, \tilde{b}), \tilde{a}, \tilde{b} \in \{3, \ldots, n-2\}, \tilde{a} \neq \tilde{b}$.
- Tours in step (**L7**): All tours in (**L1**)–(**L3**) contain the arc $(n, n-1)$ and in the tours in (**L4**)–(**L6**) at least two nodes lie between $n-1$ and n in both directions because $n \geq 8$.
- Tours in step (**L8**): All tours in (**L1**)–(**L3**) contain the arc $(n, n-1)$, in the tours of (**L4**)–(**L6**) at least two nodes lie between nodes $n-1, n$ in both directions because $n \geq 8$ and (**L7**) contains 2-arcs of type (**Type-L5**) conflicting with 2-arcs of (**Type-L6**).

Claim 3: It holds $|C_{dim}^{\bar{n}}| = f(n) + 1$.

Proof of Claim 3. We determine $|C_{dim}^{\bar{n}}| = |C_{dim}^{\bar{n},1} \dot{\cup} C_{dim}^{\bar{n},2} \dot{\cup} C_{dim}^{\bar{n},3}| = |C_{dim}^{\bar{n},1}| + |C_{dim}^{\bar{n},2}| + |C_{dim}^{\bar{n},3}|$ with

$$
\begin{aligned}
|C_{dim}^{\bar{n},1}| &= r_{\bar{n}}, \\
|C_{dim}^{\bar{n},2}| &= \sum_{k=\bar{n}+1}^{n-2} |T_k| = \sum_{k=\bar{n}+1}^{n-2} \Bigg(\underbrace{(k-1)(k-2)}_{(\mathbf{I1})} + \underbrace{(k-3)(k-4)}_{(\mathbf{I2})} + \underbrace{(k-1)(k-2)-1}_{(\mathbf{I3})} \\
&\quad + \underbrace{4(k-3)+2}_{(\mathbf{I4})} + \underbrace{2(k-1)}_{(\mathbf{I5})} \Bigg) \\
&= \sum_{k=\bar{n}+1}^{n-2} (3(k-1)(k-2) + 2(k-1) - 1) = n^3 - 8n^2 + 20n - 16 - \bar{n}^3 + 2\bar{n}^2, \\
|C_{dim}^{\bar{n},3}| &= \underbrace{(n-4)(n-5)}_{(\mathbf{L1})} + \underbrace{(n-2)(n-3)-1}_{(\mathbf{L2})} + \underbrace{4(n-4)+2}_{(\mathbf{L3})} + \underbrace{(n-4)(n-5)}_{(\mathbf{L4})} \\
&\quad + \underbrace{(n-2)(n-3)-1}_{(\mathbf{L5})} + \underbrace{4(n-4)+2}_{(\mathbf{L6})} + \underbrace{(n-2)}_{(\mathbf{L7})} + \underbrace{(n-2)}_{(\mathbf{L8})} = 4n^2 - 18n + 18
\end{aligned}
$$

For $\bar{n}$ and $n \geq \bar{n} + 2$ the described constructions are possible and we get $|C_{dim}^{\bar{n}}| = n^3 - 4n^2 + 2n + 2 + r_{\bar{n}} - \bar{n}^3 + 2\bar{n}^2$ affinely independent tours for $\bar{n} \geq 6$. Choosing $\bar{n} = 6, 7, 8$ Claim 3 and Theorem 2.2 follow because the term $r_{\bar{n}} - \bar{n}^3 + 2\bar{n}^2$ evaluates to zero in all three cases, $r_6 - 6^3 + 2 \cdot 6^2 = 144 - 216 + 72 = 0$, $r_7 - 7^3 + 2 \cdot 7^2 = 245 - 343 + 98 = 0$ resp. $r_8 - 8^3 + 2 \cdot 8^2 = 384 - 512 + 128 = 0$. □

The dimensions of $P_{\mathbf{AQTSP}_n}$ for small n are 1 for $n = 3$, 5 for $n = 4$, 22 for $n = 5$, 80 for $n = 6$ and 162 for $n = 7$. All these values were determined by a computer algebra system.

Remark 2.3 The Asymmetric Quadratic Cycle Cover Problem $\mathbf{AQCCP}_n$ asks for a set of directed 2-cycles of length at least three in a complete directed 2-graph $\tilde{G} = (\tilde{V}, \tilde{A}), |\tilde{V}| = n$. In contrast to the Asymmetric Cycle Cover Problem on a graph it is **NP**-complete because the minimum reload cost cycle cover problem [12] can be reduced to it. Its corresponding polytope reads

$$P_{\mathbf{AQCCP}_n} := \operatorname{conv}\left\{(x,y) \in \mathbb{R}^{V^{(2)} \cup V^{(3)}} : (x,y) \text{ fulfills } (1),(2),(4)\right\}.$$

Lemma 2.1 and Theorem 2.2 also prove that the dimension of $P_{\mathbf{AQCCP}_n}$ equals $f(n)$.

3 Valid inequalities and facets of $P_{\mathbf{AQTSP}_n}$

In this section we present valid inequalities of $P_{\mathbf{AQTSP}_n}$ and prove the facetness of most of them. The inequalities can be classified into three types. Class one includes inequalities that are related to the Boolean Quadric Polytope (BQP) [23]. After that two different exponential families of *conflicting arcs inequalities* are introduced and the complexity of the corresponding separation problems is investigated. Finally, we introduce a general strengthening approach that enables us to improve almost all valid inequalities of $P_{\mathbf{ATSP}_n}$ to get stronger inequalities for $P_{\mathbf{AQTSP}_n}$. Applying the presented procedure we derive strengthened versions of the subtour elimination constraints (3) and prove that these define facets of $P_{\mathbf{AQTSP}_n}$ but finding a maximally violated constraint of that type is **NP**-complete. Furthermore we apply the presented strengthening approach to some D_k-inequalities [16, 17]. For $k = 3, 4$ the approach is not sufficient to obtain facets of $P_{\mathbf{AQTSP}_n}$, here further strengthenings are possible.

3.1 Facets related to the Boolean Quadric Polytope

The polytope $P_{\mathbf{AQTSP}_n}$ arises as a linearization of the quadratic zero-one problem (5). Therefore it is natural to consider relations to facet defining inequalities of the BQP. We start with the sign constraints.

Corollary 3.1 *For $n \geq 3$ the inequalities*

$$y_{ijk} \geq 0$$

define facets of $P_{\mathbf{AQTSP}_n}$ for all $ijk \in V^{(3)}$.

Proof. For $3 \leq n \leq 7$ we verified the statement by a computer algebra system determining the rank of all incidence vectors of tours not containing, w. l. o. g., the 2-arc $(n, n-2, n-1)$. For $n \geq 8, \bar{n} = 6$ it follows directly from the proof of Theorem 2.2. In tours constructed in all steps before (**L8**) the 2-arc $(n, n-2, n-1)$ is not contained, see the triangular structure of the incidence vectors of tours in $C^{\bar{n},2}_{dim} \cup C^{\bar{n},3}_{dim}$. Restricting the construction to $a \in \{1, \ldots, n-3\}$ in (**L8**) we get $f(n)$ affinely independent tours and none of these contains $(n, n-2, n-1)$. □

Furthermore, the *triangle inequalities* [23] are known to be facet defining for BQP. Using our notation some of them read $-x_{ij} + y_{kij} + y_{ijk} - y_{jki} \leq 0, i, j, k \in V, |\{i, j, k\}| = 3$, but this can be strengthened as shown next.

Theorem 3.2 *For $n \geq 5$ the inequalities*

$$y_{ijk} + y_{kij} \leq x_{ij} \tag{9}$$

define facets of $P_{\mathbf{AQTSP}_n}$ for all $ij \in V^{(2)}$ and all $k \in V \setminus \{i, j\}$.

Proof. The inequalities are valid for $P_{\mathbf{AQTSP}_n}, n \geq 4$, because the left-hand side is at most one by (2), (3) and the presence of one of the 2-arcs $(i, j, k), (k, i, j)$ implies the presence of arc (i, j) in a tour by (2). We set, w. l. o. g., $i = n-2, j = n, k = n-1$. All tours that give rise to roots of (9), *i. e.*, $y_{(n-2,n,n-1)} + y_{(n-1,n-2,n)} = x_{(n-2,n)}$, either do not contain the arc $(n-2, n)$ or they contain with this one of the arcs $(n, n-1), (n-1, n-2)$, too. For $5 \leq n \leq 7$ we proved the statement by a computer algebra system and for $n \geq 8, \bar{n} = 6$ the construction of $f(n)$ affinely independent tours is similar to the construction in the proof of Theorem 2.2. We only mention the differences. The tours in $C_{dim}^{\bar{n},1} \cup C_{dim}^{\bar{n},2}$ do not contain the arc $(n-2, n)$ by (7). So we only have to adapt the tour construction in step three. The tours built in (**L1**) do not contain the arc $(n-2, n)$ and in steps (**L2**)-(**L3**) the arc $(n-2, n)$ implies a 2-arc $(n-2, n, n-1)$. We slightly change the further steps and add step **(L9')** that uses 2-arcs originally contained in tours of (**L4**), (**L6**).

- **(L4')** $\ldots 1\,(n-1)\,2\,\underline{a\,n\,b} \ldots$, for $a, b \in \{3, \ldots, n-2\}, a \neq b, a \neq n-2$ (the 2-arc $(1, n-1, 2)$ is not used as an a_L^i),
- **(L5')** $\ldots \underline{a\,(n-1)\,b}\,m\,n\,o \ldots$, for $a, b \in \{1, \ldots, n-2\}, (a, b) \neq (1, 2)$, with $m, o \in \{3, \ldots, n-2\}, m \neq n-2, |\{a, b, m, o\}| = 4$,
- **(L6')** $\ldots (n-1)\,m\,\underline{a\,n\,b} \ldots$, for $a, b \in \{1, \ldots, n-2\}, a \neq n-2, \{a, b\} \cap \{1, 2\} \neq \emptyset$, with $m \in \{1, \ldots, n-2\}, |\{a, b, m\}| = 3$,
- **(L7')** $\ldots \underline{(n-1)\,a\,n} \ldots$, for $a \in \{1, \ldots, n-3\}$,
- **(L8')** $\ldots m\,\underline{n\,a\,(n-1)} \ldots$, for $a \in \{1, \ldots, n-2\}$ with $m \in \{1, \ldots, n-3\}, m \neq a$,
- **(L9')** $\ldots (n-1)\,\underline{(n-2)\,n\,a} \ldots$, for $a \in \{1, \ldots, n-3\}$ (the 2-arc $(n-1, n-2, n)$ is not used as an a_L^i).

One can easily check that all tours in **(L4')**–**(L9')** are roots of (9). For $n \geq 8$ all constructions are possible. It remains to prove that all underlined 2-arcs a_L^i in **(L4')**–**(L9')** fulfill (8). The statement is clear for **(L4')**–**(L8')** by the proof of Theorem 2.2 because merely the use of some 2-arcs is deferred to **(L9')**. The a_L^i of **(L9')** are unused because all tours in (**L1**)–(**L3**) contain arc $(n, n-1)$ and in **(L4')**–**(L8')** arc $(n-2, n)$ is explicitly forbidden. In comparison to the proof of Theorem 2.2 the adapted construction generates exactly one tour less, namely that containing $(n-1, n-2, n)$, and so the inequality is facet defining for $P_{\mathbf{AQTSP}_n}, n \geq 5$. □

A further way to derive (9) is to lift the subtour elimination constraint (3) resp. the equivalent constraints $\sum_{ij \in S^{(2)}} x_{ij} \leq |S| - 1, S \subset V, 2 \leq |S| \leq n-2$, on three nodes, *i. e.*, $x_{ij} + x_{ik} + x_{ji} + x_{jk} + x_{ki} + x_{kj} \leq 2,\ i, j, k \in V, |\{i, j, k\}| = 3$, by multiplying it with x_{ij} and using (1), that cancels out the products $x_{ik} \cdot x_{ij}, x_{ji} \cdot x_{ij}, x_{kj} \cdot x_{ij}$, as well as $x_{ij}^2 = x_{ij}$.

In order to illustrate the usefulness of (9) consider Figure 1 which displays a fractional solution of a relaxation fulfilling (1), (2), (3) as well as $x_{ij} \in [0, 1]$ for all $ij \in V^{(2)}$ and $y_{ijk} \in [0, 1]$ for all $ijk \in V^{(3)}$. The x-variables correspond to a convex combination of two tours, each with value $\frac{1}{2}$, but the coupling between the x- and the y-variables is so weak that the y-variables correspond to a convex combination of three 2-cycles of length less than five. Clearly, each of the triangles of the y-variables gives rise to violated inequalities (9).

Further triangle inequalities known for BQP are $x_{ij} + x_{jk} + x_{ki} - y_{ijk} - y_{jki} - y_{kij} \leq 1, i, j, k \in V, |\{i, j, k\}| = 3$. Again we can strengthen them and get stronger inequalities because by (2) it holds $x_{ij} \geq y_{ijk}$ for all $i, j, k \in V, |\{i, j, k\}| = 3$.

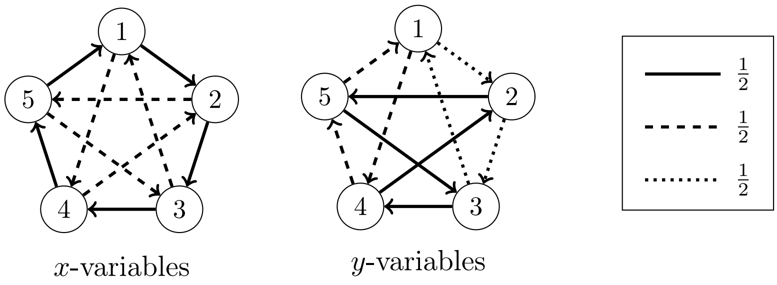


Figure 1: These fractional solutions are cut off via inequalities (9) and (14) for $n = 5$. Consider, *e. g.*, $x_{12} = y_{312} = y_{123} = \frac{1}{2}$ for $i = 1, j = 2, k = 3$ in (9).

Theorem 3.3 *For $n \geq 7$ the inequalities*

$$\sum_{ij \in D^{(2)}} x_{ij} - \sum_{ijk \in D^{(3)}} y_{ijk} \leq 1 \tag{10}$$

define facets of $P_{\mathbf{AQTSP}_n}$ for all $D \subset V, |D| = 3$.

Proof. The inequalities are valid for $P_{\mathbf{AQTSP}_n}, n \geq 7$, because by (3) at most two of the arcs ij, ik, ji, jk, ki, kj can be present in a tour. If two arcs $uv, vw \in D^{(2)}$ are contained in a tour also the 2-arc uvw is present. For $n = 7$ we checked the statement by a computer algebra system. For $n \geq 8, \bar{n} = 6$, we set, w. l. o. g., $i = 1, j = n-1, k = n$. An incidence vector of a tour fulfills (10) with equality if at least one of the arcs $(1, n-1), (1, n), (n-1, 1), (n-1, n), (n, 1), (n, n-1)$ is present. The proof is similar to the proof of Theorem 2.2 and we only outline the differences. All tours in $C_{dim}^{\bar{n},1} \cup C_{dim}^{\bar{n},2}$ contain the arc $(n-1, n)$ and the ones in steps (**L1**)-(**L4**) the arc $(n, n-1)$ or $(1, n-1)$. The construction of tours using (**Type-L4**)–(**Type-L6**) and some 2-arcs of (**Type-L3**) is divided into steps (**Δ5**)–(**Δ9**) replacing (**L5**)–(**L8**).

$$(\boldsymbol{\Delta 5}) \begin{cases} \ldots \underline{a\,(n-1)\,b}\,m\,n\,o \ldots, & \text{for } a, b \in \{1, \ldots, n-2\}, 1 \in \{a, b\}, (a, b) \neq (1, 2), \\ & \text{with } m, o \in \{3, \ldots, n-2\}, |\{a, b, m, o\}| = 4, \\ \ldots \underline{a\,(n-1)\,b}\,1\,n\,2, \ldots & \text{for } a, b \in \{3, \ldots, n-2\}, a \neq b, \end{cases}$$

(the 2-arc $(1, n, 2)$ is not used as an a_L^i),

$$(\boldsymbol{\Delta 6}) \begin{cases} \ldots (n-1)\,1\,\underline{a\,n\,b} \ldots, & \text{for } a, b \in \{2, \ldots, n-2\}, 2 \in \{a, b\}, \\ \ldots m\,(n-1)\,o\,\underline{a\,n\,b} \ldots, & \text{for } a, b \in \{1, \ldots, n-2\}, 1 \in \{a, b\}, (a, b) \neq (1, 2), \\ & \text{with } m, o \in \{3, \ldots, n-2\}, |\{a, b, m, o\}| = 4, \end{cases}$$

$$(\boldsymbol{\Delta 7}) \begin{cases} \ldots 2\,1\,\underline{(n-1)\,a\,n} \ldots, & \text{for } a \in \{3, \ldots, n-2\}, \\ \ldots \underline{(n-1)\,1\,n} \ldots, & \\ \ldots 1\,\underline{(n-1)\,2\,n} \ldots, & \end{cases}$$

$$(\boldsymbol{\Delta 8}) \begin{cases} \ldots 2\,1\,\underline{n\,a\,(n-1)} \ldots, & \text{for } a \in \{3, \ldots, n-2\}, \\ \ldots \underline{n\,1\,(n-1)} \ldots, & \\ \ldots \underline{n\,2\,(n-1)}\,1 \ldots, & \end{cases}$$

$$(\boldsymbol{\Delta 9}) \quad \ldots 1\,n\,\underline{a\,(n-1)\,b} \ldots, \text{ for } a, b \in \{2, \ldots, n-2\}, 2 \in \{a, b\}, a \neq b.$$

The tours constructed in steps (**Δ5**)–(**Δ9**) define roots of (10) because at least one of the six arcs described above is present in each tour. Again we have to prove that all underlined 2-arcs in (**Δ5**)–(**Δ9**) fulfill (8). With the proof of Theorem 2.2 it is sufficient to check that the 2-arcs of step (**Δ9**) are not contained in (**L1**)–(**L4**) and **(**$\triangle\, l$**)**, $5 \leq l \leq 8$.

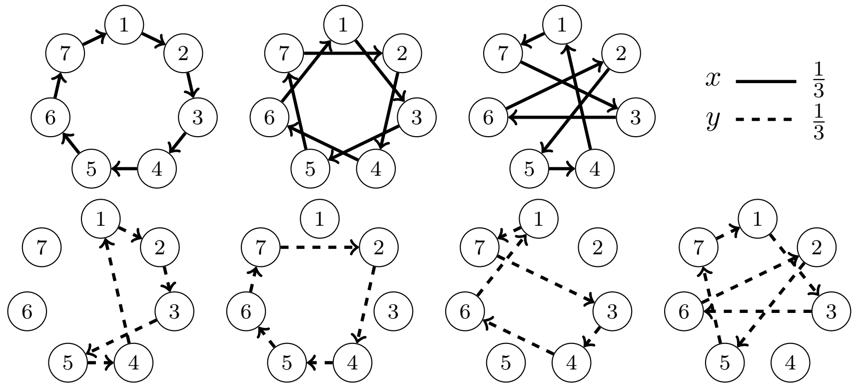


Figure 2: These fractional solutions are cut off via inequalities (10) for $n = 7$ because for $D = \{1, 2, 7\}$ it holds $x_{12} = x_{17} = x_{71} = x_{72} = \frac{1}{3}, x_{21} = x_{27} = 0$ but $y_{712} = y_{172} = 0$.

- Tours in (**Δ9**): The tours in (**L1**)–(**L3**) contain arc $(n, n-1)$ and the ones in (**L4**) 2-arc $(1, n-1, 2)$. If one of the arcs $(2, n-1), (n-1, 2)$ is not forbidden in (**Δ5**)–(**Δ8**) the tour contains one of the 2-arcs $(1, n-1, 2), (2, n-1, 1)$.

The adapted construction works out for $n \geq 8, \bar{n} = 6$, and generates exactly one tour less than in the proof of Theorem 2.2. So the inequality is facet defining for $P_{\mathbf{AQTSP}_n}, n \geq 7$.□

Figure 2 displays an example of a fractional solution (the values of each drawn arc and 2-arc equal $\frac{1}{3}$) that is cut off by inequalities (10) for $n = 7$ and $D = \{1, 2, 7\}$ because $x_{17} = x_{12} = x_{71} = x_{72} = \frac{1}{3}, x_{21} = x_{27} = 0, \sum_{ijk \in D^{(3)}} y_{ijk} = 0$. The x-variables correspond to a convex combination of the three tours in the first line of Figure 2, each with value $\frac{1}{3}$, and the y-variables to a convex combination of the four 2-cycles of lengths five resp. six in the second line. One can easily check that all inequalities of type (9) are fulfilled.

3.2 Conflicting arcs inequalities

We now introduce *conflicting arcs inequalities*. These forbid T-structures and short subtours by allowing the selection of at most one element out of a set of arcs and 2-arcs of which any tour may contain at most one, see Figure 3 for a possible partition of $V \setminus \{i, j\} = S \dot{\cup} T, S \cap T = \emptyset$ for fixed $i, j \in V, i \neq j$, and the counted arcs and 2-arcs. Depending on the size of S, T this main idea has to be extended to get facet defining inequalities. In total we consider five different cases for the sizes of S resp. T. Apart from that further inequalities are presented that are based on the same idea but we will see that the separation problem of the first classes can be solved in polynomial time. It is **NP**-complete for the last class as long as the nodes $i, j \in V, i \neq j$, are fixed in advance.

We first consider the case $T = \emptyset$ for the inequalities displayed in Figure 3.

Theorem 3.4 *For $n \geq 7$ the inequalities*

$$x_{ij} + x_{ji} + \sum_{ikj \in V^{(3)}} y_{ikj} + \sum_{jki \in V^{(3)}} y_{jki} \leq 1 \tag{11}$$

define facets of $P_{\mathbf{AQTSP}_n}$ for all $i, j \in V, i \neq j$.

Proof. The inequalities are valid because for $n \geq 5$ the presence of two of the counted arcs or 2-arcs implies a subtour or a forbidden T-structure. We set, w. l. o. g., $i = 1, j = 2$.

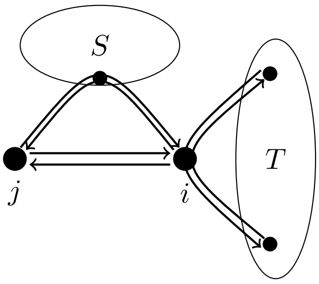


Figure 3: A tour can contain at most one out of these arcs (straight lines) and 2-arcs (curved lines).

An incidence vector of a tour fulfills (11) with equality, *i. e.*, $x_{12} + x_{21} + \sum_{1k2 \in V^{(3)}} y_{1k2} + \sum_{2k1 \in V^{(3)}} y_{2k1} = 1$, if nodes 1 and 2 lie next to each other or exactly one node lies between them. For $n = 7, 8$ we validated the statement with a computer algebra system and for $n \geq 9$ the idea of the proof is similar to the proof of Theorem 2.2. Therefore we use the same notation. For the first step, determining the tours of $C_{dim}^{\bar{n},1}$, we have to adapt the $\bar{n}$-permutation block because nodes 1,2 belong to the nodes permuted in this block. Restricting the selectable tours to $\tilde{C}_{dim}^{\bar{n},1} = \{C \in \mathcal{C}_n : (\{(1,2),(2,1)\} \cup \{(1,k,2),(2,k,1) : k = 3,\dots,\bar{n}\}) \cap (C \cup C^{(2)}) \neq \emptyset, (i,i+1) \in C^{(2)}, i = \bar{n}+1,\dots,n-1\}$ reduces the rank by one for $\bar{n} = 7$. So the following constructions work out because no further 2-arc a_k^i, a_L^i is lost in steps two and three in comparison to the proof of Theorem 2.2.

For achieving the desired structure of the tours $t \in \bar{C}_{dim}^{\bar{n},2}$ we introduce the new steps $(\boldsymbol{ST_0.1})$–$(\boldsymbol{ST_0.8})$.

$(\boldsymbol{ST_0.1})$ $\dots \underline{a\,k\,b}\,(k+1)\,\varpi_k\,n \dots$, for $a, b \in \{1,\dots,k-1\}, b \geq 3, a \neq b$,

$(\boldsymbol{ST_0.2})$ $\dots k\,3\,4\,m\,\overline{a\,b\,(k+1)}\,\varpi_k\,n \dots$, for $a, b \in \{1,2,5,6,\dots,k-1\}$ with $m \in \{1,2,5,6,\dots,k-1\}, |\{a,b,m\}| = 3$
(the 2-arc $(k,3,4)$ is not used as an a_k^i),

$(\boldsymbol{ST_0.3})$ $\dots \underline{k\,a\,b}\,m\,o\,(k+1)\,\varpi_k\,n \dots$, for $a, b \in \{1,\dots,k-1\}, a \geq 3, (a,b) \neq (3,4)$, with $m, o \in \{1,2,5,6\dots,k-1\}, |\{a,b,m,o\}| = 4$,

$(\boldsymbol{ST_0.4})$ $\dots k\,m\,o\,\overline{a\,b\,(k+1)}\,\varpi_k\,n \dots$, for $a, b \in \{1,\dots,k-1\}, \{a,b\} \cap \{3,4\} \neq \emptyset$, with $m \in \{3,\dots,k-1\}, o \in \{1,\dots,k-1\}, |\{a,b,m,o\}| = 4$,

$(\boldsymbol{ST_0.5})$ $\left\{ \begin{array}{l} \dots \underline{a\,k\,1}\,2\,(k+1)\,\varpi_k\,n \dots, \\ \dots \underline{a\,k\,2}\,1\,(k+1)\,\varpi_k\,n \dots, \end{array} \right.$ for $a \in \{3,\dots,k-1\}$
(the 2-arcs $(k,1,2),(k,2,1)$ are not used as a_k^i),

$(\boldsymbol{ST_0.6})$ $\left\{ \begin{array}{l} \dots \underline{k\,1\,a}\,2\,(k+1)\,\varpi_k\,n \dots, \\ \dots \underline{k\,2\,a}\,1\,(k+1)\,\varpi_k\,n \dots, \end{array} \right.$ for $a \in \{3,\dots,k-1\}$,

$(\boldsymbol{ST_0.7})$ $\left\{ \begin{array}{l} \dots \underline{1\,k\,2}\,3\,(k+1)\,\varpi_k\,n \dots, \\ \dots \underline{2\,k\,1}\,3\,(k+1)\,\varpi_k\,n \dots, \end{array} \right.$

$(\boldsymbol{ST_0.8})$ $\left\{ \begin{array}{l} \dots 2\,\underline{k\,1\,(k+1)}\,\varpi_k\,n \dots, \\ \dots 1\,\underline{k\,2\,(k+1)}\,\varpi_k\,n \dots, \end{array} \right.$
(In comparison to the proof of Theorem 2.2 the 2-arcs $(k,1,k+1),(k,2,k+1)$ are used as a_k^i. This allows us not to use 2-arcs $(k,1,2),(k,2,1)$ in $(\boldsymbol{ST_0.5})$.).

After that (**I5**) is performed. In steps ($\boldsymbol{ST_0}$**.5**)-($\boldsymbol{ST_0}$**.8**) the desired structure described above is given explicitly.

Claim 1: The desired structure concerning nodes 1,2 can be achieved in ($\boldsymbol{ST_0}$**.1**)–($\boldsymbol{ST_0}$**.4**), (**I5**) and in (**L1**)–(**L8**).

Proof of Claim 1. For $\{1,2\}\cap\{a,b\}=\emptyset$ we can force the 2-arc $(n,1,2)$ in ($\boldsymbol{ST_0}$**.1**)–($\boldsymbol{ST_0}$**.4**). The remaining cases for ($\boldsymbol{ST_0}$**.1**)–($\boldsymbol{ST_0}$**.4**) are:

- In ($\boldsymbol{ST_0}$**.1**), w. l. o. g. $a=1$, we place node 2 next to 1, *i. e.*, we enforce arc $(2,1)$.
- If $\{1,2\}\cap\{a,b\}\neq\emptyset$ in ($\boldsymbol{ST_0}$**.2**)–($\boldsymbol{ST_0}$**.3**) we choose $m\in\{1,2,5,6,\dots,k-1\}$ with $\{1,2\}\subset\{a,b,m\}, |\{a,b,m\}|=3$.
- If $\{1,2\}\cap\{a,b\}\neq\emptyset$ in ($\boldsymbol{ST_0}$**.4**) we choose $o\in\{1,\dots,k-1\}$ with $\{1,2\}\subset\{a,b,o\}, |\{a,b,o\}|=3$.

Because there is lot of freedom for the completion of tours in (**I5**), (**L1**)–(**L4**), (**L6**)–(**L8**) we can force the desired structure using similar techniques. If, w. l. o. g., $a\neq 2, b=1$ in (**L5**) we increase the number of nodes between $n-1$ and n placing node 2 next to 1, *i. e.* $\dots a\,(n-1)\,1\,2\,m\,n\,o\dots, |\{a,m,o,1,2\}|=5$.

Claim 2: The tours in step 2 fulfill $\bar{C}^{\bar{n},2}_{dim}=C^{\bar{n},2}_{dim}$ and condition (6).

Proof of Claim 2.

- (**Type-I1**): We use all 2-arcs of this type as a^i_k in ($\boldsymbol{ST_0}$**.1**), ($\boldsymbol{ST_0}$**.5**) and ($\boldsymbol{ST_0}$**.7**).
- (**Type-I2**): The role of nodes 1,2 and 3,4 changed comparing (**I2**) and ($\boldsymbol{ST_0}$**.2**) and so we do not use 2-arc $(k,3,4)$ as an a^i_k here. But, in addition the 2-arcs $(k,1,2),(k,2,1)$ are lost for building the triangular structure, see ($\boldsymbol{ST_0}$**.5**).
- (**Type-I3**): We use all 2-arcs of this type as a^i_k in ($\boldsymbol{ST_0}$**.2**) and ($\boldsymbol{ST_0}$**.4**).
- (**Type-I4**): We use all 2-arcs of this type as a^i_k in (**I5**).
- additional 2-arcs: In ($\boldsymbol{ST_0}$**.8**) the 2-arcs $(k,1,k+1),(k,2,k+1)$ are used as a^i_k. With these we can compensate the missing two a^i_k of type (**Type-I2**) and so $\bar{C}^{\bar{n},2}_{dim}=C^{\bar{n},2}_{dim}$.

Furthermore one can easily check that the underlined a^i_k fulfill condition (6). This proves Claim 2. All in all we constructed exactly one tour less than in the proof of Theorem 2.2, thus the inequality defines a facet of $P_{\mathbf{AQTSP}_n}, n\geq 7$. $\square$

In the case $|T|=2$ further strengthenings are possible in comparison to the 2-arcs shown in Figure 3.

Theorem 3.5 *For $n\geq 7$ the inequalities*

$$x_{ij}+x_{ji}+y_{kil}+y_{kjl}+y_{lik}+y_{ljk}+\sum_{imj\in V^{(3)}:\, m\in S} y_{imj}+\sum_{jmi\in V^{(3)}:\, m\in S} y_{jmi}\leq 1 \qquad (12)$$

define facets of $P_{\mathbf{AQTSP}_n}$ for all $i,j\in V, i\neq j$, and all $S\dot\cup T=V\setminus\{i,j\}, T=\{k,l\}, S\cap T=\emptyset$.

Proof. These inequalities are valid because all arcs and 2-arcs are in pairwise conflict for $n\geq 5$. We set, w. l. o. g., $i=1, j=2, T=\{n-1,n\}, S=\{3,\dots,n-2\}$. In all tours whose incidence vectors fulfill (12) with equality either nodes 1, 2 lie next to each other or there is one node between them and this node belongs to S or one of the nodes 1,2 lies between the nodes $n-1, n$. For $n = 7, 8$ we verified the statement with a computer algebra system and for $n \geq 9$ the proof is similar to the proof of Theorem 2.2 and Theorem 3.4. In step one and two the tours are constructed as in Theorem 3.4 with $\bar{n} = 7$. These tours are roots of (12) because all nodes $\bar{n}+1 \leq k \leq n-2$ belong to set S and we have $i = 1, j = 2$ as before. It remains to adapt the steps for the tours of $C_{dim}^{\bar{n},3}$. The desired tour structure can easily be achieved in steps (**L1**)-(**L3**) but the further steps have to be adapted.

($\boldsymbol{ST_2.4}$) $\ldots 3\,(n-1)\,4\,\underline{a\,n\,b}\ldots$, for $a, b \in \{1, 2, 5, 6, \ldots, n-2\}, a \neq b, \{a, b\} \neq \{1, 2\}$ (the 2-arc $(3, n-1, 4)$ is not used as an a_L^i – here the role of nodes 1,2 and 3,4 are changed),

($\boldsymbol{ST_2.5}$) $\ldots \underline{a\,(n-1)\,b}\,m\,n\,o\ldots$, for $a, b \in \{1, \ldots, n-2\}, (a, b) \notin \{(1, 2), (2, 1), (3, 4)\}$, with $m, o \in \{1, 2, 5, 6, \ldots, n-2\}, \{m, o\} \neq \{1, 2\}, |\{a, b, m, o\}| = 4$,

($\boldsymbol{ST_2.6}$) $\ldots m\,(n-1)\,o\,\underline{a\,n\,b}\ldots$, for $a, b, \in \{1, \ldots, n-2\}, \{a, b\} \cap \{3, 4\} \neq \emptyset$, with $m, o \in \{1, \ldots, n-2\}, \{m, o\} \neq \{1, 2\}, |\{a, b, m, o\}| = 4$,

($\boldsymbol{ST_2.7}$) $\begin{cases} \ldots \underline{(n-1)\,a\,n}\,1\,2\ldots, & \text{for } a \in \{3, \ldots, n-2\}, \\ \ldots 3\,\underline{(n-1)\,1\,n}\,4\ldots, & \ldots 3\,\underline{(n-1)\,2\,n}\,4\ldots, \end{cases}$

($\boldsymbol{ST_2.8}$) $\begin{cases} \ldots \underline{n\,a\,(n-1)}\,1\,2\ldots, & \text{for } a \in \{3, \ldots, n-2\}, \\ \ldots 3\,\underline{n\,1\,(n-1)}\,4\ldots, & \ldots 3\,\underline{n\,2\,(n-1)}\,4\ldots, \end{cases}$

($\boldsymbol{ST_2.9}$) $\begin{cases} \ldots (n-1)\,\underline{1\,n\,2}\ldots, & \ldots (n-1)\,\underline{2\,n\,1}\ldots, \\ \ldots n\,\underline{1\,(n-1)\,2}\ldots, & \ldots n\,\underline{2\,(n-1)\,1}\ldots. \end{cases}$

One can easily check that the constructed tours can be completed in such a way that they are indeed roots of (12), partially by increasing the distance between $(n-1)$ and n, *e. g.*, in ($\boldsymbol{ST_2.4}$) with $a = 1, b = 5$ we use $\ldots 3\,(n-1)\,4\,2\,1\,n\,5\ldots$. With similar arguments as in the proof of Theorem 2.2 we get that all underlined 2-arcs fulfill (8). Actually, only the roles of nodes 1,2 and 3,4 are changed in step (**L4**) resp. ($\boldsymbol{ST_2.4}$) and the use of some specific 2-arcs as a_L^i is postponed to step ($\boldsymbol{ST_2.9}$). Hence we created exactly one tour less in step one and the same number of tours in steps two and three in comparison to the proof of Theorem 2.2 and so the inequality is facet defining for $P_{\mathbf{AQTSP}_n}, n \geq 7$. □

Until now we have considered the two cases $|T| = 0$ and $|T| = 2$ with small sets T. For small sets S, *i. e.* $|S| = 1, 2$, the inequality visualized in Figure 3 can be strengthened. Considering the case $|S| = 1$ this leads to inequalities

$$x_{ij} + x_{ji} + y_{isj} + \sum_{kil \in V^{(3)}:\, k,l \in T} y_{kil} + \sum_{sik \in V^{(3)}:\, k \in T} y_{sik} \leq 1 \tag{13}$$

for all $i, j \in V, i \neq j$, and all $s \in V \setminus \{i, j\}, S = \{s\}, T = V \setminus \{i, j, s\}$ with $n \geq 5$, but these are equivalent to the facets (9), because

$$\begin{aligned} & x_{ij} + x_{ji} + y_{isj} + \underbrace{\sum_{kil \in V^{(3)}:\, k,l \in T} y_{kil} + \sum_{sik \in V^{(3)}:\, k \in T} y_{sik}}_{=1-\sum_{kis \in V^{(3)}:\, k \in T} y_{kis} - \sum_{jil \in V^{(3)}} y_{jil} - \sum_{lij \in V^{(3)}} y_{lij} \text{ by (1),(2)}} \leq 1 \\ \Leftrightarrow\ & x_{ij} + x_{ji} + y_{isj} - \underbrace{\sum_{kis \in V^{(3)}:\, k \in T} y_{kis}}_{x_{is} - y_{jis} \text{ by (2)}} - \underbrace{\sum_{jil \in V^{(3)}} y_{jil}}_{=x_{ji} \text{ by (2)}} - \underbrace{\sum_{lij \in V^{(3)}} y_{lij}}_{=x_{ij} \text{ by (2)}} \leq 0 \\ \Leftrightarrow\ & y_{isj} + y_{jis} \leq x_{is}. \end{aligned}$$

The case $|S| = 2$ is considered next.

Theorem 3.6 *For $n \geq 5$ the inequalities*

$$x_{ij} + x_{ji} + y_{is_1j} + y_{js_2i} + y_{s_1is_2} + \sum_{kil \in V^{(3)}:\, k,l \in T} y_{kil} + \sum_{s_1ik, kis_2 \in V^{(3)}:\, k \in T} (y_{s_1ik} + y_{kis_2}) \leq 1 \tag{14}$$

define facets of $P_{\mathbf{AQTSP}_n}$ for all $i, j \in V, i \neq j$, and all $S, T \subset V \setminus \{i, j\}, S = \{s_1, s_2\}, s_1 \neq s_2, T = V \setminus \{i, j, s_1, s_2\}$.

Proof. Validity holds for $n \geq 5$ because then the corresponding arcs and 2-arcs are in pairwise conflict. We set, w.l.o.g., $i = n-1, j = n, s_1 = n-3, s_2 = n-2$. To get tours whose incidence vectors fulfill (14) with equality, we have to ensure that exactly one of the arcs or 2-arcs is contained in each tour. For $5 \leq n \leq 7$ we checked the statement with a computer algebra package. For $n \geq 8$ we slightly adapt the proof of Theorem 2.2. In all tours $t \in C_{dim}^{\bar{n},1} \cup C_{dim}^{\bar{n},2}$ and all tours built in steps (**L1**)-(**L3**) with $\bar{n} = 6$ the nodes $n-1, n$ are adjacent and in (**L4**) node $n-1$ lies between two nodes belonging to T. So (14) automatically holds with equality. To achieve the root property of the tours for (**L5**)–(**L8**) slight adaptations are needed and (**L5**) is split into the three steps ($\boldsymbol{S_2T}$.**5**), ($\boldsymbol{S_2T}$.**7**) and ($\boldsymbol{S_2T}$.**10**).

($\boldsymbol{S_2T}$.**5**) $\ldots \underline{a\,(n-1)\,b}\, m\, n\, o \ldots$, for $a, b \in \{1, \ldots, n-2\}$, $(a, b) \notin (\{(1, 2), (s_2, s_1)\} \cup \{(t, s_1) : t \in T\} \cup \{(s_2, t) : t \in T\})$, with $m, o \in \{3, \ldots, n-2\}, |\{a, b, m, o\}| = 4$,

($\boldsymbol{S_2T}$.**6**) $\ldots m\,(n-1)\,o\,\underline{a\,n\,b} \ldots$, for $a, b \in \{1, \ldots, n-2\}, \{a, b\} \cap \{1, 2\} \neq \emptyset$, with $m, o \in T, |\{a, b, m, o\}| = 4$,

($\boldsymbol{S_2T}$.**7**) $\ldots \underline{a\,(n-1)\,s_1}\, n \ldots$, for $a \in T \cup \{s_2\}$
(the 2-arc $(n-1, s_1, n)$ is not used as an a_k^i),

($\boldsymbol{S_2T}$.**8**) $\begin{cases} \ldots m\,\underline{(n-1)\,a\,n} \ldots, & \text{for } a \in T \text{ with } m \in T, m \neq a, \\ \ldots s_1\,\underline{(n-1)\,s_2\,n} \ldots, & \end{cases}$

($\boldsymbol{S_2T}$.**9**) $\begin{cases} \ldots \underline{n\,a\,(n-1)}\, m \ldots, & \text{for } a \in T \text{ with } m \in T, m \neq a, \\ \ldots \underline{n\,s_1\,(n-1)}\, 1 \ldots, & \\ \ldots \underline{n\,s_2\,(n-1)}\, s_1 \ldots, & \end{cases}$

($\boldsymbol{S_2T}$.**10**) $\ldots n\,\underline{s_2\,(n-1)\,a} \ldots$, for $a \in T$.

This construction works out for $\bar{n} = 6, n \geq 8$ and the tours form roots of (14). The proof that the underlined 2-arcs have not been used in previous tours is analogous to the proof of Claim 2 in the proof of Theorem 2.2. In comparison to Claim 3 we only lost 2-arc $(n-1, s_1, n)$ for forming the desired matrix structure and thus the inequalities define facets of $P_{\mathbf{AQTSP}_n}, n \geq 5$. □

Note, Figure 1 also displays an example of a fractional solution that violates inequality (14) for $n = 5$ and $i = 4, j = 5, s_1 = 2, s_2 = 3$ because $x_{45} = y_{425} = y_{534} = \frac{1}{2}$.

Until now we have limited either the cardinality of S or T. Now, we consider the more general case with $|S| \geq 3, |T| \geq 3$.

Theorem 3.7 *For $n \geq 8$ the inequalities*

$$x_{ij} + x_{ji} + \sum_{isj \in V^{(3)}:\, s \in S} y_{isj} + \sum_{jsi \in V^{(3)}:\, s \in S} y_{jsi} + \sum_{kil \in V^{(3)}:\, k,l \in T} y_{kil} \leq 1 \tag{15}$$

define facets of $P_{\mathbf{AQTSP}_n}$ for all $i, j \in V, i \neq j$, and all $S \dot{\cup} T = V \setminus \{i, j\}, S \cap T = \emptyset, |S| \geq 3, |T| \geq 3$.

Proof. Validity of (15) follows because all arcs and 2-arcs are in pairwise conflict. We set, w. l. o. g., $T = \{t_1 = 1, t_2 = 2, \dots, t_{|T|} = |T|\}, S = \{s_1 = |T| + 1, \dots, s_{|S|} = n - 2\}, i = n - 1, j = n$. The proof uses the notation and the ideas of the proof of Theorem 2.2. Therefore we only mention the differences. Steps one and two of the construction process can be performed without any changes setting $\bar{n} = 6$ because the nodes $n-1, n$ are adjacent and thus the corresponding tours define roots of (15). The same is true for the tours built in (**L1**)–(**L3**), and because $|T| \geq 3$ node $n - 1$ lies between the two nodes $1, 2 \in T$ in (**L4**). It remains to adapt steps (**L5**)–(**L8**) to achieve the desired root property of the tours. To emphasize the correspondence between (**L5**)–(**L8**) and the steps here these are denoted by the original step numbers and an additional counter each.

(ST.5a) $\dots \underline{a\,(n-1)\,s_1}\,n\,m \dots$, for $a \in \{1, \dots, n-2\}$ with $m \in \{3, \dots, n-2\}$, $|\{a, s_1, m\}| = 3$
(the 2-arc $(n-1, s_1, n)$ is not used as an a^i_L),

(ST.8a) $\dots 3\,\underline{n\,a\,(n-1)}\,s_1 \dots$, for $a \in S \setminus \{s_1\}$,

(ST.5b) $\dots m\,n\,\underline{s_2\,(n-1)\,a} \dots$, for $a \in \{1, \dots, n-2\} \setminus \{s_1, s_2\}$ with $m \in \{3, 4\}, m \neq a$,

(ST.7a) $\dots s_2\,\underline{(n-1)\,a\,n}\,3 \dots$, for $a \in S \setminus \{s_1, s_2\}$,

(ST.5c) $\dots \underline{a\,(n-1)\,b}\,n\,3$, for $a, b \in S \setminus \{s_2\}, b \neq s_1$,

(ST.8b) $\dots 3\,\underline{n\,s_1\,(n-1)}\,s_3 \dots$,

(ST.5d) $\dots 3\,n\,\underline{a\,(n-1)\,s_2} \dots$, for $a \in S \setminus \{s_2\}$,

(ST.7b) $\dots s_1\,\underline{(n-1)\,s_2\,n}\,3 \dots$,

(ST.6a) $\begin{cases} \dots m\,(n-1)\,\underline{a\,n\,b} \dots, & \text{for } a \in S, b \in \{1,2\} \text{ with } m \in S, m \neq a, \\ \dots \underline{a\,n\,b}\,(n-1)\,m \dots, & \text{for } a \in \{1,2\}, b \in S \text{ with } m \in S, m \neq b, \end{cases}$

(ST.5e) $\begin{cases} \dots \underline{a\,(n-1)\,b}\,n \dots, & \text{for } a \in T, b \in S \setminus \{s_1\}, \\ \dots n\,\underline{a\,(n-1)\,b} \dots, & \text{for } a \in S \setminus \{s_2\}, b \in T, \\ \dots \underline{a\,(n-1)\,b}\,s_1\,n \dots, & \text{for } a, b \in T, a \neq b, \end{cases}$

(ST.8c) $\dots s_1\,\underline{n\,a\,(n-1)}\,m \dots$, for $a \in T$ with $m \in T, m \neq a$,

(ST.7c) $\dots m\,\underline{(n-1)\,a\,n}\,s_1 \dots$, for $a \in T$ with $m \in T, m \neq a$,

(ST.6b) $\dots \underline{a\,n\,b}\,(n-1)\,m \dots$, for $a, b \in T, \{a, b\} \cap \{1, 2\} \neq \emptyset$, with $m \in T, |\{a, b, m\}| = 3$.

For $\bar{n} = 6, n \geq 8$ the construction works out and the generated tours define roots of (15) because in **(ST.5a)**–**(ST.6a)** and in the first two lines of **(ST.5e)**, there is exactly one node $s \in S$ between nodes $n-1, n$ and in last line of **(ST.5e)** as well as in **(ST.8c)**–**(ST.6b)** node $n-1$ is positioned between two nodes belonging to T. One can check straightforward that only the 2-arc $(n-1, s_1, n)$ is not used as an a^i_L in comparison to the proof of Theorem 2.2 and that the 2-arcs underlined in the steps above fulfill (8). With the considerations in the proof of Theorem 2.2 Theorem 3.7 follows. $\square$

The number of inequalities of type (15) is exponential, nonetheless the corresponding separation problem can be solved in polynomial time. The proof of this result is similar to the proof for the inequality-counterpart for the SQTSP [10]. To keep the presentation self-contained and to allow the comparison to the inequalities presented next we present the proof anyhow.

Theorem 3.8 *The separation problem for the conflicting arcs inequalities of type* (15) *can be solved in polynomial time.*

Proof. Given a fractional solution $(\bar{x}, \bar{y})$ of $\mathbf{AQTSP}_n$ and fixing $i, j \in V, i \neq j$, we want to find sets $S, T \subset V \setminus \{i, j\}$ that maximize the sum $(\sum_{isj \in V^{(3)}, s \in S} \bar{y}_{isj} + \sum_{jsi \in V^{(3)}, s \in S} \bar{y}_{jsi} + \sum_{kil \in V^{(3)}, k,l \in T} \bar{y}_{kil})$. For this we construct a node-weighted undirected bipartite graph $\tilde{G} = (\tilde{V}, \tilde{E}, w)$ with node set $\tilde{V} = \tilde{V}_1 \cup \tilde{V}_2, \tilde{V}_1 = V \setminus \{i, j\}, \tilde{V}_2 = \{\{k, l\} \colon k, l \in V \setminus \{i, j\}, k \neq l\}$ and set of edges $\tilde{E} = \{\{m, \{k, l\}\} \colon m \in \{k, l\} \in \tilde{V}_2\}$. The weights of nodes $v \in \tilde{V}_1$ are fixed to $w_v = \bar{y}_{ivj} + \bar{y}_{jvi}$ and for nodes $\{k, l\} \in \tilde{V}_2$ to $w_{\{k,l\}} = \bar{y}_{kil} + \bar{y}_{lik}$. With this construction the separation problem reduces to the polynomial-time solvable problem of finding a maximum weight independent set in a bipartite graph, see, *e. g.*, [9], because choosing node $v \in \tilde{V}_1$ in the solution of this corresponds to the assignment of v to S and selecting node $\{k, l\} \in \tilde{V}_2$ to the assignment of k, l to T. $\square$

The idea of summing up arcs and 2-arcs that are in pairwise conflict and to restrict the number of them contained in a tour to one can be found in the following theorem, too.

Theorem 3.9 *For $n \geq 6$ the inequalities*

$$x_{ij} + x_{ji} + \sum_{ikj \in V^{(3)} \colon k \in S} y_{ikj} + \sum_{jki \in V^{(3)} \colon k \in T} y_{jki} + \sum_{kil \in V^{(3)} \colon k \in S, l \in T} y_{kil} \leq 1 \tag{16}$$

define facets of $P_{\mathbf{AQTSP}_n}$ for all $i, j \in V, i \neq j$, and $S \dot{\cup} T = V \setminus \{i, j\}, S \cap T = \emptyset, |S| \geq 2, |T| \geq 2$.

Proof. Validity of (16) follows because all arcs and 2-arcs are in pairwise conflict. For $n = 6, 7$ we checked the statement with a computer algebra system. So we assume $n \geq 8$. We set, w. l. o. g., $S = \{s_1 = 1, \ldots, s_{|S|} = |S|\}, T = \{t_1 = |S| + 1, t_{|T|} = n - 2\}, i = n - 1, j = n$. The proof is similar to the proof of Theorem 2.2 and uses the notation therein. Steps one and two of the construction and (**L1**)–(**L3**) of step three can be performed analogously because the nodes $n - 1$ and n are adjacent and therefore the tours define roots of (16). It remains to adapt steps (**L4**)–(**L8**). To emphasize the correspondence of (**L4**)–(**L8**) and the steps here these are denoted by the original step numbers and an additional counter each.

($\overline{\mathsf{ST}}$.4) $\ldots s_1\,(n-1)\,t_1\,\underline{a\,n\,b} \ldots$, for $a, b \in \{1, \ldots, n-2\} \setminus \{s_1, t_1\}, a \neq b$
(the 2-arc $(s_1, n-1, t_1)$ is not used as an a^i_L, in (**L4**) we do not use $(1, n-1, 2)$ instead),

($\overline{\mathsf{ST}}$.5a) $\ldots \underline{a\,(n-1)\,b}\,m\,n\,o \ldots$, for $a \in S, b \in T, (a, b) \neq (s_1, t_1)$, with $m, o \in \{1, \ldots, n-2\} \setminus \{s_1, t_1\}, |\{a, b, m, o\}| = 4$,

($\overline{\mathsf{ST}}$.6) $\ldots m\,(n-1)\,o\,\underline{a\,n\,b} \ldots$, for $a, b \in \{1, \ldots, n-2\}, \{a, b\} \cap \{s_1, t_1\} \neq \emptyset$, with $m \in S, o \in T, |\{a, b, m, o\}| = 4$
(in this step it is important that both sets S, T contain at least two nodes),

($\overline{\mathsf{ST}}$.7a) $\ldots m\,\underline{(n-1)\,a\,n} \ldots$, for $a \in T$ with $m \in S$,

($\overline{\mathsf{ST}}$.8a) $\ldots \underline{n\,a\,(n-1)}\,m \ldots$, for $a \in S$ with $m \in T$,

($\overline{\mathsf{ST}}$.5b) $\ldots n\,\underline{t_1\,(n-1)\,a} \ldots$, for $a \in (S \cup T) \setminus \{t_1\}$
(the 2-arc $(n, t_1, n-1)$ is not used as an a^i_L),

($\overline{\mathsf{ST}}$.7b) $\ldots t_1\underline{(n-1)\,a\,n} \ldots$, for $a \in S$,

($\overline{\mathsf{ST}}$.5c) $\ldots \underline{a\,(n-1)\,b}\,n \ldots$, for $a \in (S \cup T) \setminus \{t_1\}, b \in S, a \neq b$,

($\overline{\mathsf{ST}}$.8b) $\ldots \underline{n\,a\,(n-1)}\,m \ldots$, for $a \in T \setminus \{t_1\}$ with $m \in S$,

($\overline{\mathsf{ST}}$.5d) $\ldots n\,\underline{a\,(n-1)\,b} \ldots$, for $a, b \in T, a \neq t_1, a \neq b$.

Obviously all tours define roots of (16) and the underlined 2-arcs fulfill (8). In comparison to the proof of Theorem 2.2 only 2-arc $(n, t_1, n-1)$ is not used as an a_L^i and so exactly $f(n)$ tours are constructed. This proves Theorem 3.9. □

Next we will see that under certain conditions the separation problem for (16) is **NP**-complete.

Theorem 3.10 *The separation problem for* (16) *is* **NP***-complete for fixed nodes $i, j \in V, i \neq j$, even if the x-variables correspond to a convex combination of tours, equalities* (2) *are fulfilled and it holds $y_{ijk} \in [0,1]$ for $ijk \in V^{(3)}$.*

Proof. We prove this by a reduction from MAX-CUT (ND16 in [13]). Given an undirected graph $\tilde{G} = (\tilde{V}, \tilde{E})$ and $\delta \in \mathbb{N}_0$ we ask if there is a partition of node set $\tilde{V}$ into $\tilde{S}, \tilde{V} \setminus \tilde{S}$ such that $|\{\{v, w\} \in \tilde{E} \colon v \in \tilde{S}, w \in \tilde{V} \setminus \tilde{S}\}| \geq \delta$.

The idea is to construct a directed 2-graph $G' = (V', A')$ whose node set contains the nodes $\tilde{V}$, the special nodes i, j and further nodes. The x- and y-variables that correspond to A' are constructed in such way that (16) is violated if and only if $\tilde{G}$ contains a cut with value at least δ. In the case that (16) is violated, *i. e.*, there exist sets $S, T \subset V \setminus \{i, j\}, S \cap T = \emptyset, V' = \{i, j\} \cup S \cup T$ such that the left-hand side of (16) sums up to more than one for these sets, the partition of $\tilde{V}$ in $\tilde{S}, \tilde{V} \setminus \tilde{S}$ can be read off the sets S, T via $\tilde{S} = S \cap \tilde{V}$.

We construct a directed 2-graph $G' = (V', A')$ with node set $V' = \tilde{V} \dot\cup \{i, j\} \dot\cup \dot\bigcup_{e \in \tilde{E}} \{u_e\} \dot\cup \{v_1^\eta, v_2^\eta, u_{\{v_1^\eta, v_2^\eta\}}\}$ that contains the nodes $\tilde{V}$ of the max-cut problem and the special nodes i, j, and for each arc $e \in \tilde{E}$ a node u_e that will be used in the construction presented below. The construction steps are performed twice for each arc in $\tilde{A}$ and once for the extra nodes $\{v_1^\eta, v_2^\eta, u_{\{v_1^\eta, v_2^\eta\}}\}$. These are introduced to increase the left-hand side of (16) in such a way that the inequality is violated if and only if $\tilde{G}$ contains a cut $\geq \delta$. For the 2-arcs $a \in A'$ and the arcs $b \in V'^{(2)}$ we algorithmically specify weights w_a^y and w_b^x that correspond to the values of the y-variables resp. of the x-variables of $P_{\mathbf{AQTSP}_n}, n = |V'|$. At the beginning we set $w_a^y = 0$ for all $a \in A'$ and $w_b^x = 0$ for all $b \in V'^{(2)}$.

First we insert directed 2-cycles that enlarge the weight w_a^y of each of the corresponding 2-arcs $a \in A'$ on the 2-cycles by a given value and later we construct the values $w_b^x, b \in V'^{(2)}$ according to the requirements of Theorem 3.10. Set $M := 4|\tilde{E}| - 2\delta + 1$. For each arc $e = (e_1, e_2) \in \{(v, w), (w, v) \colon \{v, w\} \in \tilde{E}\} \cup \{(v_1^\eta, v_2^\eta)\}$ we use the following 2-cycles that are visualized on the left side of Figure 4, where the expression γ_e with $\gamma_e = (\gamma_e^1, \ldots, \gamma_e^{|V'|-5})$ denotes an arbitrary but fixed order of all nodes $V' \setminus \{i, j, e_1, e_2, u_{\{e_1, e_2\}}\}$, *i. e.*, $V' = \{i, j, e_1, e_2, u_{\{e_1, e_2\}}\} \cup \bigcup_{i=1}^{|V'|-5} \{\gamma_e^i\}$.

(Y1) $C_e^1 = \{iju_{\{e_1,e_2\}}, ju_{\{e_1,e_2\}}i, u_{\{e_1,e_2\}}ij\}, C_e^2 = \{u_{\{e_1,e_2\}}ji, jiu_{\{e_1,e_2\}}, iu_{\{e_1,e_2\}}j\}$, the weights of all corresponding 2-arcs are enlarged by $\frac{1}{4M}$ for each 2-cycle if $e \neq (v_1^\eta, v_2^\eta)$ and by $\frac{2|\tilde{E}|-2\delta+1}{4M}$ if $e = (v_1^\eta, v_2^\eta)$,

(Y2) $C_e^3 = \{ie_2\gamma_e^1, e_2\gamma_e^1\gamma_e^2, \ldots, \gamma_e^{|V'|-5}e_1i, e_1ie_2\}, C_e^4 = \{j\gamma_e^1\gamma_e^2, \ldots, \gamma_e^{|V'|-5}e_1u_{\{e_1,e_2\}}, e_1u_{\{e_1,e_2\}}e_2, u_{\{e_1,e_2\}}e_2j, e_2j\gamma_e^1\}$, the weights of all corresponding 2-arcs are enlarged by $\frac{1}{2M}$ for each 2-cycle if $e \neq (v_1^\eta, v_2^\eta)$ and by $\frac{2|\tilde{E}|-2\delta+1}{2M}$ if $e = (v_1^\eta, v_2^\eta)$.

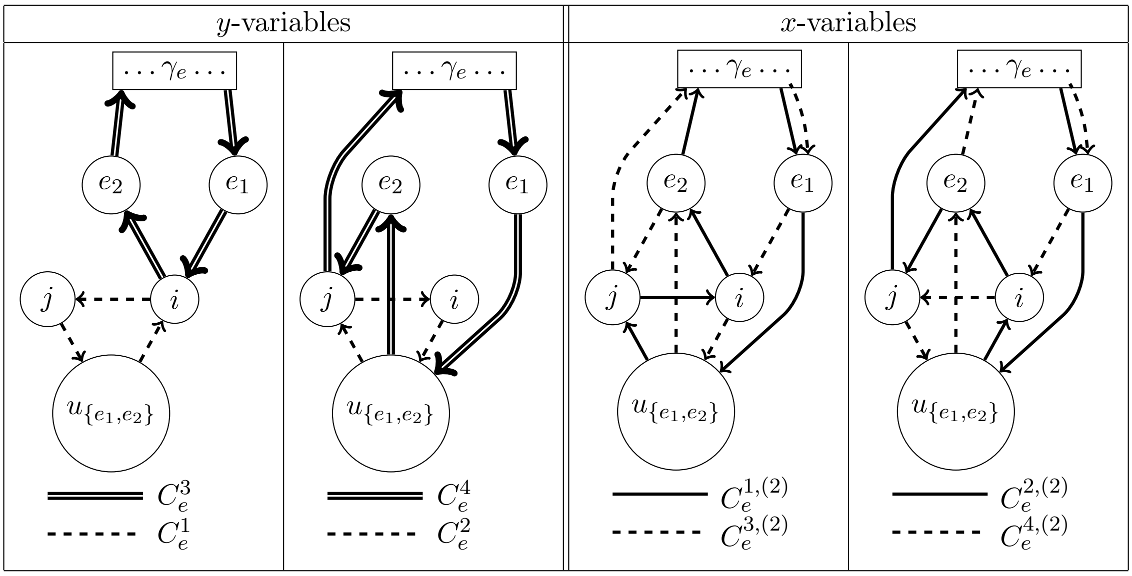


Figure 4: Visualization of the constructions for the weights corresponding to the y- and the x-variables used in the proof of Theorem 3.10 for an arc $e = (e_1, e_2)$.

After (Y1) and (Y2) the weights $w_a^y, a \in A'$, trivially fulfill $w_a^y \geq 0$ and

$$\sum_{k \in V' \colon ijk \in V'^{(3)}} w_{ijk}^y = \sum_{k \in V' \colon kij \in V'^{(3)}} w_{kij}^y \text{ for all } ij \in V^{(2)}$$

because we included only 2-cycles with weights greater than zero. In order to set up weights w_b^x satisfying (1) and (2) in the following we next assert that the conditions $w_a^y \leq 1$ for $a \in A'$ and $\sum_{i,k \in V' \colon ijk \in V'^{(3)}} w_{ijk}^y = 1$ for all $j \in V'$ hold.

During steps (Y1) and (Y2) for each arc $e = (e_1, e_2) \in \{(v,w), (w,v) \colon \{v,w\} \in \tilde{E}\} \cup \{(v_1^\eta, v_2^\eta)\}$ the value $\sum_{i,k \in V' \colon ijk \in V'^{(3)}} w_{ijk}^y$ is raised for all $j \in V'$ by $\frac{1}{M}$ for $e \neq (v_1^\eta, v_2^\eta)$ and by $\frac{2|\tilde{E}|-2\delta+1}{M}$ for $e = (v_1^\eta, v_2^\eta)$. For example for $e = (e_1, e_2) \in \{(v,w), (w,v) \colon \{v,w\} \in \tilde{E}\}$ we get $\frac{1}{2M}$ by C_e^3 and $\frac{1}{2M}$ by C_e^4 for node e_1 that sums up to $\frac{1}{M}$. Altogether the value $\sum_{i,k \in V' \colon ijk \in V'^{(3)}} w_{ijk}^y$ sums up to $2|\tilde{E}|\frac{1}{M} + \frac{2|\tilde{E}|-2\delta+1}{M} = 1$ for all $j \in V'$.

For the x-variables the weights $w_b^x, b \in V'^{(2)}$, are raised for each arc $e = (e_1, e_2) \in \{(v,w), (w,v) \colon \{v,w\} \in \tilde{E}\} \cup \{(v_1^\eta, v_2^\eta)\}$ by the following tours.

(X1) $C_e^{1,(2)} = \{ie_2, e_2\gamma_e^1, \gamma_e^1\gamma_e^2, \ldots, \gamma_e^{|V'|-5}e_1, e_1u_{\{e_1,e_2\}}, u_{\{e_1,e_2\}}j, ji\}$,
$C_e^{2,(2)} = \{ie_2, e_2j, j\gamma_e^1, \gamma_e^1\gamma_e^2, \ldots, \gamma_e^{|V'|-5}e_1, e_1u_{\{e_1,e_2\}}, u_{\{e_1,e_2\}}i\}$,
$C_e^{3,(2)} = \{iu_{\{e_1,e_2\}}, u_{\{e_1,e_2\}}e_2, e_2j, j\gamma_e^1, \gamma_e^1\gamma_e^2, \ldots, \gamma_e^{|V'|-5}e_1, e_1i\}$,
$C_e^{4,(2)} = \{ij, ju_{\{e_1,e_2\}}, u_{\{e_1,e_2\}}e_2, e_2\gamma_e^1, \gamma_e^1\gamma_e^2, \ldots, \gamma_e^{|V'|-5}e_1, e_1i\}$,
each of these cycles enlarges the weight of the corresponding arcs by $\frac{1}{4M}$ if $e \neq (v_1^\eta, v_2^\eta)$ and by $\frac{2|\tilde{E}|-2\delta+1}{4M}$ if $e = (v_1^\eta, v_2^\eta)$.

One can easily check in Figure 4 (2-cycles with double weight are drawn with a double line) that (2) is fulfilled for each arc $e = (e_1, e_2) \in \{(v,w), (w,v) \colon \{v,w\} \in \tilde{E}\} \cup \{(v_1^\eta, v_2^\eta)\}$ considering C_e^1–C_e^4 and $C_e^{1,(2)}$–$C_e^{4,(2)}$. Hence (2) holds for the whole construction and the x-variables correspond to a convex combination of tours.

It remains to show that the result of the separation problem for inequalities (16) for fixed $i, j \in V', i \neq j$, leads to a cut in $\tilde{G}$ of value greater than or equal to δ if the inequality is violated and that we can find a violated inequality of type (16) if there exists a cut of value greater than or equal to δ in $\tilde{G}$. Assume that for fixed $i, j \in V', i \neq j$, the separation oracle returns *violated*. With the construction above each $e = \{e_1, e_2\} \in \tilde{E}$ contributes at least a value of $\frac{3}{2M}$ to the left-hand side of (16), $\frac{1}{M}$ via arcs $(i,j), (j,i)$ and $\frac{1}{2M}$ via 2-arcs $(i, u_{\{e_1,e_2\}}, j)$ or $(j, u_{\{e_1,e_2\}}, i)$ (these costs are caused in both cases $u_{\{e_1,e_2\}} \in S$ or $u_{\{e_1,e_2\}} \in T$). It follows by the same arguments that for $e = (v_1^\eta, v_2^\eta)$ the left-hand side of (16) is enlarged by at least $\frac{3(2|\tilde{E}|-2\delta+1)}{4M}$. Furthermore the value of the left-hand side of (16) depends on the assignment of the nodes to S. For an arc $e = (e_1, e_2) \in \{(v,w), (w,v) \colon \{v,w\} \in \tilde{E}\} \cup \{(v_1^\eta, v_2^\eta)\}$ with $e_1 \in S, e_2 \notin T$ the 2-arc (e_1, i, e_2) enlarges the value of the left-hand side of (16) by $\frac{1}{2M}$ for $e \neq (v_1^\eta, v_2^\eta)$ and by $\frac{2|\tilde{E}|-2\delta+1}{2M}$ for $e = (v_1^\eta, v_2^\eta)$. If $e_1 \in T$ or $e_2 \in S$ 2-arc (e_1, i, e_2) does not contribute to the left-hand side of (16). Assume $v_1^\eta \in T$ or $v_2^\eta \in S$ then the left-hand side of (16) is less than one independent of the assignment of all other nodes to S, T because

$$\underbrace{2|\tilde{E}| \cdot \frac{3}{4M} + \frac{3(2|\tilde{E}|-2\delta+1)}{4M}}_{\text{fix costs}} + |\tilde{E}|\frac{1}{2M} = \frac{7|\tilde{E}|-3\delta+1.5}{2M} < 1.$$

So we know $v_1^\eta \in S, v_2^\eta \in T$ if (16) is violated.

With these observation we get that if the separation oracle for (16) returns *violated* $\tilde{G}$ contains a cut with value greater than or equal to δ because we get maximally

$$\underbrace{\frac{3|\tilde{E}|}{2M}}_{\text{fix costs for } |\tilde{E}|} + \underbrace{\frac{5(2|\tilde{E}|-2\delta+1)}{4M}}_{\text{fix costs for } (v_1^\eta, v_2^\eta)} = \frac{8|\tilde{E}|-5\delta+2.5}{2M} = \frac{2M-\delta+0.5}{2M}$$

via fix costs and adding at least $\delta \cdot \frac{1}{2M}$ to these fix costs enlarges the left-hand side to a value greater one. So at least δ of the y-variables of 2-arcs $(e_1, i, e_2), (e_1, e_2) \in \{(v,w), (w,v) \colon \{v,w\} \in \tilde{E}\}$ have to be counted for (16). Setting $\tilde{S} = (\tilde{V} \cap S), V \setminus \tilde{S} = (\tilde{V} \cap T)$ leads to a cut of value at least δ.

If otherwise the set $\tilde{S}$ with cut value $\geq \delta$ is given inequality (16) is violated setting $S = (\tilde{S} \cup \{v_1^\eta\}), T = (\tilde{V} \setminus \tilde{S}) \cup \{v_2^\eta\} \cup \bigcup_{e \in \tilde{E}} \{u_e\} \cup \{u_{\{v_1^\eta, v_2^\eta\}}\}$ because

$$\frac{3|\tilde{E}|}{2M} + \frac{5(2|\tilde{E}|-2\delta+1)}{4M} + \frac{\delta}{2M} = \frac{8|\tilde{E}|-4\delta+2.5}{2M} > 1. \qquad \square$$

Remark 3.11 The construction used in the proof of Theorem 3.10 resp. the corresponding x- and y-variables violate (9), *i. e.*, it holds $y_{pqr} + y_{rpq} > x_{pq}$ for some $p, q, r \in V', |\{p,q,r\}| = 3$. For example for $p = u_{\{v_1^\eta, v_2^\eta\}}, q = i, r = j$ we have $x_{u_{\{v_1^\eta, v_2^\eta\}} i} = y_{u_{\{v_1^\eta, v_2^\eta\}} ij} = y_{j u_{\{v_1^\eta, v_2^\eta\}} i} > 0$. Extending the presented construction by convex combining it with additional specially chosen tours on the x- and the y-variables allows to prove Theorem 3.10 with the additional requirement that all inequalities (9) are fulfilled.

3.3 Strengthened valid inequalities of $P_{\mathbf{ATSP}_n}$

In general all inequalities that are valid for $P_{\mathbf{ATSP}_n}$ remain valid for $P_{\mathbf{AQTSP}_n}$ because $P_{\mathbf{ATSP}_n}$ is a projection of $P_{\mathbf{AQTSP}_n}$. But in most cases the inequalities can be strengthened using the general approach presented next. This is motivated by the idea that lead to inequalities (11). Starting with the subtour elimination constraints on two nodes, $x_{ij} + x_{ji} \leq 1, i, j \in V, i \neq j$, we can add all variables corresponding to 2-arcs $(i,k,j), (j,k,i), k \in V \setminus \{i,j\}$, to the left-hand side and for $n \geq 5$ the inequality remains valid because any two of these arcs and 2-arcs would imply a subtour. So a 2-arc, w. l. o. g., (i,k,j) acts as the arc (i,j) itself expressing that the two nodes i,j are closely related. Generalizing this idea leads to the following approach that was introduced in a similar way to strengthen valid inequalities of $P_{\mathbf{STSP}_n}$ for $P_{\mathbf{SQTSP}_n}$ in [10].

Definition 3.12 *For a given set $A' \subseteq V^{(2)}$, a family $\mathcal{F} = \{(F_a^2, F_a^3)\}_{a\in A'}$ of pairs of sets $F_a^2 \subseteq V^{(2)}, F_a^3 \subseteq V^{(3)}$ for $a \in A'$ is* A'-dominated *if for each tour $C \in \mathcal{C}_n$ there is a tour $\bar{C} \in \mathcal{C}_n$ with $\sum_{f\in F_a^2} x_f^C + \sum_{f\in F_a^3} y_f^C \leq x_a^{\bar{C}}$ for all $a \in A'$. It is* improving, *if $a \in F_a^2$ for $a \in A'$ and there is an $a \in A'$ with $F_a^2 \neq \{a\}$ or $F_a^3 \neq \emptyset$.*

Starting with a valid inequality of $P_{\mathbf{ATSP}_n}$ with nonnegative coefficients this can be strengthened for $P_{\mathbf{AQTSP}_n}$ using any improving support-dominated family.

Observation 3.13 *Suppose $\sum_{a\in A'} d_a x_a \leq b$ is a valid inequality for $P_{\mathbf{ATSP}_n}$ with $d_a \geq 0, a \in A'$, and let $\mathcal{F} = \{(F_a^2, F_a^3)\}_{a\in A'}$ be A'-dominated. Then the inequality*

$$\sum_{a\in A'} d_a\Big(\sum_{f\in F_a^2} x_f + \sum_{f\in F_a^3} y_f\Big) \leq b$$

is valid for $P_{\mathbf{AQTSP}_n}$.

Proof. Because $\mathcal{F}$ is A'-dominated there exists by Definition 3.12 for each tour $C \in \mathcal{C}_n$ a tour $\bar{C} \in \mathcal{C}_n$ with

$$\sum_{a\in A'} d_a\Big(\sum_{f\in F_a^2} x_f^C + \sum_{f\in F_a^3} y_f^C\Big) \leq \sum_{a\in A'} d_a x_a^{\bar{C}} \leq b.$$

□

Depending on the number of nodes $|V(A')|$ for $A' \subset V^{(2)}$ there are two different ways to derive an improving support-dominated family.

Observation 3.14 *Given $A' \subset V^{(2)}$, suppose $|V(A')| < \frac{n}{2}$. Then*

$$\mathcal{F} = \Big\{\big(F_{ij}^2 := \{ij\}, F_{ij}^3 := \{ikj \in V^{(3)} \colon ik \notin A', kj \notin A'\}\big)\Big\}_{ij\in A'}$$

is A'-dominated. It is improving whenever $A' \neq \emptyset$.

Proof. If $\mathcal{F}$ is A'-dominated and $A' \neq \emptyset$ then it is improving because any node $k \in V\backslash V(A')$ allows for a 2-arc $ikj \in F_{ij}^3$ for each $ij \in A'$.

So it remains to show that $\mathcal{F}$ is A'-dominated. In the case $A' = \emptyset$ there is nothing to show so we may assume $A' \neq \emptyset$ and thus $n \geq 5$ by $|V(A')| < \frac{n}{2}$. Let $C \in \mathcal{C}_n$ be a tour. We have to show that there exists a tour $\bar{C} \in \mathcal{C}_n$ satisfying the requirements of Definition 3.12. Therefore we define $F_C^2 = A' \cap C^{(2)}, F_C^3 = \{ij \in A' : F_{ij}^3 \cap C \neq \emptyset\}$. By $C \in \mathcal{C}_n$ and $n \geq 5$ it holds $F_C^2 \cap F_C^3 = \emptyset$ and on the one hand for each node $k \in V$ there exists at most one $ij \in F_C^3$ with $ikj \in C$ and on the other hand there exists exactly one $k_{ij} \in V$ for each $ij \in F_C^3$ with $ik_{ij}j \in C$. Furthermore we know by the definition of $\mathcal{F}$ that $ik_{ij}j \in C$ with $ij \in F_C^3$ implies $ik_{ij}, k_{ij}j \notin F_C^2 \subset A'$. Hence $C \in \mathcal{C}_n$ implies that all nodes $i \in V$ of the graph $G_C^{\mathcal{F}} = (V, F_C^2 \cup F_C^3)$ fulfill $|\{(i,j) \colon j \in V, i \neq j\}| \leq 1, |\{(j,i) \colon j \in V, i \neq j\}| \leq 1$. Furthermore $G_C^{\mathcal{F}}$ cannot contain a cycle $\tilde{C}$ because this would imply a cycle in C of length less than n. So all components of $G_C^{\mathcal{F}}$ are isolated nodes or paths and we may complete $F_C^2 \cup F_C^3$ to $\bar{C}$ with $F_C^2 \cup F_C^3 \subset \bar{C}^{(2)}$ by adding appropriate arcs. This $\bar{C}$ fulfills the requirements of Definition 3.12 because if either $ij \in C$ or $ik_{ij}j \in C$ for a unique k_{ij} $(n \geq 5)$ we have $ij \in F_C^2 \cup F_C^3 \subset \bar{C}^{(2)}$ and so $1 = x_{ij}^{\bar{C}} = \sum_{f\in F_{ij}^2} x_f^C + \sum_{f\in F_{ij}^3} y_f^C$. □

Observation 3.15 *Given $A' \subset V^{(2)}$, suppose $|V(A')| \geq \frac{n}{2}$ with some $\bar{t} \in V \setminus V(A')$. Then*

$$\mathcal{F} = \Big\{(F_{ij}^2 := \{ij\}, F_{ij}^3 := \{ikj \in V^{(3)} \colon k \neq \bar{t}, ik \notin A', kj \notin A'\})\Big\}_{ij \in A'}$$

is A'-dominated. It is improving if and only if $F_{ij}^3 \neq \emptyset$ for some $ij \in A'$. In particular, if $|V(A')| \leq n-2$, then $\mathcal{F}$ is improving.

Proof. First we show that $\mathcal{F}$ is A'-dominated. For $n = 3$ the two possibilities are $A' = \{ij\}$ for an arc $ij \in V^{(2)}$ or $A' = \{ij, ji\}$ with $ij, ji \in V^{(2)}$ but then in both cases $\mathcal{F} = \{(ij, \emptyset)\}_{ij \in A'}$ and each $C \in \mathcal{C}_3$ serves as its own $\bar{C}$. The same is true if $n = 4$ and $|V(A')| = 3$. For $n = 4, |V(A')| = 2$ we know $A' \subset \{ij, ji\}$ for some $i, j \in V, i \neq j$, and for a tour $C \in \mathcal{C}_4$ it holds $|\{ij, ji\} \cap C^{(2)}| + |\{ikj, jki\} \cap C| \leq 1$ by (3). By the choice of F_{ij}^2, F_{ij}^3 the tour $\bar{C}$ can easily be constructed. For $n \geq 5$ the proof is almost identical to the proof of Observation 3.14 and we use the same notation. Given a tour $C \in \mathcal{C}_n$ we may construct the graph $G_{\mathcal{F}}^C = (V, F_2^C \cup F_3^C)$ and prove that all its nodes have indegree and outdegree at most one in exactly the same way. This time, however, $G_{\mathcal{F}}^C$ cannot contain a cycle, because it would induce a subcycle of C that does not visit $\bar{t}$ as $\bar{t} \notin V(F_{ij}^3)$ for $ij \in A'$. From this point on the proof of $\mathcal{F}$ being A'-dominated follows by analogous arguments as for Observation 3.14.

By definition, $\mathcal{F}$ is improving if $F_{ij}^3 \neq \emptyset$ for some $ij \in A'$. In the case $|V(A')| \leq n-2$ we have $\emptyset \neq \{ikj \colon k \in V \setminus (V(A') \cup \{\bar{t}\})\} \subset F_{ij}^3$. $\square$

Next, we show how these two approaches can be applied to the subtour elimination constraints (3) that read $\sum_{ij \in S^{(2)}} x_{ij} \leq |S| - 1$ for all $S \subset V, 2 \leq |S| \leq n-2$. Using Observation 3.14 these can be improved to the *extended subtour elimination constraints*

$$\sum_{ij \in S^{(2)}} x_{ij} + \sum_{ikj \in V^{(3)} \colon i,j \in S, k \in V \setminus S} y_{ikj} \leq |S| - 1, \qquad \text{for } 2 \leq |S| < \tfrac{n}{2},$$

respectively using Observation 3.15 to

$$\sum_{ij \in S^{(2)}} x_{ij} + \sum_{ikj \in V^{(3)} \colon i,j \in S, k \in (V \setminus S) \setminus \{\bar{t}\}} y_{ikj} \leq |S| - 1, \quad \text{for } \tfrac{n}{2} \leq |S| < n-2, \bar{t} \in V \setminus S.$$

With (1), (2) these can be transformed to

$$\sum_{ijk \in V^{(3)} \colon i \in S, j,k \in V \setminus S} y_{ikj} \geq 1, \tag{17}$$

for $2 \leq |S| < \frac{n}{2}$ respectively to

$$\sum_{ijk \in V^{(3)} \colon i \in S, j,k \in V \setminus S} y_{ijk} + \sum_{i\bar{t}j \in V^{(3)} \colon i,j \in S} y_{i\bar{t}j} \geq 1, \tag{18}$$

for $\frac{n}{2} \leq |S| < n-2, \bar{t} \in V \setminus S$. With Observation 3.13 we get that the inequalities above are valid for $P_{\mathbf{AQTSP}_n}$, their facetness for certain cardinalities of S is shown next.

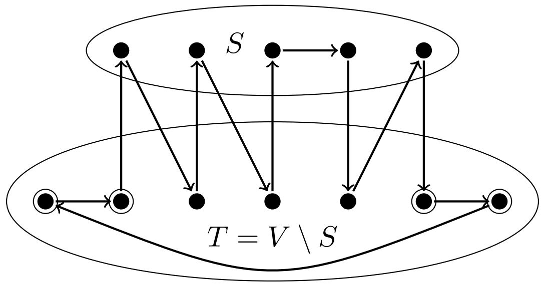


Figure 5: The incidence vector of the shown tour fulfills $\sum_{ijk\in V^{(3)}:\, i\in S, j,k\in V\setminus S} y_{ijk} = 1$. The marked nodes belong to the only block of nodes in $V \setminus S$ with more than one node.

Remark 3.16 1. For $n \geq 7$ inequalities (17) define facets of $P_{\mathbf{AQTSP}_n}$ for all $S \subset V, |S| = 2$, because with $S = \{i,j\}, i \neq j$, they are equivalent to (11),

$$\sum_{ikl\in V^{(3)}:\, j\notin\{k,l\}} y_{ikl} + \sum_{jkl\in V^{(3)}:\, i\notin\{k,l\}} y_{jkl} \geq 1$$

$$\overset{(2)}{\Leftrightarrow} \underbrace{\sum_{ik\in V^{(2)}:\, k\neq j} x_{ik}}_{=1-x_{ij}\text{ (by (1))}} - \sum_{ikj\in V^{(3)}} y_{ikj} + \underbrace{\sum_{jk\in V^{(2)}:\, k\neq i} x_{jk}}_{=1-x_{ji}\text{ (by (1))}} - \sum_{jki\in V^{(3)}} y_{jki} \geq 1$$

$$\Leftrightarrow x_{ij} + x_{ji} + \sum_{ikj\in V^{(3)}} y_{ikj} + \sum_{jki\in V^{(3)}} y_{jki} \leq 1.$$

So the statement follows from Theorem 3.4.

2. If for a tour C and $S \subset V, 2 \leq |S| < \frac{n}{2}$, equality holds in (17), *i. e.*, deleting all nodes S from C decomposes C into single nodes and exactly one directed path of more than one node. For example, such a tour may look like in Figure 5.

Lemma 3.17 *For $n \geq 11$ the inequalities*

$$\sum_{ijk\in V^{(3)}:\, i\in S, j,k\in V\setminus S} y_{ijk} \geq 1$$

define facets of $P_{\mathbf{AQTSP}_n}$ for all $S \subset V, 5 \leq |S| < \frac{n}{2}$.

Proof. In the proof the same ideas and the same structure are used as in the proof of Theorem 2.2. But this time some of the constructions are quite involved ensuring that all tour are roots of (17), see Remark 3.16, 2. To emphasize the correspondence between the construction used in the proof of Theorem 2.2 and the construction steps here these are denoted by the original step numbers and an additional counter each. In the following $\bar{S}$ denotes all those nodes of S that are not explicitly mentioned.

The case $|S| = 5, n = 11$ was checked with a computer algebra system. For $n \geq 12$ we set, w. l. o. g., $T := V \setminus S = \{t_1 = 1, t_2 = 2, \ldots, t_{|T|} = |T|\}, S = \{s_1 = |T|+1, \ldots, s_{|S|-1} = n-1, s_{|S|} = n\}$. The set $C_{dim}^{\bar{n},1}$ is constructed in the same way as in the proof of Theorem 2.2 using $\bar{n} = 7$. Since the nodes 1 to 7 belong to set T ($n \geq 12, |S| \geq 5, |S| < \frac{n}{2}$) the desired T-block-structure is obtained automatically. The same is true if in the inductive part k fulfills $k \in T$ using steps (**I1**)-(**I5**).

For $k = s_1$ the tours in the original steps (**I2**)–(**I4**) violate the root property so adaptations are needed.

($\mathsf{I}^{s_1}_{\mathsf{ex}}$.1) $\ldots \underline{a\, s_1\, b}\, s_2\, \varpi_k\, n \ldots,$ for $a, b \in T, a \neq b,$

($\mathsf{I}^{s_1}_{\mathsf{ex}}$.2) $\ldots \underline{a\, b\, s_2}\, \varpi_k\, n\, 1\, s_1\, 2\, 3 \ldots,$ for $a, b \in T \setminus \{1, 2, 3\}, a \neq b$
(the 2-arcs $(n, 1, s_1), (s_1, 2, 3)$ are not used as a^i_k),

($\mathsf{I}^{s_1}_{\mathsf{ex}}$.3a) $\ldots m\, o\, s_2\, \varpi_k\, n\, 1\, \underline{s_1\, a\, b} \ldots,$ for $a, b \in T \setminus \{1\}, (a, b) \neq (2, 3)$, with $m, o \in T \setminus \{1, 2, 3\}$, $|\{a, b, m, o\}| = 4,$

($\mathsf{I}^{s_1}_{\mathsf{ex}}$.4a) $\ldots \underline{a\, b\, s_2}\, \varpi_k\, n\, 1\, s_1 \ldots,$ for $a, b \in T \setminus \{1\}, \{a, b\} \cap \{2, 3\} \neq \emptyset, a \neq b,$

($\mathsf{I}^{s_1}_{\mathsf{ex}}$.5a) $\ldots m\, o\, s_2\, \varpi_k\, \underline{n\, s_1\, a}\, p \ldots,$ for $a \in T \setminus \{1\}$ with $m, o, p \in T \setminus \{1\}, |\{a, m, o, p\}| = 4,$

($\mathsf{I}^{s_1}_{\mathsf{ex}}$.4b) $\ldots \underline{a\, b\, s_2}\, \varpi_k\, n\, s_1 \ldots,$ for $a, b \in T, 1 \in \{a, b\}, a \neq b,$

($\mathsf{I}^{s_1}_{\mathsf{ex}}$.5b) $\ldots s_2\, \varpi_k\, \underline{n\, a\, s_1}\, m\, o \ldots,$ for $a \in T \setminus \{1\}$ with $m, o \in T \setminus \{1\}, |\{a, m, o\}| = 3,$

($\mathsf{I}^{s_1}_{\mathsf{ex}}$.3b) $\ldots s_2\, \varpi_k\, n\, m\, \underline{s_1\, a\, b} \ldots,$ for $a, b \in T, 1 \in \{a, b\}$, with $m \in T \setminus \{1\}, |\{a, b, m\}| = 3,$

($\mathsf{I}^{s_1}_{\mathsf{ex}}$.5c) $\ldots s_2\, \varpi_k\, \underline{n\, s_1\, 1} \ldots.$

In comparison to the proof of Theorem 2.2 we constructed exactly one tour less. It is easy to check that all underlined 2-arcs have not been used in previous tours and that the construction is possible for $\bar{n} = 7$ (in **($\mathsf{I}^{s_1}_{\mathsf{ext}}$.3a)** we need that $s_1 \geq 8$). For $k = s_2$ the 2-arcs are restricted to some specific types in order to build the triangular matrix structure.

($\mathsf{I}^{s_2}_{\mathsf{ex}}$.1) $\ldots \underline{a\, s_2\, b}\, s_3\, \varpi_k\, n\, \bar{S} \ldots,$ for $a, b \in \{1, \ldots, s_1\}, a \neq b,$

($\mathsf{I}^{s_2}_{\mathsf{ex}}$.2) $\ldots \underline{a\, b\, s_3}\, \varpi_k\, n\, m\, s_1\, s_2\, 1\, 2 \ldots,$ for $a, b \in T \setminus \{1, 2\}$ with $m \in T \setminus \{1, 2\}, |\{a, b, m\}| = 3$
(the 2-arc $(s_2, 1, 2)$ is not used as an a^i_k),

($\mathsf{I}^{s_2}_{\mathsf{ex}}$.5a) $\ldots s_3\, \varpi_k\, \underline{n\, s_1\, s_2}\, 1\, 2 \ldots,$

($\mathsf{I}^{s_2}_{\mathsf{ex}}$.3a) $\ldots m\, o\, s_3\, \varpi_k\, n\, s_1\, \underline{s_2\, a\, b} \ldots,$ for $a, b \in T, (a, b) \neq (1, 2)$, with $m, o \in T \setminus \{1, 2\}, |\{a, b, m, o\}| = 4,$

($\mathsf{I}^{s_2}_{\mathsf{ex}}$.4a) $\ldots \underline{a\, b\, s_3}\, \varpi_k\, n\, s_1\, s_2 \ldots,$ for $a, b \in T, \{a, b\} \cap \{1, 2\} \neq \emptyset, a \neq b,$

The 2-arcs $(s_1, a, s_3), (a, s_1, s_3), (s_2, s_1, a), (s_2, a, s_1), (n, s_2, a), (n, a, s_2), a \in T$, as well as the 2-arc (n, s_2, s_1) are not contained in any of the tours presented above. Their usage is deferred to the case $k = s_3$ that follows next.

($\mathsf{I}^{s_3}_{\mathsf{ex}}$.1a)
$$\begin{cases} \ldots \underline{a\, s_3\, b}\, s_4\, \varpi_k\, n\, s_1\, \bar{S} \ldots, & \text{for } a, b \in T \cup \{s_2\}, a \neq b, \\ \ldots \bar{S}\, \underline{a\, s_3\, s_1}\, s_4\, \varpi_k\, n \ldots, & \text{for } a \in T \cup \{s_2\}, \\ \ldots s_2\, \underline{s_1\, s_3\, a}\, s_4\, \varpi_k\, n \ldots, & \text{for } a \in T, \end{cases}$$

($\mathsf{I}^{s_3}_{\mathsf{ex}}$.2) $\ldots \underline{a\, b\, s_4}\, \varpi_k\, n\, s_1\, s_2\, s_3\, 1\, 2 \ldots,$ for $a, b \in T \setminus \{1, 2\}, a \neq b$
(the 2-arc $(s_3, 1, 2)$ is not used as an a^i_k),

($\mathsf{I}^{s_3}_{\mathsf{ex}}$.3a) $\ldots m\, o\, s_4\, \varpi_k\, n\, s_1\, s_2\, \underline{s_3\, a\, b} \ldots,$ for $a, b \in T, (a, b) \neq (1, 2)$, with $m, o \in T \setminus \{1, 2\}$, $|\{a, b, m, o\}| = 4,$

($\mathsf{I}^{s_3}_{\mathsf{ex}}$.4a) $\ldots \underline{a\, b\, s_4}\, \varpi_k\, n\, s_1\, s_2\, s_3 \ldots,$ for $a, b \in T, \{a, b\} \cap \{1, 2\} \neq \emptyset, a \neq b,$

($\mathsf{I}^{s_2}_{\mathsf{ex}}$.5b) $\ldots s_4\, \varpi_k\, \underline{n\, s_2\, s_1}\, s_3 \ldots,$

($\mathsf{I}^{s_2}_{\mathsf{ex}}$.3b) $\ldots s_3\, s_4\, \varpi_k\, n\, \underline{s_2\, s_1\, a} \ldots,$ for $a \in T,$

($\mathsf{I}^{s_2}_{\mathsf{ex}}$.5c) $\ldots s_3\, s_4\, \varpi_k\, \underline{n\, a\, s_2}\, s_1 \ldots,$ for $a \in T,$

($\mathbf{I}^{s_2}_{\mathbf{ex}}$**.3c**) $\ldots s_3\, s_4\, \varpi_k\, n\, m\, \underline{s_2\, a\, s_1} \ldots,$ for $a \in T$ with $m \in T, m \neq a$,

($\mathbf{I}^{s_2}_{\mathbf{ex}}$**.5d**) $\ldots s_3\, s_4\, \varpi_k\, \underline{n\, s_2\, a}\, s_1 \ldots,$ for $a \in T$,

($\mathbf{I}^{s_3}_{\mathbf{ex}}$**.4b**) $\ldots \underline{a\, b\, s_4}\, \varpi_k\, n\, m\, s_2\, s_3 \ldots,$ for $a, b \in (T \cup \{s_1\}), s_1 \in \{a, b\}$ with $m \in T, |\{a, b, m\}| = 3$,

($\mathbf{I}^{s_3}_{\mathbf{ex}}$**.5a**) $\begin{cases} \ldots s_2\, m\, s_1\, s_4\, \varpi_k\, \underline{n\, s_3\, a} \ldots, & \text{for } a \in T \text{ with } m \in T, m \neq a, \\ \ldots s_2\, m\, s_1\, s_4\, \varpi_k\, \underline{n\, a\, s_3} \ldots, & \text{for } a \in T \text{ with } m \in T, m \neq a, \end{cases}$

($\mathbf{I}^{s_3}_{\mathbf{ex}}$**.4c**) $\ldots s_1\, \underline{a\, b\, s_4}\, \varpi_k\, n\, s_3 \ldots,$ for $a, b \in (T \cup \{s_2\}), s_2 \in \{a, b\}, a \neq b$,

($\mathbf{I}^{s_3}_{\mathbf{ex}}$**.4d**) $\begin{cases} \ldots \underline{s_1\, s_2\, s_4}\, \varpi_k\, n\, s_3 \ldots, \\ \ldots \underline{s_2\, s_1\, s_4}\, \varpi_k\, n\, s_3 \ldots, \end{cases}$

($\mathbf{I}^{s_2}_{\mathbf{ex}}$**.4b**) $\ldots s_2\, s_4\, \varpi_k\, n\, \underline{s_1\, a\, s_3} \ldots,$ for $a \in T$,

($\mathbf{I}^{s_2}_{\mathbf{ex}}$**.4c**) $\ldots s_2\, s_4\, \varpi_k\, n\, \underline{a\, s_1\, s_3} \ldots,$ for $a \in T$,

($\mathbf{I}^{s_3}_{\mathbf{ex}}$**.1b**) $\ldots \underline{s_1\, s_3\, s_2}\, s_4\, \varpi_k\, n \ldots,$

($\mathbf{I}^{s_3}_{\mathbf{ex}}$**.3b**) $\ldots \bar{S} \underline{s_3\, a\, b}\, s_4\, \varpi_k\, n \ldots,$ for $a, b \in \{1, \ldots s_2\}, \{a, b\} \cap \{s_1, s_2\} \neq \emptyset, a \neq b$,

($\mathbf{I}^{s_3}_{\mathbf{ex}}$**.5b**) $\begin{cases} \ldots s_4\, \varpi_k\, \underline{n\, s_3\, a}\, \bar{S} \ldots, & \text{for } a \in \{s_1, s_2\}, \\ \ldots s_4\, \varpi_k\, \underline{n\, a\, s_3}\, \bar{S} \ldots, & \text{for } a \in \{s_1, s_2\}. \end{cases}$

During step two for $k = \bar{n} + 1, \ldots, s_3$ we build exactly one tour less than in the proof of Theorem 2.2. For $k \geq s_4$ in the second step the procedure presented in the proof of Theorem 2.2 has to be adapted slightly mentioning the position of $\bar{S}$.

($\mathbf{I}^{s_i}_{\mathbf{ex}}$.**1**) $\ldots \underline{a\, s_i\, b}\, s_{i+1}\, \varpi_k\, n\, \bar{S} \ldots,$ for $a, b \in \{1, \ldots, s_{i-1}\}, a \neq b$,

($\mathbf{I}^{s_i}_{\mathbf{ex}}$.**2**) $\ldots \underline{a\, b\, s_{i+1}}\, \varpi_k\, n\, m\, \bar{S}\, s_i\, 1\, 2 \ldots,$ for $a, b \in \{3, \ldots, s_{i-1}\}$ with $m \in T, |\{1, 2, a, b, m\}| = 5$
(the 2-arc $(s_1, 1, 2)$ is not used as an a^i_k),

($\mathbf{I}^{s_i}_{\mathbf{ex}}$.**3**) $\ldots m\, o\, s_{i+1}\, \varpi_k\, n\, p\, \bar{S}\, \underline{s_i\, a\, b} \ldots,$ for $a, b \in \{1, \ldots, s_{i-1}\}, (a, b) \neq (1, 2)$, with $m, o, p \in T, |\{a, b, m, o, p\}| = 5$,

($\mathbf{I}^{s_i}_{\mathbf{ex}}$.**4**) $\ldots \underline{a\, b\, s_{i+1}}\, \varpi_k\, n\, m\, \bar{S}\, s_i \ldots,$ for $a, b \in \{1, \ldots, s_{i-1}\}, \{a, b\} \cap \{1, 2\} \neq \emptyset$, with $m \in T, |\{a, b, m\}| = 3$,

($\mathbf{I}^{s_i}_{\mathbf{ex}}$.**5**) $\begin{cases} \ldots s_{i+1}\, \varpi_k\, \underline{n\, s_i\, a}\, \bar{S} \ldots, & \text{for } a \in \{1, \ldots, s_{i-1}\}, \\ \ldots s_{i+1}\, \varpi_k\, \underline{n\, a\, s_i}\, \bar{S} \ldots, & \text{for } a \in \{1, \ldots, s_{i-1}\}. \end{cases}$

This finishes the second step. Note, if $|S| \geq 5$ it is possible to perform the presented steps for s_1, s_2, s_3 and $s_i, 4 \leq i \leq |S| - 2$.

For the third step we again specify the position of $\bar{S}$.

($\mathbf{L}_{\mathbf{ex}}$.**1**) $\ldots \bar{S}\, s_1\, s_2\, n\, \underline{n-1\, a\, b} \ldots,$ for $a, b \in \{1, \ldots, n-2\} \setminus \{s_1, s_2\}, a \neq b$
(the 2-arc (s_1, s_2, n) is not used as an a^i_L),

($\mathbf{L}_{\mathbf{ex}}$.**2**) $\begin{cases} \ldots \underline{a\, b\, n}\, (n-1)\, m\, s_3\, \bar{S} \ldots, & \text{for } a, b \in T, a \neq b, \text{ with } m \in T, |\{a, b, m\}| = 4, \\ \ldots \bar{S}\, \underline{a\, b\, n}\, (n-1) \ldots, & \text{for } a, b \in \{1, \ldots, n-2\}, a \neq b, \{a, b\} \not\subset T, \\ & (a, b) \neq (s_1, s_2), \end{cases}$

($\mathbf{L}_{\mathbf{ex}}$.**3**) $\ldots n\, \underline{n-1\, a\, b}\, \bar{S} \ldots,$ for $a, b \in \{1, \ldots, n-2\}, a \neq b, \{a, b\} \cap \{s_1, s_2\} \neq \emptyset$,

($\mathsf{L_{ex}.4}$) $\ldots s_1\,(n-1)\,s_2\,\bar{S}\,\underline{a\,n\,b}\ldots,$ for $a,b \in \{1,\ldots,n-2\} \setminus \{s_1,s_2\}, a \neq b$
(the 2-arc $(s_1, n-1, s_2)$ is not used as an a_L^i),

($\mathsf{L_{ex}.5}$) $\ldots \underline{a\,(n-1)\,b}\,\bar{S}\,m\,n\ldots,$ for $a,b \in \{1,\ldots,n-2\}, (a,b) \neq (s_1,s_2)$, with $m \in T, |\{a, b, m\}| = 3$,

($\mathsf{L_{ex}.6}$) $\ldots (n-1)\,\bar{S}\,\underline{a\,n\,b}\ldots,$ for $a,b \in \{1,\ldots,n-2\}, \{a,b\} \cap \{s_1,s_2\} \neq \emptyset, a \neq b$,

($\mathsf{L_{ex}.7}$) $\ldots \underline{(n-1)\,a\,n}\,\bar{S}\ldots,$ for $a \in \{1,\ldots,n-2\}$,

($\mathsf{L_{ex}.8}$) $\ldots \underline{n\,a\,(n-1)}\,\bar{S}\ldots,$ for $a \in \{1,\ldots,n-2\}$.

One can easily check that all construction are possible for $n \geq 12, \bar{n} = 7, |S| \geq 5$. Having built exactly $f(n)$ affinely independent tours, Lemma 3.17 follows. $\square$

It remains to consider the case $3 \leq |S| \leq 4$ with $|S| < \frac{n}{2}$ for inequalities (18).

Lemma 3.18 *For $n \geq 7$ the inequalities*

$$\sum_{ijk \in V^{(3)}:\; i \in S, j,k \in V \setminus S} y_{ijk} \geq 1$$

define facets of $P_{\mathbf{AQTSP}_n}$ for all $S \subset V, 3 \leq |S| \leq 4, |S| < \frac{n}{2}$.

The proof of this result is deferred to the appendix.

The following theorem summarizes the last results.

Theorem 3.19 *For $n \geq 7$ the inequalities*

$$\sum_{ijk \in V^{(3)}:\; i \in S, j,k \in V \setminus S} y_{ijk} \geq 1$$

define facets of $P_{\mathbf{AQTSP}_n}$ for all $S \subset V, 2 \leq |S| < \frac{n}{2}$.

Proof. Follows directly from Remark 3.16, Lemma 3.17 and Lemma 3.18. $\square$

It is well-known that the subtour elimination constraints (3) can be separated in polynomial time [20]. It requires, *e. g.*, the solution of a minimum s-t-cut problem between each two nodes of G. The minimum s-t-cut problem asks for a partition of node set $\tilde{V}$ of a arc-weighted directed graph $\tilde{G} = (\tilde{V}, \tilde{A}, w)$ into $\tilde{V} = \tilde{S} \cup \tilde{T}, \tilde{S} \cap \tilde{T} = \emptyset, s \in \tilde{S}, t \in T$ for fixed nodes $s,t \in V$ such that the sum $\sum_{ij \in \tilde{E}, i \in \tilde{S}, j \in \tilde{T}} w_{ij}$ is minimal. For the extended subtour elimination constraints (17) it is **NP**-complete to determine a maximally violated one. Because in (17) we sum up over all 2-arcs ijk with $i \in S, j,k \in V \setminus S$, *i. e.*, all 2-arcs that leave set S without immediately returning to S, we also speak of a cut in the 2-graph. Note, using y-variables (3) equals $\sum_{ijk \in V^{(3)}:\; i \in S, j \in V \setminus S} y_{ijk}$.

Theorem 3.20 *It is* **NP***-complete to determine a maximally violated inequality of type* (17) *for points $(\bar{x}, \bar{y})$ satisfying equality constraints* (1), (2), *$x_{ij} \in [0,1]$ for all $ij \in V^{(2)}$ and $y_{ijk} \in [0,1]$ for all $ijk \in V^{(3)}$.*

Proof. We prove this by a reduction from 3-SAT (LO2 in [13]). Given a 3-SAT-formula with m variables and $|C|$ clauses, the task is to find a truth assignment for the variables that satisfies all clauses.

We build a 2-graph $\tilde{G} = (\tilde{V}, \tilde{A})$ with node set

$$\tilde{V} = \underbrace{\{s_1, \dots, s_{m+1}\}}_{=:\tilde{S}} \cup \underbrace{\{t_1, \dots, t_{m+2}\}}_{=:\tilde{T}} \cup \underbrace{\{x_i, \neg x_i \colon i = 1, \dots, m\}}_{=:\tilde{V}_x}$$

and set of 2-arcs $\tilde{A}$ to be defined next.

The idea is to assign weights to the 2-arcs of $\tilde{G}$ in such a way that the requirements of Theorem 3.20 are fulfilled and that a maximally violated inequality fulfills $\tilde{S} \subset S, \tilde{T} \subset \tilde{V} \setminus S =: T$ as well as $\{x_i, \neg x_i\} \cap S \neq \emptyset, \{x_i, \neg x_i\} \cap T \neq \emptyset$ for all $i = 1, \dots, m$, *i. e.*, the variables of the 3-SAT-formula belong to a truth assignment. We use the interpretation that all literals that are contained in S correspond to *true* and in T to *false*.

To achieve this we algorithmically specify weights w_a for all 2-arcs $a \in \tilde{A}$ that correspond to scaled values of the y-variables in the relaxation of $P_{\mathbf{AQTSP}_n}$, $n = 4m + 3$. In the beginning $w_a = 0$ for all $a \in \tilde{A}$. Then we insert 2-cycles that enlarge the weight of each of the corresponding 2-arcs by a given value. At the end the y-variables $y_a, a \in \tilde{A}$, have to fulfill $0 \leq y_a \leq 1$, $\sum_{ijk \in \tilde{V}^{(3)}} y_{ijk} = 1$ for all $j \in \tilde{V}$ and $\sum_{ijk \in \tilde{V}^{(3)}} y_{ijk} = \sum_{kij \in \tilde{V}^{(3)}} y_{kij}$ for all $ij \in V^{(2)}$. We will achieve this by bringing all *node degrees* $w_v := \sum_{i,k \colon ivk \in \tilde{A}} w_{ivk}, v \in V$, to a common level and normalize them by the same value at the end. The construction runs as follows.

(S1) For each clause $(a \vee b \vee c)$ we insert a 2-cycle $\{\neg abc, bcs_1, cs_1s_2, s_1s_2s_3, \dots, s_{m+1}t_1t_2, t_1t_2t_3, \dots, t_{m+1}t_{m+2}\neg a\}$ with weight 1. Thus, the 2-arc $(\neg a, b, c)$ is contained in the cut, *i. e.*, it holds $\neg a \in S$ and $b, c \in T$, if and only if all literals a, b, c are set to *false*, given that the solution corresponds to a truth assignment. After the insertion of the 2-cycles we set $u_v := w_v$ for $v \in \tilde{V}_x$ and $u_{\max} := \max\{u_v \colon v \in \tilde{V}_x\}, u_\Sigma := \sum_{v \in \tilde{V}_x} (u_{\max} - u_v)$.

(S2) We want to enforce $\tilde{S} \subset S$ in any maximally violated inequality (17). For this we add the 2-cycles $\{s_i s_{i+1} v, s_{i+1} v s_i, v s_i s_{i+1}\}, v \in \tilde{V} \setminus \{s_i, s_{i+1}\}, i = 1, \dots, m$, and $\{s_{m+1} s_1 v, s_1 v s_{m+1}, v s_{m+1} s_1\}, v \in \tilde{V} \setminus \{s_1, s_{m+1}\}$, with weight $D := 2|C| + 1$ for all nodes $v \in \tilde{V} \setminus \tilde{V}_x$ and with weight $D + \frac{u_{\max} - u_v}{m+1}$ for $v \in \tilde{V}_x$.

So the node degree of all nodes $v \in \tilde{V} \setminus \tilde{S}$ is enlarged $|\tilde{S}|$ times by the value D. For nodes $v \in \tilde{S}$ we have to distinguish two cases. Consider, w. l. o. g., node $s_2 \in \tilde{S}$. First, a 2-cycle $\{s_1 s_2, v, s_2 v s_1, v s_1 s_2\}$, $v \in \tilde{V} \setminus \{s_1, s_2\}$, enlarges the node degree of s_2 by D for $v \in \tilde{T} \cup (\tilde{S} \setminus \{s_1, s_2\})$ with $|\tilde{T} \cup (\tilde{S} \setminus \{s_1, s_2\})| = 2m + 1$ and by $(D + \frac{u_{\max} - u_v}{m+1})$ for $v \in \tilde{V}_x$. A similar result holds for the 2-cycles $\{s_2 s_3, v, s_3 v s_2, v s_2 s_3\}, v \in \tilde{V} \setminus \{s_2, s_3\}$. Second, the node degree of s_2 is increased $(|\tilde{S}| - 2)$ times by D by the $|\tilde{S}| - 2$ 2-cycles $\{s_i s_{i+1} s_2, s_{i+1} s_2 s_i, s_2 s_i s_{i+1}\}, i = 3, \dots, m$, and $\{s_{m+1} s_1 s_2, s_1 s_2 s_{m+1}, s_2 s_{m+1} s_1\}$.

(S3) In order to ensure $\{x_i, \neg x_i\} \not\subset T, i = 1, \dots, m$, in each maximally violated inequality (17), we insert the 2-cycles $\{s_j x_i \neg x_i, x_i \neg x_i s_j, \neg x_i s_j x_i\}$, $i = 1, \dots, m, j = 1, \dots, m+1$, each with weight

$$D + \underbrace{|C| + \tfrac{2u_\Sigma}{m+1} - u_{\max} + 8mD - D}_{=:u_T \geq 0}.$$

(S4) For the degree compensation of nodes in $\tilde{T}$ we insert the 2-cycle $\{t_1 t_2 t_3, t_2 t_3 t_4, \dots, t_{m+2} t_1 t_2\}$ with weight $\frac{2u_\Sigma}{m+1} + m u_T + 9mD$.

We set $M := |C| + \frac{2u_\Sigma}{m+1} + mu_T + 10mD + D$. Let us calculate $\sum_{ijk \in \tilde{V}^{(3)}} w_{ijk}$ for all $j \in \tilde{V}$.

$$\begin{aligned}
v \in \tilde{S}\colon\ & \underbrace{|C|}_{(S1)} + \underbrace{2\big(D(2m+1) + \sum_{v \in \tilde{V}_x} (D + \tfrac{u_{\max} - u_v}{m+1})\big) + D(|\tilde{S}| - 2)}_{(S2)} + \underbrace{(D + u_T)m}_{(S3)} \\
& = |C| + 4mD + 2D + 4mD + \tfrac{2u_\Sigma}{m+1} + Dm - D + Dm + u_T m \\
& = |C| + \tfrac{2u_\Sigma}{m+1} + mu_T + 10mD + D = M, \\
v \in \tilde{T}\colon\ & \underbrace{|C|}_{(S1)} + \underbrace{D|\tilde{S}|}_{(S2)} + \underbrace{(\tfrac{2u_\Sigma}{m+1} + mu_T + 9mD)}_{(S4)} = M, \\
v \in \tilde{V}_x\colon\ & \underbrace{u_v}_{(S1)} + \underbrace{(D + \tfrac{u_{\max} - u_v}{m+1})|\tilde{S}|}_{(S2)} + \underbrace{(D + u_T)|\tilde{S}|}_{(S3)} \\
& = D(m+1) + u_{\max} + D(m+1) + mu_T + |C| + \tfrac{2u_\Sigma}{m+1} - u_{\max} + 8mD - D \\
& = |C| + \tfrac{2u_\Sigma}{m+1} + mu_T + 10mD + D = M.
\end{aligned}$$

Then with $\bar{y}_a = \frac{w_a}{M}, a \in \tilde{V}^{(3)}$, and $\bar{x}_{ij} = \sum_{ijk \in \tilde{V}^{(3)}} \bar{y}_{ijk}, ij \in V^{(2)}$, the point $(\bar{x}, \bar{y})$ fulfills the requirements of Theorem 3.20.

It remains to show the correctness of the construction. First observe that with $\tilde{S} \subset S, \tilde{T} \subset T, |\{x_i, \neg x_i\} \cap S| = 1, |\{x_i, \neg x_i\} \cap T| = 1$, *i. e.*, the $x_i, \neg x_i, i = 1, \ldots, m$, correspond to a proper truth assignment, the left-hand side of (17) is less than or equal to $\frac{2|C|}{M}$. Indeed, in (S1) for each clause the 2-arc $s_{m+1}t_1t_2$ causes costs of $\frac{1}{M}$ and if a clause $(a \vee b \vee c)$ is not fulfilled, the 2-arc $(\neg a, b, c)$ causes costs of $\frac{1}{M}$. The 2-arcs introduced in (S2)–(S4) do not contribute to the cut in this case. For solutions that observe the described structure the cut value is therefore minimal if a minimal number of clauses of the 3-SAT-formula is violated. In particular, for a satisfying truth assignment of a feasible 3-SAT-instance none of the 2-arcs $(\neg a, b, c)$ is contained in the cut and therefore the cut value equals $z = \frac{|C|}{M}$. Let $z < \frac{D}{M}$ denote the optimal value of the cut problem. We show next that all solutions having not this structure have higher objective value.

- $\tilde{S} \subset S$: If, w. l. o. g., $s_1 \in V \setminus S$ then with $1 \le |S| \le 2m+1$ at least one of the 2-cycles including node s_1 causes costs of $\frac{D}{M} > z$ in (S2) and this cannot be optimal. So we know $\tilde{S} \subset S$ in all optimal solutions.

- $|\{x_i, \neg x_i\} \cap T| \le 1$ for $i = 1, \ldots, m$: Assume $\{x_i, \neg x_i\} \subset T$ for $i \in \{1, \ldots, m\}$ then each 2-arc $(s_j, x_i, \neg x_i), j = 1, \ldots, m+1$, causes costs of $\frac{D}{M} > z$.

- $\tilde{T} \subset T$ and $|\{x_i, \neg x_i\} \cap S| = 1, |\{x_i, \neg x_i\} \cap T| = 1$ for $i = 1, \ldots, m$: With the considerations above we know that an optimal solution fulfills $\tilde{S} \subset S, |\tilde{S}| = m+1$ and $|\bigcup_{i=1}^m \{x_i, \neg x_i\} \cap S| \ge m$ and by $|S| \le 2m+1$ the statement follows.

Thus any solution with objective value at most z has the desired structure and if $z = |C|$ the 3-SAT-formula can be fulfilled. Conversely, given an optimal solution with value $z = |C|$ we can construct a truth assignment for a 3-SAT-formula with all literals $a \in S$ set to *true* and vice versa. $\square$

The inequalities of type (18) define facets for $S \subset V, \frac{n}{2} \le |S| \le n-5$. Note, for $S \subset V, |S| = n-3, T = \{t_1, t_2, \bar{t}\}$ the inequality is dominated by inequalities (9) because by (1) and (2) inequality (18) can be transformed to $y_{t_1t_2\bar{t}} + y_{\bar{t}t_1t_2} + y_{t_2t_1\bar{t}} + y_{\bar{t}t_2t_1} \le x_{t_1t_2} + x_{t_2t_1}$.

Theorem 3.21 *For $n \geq 10$ the inequalities*

$$\sum_{ijk \in V^{(3)}:\, i \in S, j,k \in V \setminus S} y_{ijk} + \sum_{i\bar{t}j \in V^{(3)}:\, i,j \in S} y_{i\bar{t}j} \geq 1$$

define facets of $P_{\mathbf{AQTSP}_n}$ for all $S \subset V, \frac{n}{2} \leq |S| \leq n-5, \bar{t} \in V \setminus S$.

The proof of Theorem 3.21 has a similar structure as the proof of Lemma 3.17 but this time the position of node $\bar{t} \in V \setminus S$ has to be chosen carefully in order to get roots of (18). As the whole proof is quite involved we defer it to the appendix.

A further well-known class of inequalities known for $\mathbf{ATSP}_n$ are the so called D_k-inequalities [15, 16]

$$\begin{aligned} D_k^-: &\sum_{j=1}^{k-1} x_{i_j i_{j+1}} + x_{i_k i_1} + 2 \cdot \sum_{j=2}^{k-1} x_{i_j i_1} + \sum_{j=3}^{k-1} \sum_{h=2}^{j-1} x_{i_j i_h} \leq k-1, \\ D_k^+: &\sum_{j=1}^{k-1} x_{i_j i_{j+1}} + x_{i_k i_1} + 2 \cdot \sum_{j=3}^{k} x_{i_1 i_j} + \sum_{j=4}^{k} \sum_{h=3}^{j-1} x_{i_j i_h} \leq k-1, \\ &\text{for all subsets } \{i_1, \ldots, i_k\} \subset V, 3 \leq k \leq n-1. \end{aligned}$$

That these are facets of $\mathbf{ATSP}_n$ for $k = 3, 4$ is shown in [15]. It is clear that these can be strengthened by the general procedure introduced above (Observation 3.13, Observation 3.14). Unfortunately, this strengthening is not yet sufficient to guarantee facetness. For D_3, *e. g.*, the coefficient of one of the y-variables can be raised by one further unit.

Theorem 3.22 *For $n \geq 7$ the inequalities*

$$x_{i_1 i_2} + x_{i_2 i_3} + x_{i_3 i_1} + 2x_{i_2 i_1} + \sum_{j \in V \setminus \{i_1, i_2, i_3\}} (y_{i_1 j i_2} + y_{i_2 j i_3} + y_{i_3 j i_1} + 2y_{i_2 j i_1}) + 2y_{i_1 i_3 i_2} \leq 2 \tag{19}$$

define facets of $P_{\mathbf{AQTSP}_n}$ for all $i_1, i_2, i_3 \in V, |\{i_1, i_2, i_3\}| = 3$.

Proof. If $y_{i_1 i_3 i_2} = 0$ inequality (19) is valid for $P_{\mathbf{AQTSP}_n}, n \geq 7$, by Observation 3.13, Observation 3.14 and the validity of the D_k-inequalities. In the case $y_{i_1 i_3 i_2} = 1$ all other x- and y-variables have to be zero for $n \geq 7$ because otherwise a subtour is contained in the graph or (1) is violated.

For $n = 7$ we checked the statement by a computer algebra system. For $n \geq 8, \bar{n} = 6$, we set, w. l. o. g., $i_1 = n, i_2 = n-1, i_3 = 2$. The proof is similar to the proof of Theorem 2.2 and we only notice the differences. Steps one and two remain unchanged because all tours $t \in C_{dim}^{\bar{n},1} \cup C_{dim}^{\bar{n},2}$ fulfill condition (7) and so are roots of (19). We only adapt step three. To emphasize the correspondence between the construction used in the proof of Theorem 2.2 and the construction steps here these are denoted by the original step numbers and an additional counter each.

($\mathsf{L}_{\mathsf{D_3}}$.1) $\ldots 1\,2\,n\,\underline{n-1\,a\,b} \ldots$, for $a, b \in \{3, \ldots, n-2\}, a \neq b$
(the 2-arc $(1, 2, n)$ is not used as an a_L^i),

($\mathsf{L}_{\mathsf{D_3}}$.2a) $\begin{cases} \ldots \underline{a\,b\,n}\,(n-1)\,m\,o \ldots, & \text{for } a, b \in \{1, \ldots, n-2\}, 2 \in \{a, b\}, (a, b) \neq (1, 2), \\ & \text{with } m, o \in \{3, \ldots, n-2\}, |\{a, b, m, o\}| = 4, \\ \ldots \underline{a\,b\,n}\,(n-1)\,2\,1 \ldots, & \text{for } a, b \in \{3, \ldots, n-2\}, a \neq b, \end{cases}$
(the 2-arc $(n-1, 2, 1)$ is not used as an a_L^i),

($\mathbf{L_{D_3}}$.3) $\begin{cases} \ldots m\,o\,n\,\underline{(n-1)\,a\,b}\ldots, & \text{for } a,b \in \{1,\ldots,n-2\}, 2 \in \{a,b\}, (a,b) \neq (2,1), \\ & \text{with } m,o \in \{3,\ldots,n-2\}, |\{a,b,m,o\}| = 4, \\ \ldots 2\,n\,\underline{(n-1)\,a\,b}\ldots, & \text{for } a,b \in \{1,\ldots,n-2\} \setminus \{2\}, 1 \in \{a,b\}, \end{cases}$

($\mathbf{L_{D_3}}$.2b) $\ldots\underline{a\,b\,n}\,(n-1)\,2\ldots,$ for $a,b \in \{1,3,4,\ldots,n-2\}, 1 \in \{a,b\}, a \neq b,$

($\mathbf{L_{D_3}}$.4) $\ldots 1\,(n-1)\,2\,\underline{a\,n\,b}\ldots,$ for $a,b \in \{3,\ldots,n-2\}, a \neq b$
(the 2-arc $(1,n-1,2)$ is not used as an a_L^i),

($\mathbf{L_{D_3}}$.5a) $\begin{cases} \ldots\underline{a\,(n-1)\,b}\,2\,m\,n\,o\ldots, & \text{for } a,b \in \{1,\ldots,n-2\} \setminus \{2\}, 1 \in \{a,b\}, a \neq b, \\ \ldots\underline{a\,(n-1)\,b}\,m\,n\,o\ldots, & \text{for } a \in \{3,\ldots,n-2\}, b = 2, \end{cases}$
with $m,o \in \{3,\ldots,n-2\}, |\{a,b,m,o\}| = 4,$

($\mathbf{L_{D_3}}$.6a) $\begin{cases} \ldots(n-1)\,m\,2\,\underline{a\,n\,b}\ldots, & \text{for } a,b \in \{1,\ldots,n-2\} \setminus \{2\}, 1 \in \{a,b\}, a \neq b, \\ \ldots(n-1)\,m\,\underline{a\,n\,b}\ldots, & \text{for } a = 2, b \in \{1,\ldots,n-2\} \setminus \{2\}, \end{cases}$
with $m \in \{3,\ldots,n-2\}, |\{a,b,m\}| = 3,$

($\mathbf{L_{D_3}}$.7) $\ldots m\,\underline{(n-1)\,a\,n}\,o\ldots,$ for $a \in \{1,\ldots,n-2\}$ with $m,o \in \{1,\ldots,n-2\} \setminus \{2\}, |\{a, m,o\}| = 3,$

($\mathbf{L_{D_3}}$.5b) $\ldots\underline{2\,(n-1)\,a}\,n\ldots,$ for $a \in \{1,\ldots,n-2\} \setminus \{2\},$

($\mathbf{L_{D_3}}$.6b) $\ldots(n-1)\,\underline{a\,n\,2}\ldots,$ for $a \in \{1,\ldots,n-2\} \setminus \{2\},$

($\mathbf{L_{D_3}}$.8) $\begin{cases} \ldots 2\,\underline{n\,a\,(n-1)}\ldots, & \text{for } a \in \{1,\ldots,n-2\} \setminus \{2\}, \\ \ldots\underline{n\,2\,(n-1)}\ldots. \end{cases}$

One can easily check that all $|C_{dim}^{\bar{n},3}| - 1$ tours are roots of (19) and that each underlined 2-arc is not used in a previous tour. All in all we get the same number of tours in steps one and two and one tour less in step three in comparison to the proof of Theorem 2.2 and so inequalities (19) define facets of $P_{\mathbf{AQTSP}_n}$. □

Until now only strengthenings of facets of $P_{\mathbf{ATSP}_n}$ are used that add y-variables corresponding to 2-arcs $(i,j,k) \in V^{(3)}$ if the x-variables corresponding to the arcs (i,j) and (j,k) have zero-coefficients. This changes for the next two classes, strengthenings of D_4^-, D_4^+ [16].

Theorem 3.23 *For $n \geq 9$ the inequalities*

$$x_{i_1i_2} + x_{i_2i_3} + x_{i_3i_4} + x_{i_4i_1} + 2x_{i_2i_1} + 2x_{i_3i_1} + x_{i_3i_2} + y_{i_1i_4i_2} + y_{i_2i_4i_3} + y_{i_1i_4i_3} + y_{i_1i_3i_2}$$
$$+ y_{i_2i_4i_1} + \sum_{j \in V \setminus \{i_1,i_2,i_3,i_4\}} (y_{i_1ji_2} + y_{i_2ji_3} + y_{i_3ji_4} + y_{i_4ji_1} + 2y_{i_2ji_1} + 2y_{i_3ji_1} + y_{i_3ji_2}) \leq 3 \tag{20}$$

define facets of $P_{\mathbf{AQTSP}_n}$ for all $i_1, i_2, i_3, i_4 \in V, |\{i_1,i_2,i_3,i_4\}| = 4$.

The validity of (20) for $P_{\mathbf{AQTSP}_n}, n \geq 9$, is proved next but the proof of its facetness is deferred to the appendix.

Proof. The validity of (20) except for the term $y_{i_1i_4i_3} + y_{i_1i_3i_2} + y_{i_2i_4i_1}$ follows from Observation 3.13, Observation 3.14 and the validity of the D_k-inequalities. For $n \geq 5$ the presence of more than one of the corresponding 2-arcs $i_1i_4i_3, i_1i_3i_2$ and $i_2i_4i_1$ would imply a subtour or (1) would be violated. Let $n \geq 9$. There remain three cases:

- If $y_{i_1i_4i_3} = 1$ it holds $x_{i_1i_2} = x_{i_2i_3} = x_{i_3i_4} = x_{i_4i_1} = x_{i_3i_1} = y_{i_1i_4i_2} = y_{i_2i_4i_3} = \sum_{j\in V\setminus\{i_1,i_2,i_3,i_4\}}(y_{i_1ji_2} + y_{i_2ji_3} + y_{i_3ji_4} + y_{i_4ji_1} + y_{i_3ji_1}) = 0$ by (1), (3) and $2x_{i_2i_1} + x_{i_3i_2} + \sum_{j\in V\setminus\{i_1,i_2,i_3,i_4\}}(2y_{i_2ji_1} + y_{i_3ji_2}) \leq 2$ because otherwise a subtour would be implied. So the left-hand side of (20) is lower than or equal to three.

- If $y_{i_1i_3i_2} = 1$ it holds $x_{i_1i_2} = x_{i_2i_3} = x_{i_3i_4} = x_{i_2i_1} = x_{i_3i_1} = y_{i_1i_4i_2} = y_{i_2i_4i_3} = y_{i_1i_4i_3} = y_{i_2i_4i_1} = \sum_{j\in V\setminus\{i_1,i_2,i_3,i_4\}}(y_{i_1ji_2} + y_{i_2ji_3} + y_{i_3ji_4} + y_{i_2ji_1} + y_{i_3ji_1} + y_{i_3ji_2}) = 0$ and $x_{i_3i_2} = 1$. By $x_{i_4i_1} + \sum_{j\in V\setminus\{i_1,i_2,i_3,i_4\}} y_{i_4ji_1} \leq 1$ (see (11)) the validity of (20) follows in the considered case.

- If $y_{i_2i_4i_1} = 1$ it holds $x_{i_1i_2} = x_{i_2i_3} = x_{i_3i_4} = x_{i_2i_1} = x_{i_3i_1} = y_{i_1i_4i_2} = y_{i_2i_4i_3} = y_{i_1i_4i_3} = y_{i_1i_3i_2} = \sum_{j\in V\setminus\{i_1,i_2,i_3,i_4\}}(y_{i_1ji_2} + y_{i_2ji_3} + y_{i_3ji_4} + y_{i_4ji_1} + y_{i_2ji_1} + y_{i_3ji_1}) = 0$ and $x_{i_4i_1} = 1$. By $x_{i_3i_2} + \sum_{j\in V\setminus\{i_1,i_2,i_3,i_4\}} y_{i_3ji_2} \leq 1$ (see (11)) the validity of (20) follows. □

Similar adaptation are possible for D_4^+ to receive facets of $P_{\mathbf{AQTSP}_n}$.

Theorem 3.24 *For $n \geq 9$ the inequalities*

$$\begin{aligned} &x_{i_1i_2} + x_{i_2i_3} + x_{i_3i_4} + x_{i_4i_1} + 2x_{i_1i_4} + 2x_{i_1i_3} + x_{i_4i_3} + y_{i_3i_2i_4} + y_{i_4i_2i_1} + y_{i_3i_2i_1} + y_{i_3i_1i_2} \\ &+y_{i_2i_4i_3} + \sum_{j\in V\setminus\{i_1,i_2,i_3,i_4\}} (y_{i_1ji_2} + y_{i_2ji_3} + y_{i_3ji_4} + y_{i_4ji_1} + 2y_{i_1ji_4} + 2y_{i_1ji_3} + y_{i_4ji_3}) \leq 3 \end{aligned} \tag{21}$$

define facets of $P_{\mathbf{AQTSP}_n}$ for all $i_1, i_2, i_3, i_4 \in V, |\{i_1, i_2, i_3, i_4\}| = 4$.

The proof of this result is similar to the proof of Theorem 3.23 and can be found in the appendix.

4 Computational results

In order to show that the new inequalities are useful in practical cutting plane approaches we present a few computational results for random Angle-TSP instances with additional 2-arc dependent costs and for randomly generated reload cost instances. Furthermore we tested real-world instances from biology that arise in the solution of Permuted Variable Length Markov models. We used CPLEX 12.1 [1] as a branch-and-cut solver on an Intel Core i7 CPU 920 with 2.67 GHz and 12 GB RAM in single processor mode. In the basic relaxation, indicated with (I) in the following, we only separate the subtour elimination constraints (3) using the LEMON Graph Library 1.2.2 [2] that solves the problem of finding a minimum cut in a directed graph by the algorithm of Hao and Orlin [19]. In (II) this is extended by separating (9), (10), all facets of the first class of conflicting arcs inequalities ((11), (12), (14), (15)), (17) with $|S| = 3$ and using the strengthened versions (17), (18) instead of (3) if a violated inequality (3) is found. The separation problem of (15) (this includes (11)) was solved as a linear program with CPLEX which works because the corresponding constraint matrix is totally unimodular. Here we took advantage of the warm-start properties of the simplex-algorithm because the cost coefficients change only slightly if node i is fixed and we vary only j.

The Angle-TSP instances with $5 \leq n \leq 25$ were generated by choosing n points $p_1, \dots, p_n$ uniformly at random out of $\{0, 1000\}^3$ with $p_i = (x_i, y_i, z_i)^T$. The coefficients $c_{ijk}, ijk \in V^{(3)}$, depend on the angle if we go from node i over node j to node k, on the distances

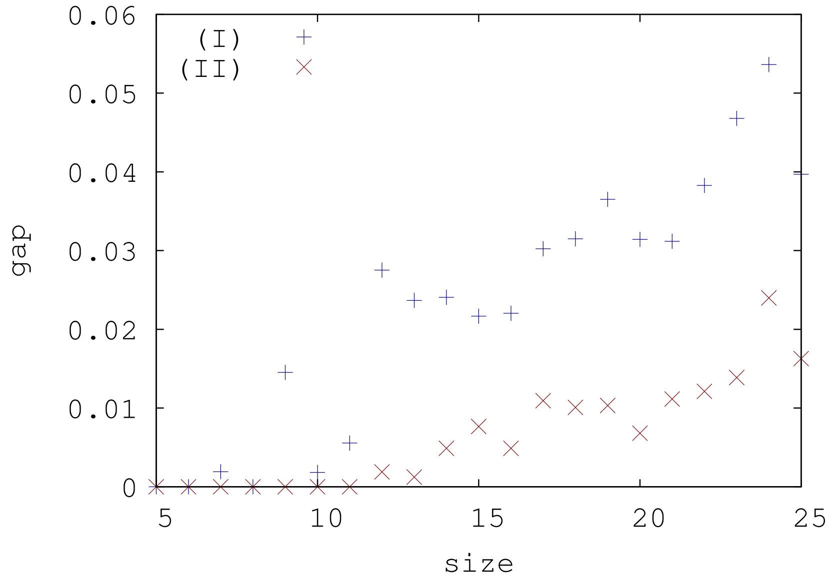


Figure 6: Comparison of the gaps at the root node for random Angle-TSP instances with additional arc- and 2-arc dependent costs.

between i and j as well as j and k and they also depend on the change in the z-coordinate with the interpretation that gaining and loosing height have different energy demand,

$$c_{ijk} = \left\lfloor \frac{18000}{\pi} \arccos\left(\left(\frac{p_j - p_i}{\|p_j - p_i\|} \right)^T \left(\frac{p_k - p_j}{\|p_k - p_j\|} \right) \right) \right\rfloor + \tfrac{1}{10}(\|p_i - p_j\| + \|p_j - p_k\|)$$
$$+ \begin{cases} |z_i - z_k| & z_k - z_i \leq 0 \\ 5 \cdot |z_i - z_k| & z_k - z_i > 0. \end{cases}$$

A comparison of the average root gaps $(c^* - c_{relax})/c_{relax}$ over 10 instances, can be found in Figure 6.

Furthermore we considered instances with reload costs. The random graphs $\tilde{G} = (\tilde{V}, \tilde{A})$ were generated by inserting each arc $ij \in V^{(2)}$ with probability p. If an arc ij is present it is colored randomly with one of the colors in $D = \{1, \ldots, d\}$. We tested two different cost types. A color change between color i and j, $i, j \in D, i \neq j$, caused costs of one for all instances RI_1 and costs of d_{ij} with d_{ij} chosen uniformly at random in $\{1, \ldots, 10\}$ for instances RI_2. Because the value of the optimal solution is always zero if the graph $\tilde{G}$ contains a monochromatic Hamiltonian directed cycle gaps are meaningless and so Table 1 shows the average optimal value and relaxation value at the root node over ten instances for each parameter setting. Exploiting the fact that with the used cost structures the optimal value is either zero or at least two we could prove the optimality of 147 instances with (I) and of 187 instances with approach (II) at the root node.

Regarding the instances arising in biology the presented inequalities are very effective. All instances could be solved without branching separating only (9), (10), (11), (17) with $|S| = 3$ and using the strengthened versions (17), (18) instead of (3) if a violated inequality (3) is found. Therefore each instance could be solved in less than 605 seconds, all instances with $n \leq 39$ in less than 10 seconds. Figure 7 shows the average gaps over three instances for each n in 6 to 41, denoted by B1-B3, and for a second test set for n in 6 to 100, denoted by B4-B6, for approach (I). For the first test set Jäger and Molitor [21] reported that they

RI_1						RI_2					
p	d	n	opt.	(I)	(II)	p	d	n	opt.	(I)	(II)
$\frac{1}{2}$	5	10	5.200	5.200	5.200	$\frac{1}{2}$	5	10	16.100	15.573	16.100
		15	4.100	3.050	3.583			15	17.700	12.309	13.784
		20	3.200	1.239	1.454			20	8.300	3.513	4.086
	10	10	5.800	5.800	5.800		10	10	19.800	19.498	19.700
		15	6.500	5.919	6.123			15	23.300	18.339	19.175
		20	6.100	4.622	5.040			20	19.200	12.861	14.369
	20	10	7.700	7.700	7.700		20	10	27.100	25.955	26.283
		15	8.500	8.244	8.500			15	27.200	22.297	23.937
		20	9.300	8.471	8.951			20	26.800	20.759	21.701
1	5	10	2.000	0.550	1.368	1	5	10	4.600	1.006	2.003
		15	0.400	0.000	0.000			15	2.100	0.000	0.000
		20	0.000	0.000	0.000			20	0.000	0.000	0.000
	10	10	3.400	2.484	3.102		10	10	8.400	5.975	7.588
		15	2.900	0.596	1.291			15	6.500	1.363	2.615
		20	2.400	0.000	0.052			20	4.500	0.000	0.087
	20	10	4.500	4.160	4.500		20	10	11.600	9.981	11.194
		15	5.100	3.956	4.302			15	11.700	6.285	7.697
		20	4.800	2.095	2.796			20	9.800	4.043	5.449

Table 1: Average optimal values and values at the root node for random instances with reload costs with arc-probability p, d colors and n nodes.

were able to solve instances with n up to 26, but for $n = 26$ the running times are about three weeks. Using the basic relaxation most instances could not be solved in the root node and thus the running times increased. For example, not all instances with $n \leq 18$ could be solved in less than 605 seconds. It remains for future work to further investigate the structure of the instances from biology. We thank Ivo Grosse and Jens Keilwagen at the Leibniz Institute of Plant Genetics and Crop Plant Research for providing us these test instances.

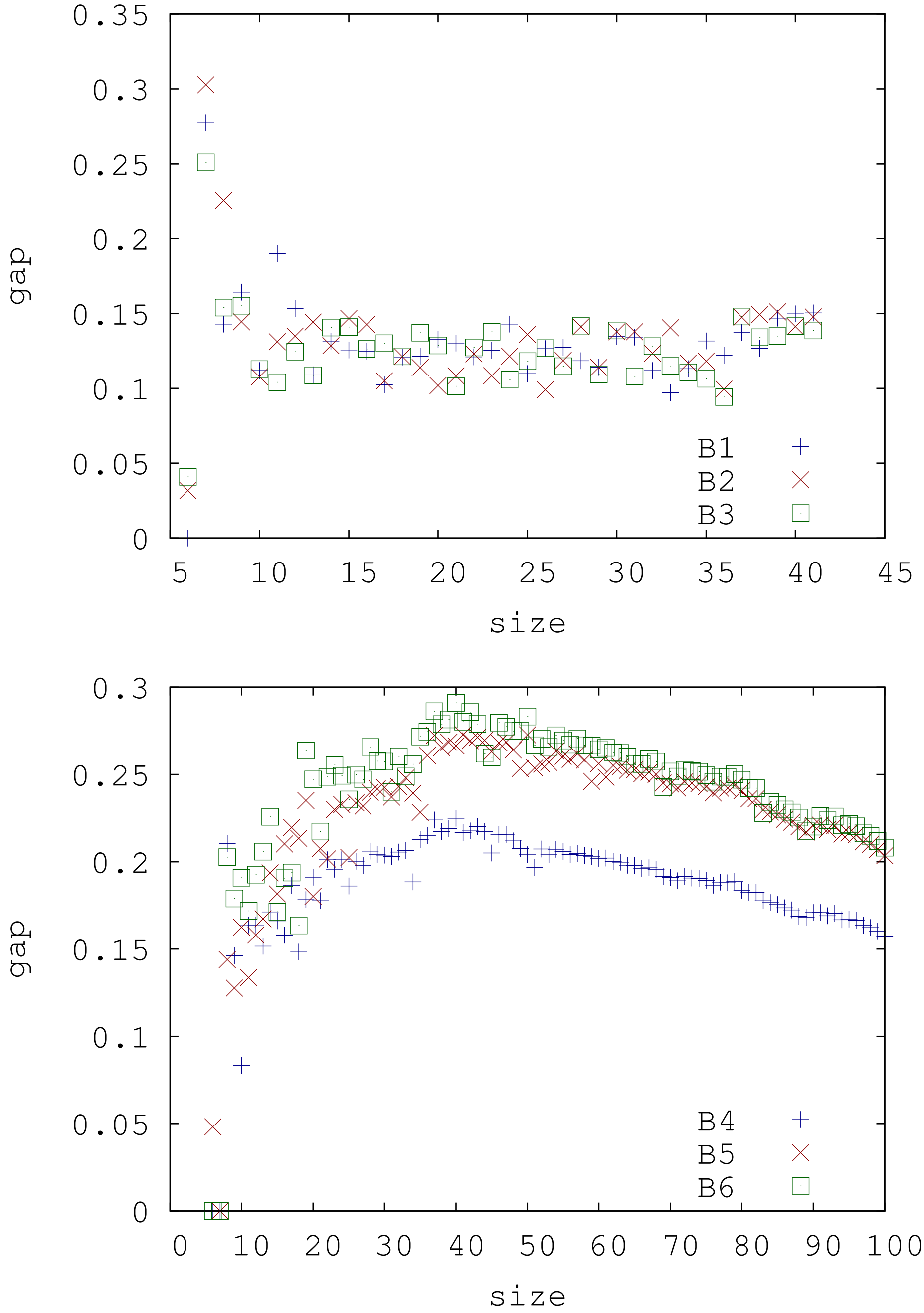


Figure 7: Gaps at the root node for instances from biology with approach (I).

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Appendix

Proof of Lemma 3.18. In the proof the same ideas and the same structure are used as in the proof of Theorem 2.2. But this time some of the constructions are quite involved ensuring that all tours are roots of (17). In the following $\bar{S}$ denotes all those nodes of set S that are not explicitly mentioned.

For $|S| = 3, 7 \le n \le 9$ and $|S| = 4, 9 \le n \le 10$ we checked the facetness of (17) with the computer. We set for all other cases, w. l. o. g., $S = \{s_1 = 1, \dots, s_{|S|} = |S|\}, T = \{t_1 = |S| + 1, \dots, t_{|T|} = n\}$. Using $\bar{n} = 7$ for $|S| = 3$ and $\bar{n} = 8$ for $|S| = 4$ the cardinality of $C^{\bar{n},1}_{dim}$ reduces by one if the desired structure of roots of (17) is taken into account in the $\bar{n}$-permutation block. The iterative steps have to be adapted, too. To emphasize the correspondence between the construction used in the proof of Theorem 2.2 and the construction steps here these are denoted by the original step numbers and an additional counter each.

($\mathbf{I^{34}_{ex}}$.1a) $\dots \bar{S}\, \underline{a\, k\, b}\, (k+1)\, \varpi_k\, n \dots,$ for $a, b \in \{1, \dots, k-1\}, (a, b) \notin T \times S, a \neq b,$

($\mathbf{I^{34}_{ex}}$.2a) $\begin{cases} \dots k\, t_1\, t_2\, \bar{S}\, \underline{a\, b\, (k+1)}\, \varpi_k\, n \dots, & \text{for } a, b \in S, a \neq b, \\ \dots k\, t_1\, t_2\, \underline{a\, b\, (k+1)}\, \varpi_k\, n\, \bar{S} \dots, & \text{for } a, b \in \{t_3, \dots, k-1\}, a \neq b, \\ \dots k\, \underline{t_1\, t_2\, (k+1)}\, \varpi_k\, n\, \bar{S} \dots, & \end{cases}$

(the 2-arc (k, t_1, t_2) is not used as an a^i_k),

($\mathbf{I^{34}_{ex}}$.3a) $\begin{cases} \dots \bar{S}\, \underline{k\, a\, b}\, (k+1)\, \varpi_k\, n\, t_1\, t_2 \dots, & \text{for } a, b \in S, a \neq b, \\ \dots \underline{k\, a\, b}\, (k+1)\, \varpi_k\, n\, \bar{S} \dots, & \text{for } a, b \in \{t_3, \dots, k-1\}, a \neq b, \end{cases}$

($\mathbf{I^{34}_{ex}}$.5a) $\begin{cases} \dots t_3\, t_4\, (k+1)\, \varpi_k\, \underline{n\, a\, k}\, \bar{S} \dots, & \text{for } a \in S, \\ \dots t_3\, t_4\, (k+1)\, \varpi_k\, \underline{n\, k\, a}\, \bar{S} \dots, & \text{for } a \in S, \end{cases}$

($\mathbf{I^{34}_{ex}}$.2b) $\dots \underline{a\, b\, (k+1)}\, \varpi_k\, n\, k\, \bar{S} \dots,$ for $a, b \in \{t_1, \dots, k-1\}, \{a, b\} \cap \{t_1, t_2\} \neq \emptyset, (a, b) \neq (t_1, t_2), a \neq b,$

($\mathbf{I^{34}_{ex}}$.3b) $\dots \underline{k\, a\, b}\, (k+1)\, \varpi_k\, n\, \bar{S} \dots,$ for $a, b \in \{t_1, \dots, k-1\}, \{a, b\} \cap \{t_1, t_2\} \neq \emptyset, (a, b) \neq (t_1, t_2), a \neq b,$

($\mathbf{I^{34}_{ex}}$.3c) $\begin{cases} \dots (k+1)\, \varpi_k\, n\, \underline{k\, a\, b}\, \bar{S} \dots, & \text{for } a \in S, b \in \{t_1, \dots, k-1\}, \\ \dots \underline{k\, a\, b}\, \bar{S}\, (k+1)\, \varpi_k\, n\, \dots, & \text{for } a \in \{t_1, \dots, k-1\}, b \in S, \end{cases}$

($\mathbf{I^{34}_{ex}}$.1b) $\dots \underline{a\, k\, b}\, \bar{S}\, (k+1)\, \varpi_k\, n \dots,$ for $a \in \{t_1, \dots, k-1\}, b \in S,$

($\mathbf{I^{34}_{ex}}$.5b) $\begin{cases} \dots (k+1)\, \varpi_k\, \underline{n\, k\, a}\, \bar{S} \dots, & \text{for } a \in \{t_1, \dots, k-1\}, \\ \dots (k+1)\, \varpi_k\, \underline{n\, a\, k}\, \bar{S} \dots, & \text{for } a \in \{t_1, \dots, k-1\}, \end{cases}$

($\mathbf{I^{34}_{ex}}$.2c) $\begin{cases} \dots \bar{S}\, \underline{a\, b\, (k+1)}\, \varpi_k\, n\, k \dots, & \text{for } a \in S, b \in \{t_1, , \dots, k-1\}, \\ \dots \bar{S}\, \underline{a\, b\, (k+1)}\, \varpi_k\, n\, k \dots, & \text{for } a \in \{t_1, , \dots, k-1\}, b \in S. \end{cases}$

One can easily check that for $|S| = 3, \bar{n} = 7$ and $|S| = 4, \bar{n} = 8$ exactly $|C^{\bar{n},2}_{dim}|$ tours are built and that all underlined 2-arcs are not used in a previous tour in steps **($\mathbf{I^{34}_{ex}}$.1a)**–**($\mathbf{I^{34}_{ex}}$.2c)** above. It remains to adapt steps (**L1**)–(**L8**) by specifying the position of $\bar{S}$ and exchanging the role of nodes $1, 2$ with $n-3, n-2$. To simplify the presentation (**L2**) and (**L4**)–(**L6**) are divided into two steps each.

($\mathbf{L^{34}_{ex}}$.1) $\dots (n-3)\, (n-2)\, n\, \underline{n-1\, a\, b}\, \bar{S} \dots,$ for $a, b \in \{1, \dots, n-4\}, a \neq b$

(the 2-arc $(n-3, n-2, n)$ is not used as an a^i_L),

(L^{34}_{ex}.2a) $\ldots \bar{S}\, \underline{a\, b\, n}\, (n-1)\, m\, o \ldots,$ for $a, b \in \{1, \ldots, n-2\}, (a, b) \neq (n-3, n-2), \{a, b\} \cap S \neq \emptyset$, with $m, o \in \{t_1, \ldots, n-4\}, |\{a, b, m, o\}| = 4$,

(L^{34}_{ex}.2b) $\ldots \underline{a\, b\, n}\, (n-1)\, \bar{S} \ldots,$ for $a, b \in \{t_1, \ldots, n-2\}, a \neq b, (a, b) \neq (n-3, n-2)$,

(L^{34}_{ex}.3) $\ldots n\, \underline{n-1\, a\, b}\, \bar{S} \ldots,$ for $a, b \in \{1, \ldots, n-2\}, a \neq b, \{a, b\} \cap \{n-3, n-2\} \neq \emptyset$,

(L^{34}_{ex}.4a) $\ldots (n-3)\, (n-1)\, (n-2)\, \underline{a\, n\, b}\, \bar{S} \ldots,$ for $a, b \in \{1, \ldots, n-4\}, a \notin S, a \neq b$ (the 2-arc $(n-3, n-1, n-2)$ is not used as an a^i_L),

(L^{34}_{ex}.4b) $\ldots (n-3)\, (n-1)\, (n-2)\, \bar{S}\, \underline{a\, n\, b} \ldots,$ for $a \in S, b \in \{1, \ldots, n-4\}, a \neq b$,

(L^{34}_{ex}.5a) $\ldots \underline{a\, (n-1)\, b}\, \bar{S}\, n\, m \ldots,$ for $a \in \{1, \ldots, n-2\}, b \in S$ with $m \in \{t_1, \ldots, n-4\}, |\{a, b, m\}| = 3$,

(L^{34}_{ex}.5b) $\ldots m\, n\, \bar{S}\, \underline{a\, (n-1)\, b} \ldots,$ for $a, b \in \{1, \ldots, n-2\}, b \notin S, (a, b) \neq (n-3, n-2)$ with $m \in S, |\{a, b, m\}| = 4$,

(L^{34}_{ex}.6a) $\ldots (n-1)\, \bar{S}\, \underline{a\, n\, b} \ldots,$ for $a \in S, b \in \{n-3, n-2\}$,

(L^{34}_{ex}.6b) $\ldots \underline{a\, n\, b}\, \bar{S}\, (n-1) \ldots,$ for $a, b \in \{1, \ldots, n-2\}, a \notin S, \{a, b\} \cap \{n-3, n-2\} \neq \emptyset, a \neq b$,

(L^{34}_{ex}.7) $\ldots \underline{(n-1)\, a\, n}\, \bar{S} \ldots,$ for $a \in \{1, \ldots, n-2\}$,

(L^{34}_{ex}.8) $\ldots \underline{n\, a\, (n-1)}\, \bar{S} \ldots,$ for $a \in \{1, \ldots, n-2\}$.

In comparison to the proof of Theorem 2.2 we build exactly one tour less in step one and the same number of tours in step two and three. This finishes the proof of Lemma 3.18.□

Proof of Theorem 3.21. The proof consists of the same structure and the same notation is used as in the proofs of theorems 2.2 and 3.19. Fulfilling $\sum_{ijk \in V^{(3)}:\, i \in S, j,k \in V \setminus S} y_{ijk} + \sum_{i\bar{t}j \in V^{(3)}:\, i,j \in S} y_{i\bar{t}j} = 1$ means that if all nodes that belong to set S are deleted from a tour, the tour decomposes into single nodes and at most one directed path of more than one node and node $\bar{t}$ belongs to that path. Again $\bar{S}$ denotes all nodes of set S that are not explicitly mentioned.

First we consider the case $|V \setminus S| \geq 6$. We set, w. l. o. g., $T := V \setminus S = \{t_1 = 1, \ldots, t_{|T|-1}, t_{|T|} = \bar{t}\}$ and $S = \{s_1 = |T|+1, \ldots, s_{|S|} = n\}$. Set $C^{\bar{n},1}_{dim}$ is constructed in the same way as in the proof of Theorem 2.2 setting $\bar{n} = 6$. As long as $k \in T$ in the inductive part of step two the desired structure is obtained automatically if steps (**I1**)–(**I5**) are used. But adaptations are needed for $k \in S$. More precisely, we introduce, similar to the proof of Theorem 3.19, specific steps for $k = s_1, k = s_2, k = s_3$ and $k = s_i, i \geq 4$. To emphasize the correspondence between the construction used in the proof of Theorem 2.2 and the construction steps here these are denoted by the original step numbers and an additional counter each.

($I^{s_1}_{ext}$.1a) $\ldots \underline{a\, s_1\, b}\, s_2\, \varpi_k\, n \ldots,$ for $a, b \in T, b \neq \bar{t}, a \neq b$
(The missing 2-arcs $(a, s_1, \bar{t}), a \in T \setminus \{1\}$, are compensated in **($I^{s_1}_{ext}$.1b)** and 2-arc $(1, s_1, \bar{t})$ is used for patching in **($I^{s_1}_{ext}$.3a)**. Furthermore the 2-arc $(s_1, \bar{t}, s_2)$ is not used in any of the constructed tours. We will see at the end of this proof that the tours can be constructed in such a way that all 2-arcs $(s_i, \bar{t}, s_j), i, j = 1, \ldots, |S|, i \neq j$, appear only once in the whole process and that they can so be used for building up the triangular matrix structure of the corresponding incidence vectors.),

($\mathsf{I}^{s_1}_{\mathsf{ext}}$.2) $\ldots \underline{a\,b\,s_2}\,\varpi_k\, n\, 1\, s_1\, 2\, 3 \ldots,$ for $a, b \in T \setminus \{1,2,3\}, a \neq b$
(the 2-arcs $(n,1,s_1),(s_1,2,3)$ are not used as a^i_k),

($\mathsf{I}^{s_1}_{\mathsf{ext}}$.3a) $\ldots m\, o\, s_2\, \varpi_k\, n\, 1\, \underline{s_1\, a\, b} \ldots,$ for $a, b \in T \setminus \{1\}, |\{a,b\} \cap \{2,3\}| \geq 1, (a,b) \neq (2,3)$, with $m, o \in T \setminus \{1,2,3\}, |\{a,b,m,o\}| = 4$
(the 2-arc $(1, s_1, \bar{t})$ is not used as an a^i_k),

($\mathsf{I}^{s_1}_{\mathsf{ext}}$.4a) $\ldots \underline{a\,b\,s_2}\,\varpi_k\, n\, 1\, s_1\, m\, o \ldots,$ for $a, b \in T \setminus \{1\}, |\{a,b\} \cap \{2,3\}| = 1$, with $m, o \in T \setminus \{1\}, |\{m,o\} \cap \{2,3\}| = 1, |\{a,b,m,o\}| = 4$,

($\mathsf{I}^{s_1}_{\mathsf{ext}}$.3b) $\ldots m\, o\, s_2\, \varpi_k\, n\, 1\, \underline{s_1\, a\, b} \ldots,$ for $a, b \in T \setminus \{1,2,3\}$ with $m, o \in T \setminus \{1\}, \{m,o\} \neq \{2,3\}, |\{a,b,m,o\}| = 4$,

($\mathsf{I}^{s_1}_{\mathsf{ext}}$.4b) $\ldots \underline{a\,b\,s_2}\,\varpi_k\, n\, 1\, s_1\, 4\, 5 \ldots,$ for $a, b \in \{2,3\}, a \neq b$,

($\mathsf{I}^{s_1}_{\mathsf{ext}}$.5a) $\ldots m\, o\, s_2\, \varpi_k\, \underline{n\, s_1\, a}\, p \ldots,$ for $a \in T \setminus \{1\}$ with $m, o, p \in T \setminus \{1\}, |\{a,m,o,p\}| = 4$,

($\mathsf{I}^{s_1}_{\mathsf{ext}}$.4c) $\ldots \underline{a\,b\,s_2}\,\varpi_k\, n\, s_1 \ldots,$ for $a, b \in T, 1 \in \{a,b\}, a \neq b$,

($\mathsf{I}^{s_1}_{\mathsf{ext}}$.5b) $\ldots \bar{t}\, 1\, s_2\, \varpi_k\, \underline{n\, a\, s_1} \ldots,$ for $a \in T \setminus \{1, \bar{t}\}$
(the missing 2-arc $(n, \bar{t}, s_1)$ is compensated in (**$\mathsf{L}_{\bar{t}}$.1**)),

($\mathsf{I}^{s_1}_{\mathsf{ext}}$.1b) $\ldots 1\, s_2\, \varpi_k\, n\, \underline{a\, s_1\, \bar{t}} \ldots,$ for $a \in T \setminus \{1, \bar{t}\}$,

($\mathsf{I}^{s_1}_{\mathsf{ext}}$.3c) $\ldots s_2\, \varpi_k\, n\, m\, \underline{s_1\, a\, b} \ldots,$ for $a, b \in T, 1 \in \{a,b\}$, with $m \in T \setminus \{1, \bar{t}\}, |\{a,b,m\}| = 3$,

($\mathsf{I}^{s_1}_{\mathsf{ext}}$.5c) $\ldots s_2\, \varpi_k\, \underline{n\, s_1\, 1} \ldots.$

In comparison to the proof of Theorem 2.2 we constructed exactly one tour less if we attribute one tour additionally for the 2-arc $(s_1, \bar{t}, s_2)$ that is preserved here for later steps and used in (**$\mathsf{L}_{\bar{t}}$.1**). It is easy to check that all underlined 2-arcs have not been used in previous tours and that the construction is possible for $\bar{n} = 6$. If $|T| = 5$ node $6 \in S$ belongs to the $\bar{n}$-permutation block and so these steps for $k = s_1$ can be neglected. But due to the desired structure the cardinality of $C^{\bar{n},1}_{dim}$ reduces by three. Fortunately the extra step (**$\mathsf{L}_{\bar{t}}$.1**) at the end of the whole construction described above allows to raise this number by two since the appearance of the 2-arcs $(n, \bar{t}, s_1), (s_1, \bar{t}, s_2)$ can be prevented until that step.

From now on the steps are equal for the two cases $|V \setminus S| = 5$ and $|V \setminus S| \geq 6$. For $k = s_2$ the 2-arcs that are used for building the triangular matrix structure are restricted to specific types, all missing 2-arcs that do not have the type $(\tilde{a}, \bar{t}, \tilde{b}), \tilde{a}, \tilde{b} \in S, \tilde{a} \neq \tilde{b}$, are used in the construction steps for $k = s_3$.

($\mathsf{I}^{s_2}_{\mathsf{ext}}$.1a) $\ldots \underline{a\, s_2\, b}\, s_3\, \varpi_k\, n\, \bar{S} \ldots,$ for $a, b \in \{1, \ldots, s_1\}, b \neq \bar{t}, a \neq b$
(The 2-arc $(s_2, \bar{t}, s_3)$ is not used here and will be counted in (**$\mathsf{L}_{\bar{t}}$.1**).),

($\mathsf{I}^{s_2}_{\mathsf{ext}}$.2) $\ldots \underline{a\, b\, s_3}\, \varpi_k\, n\, m\, s_1\, s_2\, 1\, 2 \ldots,$ for $a, b \in T \setminus \{1,2\}$ with $m \in T \setminus \{1, 2, \bar{t}\}, |\{a,b,m\}| = 3$
(the 2-arc $(s_2, 1, 2)$ is not used as an a^i_k),

($\mathsf{I}^{s_2}_{\mathsf{ext}}$.5a) $\ldots s_3\, \varpi_k\, \underline{n\, s_1\, s_2}\, 1\, 2 \ldots,$

($\mathsf{I}^{s_2}_{\mathsf{ext}}$.3a) $\ldots m\, o\, s_3\, \varpi_k\, n\, s_1\, \underline{s_2\, a\, b} \ldots,$ for $a, b \in T, \{a,b\} \cap \{1,2\} \neq \emptyset, (a,b) \neq (1,2)$, with $m, o \in T \setminus \{1,2\}, |\{a,b,m,o\}| = 4$
(the 2-arc $(s_1, s_2, \bar{t})$ is not used as an a^i_k),

($\mathsf{I}^{s_2}_{\mathsf{ext}}$**.4a)** $\ldots \underline{a\,b\,s_3}\,\varpi_k\,n\,s_1\,s_2\,m\,o\ldots$, for $a,b \in T, |\{a,b\} \cap \{1,2\}| = 1$, with $m,o \in T$, $|\{m, o\} \cap \{1,2\}| = 1$, $|\{a,b,m,o\}| = 4$,

($\mathsf{I}^{s_2}_{\mathsf{ext}}$**.3b)** $\ldots m\,1\,s_3\,\varpi_k\,n\,s_1\,\underline{s_2\,a\,b}\ldots$, for $a,b \in T\backslash\{1,2\}$ with $m \in T\backslash\{1,2\}, |\{a,b,m\}| = 3$,

($\mathsf{I}^{s_2}_{\mathsf{ext}}$**.4b)** $\ldots \underline{a\,b\,s_3}\,\varpi_k\,n\,s_1\,s_2\ldots$, for $a,b \in \{1,2\}, a \neq b$,

($\mathsf{I}^{s_2}_{\mathsf{ext}}$**.1b)** $\ldots s_3\,\varpi_k\,n\,s_1\,\underline{a\,s_2\,\bar{t}}\ldots$, for $a \in T \setminus \{\bar{t}\}$.

The 2-arcs $(s_1,a,s_3),(a,s_1,s_3),(s_2,s_1,a),(s_2,a,s_1),(n,s_2,a),(n,a,s_2), a \in T$, and (n, s_2,s_1) are not contained in any of the tours presented above. Their usage is deferred to the case $k = s_3$ that follows next and to ($\mathsf{L}_{\bar{t}}$**.1**).

($\mathsf{I}^{s_3}_{\mathsf{ext}}$**.1a)** $\begin{cases} \ldots \underline{a\,s_3\,b}\,s_4\,\varpi_k\,n\,s_1\,\bar{S}\ldots, & \text{for } a,b \in T \cup \{s_2\}, b \neq \bar{t}, a \neq b, \\ \ldots \bar{S}\,\underline{a\,s_3\,s_1}\,s_4\,\varpi_k\,n\ldots, & \text{for } a \in (T \cup \{s_2\}) \setminus \{\bar{t}\}, \\ \ldots s_2\,\underline{s_1\,s_3\,a}\,s_4\,\varpi_k\,n\ldots, & \text{for } a \in T \setminus \{\bar{t}\}, \end{cases}$

($\mathsf{I}^{s_3}_{\mathsf{ext}}$**.2)** $\ldots \underline{a\,b\,s_4}\,\varpi_k\,n\,s_1\,s_2\,s_3\,1\,2\ldots$, for $a,b \in T \setminus \{1,2\}, a \neq b$
(the 2-arc $(s_3,1,2)$ is not used as an a^i_k),

($\mathsf{I}^{s_3}_{\mathsf{ext}}$**.3a)** $\ldots m\,o\,s_4\,\varpi_k\,n\,s_1\,s_2\,\underline{s_3\,a\,b}\ldots$, for $a,b \in T, (a,b) \neq (1,2), \{a,b\} \cap \{1,2\} \neq \emptyset$, with $m,o \in T \setminus \{1,2\}, |\{a,b,m,o\}| = 4$,

($\mathsf{I}^{s_3}_{\mathsf{ext}}$**.4a)** $\ldots \underline{a\,b\,s_4}\,\varpi_k\,n\,s_1\,s_2\,s_3\,m\,o\ldots$, for $a,b \in T, |\{a,b\} \cap \{1,2\}| = 1$, with $m,o \in T$, $|\{m,o\} \cap \{1,2\}| = 1, |\{a,b,m,o\}| = 4$,

($\mathsf{I}^{s_3}_{\mathsf{ext}}$**.3b)** $\ldots m\,1\,s_4\,\varpi_k\,n\,s_1\,s_2\,\underline{s_3\,a\,b}\ldots$, for $a,b \in T \setminus \{1,2\}$ with $m \in T \setminus \{1,2\}, |\{a,b, m\}| = 3$,

($\mathsf{I}^{s_3}_{\mathsf{ext}}$**.4b)** $\ldots \underline{a\,b\,s_4}\,\varpi_k\,n\,s_1\,s_2\,s_3\ldots$, for $a,b \in \{1,2\}, a \neq b$,

($\mathsf{I}^{s_2}_{\mathsf{ext}}$**.5b)** $\ldots s_4\,\varpi_k\,\underline{n\,s_2\,s_1}\,s_3\,1\ldots$,

($\mathsf{I}^{s_2}_{\mathsf{ext}}$**.3c)** $\ldots s_3\,s_4\,\varpi_k\,n\,\underline{s_2\,s_1\,a}\ldots$, for $a \in T$,

($\mathsf{I}^{s_2}_{\mathsf{ext}}$**.5c)** $\ldots s_3\,s_4\,\varpi_k\,\underline{n\,a\,s_2}\,s_1\ldots$, for $a \in T \setminus \{\bar{t}\}$,

($\mathsf{I}^{s_2}_{\mathsf{ext}}$**.3d)** $\ldots s_3\,s_4\,\varpi_k\,n\,m\,\underline{s_2\,a\,s_1}\ldots$, for $a \in T \setminus \{\bar{t}\}$ with $m \in T \setminus \{\bar{t}\}, m \neq a$,

($\mathsf{I}^{s_2}_{\mathsf{ext}}$**.5d)** $\ldots s_3\,s_4\,\varpi_k\,\underline{n\,s_2\,a}\,s_1\ldots$, for $a \in T \setminus \{\bar{t}\}$

($\mathsf{I}^{s_3}_{\mathsf{ext}}$**.4c)** $\ldots \underline{a\,b\,s_4}\,\varpi_k\,n\,m\,s_2\,s_3\ldots$, for $a,b \in (T \cup \{s_1\}), s_1 \in \{a,b\}, (a,b) \neq (s_1,\bar{t})$, with $m \in T \setminus \{\bar{t}\}, |\{a,b,m\}| = 3$,

($\mathsf{I}^{s_3}_{\mathsf{ext}}$**.5a)** $\begin{cases} \ldots s_2\,m\,s_1\,s_4\,\varpi_k\,\underline{n\,s_3\,a}\ldots, & \text{for } a \in T \text{ with } m \in T \setminus \{\bar{t}\}, m \neq a, \\ \ldots \bar{t}\,s_2\,m\,s_1\,s_4\,\varpi_k\,\underline{n\,a\,s_3}\ldots, & \text{for } a \in T \setminus \{\bar{t}\} \text{ with } m \in T \setminus \{\bar{t}\}, m \neq a, \end{cases}$

($\mathsf{I}^{s_3}_{\mathsf{ext}}$**.4d)** $\begin{cases} \ldots \underline{a\,b\,s_4}\,\varpi_k\,n\,s_3\ldots, \text{ for } a,b \in \{s_1,s_2\}, a \neq b, \\ \ldots s_1\,\underline{s_2\,a\,s_4}\,\varpi_k\,n\,s_3\ldots, \text{ for } a \in T \setminus \{\bar{t}\}, \\ \ldots s_1\,\underline{a\,s_2\,s_4}\,\varpi_k\,n\,s_3\ldots, \text{ for } a \in T \setminus \{\bar{t}\}, \end{cases}$

($\mathsf{I}^{s_2}_{\mathsf{ext}}$**.4c)** $\ldots m\,s_2\,s_4\,\varpi_k\,n\,\underline{s_1\,a\,s_3}\,o\ldots$, for $a \in T \setminus \{\bar{t}\}$ with $m,o \in T \setminus \{\bar{t}\}, |\{a,m,o\}| = 3$,

($\mathsf{I}^{s_3}_{\mathsf{ext}}$**.4e)** $\ldots \underline{\bar{t}\,s_2\,s_4}\,\varpi_k\,n\,s_1\,1\,s_3\,2\ldots$,

($\mathsf{I}^{s_2}_{\mathsf{ext}}$**.4d)** $\ldots s_2\,s_4\,\varpi_k\,n\,\underline{a\,s_1\,s_3}\,m\ldots$, for $a \in T \setminus \{\bar{t}\}$ with $m \in T \setminus \{\bar{t}\}, m \neq a$,

($\mathsf{I}^{s_3}_{\mathsf{ext}}$.1b) $\ldots 1\, \underline{s_1\, s_3\, s_2}\, s_4\, \varpi_k\, n \ldots,$

($\mathsf{I}^{s_2}_{\mathsf{ext}}$.4e) $\ldots \underline{\bar{t}\, s_1\, s_3}\, s_2\, s_4\, \varpi_k\, n \ldots,$

($\mathsf{I}^{s_3}_{\mathsf{ext}}$.3c) $\ldots \bar{S}\, \underline{s_3\, a\, b}\, s_4\, \varpi_k\, n\, \bar{t} \ldots,$ for $a, b \in \{1, \ldots s_2\} \setminus \{\bar{t}\}, \{a, b\} \cap \{s_1, s_2\} \neq \emptyset, a \neq b,$

($\mathsf{I}^{s_3}_{\mathsf{ext}}$.5b) $\begin{cases} \ldots s_4\, \varpi_k\, \underline{n\, s_3\, a}\, \bar{S} \ldots, & \text{for } a \in \{s_1, s_2\}, \\ \ldots \bar{t}\, s_4\, \varpi_k\, \underline{n\, a\, s_3}\, \bar{S} \ldots, & \text{for } a \in \{s_1, s_2\}, \end{cases}$

($\mathsf{I}^{s_3}_{\mathsf{ext}}$.1c) $\ldots \underline{\bar{t}\, s_3\, s_1}\, s_4\, \varpi_k\, n\, \bar{S} \ldots,$

($\mathsf{I}^{s_3}_{\mathsf{ext}}$.1d) $\ldots \bar{S}\, s_4\, \varpi_k\, n\, \underline{a\, s_3\, \bar{t}} \ldots,$ for $a \in (T \cup \{s_1\}) \setminus \{\bar{t}\},$

($\mathsf{I}^{s_2}_{\mathsf{ext}}$.5e) $\ldots \bar{S}\, s_4\, \varpi_k\, \underline{n\, s_2\, \bar{t}} \ldots,$

($\mathsf{I}^{s_3}_{\mathsf{ext}}$.3d) $\ldots \bar{S}\, s_4\, \varpi_k\, n\, \underline{s_3\, a\, \bar{t}} \ldots,$ for $a \in \{s_1, s_2\}.$

For $k = s_i, i \geq 4$ the usage of some 2-arcs is deferred to later steps or to ($\mathsf{L}_{\bar{t}}$.**1**) in comparison to the proof of Theorem 3.19.

($\mathsf{I}^{s_i}_{\mathsf{ext}}$.1a) $\ldots \underline{a\, s_i\, b}\, s_{i+1}\, \varpi_k\, n\, \bar{S} \ldots,$ for $a, b \in \{1, \ldots, s_{i-1}\}, b \neq \bar{t}, a \neq b$
(as unused here the 2-arc $(s_i, \bar{t}, s_{i+1})$ will be counted in ($\mathsf{L}_{\bar{t}}$.**1**)),

($\mathsf{I}^{s_i}_{\mathsf{ext}}$.2) $\ldots \underline{a\, b\, s_{i+1}}\, \varpi_k\, n\, m\, \bar{S}\, s_i\, 1\, 2 \ldots,$ for $a, b \in \{3, \ldots, s_{i-1}\}, (a, b) \notin \{(i, \bar{t}) \colon i \in S\}, \{a, b\} \not\subset T \setminus \{\bar{t}\}$, with $m \in T \setminus \{1, 2, \bar{t}\}, |\{a, b, m\}| = 3,$
(the 2-arc $(s_i, 1, 2)$ is not used as an a^j_k)

($\mathsf{I}^{s_i}_{\mathsf{ext}}$.3a) $\ldots m\, o\, s_{i+1}\, \varpi_k\, n\, p\, \bar{S}\, s_{i-1}\, \underline{s_i\, a\, b} \ldots,$ for $a, b \in \{1, \ldots, s_{i-2}\}$, $(a, b) \notin \{(\bar{t}, i) \colon i \in S\}$, $(a, b) \neq (1, 2)$, with $m, o \in \{3, \ldots, s_{i-2}\}, p \in \{1, \ldots, s_{i-2}\} \setminus \{\bar{t}\}, (m, o) \notin \{(i, \bar{t}) \colon i \in S\}, \{m, o\} \not\subset T \setminus \{\bar{t}\}, |\{a, b, m, o, p\}| = 5,$
(the 2-arc $(s_{i-1}, s_i, \bar{t})$ is not used as an a^j_k),

($\mathsf{I}^{s_i}_{\mathsf{ext}}$.4) $\begin{cases} \ldots \underline{a\, b\, s_{i+1}}\, \varpi_k\, n\, p\, \bar{S}\, s_i\, m\, o \ldots, & \text{for } a, b \in \{1, \ldots, s_{i-1}\}, 1 \in \{a, b\}, a \neq b, \\ \ldots \underline{a\, b\, s_{i+1}}\, \varpi_k\, n\, p\, \bar{S}\, s_i\, m\, o \ldots, & \text{for } a, b \in \{2, \ldots, s_{i-1}\}, 2 \in \{a, b\}, a \neq b, \\ \ldots \underline{a\, b\, s_{i+1}}\, \varpi_k\, n\, p\, \bar{S}\, s_i\, m\, o \ldots, & \text{for } a, b \in T \setminus \{1, 2, \bar{t}\}, a \neq b, \end{cases}$
with $m, o, p \in T, m \neq \bar{t}, p \neq \bar{t}, |\{a, b, m, o, p\}| = 5,$

($\mathsf{I}^{s_i}_{\mathsf{ext}}$.5a) $\begin{cases} \ldots \bar{S}\, s_{i+1}\, \varpi_k\, \underline{n\, s_i\, a} \ldots, & \text{for } a \in \{1, \ldots, s_{i-2}\}, \\ \ldots \bar{t}\, \bar{S}\, s_{i+1}\, \varpi_k\, \underline{n\, a\, s_i} \ldots, & \text{for } a \in \{1, \ldots, s_{i-1}\} \setminus \{\bar{t}\}, \end{cases}$

($\mathsf{I}^{s_i}_{\mathsf{ext}}$.1b) $\ldots \bar{S}\, s_{i+1}\, \varpi_k\, n\, \underline{a\, s_i\, \bar{t}} \ldots,$ for $a \in \{1, \ldots, s_{i-1}\} \setminus \{\bar{t}\},$

($\mathsf{I}^{s_i}_{\mathsf{ext}}$.3b) $\begin{cases} \ldots \bar{S}\, s_{i+1}\, \varpi_k\, n\, m\, \underline{s_i\, s_{i-1}\, a} \ldots, & \text{for } a \in \{1, \ldots, s_{i-2}\}, \\ \ldots \bar{S}\, s_{i+1}\, \varpi_k\, n\, m\, \underline{s_i\, a\, s_{i-1}} \ldots, & \text{for } a \in \{1, \ldots, s_{i-2}\} \setminus \{\bar{t}\}, \end{cases}$
with $m \in T \setminus \{\bar{t}\}, m \neq a,$

($\mathsf{I}^{s_i}_{\mathsf{ext}}$.5b) $\bar{S}\, s_{i+1}\, \varpi_k\, \underline{n\, s_i\, s_{i-1}}\, \ldots.$

This finishes the second step. Note, if $|V \setminus S| \geq 6$ it is possible to perform the presented steps for $k = s_1, s_2, s_3$ and $k = s_i, 4 \leq i \leq |S| - 2$. If $|V \setminus S| = 5$ one only needs the steps with $k = s_2, s_3$ and $k = s_i, 4 \leq i \leq |S| - 2$.

It remains to adapt (**L1**)–(**L8**) resp. (L_{ex}.**1**)–(L_{ex}.**8**) and to introduce the extra step ($\mathsf{L}_{\bar{t}}$.**1**).

($\mathsf{L}_{\mathsf{ext}}$.1) $\ldots \bar{S}\, s_1\, s_2\, n\, \underline{n-1\, a\, b} \ldots,$ for $a, b \in \{1, \ldots, n-2\} \setminus \{s_1, s_2\}, (a, b) \notin \{(\bar{t}, i) \colon i \in S\}, a \neq b$
(the 2-arc (s_1, s_2, n) is not used as an a^i_L),

($\mathsf{L_{ext}.2a}$) $\ldots \underline{a\,b\,n}\,(n-1)\,m\,s_3\,\bar{S}\ldots$, for $a, b \in T, a \neq b$, with $m \in T \setminus \{\bar{t}\}, |\{a, b, m\}| = 3$,

($\mathsf{L_{ext}.2b}$) $\ldots \bar{S}\,\underline{a\,b\,n}\,(n-1)\,m\,o\ldots$, for $a, b \in \{1, \ldots, n-2\} \setminus \{\bar{t}\}, a \neq b, \{a, b\} \cap S \neq \emptyset, (a, b) \neq (s_1, s_2)$, with $m, o \in T, |\{a, b, m, o\}| = 4$,

($\mathsf{L_{ext}.3}$) $\ldots m\,\bar{S}\,n\,\underline{n-1\,a\,b}\ldots$, for $a, b \in \{1, \ldots, n-2\}, a \neq \bar{t}, \{a, b\} \cap \{s_1, s_2\} \neq \emptyset$, with $m \in T \setminus \{\bar{t}\}, |\{a, b, m\}| = 3$,

($\mathsf{L_{ext}.2c}$) $\ldots \underline{\bar{t}\,a\,n}\,(n-1)\,\bar{S}\ldots$, for $a \in S$,

($\mathsf{L_{ext}.4}$) $\begin{cases} \ldots s_1\,(n-1)\,s_2\,\bar{S}\,\underline{a\,n\,b}\ldots, & \text{for } a, b \in \{1, \ldots, n-2\} \setminus \{s_1, s_2\}, a \neq \bar{t}, a \neq b, \\ \ldots \underline{\bar{t}\,n\,a}\,s_1\,(n-1)\,s_2\,\bar{S}\ldots, & \text{for } a \in \{1, \ldots, n-2\} \setminus \{s_1, s_2\}, \end{cases}$
(the 2-arc $(s_1, n-1, s_2)$ is not used as an a_L^i),

($\mathsf{L_{ext}.5}$) $\begin{cases} \ldots \underline{a\,(n-1)\,b}\,\bar{S}\,m\,n\,o\ldots, & \text{for } a, b \in \{1, \ldots, n-2\}, b \neq \bar{t}, (a, b) \neq (s_1, s_2), \\ \ldots m\,n\,o\,\bar{S}\,\underline{a\,(n-1)\,b}\ldots, & \text{for } a \in \{1, \ldots, n-2\} \setminus \{\bar{t}\}, b = \bar{t}, \end{cases}$
with $m, o \in T \setminus \{\bar{t}\}, |\{a, b, m, o\}| = 4$,

($\mathsf{L_{ext}.6}$) $\begin{cases} \ldots (n-1)\,\bar{S}\,\underline{a\,n\,b}\ldots, & \text{for } a, b \in \{1, \ldots, n-2\}, \{a, b\} \cap \{s_1, s_2\} \neq \emptyset, \\ & a \neq \bar{t}, a \neq b, \\ \ldots \underline{\bar{t}\,n\,a}\,\bar{S}\,(n-1)\ldots, & \text{for } a \in \{s_1, s_2\}, \end{cases}$

($\mathsf{L_{ext}.7}$) $\ldots \underline{(n-1)\,a\,n}\,\bar{S}\ldots$, for $a \in \{1, \ldots, n-2\} \setminus \{\bar{t}\}$,

($\mathsf{L_{ext}.8}$) $\ldots \underline{n\,a\,(n-1)}\,\bar{S}\ldots$, for $a \in \{1, \ldots, n-2\} \setminus \{\bar{t}\}$.

Until now all 2-arcs $(\tilde{a}, \bar{t}, \tilde{b}), \tilde{a}, \tilde{b} \in S, \tilde{a} \neq \tilde{b}$, are unused. If a tour that defines a root of (18) contains one of these 2-arcs then no two nodes in T are allowed to be adjacent. Tours with this structure are built in the following extra step.

($\mathsf{L}_{\bar{t}}$**.1**) $\ldots \underline{a\,\bar{t}\,b}\,w_{ab}\ldots$, for $a, b \in S, a \neq b$, where w_{ab} denotes an appropriately completed alternating sequence of the remaining nodes in $T \setminus \{\bar{t}\}$ and in $S \setminus \{a, b\}$, possibly with a block of nodes of set S. Such an alternating sequence always exists because $|S| \geq \frac{n}{2}$.

It remains to calculate the number of the constructed tours. Therefore we compare the used a_k^j, a_L^j with the ones used in Theorem 2.2 with assigning the tours of ($\mathsf{L}_{\bar{t}}$**.1**) to the step the 2-arcs originally belong to. We concentrate on the case $|V \setminus S| \geq 6$, the case $|V \setminus S| = 5$ is similar because there are only slight differences in the construction. We get the same number of tours in step one. During the second step the situation depends on the value of k. For $k = s_1$ we cannot use 2-arc $(s_1, 1, 2)$ as an a_k^i originally, here the role of nodes $1, 2, 3$ has changed and we lost $(s_1, 2, 3)$. Furthermore $(1, s_1, \bar{t})$ and $(n, 1, s_1)$ are used for patching, in exchange the 2-arc $(s_1, \bar{t}, s_2)$ is not used here but we can assign one tour to this step because its use is deferred to ($\mathsf{L}_{\bar{t}}$**.1**). For $k \geq s_2$ we always lost $(s_i, 1, 2)$, like in (**I2**), and exactly one more 2-arc, precisely $(s_{i-1}, s_i, \bar{t})$, that can be compensated by the 2-arc $(s_i, \bar{t}, s_{i+1})$ in ($\mathsf{L}_{\bar{t}}$**.1**). In step three only the roles of some nodes are changed, *i. e.*, instead of $(1, 2, n), (1, n-1, 2)$ we use 2-arcs $(s_1, s_2, n), (s_1, n-1, s_2)$ for patching. All in all we get exactly $f(n)$ affinely independent tours and thus inequalities (18) are facet defining for $P_{\mathbf{AQTSP}_n}, n \geq 10$. □

Proof of Theorem 3.23, facetness.
The proof of the facetness of (20) is similar to the proof of Theorem 2.2. So we only mention the differences. To emphasize the correspondence between the construction used

in the proof of Theorem 2.2 and the construction steps here these are denoted by the original step numbers and an additional counter each. We set, w. l. o. g., $i_1 = n, i_2 = n-1, i_3 = n-2, i_4 = 1, \bar{n} = 6$. The tour construction of step one can be adopted without any changes because all tours define roots of (20) with $2x_{i_2 i_1} + x_{i_3 i_2} = 3$. As long as $k < n-2$ the same is true for all tours that are built in step two. We only have to adapt the construction for $k = n-2$. We start with the identical substeps (**I1**), (**I2**) and use the following.

($\mathsf{I}^{n-2}_{D_4-}$.3a) $\ldots \underline{(n-2)\,a\,b}\,m\,o\,(n-1)\,n \ldots$, for $a, b \in \{1, \ldots, n-3\}, (a, b) \neq (1, 2), 1 \in \{a, b\}$, with $m, o \in \{3, \ldots, n-3\}, |\{a, b, m, o\}| = 4$,

($\mathsf{I}^{n-2}_{D_4-}$.4) $\begin{cases} \ldots (n-2)\,1\,\underline{a\,b\,(n-1)}\,n \ldots, & \text{for } a, b \in \{2, \ldots, n-3\}, 2 \in \{a, b\}, a \neq b, \\ \ldots (n-2)\,\underline{a\,b\,(n-1)\,n} \ldots, & \text{for } a, b \in \{1, \ldots, n-3\}, 1 \in \{a, b\}, a \neq b, \end{cases}$

($\mathsf{I}^{n-2}_{D_4-}$.5) $\begin{cases} \ldots (n-1)\,\underline{n\,(n-2)\,a}\,1 \ldots, & \text{for } a \in \{2, \ldots, n-3\}, \\ \ldots (n-1)\,\underline{n\,(n-2)\,1} \ldots, & \\ \ldots (n-1)\,\underline{n\,a\,(n-2)}\,1 \ldots, & \text{for } a \in \{2, \ldots, n-3\}, \end{cases}$

($\mathsf{I}^{n-2}_{D_4-}$.3b) $\ldots (n-1)\,n\,1\,\underline{(n-2)\,a\,b} \ldots$, for $a, b \in \{2, \ldots, n-3\}, a \neq b$ (the 2-arc $(n, 1, n-2)$ is not used as an a^i_k).

Large modifications are needed in step three in order to ensure the root property of the tours.

(L_{D_4-}.1) $\ldots (n-2)\,1\,n\,\underline{(n-1)\,a\,b} \ldots$, for $a, b \in \{2, \ldots, n-3\}, a \neq b$ (the 2-arc $(n-2, 1, n)$ is not used as an a^i_L),

(L_{D_4-}.2a) $\ldots (n-2)\,\underline{a\,b\,n}\,(n-1)\,m\,o \ldots$, for $a, b \in \{1, \ldots, n-3\}, 1 \in \{a, b\}$, with $m, o \in \{2, \ldots, n-3\}, |\{a, b, m, o\}| = 4$,

(L_{D_4-}.2b) $\ldots \underline{a\,b\,n}\,(n-1)\,m\,o \ldots$, for $a, b \in \{1, \ldots, n-2\}, (n-2) \in \{a, b\}, (a, b) \neq (n-2, 1)$, with $m, o \in \{2, \ldots, n-3\}, |\{a, b, m, o\}| = 4$,

(L_{D_4-}.3a) $\begin{cases} \ldots (n-2)\,n\,\underline{(n-1)\,a\,b} \ldots, & \text{for } a, b \in \{1, \ldots, n-3\}, 1 \in \{a, b\}, a \neq b, \\ \ldots 1\,n\,\underline{(n-1)\,a\,b} \ldots, & \text{for } a, b \in \{2, \ldots, n-2\}, (n-2) \in \{a, b\}, a \neq b, \end{cases}$

(L_{D_4-}.2c) $\ldots \underline{a\,b\,n}\,(n-1)\,(n-2)\,m\,1 \ldots$, for $a, b \in \{2, \ldots, n-3\}$ with $m \in \{2, \ldots, n-3\}, |\{a, b, m\}| = 3$,

(L_{D_4-}.3b) $\ldots n\,\underline{(n-1)\,(n-2)\,1} \ldots$,

(L_{D_4-}.4) $\ldots 1\,(n-1)\,(n-2)\,\underline{a\,n\,b} \ldots$, for $a, b \in \{2, \ldots, n-3\}, a \neq b$ (the 2-arc $(1, n-1, n-2)$ is not used as an a^i_L),

(L_{D_4-}.5a) $\begin{cases} \ldots 1\,\underline{a\,(n-1)\,b}\,(n-2)\,m\,n\,o \ldots, & \text{for } a, b \in \{2, \ldots, n-3\}, \\ \ldots \underline{a\,(n-1)\,b}\,(n-2)\,m\,n\,o \ldots, & \text{for } a = 1, b \in \{2, \ldots, n-3\}, \\ \ldots 1\,\underline{a\,(n-1)\,b}\,m\,n\,o \ldots, & \text{for } a \in \{2, \ldots, n-3\}, b = (n-2), \end{cases}$
with $m, o \in \{2, \ldots, n-3\}, |\{a, b, m, o\}| = 4$,

(L_{D_4-}.7a) $\ldots (n-2)\,m\,\underline{(n-1)\,a\,n}\,o \ldots$, for $a \in \{2, \ldots, n-3\}$ with $m, o \in \{2, \ldots, n-3\}, |\{a, m, o\}| = 3$,

(L_{D_4-}.8a) $\ldots \underline{n\,a\,(n-1)}\,m\,(n-2)\,o\,1 \ldots$, for $a \in \{2, \ldots, n-3\}$ with $m, o \in \{2, \ldots, n-3\}, |\{a, m, o\}| = 3$,

(L_{D_4-}.6a) $\dots (n-1)\, m\, \underline{(n-2)\, n\, a} \dots,$ for $a \in \{2,\dots,n-3\}$ with $m \in \{2,\dots,n-3\}, m \neq a$,

(L_{D_4-}.6b) $\dots (n-1)\,(n-2)\, \underline{a\, n\, 1} \dots,$ for $a \in \{2,\dots,n-3\}$,

(L_{D_4-}.5b) $\dots \underline{(n-2)\,(n-1)}\, a\, n \dots,$ for $a \in \{2,\dots,n-3\}$,

(L_{D_4-}.6c) $\dots \underline{1\, n\, a}\,(n-1)\, m\,(n-2) \dots,$ for $a \in \{2,\dots,n-3\}$ with $m \in \{2,\dots,n-3\}, m \neq a$

(L_{D_4-}.6d) $\dots (n-1)\, \underline{a\, n\,(n-2)}\, m\, 1 \dots,$ for $a \in \{2,\dots,n-3\}$ with $m \in \{2,\dots,n-3\}, m \neq a$

(L_{D_4-}.5c) $\dots (n-2)\, n\, \underline{a\,(n-1)\, 1} \dots,$ for $a \in \{2,\dots,n-3\}$,

(L_{D_4-}.3c) $\dots \underline{(n-1)\, 1\,(n-2)}\, n \dots,$

(L_{D_4-}.7b) $\dots 1\, \underline{(n-1)\,(n-2)\, n} \dots, \quad \dots (n-2)\, 2\, \underline{(n-1)\, 1\, n} \dots,$

(L_{D_4-}.8b) $\dots 1\, 2\, \underline{n\,(n-2)\,(n-1)} \dots, \quad \dots (n-2)\, \underline{n\, 1\,(n-1)} \dots,$

(L_{D_4-}.6e) $\dots (n-1)\, \underline{(n-2)\, n\, 1} \dots, \quad \dots \underline{1\, n\,(n-2)}\,(n-1) \dots,$

(L_{D_4-}.5d) $\dots \underline{(n-2)\,(n-1)\, 1}\, n \dots.$

One can easily check that all constructed tours are roots of (20) and that all underlined 2-arcs are not used in a previous tours. During steps one and three we build exactly as many tours as in the proof of Theorem 2.2 but for $k = n-2$ in step two we get one tour less. All in all this sums up to $f(n)$ affinely independent tours and so inequalities (20) define facets of $P_{\mathbf{AQTSP}_n}$. ☐

Proof of Theorem 3.24. The validity of (21) except for the term $y_{i_3 i_2 i_1} + y_{i_3 i_1 i_2} + y_{i_2 i_4 i_3}$ follows from Observation 3.13, Observation 3.14 and the validity of the D_k-inequalities [16]. For $n \geq 5$ the presence of more than one of the corresponding 2-arcs $i_3 i_2 i_1, i_2 i_4 i_3$ and $i_3 i_1 i_2$ would imply a subtour or (1) would be violated. Let $n \geq 9$. There remain three cases:

- If $y_{i_3 i_2 i_1} = 1$ it holds $x_{i_1 i_2} = x_{i_2 i_3} = x_{i_3 i_4} = x_{i_4 i_1} = x_{i_1 i_3} = y_{i_3 i_2 i_4} = y_{i_4 i_2 i_1} = y_{i_3 i_1 i_2} = y_{i_2 i_4 i_3} = \sum_{j \in V \setminus \{i_1,i_2,i_3,i_4\}} (y_{i_1 j i_2} + y_{i_2 j i_3} + y_{i_3 j i_4} + y_{i_4 j i_1} + y_{i_1 j i_3}) = 0$ by (1), (3) and $2x_{i_1 i_4} + x_{i_4 i_3} + \sum_{j \in V \setminus \{i_1,i_2,i_3,i_4\}} (2y_{i_1 j i_4} + y_{i_4 j i_3}) \leq 2$ because otherwise a subtour would be implied. So the left-hand side of (21) is lower than or equal to three.

- If $y_{i_3 i_1 i_2} = 1$ it holds $x_{i_2 i_3} = x_{i_3 i_4} = x_{i_4 i_1} = x_{i_1 i_4} = x_{i_1 i_3} = y_{i_3 i_2 i_4} = y_{i_4 i_2 i_1} = y_{i_3 i_2 i_1} = y_{i_2 i_4 i_3} = \sum_{j \in V \setminus \{i_1,i_2,i_3,i_4\}} (y_{i_1 j i_2} + y_{i_2 j i_3} + y_{i_3 j i_4} + y_{i_4 j i_1} + y_{i_1 j i_4} + y_{i_1 j i_3}) = 0$ and $x_{i_1 i_2} = 1$. By $x_{i_4 i_3} + \sum_{j \in V \setminus \{i_1,i_2,i_3,i_4\}} y_{i_4 j i_3} \leq 1$ (see (11)) the validity of (21) follows in the considered case.

- If $y_{i_2 i_4 i_3} = 1$ it holds $x_{i_2 i_3} = x_{i_3 i_4} = x_{i_4 i_1} = x_{i_1 i_4} = x_{i_1 i_3} = y_{i_3 i_2 i_4} = y_{i_4 i_2 i_1} = y_{i_3 i_2 i_1} = y_{i_3 i_1 i_2} = \sum_{j \in V \setminus \{i_1,i_2,i_3,i_4\}} (y_{i_2 j i_3} + y_{i_3 j i_4} + y_{i_4 j i_1} + y_{i_1 j i_4} + y_{i_1 j i_3} + y_{i_4 j i_3}) = 0$ and $x_{i_4 i_3} = 1$. By $x_{i_1 i_2} + \sum_{j \in V \setminus \{i_1,i_2,i_3,i_4\}} y_{i_1 j i_2} \leq 1$ (see (11)) the validity of (21) follows.

The proof of the facetness of (21) is similar to the proof of Theorem 2.2. So we only mention the differences. To emphasize the correspondence between the construction used in the proof of Theorem 2.2 and the construction steps here these are denoted by the original step numbers and an additional counter each. For $n = 9$ we checked the facetness with a computer algebra system. So let $n \geq 10$. We set, w. l. o. g., $i_1 = n-1, i_2 = 1, i_3 = n, i_4 = n-2, \bar{n} = 6$. The tour construction of step one can be adopted without any changes because all tours define roots of (21) with $2x_{i_1 i_3} + x_{i_4 i_1} = 3$. As long as $k < n-2$ the same is true for all tours that are built in step two. We only have to adapt the construction for $k = n-2$. We start with the identical substep (**I1**) and use the following.

(**I$^{n-2}_{D_4+}$.2**) $\ldots \underline{a\,b\,(n-1)}\,n\,(n-2)\,1\,2 \ldots,$ for $a, b \in \{3, \ldots, n-3\}, a \neq b$
(the 2-arcs $(n, n-2, 1), (n-2, 1, 2)$ are not used as a^i_k),

(**I$^{n-2}_{D_4+}$.5a**) $\ldots m\,o\,(n-1)\,\underline{n\,a\,(n-2)}\,1\,2 \ldots,$ for $a \in \{3, \ldots, n-3\}$ with $m, o \in \{3, \ldots, n-3\}, |\{a, m, o\}| = 3$,

(**I$^{n-2}_{D_4+}$.3**) $\ldots m\,o\,(n-1)\,n\,p\,\underline{(n-2)\,a\,b} \ldots,$ for $a, b \in \{1, \ldots, n-3\}, (a, b) \neq (1, 2)$, with $m, o, p \in \{3, \ldots, n-3\}, |\{a, b, m, o, p\}| = 5$,

(**I$^{n-2}_{D_4+}$.5b**) $\ldots m\,o\,(n-1)\,\underline{n\,(n-2)\,a} \ldots,$ for $a \in \{2, \ldots, n-3\}$ with $m, o \in \{3, \ldots, n-3\}, |\{a, m, o\}| = 3$,

(**I$^{n-2}_{D_4+}$.4**) $\ldots \underline{a\,b\,(n-1)}\,n\,(n-2) \ldots,$ for $a, b \in \{1, \ldots, n-3\}, \{1, 2\} \cap \{a, b\} \neq \emptyset, a \neq b$,

(**I$^{n-2}_{D_4+}$.5c**) $\begin{cases} \ldots (n-1)\,\underline{n\,1\,(n-2)} \ldots, \\ \ldots (n-1)\,\underline{n\,2\,(n-2)} \ldots. \end{cases}$

Many modifications are needed in step three in order to ensure the root property of the tours.

(**L$_{D_4+}$.2a**) $\begin{cases} \ldots \underline{1\,a\,n}\,(n-1)\,(n-2)\,2 \ldots, & \text{for } a \in \{3, 4, \ldots, n-3\}, \\ \ldots \underline{a\,1\,n}\,(n-1)\,(n-2)\,2 \ldots, & \text{for } a \in \{3, 4, \ldots, n-3\}, \end{cases}$
(the 2-arc $(n-1, n-2, 2)$ is not used as an a^i_L),

(**L$_{D_4+}$.3a**) $\begin{cases} \ldots 1\,n\,\underline{(n-1)\,(n-2)\,a} \ldots, & \text{for } a \in \{3, \ldots, n-3\}, \\ \ldots 1\,n\,\underline{(n-1)\,a\,(n-2)} \ldots, & \text{for } a \in \{2, \ldots, n-3\}, \end{cases}$

(**L$_{D_4+}$.2b**) $\begin{cases} \ldots \underline{1\,2\,n}\,(n-1)\,(n-2) \ldots, \\ \ldots \underline{2\,1\,n}\,(n-1)\,(n-2) \ldots, \end{cases}$

(**L$_{D_4+}$.4**) $\ldots 1\,\underline{a\,n\,b}\,2\,(n-1)\,(n-2) \ldots,$ for $a, b \in \{3, \ldots, n-3\}, a \neq b$,
(the 2-arc $(2, n-1, n-2)$ is not used as an a^i_L),

(**L$_{D_4+}$.6a**) $\ldots \underline{1\,n\,a}\,2\,(n-1)\,(n-2) \ldots,$ for $a \in \{3, \ldots, n-3\}$,

(**L$_{D_4+}$.5a**) $\begin{cases} \ldots 1\,n\,m\,\underline{a\,(n-1)\,b}\,(n-2) \ldots, & \text{for } a, b \in \{2, \ldots, n-3\}, a \neq b, \\ \ldots 1\,n\,m\,\underline{a\,(n-1)\,b} \ldots, & \text{for } a \in \{3, \ldots, n-3\}, b = (n-2), \end{cases}$
with $m \in \{2, \ldots, n-3\}, |\{a, b, m\}| = 3$,

(**L$_{D_4+}$.6b**) $\begin{cases} \ldots \underline{1\,n\,2}\,3\,(n-1)\,(n-2) \ldots, & \\ \ldots 1\,\underline{2\,n\,a}\,m\,(n-1)\,(n-2) \ldots, & \text{for } a \in \{3, \ldots, n-3\} \text{ with} \\ \ldots 1\,\underline{a\,n\,2}\,m\,(n-1)\,(n-2) \ldots, & m \in \{3, \ldots, n-3\}, m \neq a, \end{cases}$

(**L$_{D_4+}$.7a**) $\ldots \underline{(n-1)\,a\,n}\,m\,(n-2)\,1 \ldots,$ for $a \in \{2, \ldots, n-3\}$ with $m \in \{2, \ldots, n-3\}, m \neq a$,

(**L$_{D_4+}$.8a**) $\ldots 1\,\underline{n\,a\,(n-1)}\,(n-2) \ldots,$ for $a \in \{2, \ldots, n-3\}$,

(L_{D_4+}.1) $\ldots 1\, n\, m\, (n-2)\, o\, \underline{(n-1)\, a\, b} \ldots$, for $a, b \in \{2, \ldots, n-3\}$ with $m, o \in \{2, \ldots, n-3\}$, $|\{a, b, m, o\}| = 4$,

(L_{D_4+}.3b) $\ldots \underline{(n-1)\, a\, 1}\, n\, m\, (n-2) \ldots$, for $a \in \{2, \ldots, n-3\}$ with $m \in \{2, \ldots, n-3\}, m \neq a$,

(L_{D_4+}.6c) $\begin{cases} \ldots (n-1)\, 2\, \underline{1\, n\, (n-2)} \ldots, & \\ \ldots (n-1)\, m\, 1\, \underline{a\, n\, (n-2)} \ldots, & \text{for } a \in \{2, \ldots, n-3\} \text{ with} \\ & m \in \{2, \ldots, n-3\}, m \neq a, \end{cases}$

(L_{D_4+}.2c) $\ldots \underline{a\, b\, n}\, (n-2)\, m\, (n-1)\, o\, 1 \ldots$, for $a, b \in \{2, \ldots, n-3\}$ with $m, o \in \{2, \ldots, n-3\}$, $|\{a, b, m, o\}| = 4$,

(L_{D_4+}.2d) $\ldots m\, (n-1)\, o\, \underline{(n-2)\, a\, n}\, p \ldots$, for $a \in \{2, \ldots, n-3\}$ with $m, o, p \in \{2, \ldots, n-3\}$, $|\{a, b, m, o, p\}| = 5$,

(L_{D_4+}.5b) $\ldots \underline{1\, (n-1)\, a}\, n\, (n-2) \ldots$, for $a \in \{2, \ldots, n-3\}$,

(L_{D_4+}.5c) $\ldots \underline{(n-2)\, (n-1)\, a}\, n\, m \ldots$, for $a \in \{2, \ldots, n-3\}$ with $m \in \{2, \ldots, n-3\}, m \neq a$,

(L_{D_4+}.3c) $\ldots (n-2)\, m\, n\, \underline{(n-1)\, 1\, a} \ldots$, for $a \in \{2, \ldots, n-3\}$ with $m \in \{2, \ldots, n-3\}, m \neq a$,

(L_{D_4+}.5d) $\ldots (n-2)\, m\, n\, \underline{a\, (n-1)\, 1} \ldots$, for $a \in \{2, \ldots, n-3\}$ with $m \in \{2, \ldots, n-3\}, m \neq a$,

(L_{D_4+}.6d) $\ldots (n-2)\, (n-1)\, \underline{a\, n\, 1} \ldots$, for $a \in \{2, \ldots, n-3\}$,

(L_{D_4+}.2e) $\ldots \underline{a\, (n-2)\, n}\, (n-1)\, 1 \ldots$, for $a \in \{2, \ldots, n-3\}$,

(L_{D_4+}.6e) $\ldots \underline{(n-2)\, n\, a}\, (n-1)\, 1 \ldots$, for $a \in \{2, \ldots, n-3\}$,

(L_{D_4+}.6f) $\ldots (n-1)\, 2\, \underline{(n-2)\, n\, 1} \ldots$,

(L_{D_4+}.8b) $\ldots \underline{n\, 1\, (n-1)}\, 2\, (n-2) \ldots, \quad \ldots \underline{n\, (n-2)\, (n-1)}\, 2\, 1 \ldots$,

(L_{D_4+}.7b) $\ldots \underline{(n-1)\, 1\, n}\, 2\, (n-2) \ldots, \quad \ldots \underline{(n-1)\, (n-2)\, n}\, 2\, 1 \ldots$,

(L_{D_4+}.2e) $\ldots (n-1)\, 2\, \underline{(n-2)\, 1\, n} \ldots, \quad \ldots (n-1)\, 2\, \underline{1\, (n-2)\, n} \ldots$,

(L_{D_4+}.3d) $\ldots \underline{(n-1)\, (n-2)\, 1}\, 2\, n \ldots, \quad \ldots \underline{(n-1)\, 1\, (n-2)}\, n \ldots$,

(L_{D_4+}.5e) $\ldots \underline{1\, (n-1)\, (n-2)}\, 2\, n \ldots, \quad \ldots \underline{(n-2)\, (n-1)\, 1}\, 2\, n \ldots$.

One can easily check that all constructed tours are roots of (21) and that all underlined 2-arcs are not used in a previous tours. During steps one and three we build exactly as many tours as in the proof of Theorem 2.2 but for $k = n-2$ in step two we get one tour less. All in all this sums up to $f(n)$ affinely independent tours and so inequalities (21) define facets of $P_{\mathbf{AQTSP}_n}$. $\square$