On (N, ϵ)-pseudospectra and quasi-diagonal operators

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Abstract

Abstract. In this paper we extend the concept of the (N, ϵ)-pseudospectra of Hansen to the case of bounded linear operators on Banach spaces. We apply our results to quasi-diagonal operators.

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Introduction

The (N, ϵ)-pseudospectrum In [12] the authors write "A computer working with finite accuracy cannot distinguish between a non-invertible matrix and an invertible matrix the inverse of which has a very large norm". Therefore one replaces the spectrum $\operatorname{sp} A$ of a bounded linear operator $A \in \mathcal{L}(\mathbf{X})$ on a Banach space $\mathbf{X}$ by so-called pseudospectra which reflect finite accuracy. For some pioneering work on that topic we refer to Landau [17], [18], Reichel, Trefethen [22] and Böttcher [3], and particularly point out the comprehensive monograph [25] of Trefethen and Embree.

Definition 1. For $N \in \mathbb{N}_0$ and $\epsilon > 0$ the (N, ϵ)-pseudospectrum of a bounded linear operator A on a complex Banach space $\mathbf{X}$ is defined as the set

$$\operatorname{sp}_{\mathrm{N},\epsilon} A := \{z \in \mathbb{C} : \|(A - zI)^{-2^N}\|^{2^{-N}} \geq 1/\epsilon\}.^1$$

Notice that (for $N = 0$) this definition of the (N, ϵ)-pseudospectrum includes the definition of the (classical) ϵ-pseudospectrum

$$\operatorname{sp}_\epsilon A := \{z \in \mathbb{C} : \|(A - zI)^{-1}\| \geq 1/\epsilon\}.$$

These ϵ-pseudospectra have gained attention after it was discovered in [22] and [3] that, on the one hand they approximate the spectrum but are less sensitive to perturbations, and on the other hand the ϵ-pseudospectra of discrete convolution operators mimic exactly the ϵ-pseudospectrum of an appropriate limiting operator, which is in general not true for the "usual" spectrum. See also [2], [5], [12], [4] and the references cited there.

Later on, Hansen [14], [15] introduced the above mentioned (N, ϵ)-pseudospectra for linear operators on separable Hilbert spaces and pointed out that they share several nice properties with case $N = 0$, but offer a better insight into the approximation of the spectrum. Furthermore, he discussed how the spectrum can be approximated numerically, based on the consideration of singular values of certain finite matrices.

[1] In this text we use the convention $\|B^{-1}\| = \infty$ if B is not invertible.

More precisely, the first mentioned result can be stated as follows.

Theorem 2. *Let $A \in \mathcal{L}(\mathbf{X})$. For every $\delta > \epsilon > 0$ there is an N_0 such that, for all $N \geq N_0$,*

$$B_\epsilon(\operatorname{sp} A) \subset \operatorname{sp}_{\mathrm{N},\epsilon} A \subset B_\delta(\operatorname{sp} A), \tag{1}$$

where $B_\epsilon(S) := \{z \in \mathbb{C} : \operatorname{dist}(z, S) \leq \epsilon\}$ denotes the closed ϵ-neighborhood of the set S.

In a very recent preprint [16] Hansen and Nevanlinna pointed out that the Banach space version of this result is in force, but the Hilbert space approach for the approximate determination of the (N, ϵ)-pseudospectrum via singular values cannot be extended to the Banach space case since there is no involution available anymore. Therefore we propose a modification which replaces the singular values by the injection and surjection modulus. Here comes the precise description:

Rectangular finite sections and their contribution to the approximation of spectra
For bounded linear operators $A \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$ on Banach spaces $\mathbf{X}, \mathbf{Y}$ we denote by $B_{\mathbf{X}}$ the closed unit ball in $\mathbf{X}$ and by

$$\begin{aligned} j(A) &:= \sup\{\tau \geq 0 : \|Ax\| \geq \tau\|x\| \text{ for all } x \in \mathbf{X}\}, \\ q(A) &:= \sup\{\tau \geq 0 : A(B_{\mathbf{X}}) \supset \tau B_{\mathbf{Y}}\} \end{aligned}$$

the injection modulus and the surjection modulus, respectively. Due to [20], B.3.8, it holds that

$$j(A) = \inf\{\|Ax\| : x \in \mathbf{X}, \|x\| = 1\}, \quad j(A^*) = q(A) \quad \text{and} \quad q(A^*) = j(A). \tag{2}$$

Hence $j, q : \mathcal{L}(\mathbf{X}) \to \mathbb{R}$ are continuous functions. Furthermore, we have $j(A) = q(A) = \|A^{-1}\|^{-1}$ if A is invertible, and $j(A)$ ($q(A)$) equals zero if A is not invertible from the left (right, respectively).

If there is a sequence (L_n) of compact projections whose norms $\|L_n\|$ equal 1 and which converge $*$-strongly to the identity [2] then we define the functions

$$\gamma_N^{m,n}(z) := \Big(\min\{j((L_n(A - zI)L_n)^{2^N} L_m), q(L_m(L_n(A - zI)L_n)^{2^N})\}\Big)^{2^{-N}} \tag{3}$$

where the arguments of the moduli $j(\cdot)$ and $q(\cdot)$ are regarded as operators in $\mathcal{L}(\operatorname{im} L_m, \operatorname{im} L_n)$ or $\mathcal{L}(\operatorname{im} L_n, \operatorname{im} L_m)$, respectively. We are interested in their sublevel sets

$$\Gamma_{N,\epsilon}^{m,n} := \{z \in \mathbb{C} : \gamma_N^{m,n}(z) \leq \epsilon\}$$

because we will get

Theorem 3. *Let (L_n) be a sequence of compact projections L_n of the norm 1 which converge $*$-strongly to the identity, and let $A \in \mathcal{L}(\mathbf{X})$. For every fixed $\beta > \epsilon > \alpha > 0$ and all sufficiently large $N \geq N_0$, $m \geq m_0(N)$ [3] and $n \geq n_0(N, m)$*

$$B_\alpha(\operatorname{sp} A) \subset \Gamma_{N,\epsilon}^{m,n} \subset B_\beta(\operatorname{sp} A). \tag{4}$$

[2] That is, (L_n) as well as the adjoints (L_n^*) converge strongly.
[3] $m_0(N)$ means that m_0 depends on N.

Notice that the projections L_n are of finite rank, thus the operators which have to be considered for the evaluation of $\gamma_N^{m,n}(z)$ are operators acting on finite dimensional spaces. In the classical setting of a Hilbert space with orthogonal projections L_n which are nested in the sense $L_nL_{n+1} = L_{n+1}L_n = L_n$ for all n, this particularly means that we only have to consider the smallest singular values $\sigma_1(\cdot)$ of certain finite matrices in order to determine $\Gamma_{N,\epsilon}^{m,n}$, since [21], Theorem 4.1 (see also [23], Corollary 2.12) yields that $j(B) = \sigma_1(B)$ and $q(B) = \sigma_1(B^\star)$,[4] hence

$$\Gamma_{N,\epsilon}^{m,n} = \left\{z \in \mathbb{C} : \min\{\sigma_1((L_n(A - zI)L_n)^{2^N} L_m), \sigma_1((L_n(A^\star - \bar{z}I)L_n)^{2^N} L_m)\} \leq \epsilon^{-2^{-N}}\right\}.$$

The paper [13] of Hansen contains an extensive discussion of this case including explicit algorithms and examples. So, we omit to repeat such details here and we focus on the theoretical treatment of the Banach space case.

The first part of this piece is devoted to the proofs of Theorems 2 and 3. In the second part we apply these results to quasi-diagonal operators and their finite sections.

Remark 4. Actually, the proofs also work (with some minor modifications) in the case that the Banach space $\mathbf{X}$ possesses a uniform approximate identity $\mathcal{P} = (P_n)$ with $\sup_n \|P_n\| = 1$ such that $\mathbf{X}$ has the $\mathcal{P}$-dichotomy, and the sequence (L_n) of $\mathcal{P}$-compact projections converges $\mathcal{P}$-strongly to the identity. In such a setting the asserted relation (4) is true for $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$. This particularly enables us to treat operators and settings where the sequence (L_n) we are interested in does not consist of compact projections or does not converge $*$-strongly, such as the typical finite section projections on l^∞, or L^p-spaces with $1 \leq p \leq \infty$. For the required definitions and details of the algebraic framework which replaces compactness and strong convergence by the more general $\mathcal{P}$-compactness and $\mathcal{P}$-strong convergence see [23] and the references cited there.

1 The main theorems

1.1 A level function for the (N, ϵ)-pseudospectrum

Define

$$\gamma_N(z) := \begin{cases} \|(A - zI)^{-2^N}\|^{-2^{-N}} & \text{if } z \notin \operatorname{sp} A \\ 0 & \text{if } z \in \operatorname{sp} A \end{cases}, \quad \text{and} \quad \gamma(z) := \operatorname{dist}(z, \operatorname{sp} A).$$

The function γ is continuous everywhere, equals zero in all points $z \in \operatorname{sp} A$, and

$$\gamma(z) = \frac{1}{\rho((A - zI)^{-1})} = \frac{1}{\lim_{N\to\infty} \|(A - zI)^{-2^N}\|^{2^{-N}}} = \lim_{N\to\infty} \gamma_N(z)$$

for every $z \notin \operatorname{sp} A$. The last equation is the definition of $\gamma_N(z)$, the second one is the Spectral Radius Formula, and for the first one consider

$$(A - zI)^{-1} - yI = (A - zI)^{-1}[I - y(A - zI)] = (A - zI)^{-1} y \left[\left(z + \frac{1}{y}\right) I - A\right].$$

We see that $(A - zI)^{-1} - yI$ is not invertible if and only if $z + \frac{1}{y}$ belongs to the spectrum of A, that is $\operatorname{sp}((A - zI)^{-1}) = \left\{\frac{1}{x-z} : x \in \operatorname{sp} A\right\}$, and thus

$$\rho((A - zI)^{-1}) = \sup\left\{\left|\frac{1}{x - z}\right| : x \in \operatorname{sp} A\right\} = \frac{1}{\inf\{|x - z| : x \in \operatorname{sp} A\}} = \frac{1}{\operatorname{dist}(z, \operatorname{sp} A)}.$$

[4] Note that $A^\star$ denotes the Hilbert adjoint, whereas A^* stands for the adjoint operator which acts on the dual space.

The γ_N are continuous in every $z \notin \operatorname{sp} A$ and (pointwise) monotonically increasing w.r.t. N since

$$\|(A-zI)^{-2^{N+1}}\|^{2^{-(N+1)}} \leq (\|(A-zI)^{-2^N}\|^2)^{2^{-(N+1)}} = \|(A-zI)^{-2^N}\|^{2^{-N}}. \tag{5}$$

Combining these observations we see that $0 \leq \gamma_N(z) \leq \gamma(z)$, the functions γ_N are continuous everywhere, and $\gamma_N(z)$ converges increasingly to $\gamma(z)$ for every $z \in \mathbb{C}$. By Dini's Theorem this even gives uniform convergence on every compact subset of $\mathbb{C}$.

Fix $\delta > \epsilon > 0$. It is clear by the definition that $z \in \operatorname{sp}_{\mathrm{N},\epsilon} A$ if and only if $\gamma_N(z) \leq \epsilon$. Choose $r > 0$ large enough to guarantee that $\gamma_N(z) > \epsilon$ for all $z \in \mathbb{C} \setminus U_r(0)$ and all N.[5] This is possible, since A is bounded, by a von Neumann series argument. Then we have uniform increasing convergence of $\gamma_N(z)$ to $\gamma(z)$ on $B_r(0)$. Thus, there is an N_0 such that for every $N \geq N_0$ and every $z \in \mathbb{C}$

$$\gamma(z) \leq \epsilon \;\Rightarrow\; \gamma_N(z) \leq \epsilon \;\Rightarrow\; \gamma(z) \leq \delta,$$

which yields (1) and finishes the proof of Theorem 2.[6]

1.2 Uniform approximations for $\gamma_N(z)$

Notice that

$$\gamma_N(z) = \Big(\min\{j((A-zI)^{2^N}), q((A-zI)^{2^N})\}\Big)^{2^{-N}}.$$

We use this representation as a starting point for the definition of approximating substitutes

$$\gamma_N^m(z) := \Big(\min\{j((A-zI)^{2^N} L_m), q(L_m(A-zI)^{2^N})\}\Big)^{2^{-N}},$$

where BL_m and L_mB are regarded as operators $BL_m : \operatorname{im} L_m \to \mathbf{X}$ and $L_mB : \mathbf{X} \to \operatorname{im} L_m$.

Proposition 5. *For every N and m, the functions $\gamma_N^m(z)$ are continuous w.r.t. $z \in \mathbb{C}$. Further, in every point $z \in \mathbb{C}$, the sequence $(\gamma_N^m(z))_{m\in\mathbb{N}}$ is bounded below by $\gamma_N(z)$ and converges to $\gamma_N(z)$. The convergence is even uniform on every compact subset of $\mathbb{C}$.*

We first state an auxilliary result.

Proposition 6. *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be Banach spaces and $A : \mathbf{X} \to \mathbf{Y}$ as well as $B : \mathbf{Y} \to \mathbf{Z}$ be bounded linear operators. Then $j(BA) \geq j(B)\, j(A)$ and $q(BA) \geq q(B)\, q(A)$.*

Proof. If one of the numbers $j(A), j(B), q(A), q(B)$ is equal to zero then the respective assertion is obviously true. So, let these four numbers be strictly positive and notice that

$$\begin{aligned} j(BA) = \inf\{\|BAx\| : x \in \mathbf{X}, \|x\| = 1\} &= \inf\left\{\frac{\|BAx\|}{\|Ax\|}\|Ax\| : x \in \mathbf{X}, \|x\| = 1\right\} \\ &\geq \inf\{\|By\| : y \in \mathbf{Y}, \|y\| = 1\} \inf\{\|Ax\| : x \in \mathbf{X}, \|x\| = 1\} = j(B)\, j(A). \end{aligned}$$

Further, let $\sigma < q(B)$ and $\tau < q(A)$. Then $\sigma B_{\mathbf{Z}} \subset B(B_{\mathbf{Y}})$ and $\tau B_{\mathbf{Y}} \subset A(B_{\mathbf{X}})$. Consequently,

$$\sigma\tau B_{\mathbf{Z}} \subset \tau B(B_{\mathbf{Y}}) = B(\tau B_{\mathbf{Y}}) \subset B(A(B_{\mathbf{X}})) = BA(B_{\mathbf{X}})$$

which shows that $\sigma\tau \leq q(BA)$ and finishes the proof. □

[5] Here $U_r(y)$ (and $B_r(y)$) stand for the open (closed) ball of radius r centered at y.
[6] This already appeared in [16].

Proof of Proposition 5. The continuity is obvious by the relations (2), and we immediately conclude from the previous result that $j(BL_m) \geq j(B)$ and $q(L_m B) \geq q(B)$ for every m. This particularly holds for $B := (A - zI)^{2^N}$ and hence we get the lower bound $\gamma_N(z)$ for $(\gamma_N^m(z))_m$ in every z.

If $B \in \mathcal{L}(\mathbf{X})$ is invertible then $j(B) = q(B) = \|B^{-1}\|^{-1}$. Given $\delta > 0$ choose $x \in \mathbf{X}$, $\|x\| = 1$, such that $\|B^{-1}\|^{-1} > \|B^{-1}x\|^{-1} - \delta$ and set $y := B^{-1}x$, $w := \|y\|^{-1}y$. Then

$$j(B) = q(B) = \|B^{-1}\|^{-1} > \frac{\|x\|}{\|B^{-1}x\|} - \delta = \frac{\|By\|}{\|y\|} - \delta = \|Bw\| - \delta.$$

The latter can be further estimated by

$$\begin{aligned} \|Bw\| &\geq \|BL_m w\| - \|B(I - L_m)w\| = \frac{\|BL_m w\|}{\|L_m w\|} - \frac{1 - \|L_m w\|}{\|L_m w\|}\|BL_m w\| - \|B(I - L_m)w\| \\ &\geq j(BL_m) - \frac{\|(I - L_m)w\|}{1 - \|(I - L_m)w\|}\|BL_m w\| - \|B\|\|(I - L_m)w\| \end{aligned} \tag{6}$$

where $\|(I - L_m)w\| \to 0$ as $m \to \infty$ since (L_m) converges $*$-strongly to the identity. Also notice that (L_n) is bounded by the Uniform Boundedness Principle. Since δ was chosen arbitrarily, we find that $j(BL_m) \to j(B)$ as $m \to \infty$. Plugging this in the estimate

$$\left(j((A - zI)^{2^N} L_m)\right)^{2^{-N}} \geq \gamma_N^m(z) \geq \gamma_N(z) = \left(j((A - zI)^{2^N})\right)^{2^{-N}}, \quad z \notin \operatorname{sp} A$$

we deduce that $\gamma_N^m(z)$ converges to $\gamma_N(z)$ for every $z \notin \operatorname{sp} A$.

Now, let B be not normally solvable, or let the kernel of B be non-trivial. Fix $\delta > 0$ and choose $w \in \mathbf{X}$, $\|w\| = 1$ such that $\|Bw\| < \delta$. As in the estimate (6) we obtain

$$\delta > \|Bw\| \geq j(BL_m) - \frac{\|(I - L_m)w\|}{1 - \|(I - L_m)w\|}\|BL_m w\| - \|B\|\|(I - L_m)w\|,$$

thus we can conclude $j(BL_m) \to 0$ as $m \to \infty$, again using the strong convergence of (L_m).

If the cokernel of B is non-trivial then the kernel of B^* is non-trivial and the above yields that $j(B^* L_m^*) \to 0$ as $m \to \infty$. The mapping $(\operatorname{im} L_m)^* \to \operatorname{im}(L_m^*)$, $g \mapsto g \circ L_m$ is an isometric isomorphism and, using (2), we find $q(L_m B) = j((L_m B)^*) = j(B^* L_m^*) \to 0$ as $m \to \infty$. Hence, we get pointwise convergence of the continuous functions $\gamma_N^m(z)$ to the continuous function $\gamma_N(z)$ on the whole complex plane.

Let $M \subset \mathbb{C}$ be a compact set. It remains to prove the uniform convergence of γ_N^m to γ_N on M. For this we construct a sequence (f_m) of continuous functions defined on M such that $f_m(z) \geq \gamma_N^m(z) \geq \gamma_N(z)$ and $f_m(z)$ converges decreasingly to $\gamma_N(z)$ for every $z \in M$. Then Dini's Theorem applies at least to (f_m), provides its uniform convergence, and hence also the uniform convergence of (γ_N^m). So, let us construct the desired functions (f_m). Fix $k \in \mathbb{N}$ and conclude from (6) that for each $B \in \mathcal{R} := \{(A - zI)^{2^N} : z \in M\}$ there is a $w \in \operatorname{im} L_k$, $\|w\| = 1$, such that

$$\begin{aligned} j(BL_k) &= \inf\{\|Bx\| : x \in \operatorname{im} L_k, \|x\| = 1\} \geq \|Bw\| - \frac{1}{2k} \\ &\geq j(BL_m) - \frac{\|(I - L_m)w\|}{1 - \|(I - L_m)w\|}\|BL_m w\| - \|B\|\|(I - L_m)w\| - \frac{1}{2k} \\ &\geq j(BL_m) - \frac{\|(I - L_m)L_k\|}{1 - \|(I - L_m)L_k\|}\|BL_m\| - \|B\|\|(I - L_m)L_k\| - \frac{1}{2k}. \end{aligned}$$

Hence, there is an $m_k \in \mathbb{N}$ such that, for every operator B in the bounded set $\mathcal{R}$ and every $m \geq m_k$, the estimate $j(BL_k) \geq j(BL_m) - 1/k$ holds. WLOG we can choose these m_k such that $m_k \geq 2k$ for every k, and we conclude

$$j(BL_k) + \frac{2}{k} \geq j(BL_m) + \frac{2}{m} \quad \text{for all } k \in \mathbb{N},\ m \geq m_k \text{ and } B \in \mathcal{R}.$$

The same arguments applied to $j(B^* L_k^*)$, together with the relations (2), yield numbers $\tilde{m}_k$ with

$$q(L_k B) + \frac{2}{k} \geq q(L_m B) + \frac{2}{m} \quad \text{for all } k \in \mathbb{N},\ m \geq \tilde{m}_k \text{ and } B \in \mathcal{R}.$$

Consequently, there is a strictly increasing sequence $(l_n) \subset \mathbb{N}$ such that

$$\min\{j(BL_{l_n}), q(L_{l_n}B)\} + \frac{2}{l_n} \geq \min\{j(BL_s), q(L_s B)\} + \frac{2}{s}$$

for all $n \in \mathbb{N}$, $B \in \mathcal{R}$ and $s \geq l_{n+1}$. We define the desired functions f_m by

$$f_m(z) := \left(\min\{j((A - zI)^{2^N} L_{l_n}), q(L_{l_n}(A - zI)^{2^N})\} + \frac{2}{l_n}\right)^{2^{-N}} \quad \text{for all} \quad l_{n+1} \leq m < l_{n+2}$$

and straightforwardly check that they meet all requirements. □

Until now, we have seen that the function γ can be approximated in a sense by the functions γ_N, and further the functions γ_N^m are approximations for the γ_N. As a third step we finally approximate γ_N^m by the functions $\gamma_N^{m,n}$ given in (3)

$$\gamma_N^{m,n}(z) = \left(\min\{j((L_n(A - zI)L_n)^{2^N} L_m), q(L_m(L_n(A - zI)L_n)^{2^N})\}\right)^{2^{-N}}.$$

For $A \in \mathcal{L}(\mathbf{X})$ it is clear that the sequence $((L_n(A - zI)L_n)^{2^N})_{n \in \mathbb{N}}$ as well as the sequence of the respective adjoint operators converge strongly, hence the arguments of $j(\cdot)$ and $q(\cdot)$ converge in the norm as $n \to \infty$ for every z since L_m is compact. This convergence is even uniform with respect to z on every compact set, since these arguments are polynomials in z whose (operator-valued) coefficients converge in the norm. This provides the uniform convergence $\gamma_N^{m,n}(z) \to \gamma_N^m(z)$ as $n \to \infty$ on every compact subset of $\mathbb{C}$.

Let M be a compact set. For given $\beta > \epsilon > \alpha > 0$ we choose N_0 large enough such that $\gamma(z) - \gamma_N(z) < (\beta - \epsilon)/2$ for all $z \in M$ and all $N \geq N_0$. Furthermore, for $N \geq N_0$, we choose $m_0(N)$ such that $\gamma_N^m(z) - \gamma_N(z) < (\epsilon - \alpha)/2$ for all $z \in M$ and all $m \geq m_0(N)$. Finally, we take $n_0(N, m)$ large enough to guarantee that for all $n \geq n_0(N, m)$ and $z \in M$ the estimate $|\gamma_N^{m,n}(z) - \gamma_N^m(z)| < \min\{(\epsilon - \alpha), (\beta - \epsilon)\}/2$ and, additionally, $j(L_n L_m), q(L_m L_n) \geq 1/2$ holds. Then,

$$\gamma(z) \leq \alpha \;\Rightarrow\; \gamma_N(z) \leq \alpha \;\Rightarrow\; \gamma_N^m(z) \leq \alpha + \frac{\epsilon - \alpha}{2} \;\Rightarrow\; \gamma_N^{m,n}(z) \leq \alpha + 2\frac{\epsilon - \alpha}{2} = \epsilon$$

and

$$\gamma_N^{m,n}(z) \leq \epsilon \;\Rightarrow\; \gamma_N^m(z) \leq \epsilon + \frac{\beta - \epsilon}{2} \;\Rightarrow\; \gamma_N(z) \leq \epsilon + \frac{\beta - \epsilon}{2} \;\Rightarrow\; \gamma(z) \leq \epsilon + 2\frac{\beta - \epsilon}{2} = \beta.$$

Consequently, for suitably chosen N, m and n,

$$B_\alpha(\operatorname{sp} A) \cap M \subset \Gamma_{N,\epsilon}^{m,n} \cap M \subset B_\beta(\operatorname{sp} A) \cap M.$$

As a final step, we eliminate the M from this relation to obtain the assertion (4) in Theorem 3. For this we only need to find a compact set M, such that $\gamma(z) > \beta$ and $\gamma_N^{m,n}(z) > \beta$ for all $z \notin M$ and N, m, n as above. Actually, Proposition 6 yields for arbitrary bounded linear operators B that

$$j(BL_nL_m) = j((BL_n)(L_nL_m)) \geq j(BL_n)j(L_nL_m) \quad \text{and} \quad q(L_mL_nB) \geq q(L_mL_n)q(L_nB),$$

where $j(L_nL_m), q(L_mL_n) \geq 1/2$ for all $n \geq n_0(N, m)$. Now it is immediate from the definitions of these functions and an estimate similar to (5) that, for $|z| > \|A\|$,

$$2\gamma_N^{m,n}(z) \geq \gamma_N^{n,n}(z) \geq \gamma_0^{n,n}(z) = |z| \|(L_n - z^{-1}L_nAL_n)^{-1}\|^{-1}_{\mathcal{L}(\operatorname{im} L_n)}$$

hence the set $M := B_{2\|A\|+4\beta}(0)$ does this job and completes the proof of Theorem 3.

1.3 Convergence of sets

Definition 7. The Hausdorff distance of two compact sets $S, T \subset \mathbb{C}$ is defined as

$$d_H(S,T) := \max\left\{\max_{s\in S} \operatorname{dist}(s,T), \max_{t\in T} \operatorname{dist}(t,S)\right\},$$

where $\operatorname{dist}(s,T) := \min_{t\in T} |s-t|$.

This function d_H forms actually a metric on the set of all non-empty compact subsets of $\mathbb{C}$, and offers an alternative view on the present results. Notice that the sets $\operatorname{sp} A$, $B_\epsilon(\operatorname{sp} A)$, $\operatorname{sp}_{\mathrm{N},\epsilon} A$ and $\Gamma_{N,\epsilon}^{n,m}$ are compact.

Corollary 8. *Let $A \in \mathcal{L}(\mathbf{X})$. Then*

$$\lim_{\epsilon\to 0} d_H(\operatorname{sp} A, B_\epsilon(\operatorname{sp} A)) = 0 \quad \textit{and} \quad \lim_{N\to\infty} d_H(B_\epsilon(\operatorname{sp} A), \operatorname{sp}_{\mathrm{N},\epsilon}(A)) = 0.$$

Thus, the (N,ϵ)-pseudospectra converge to the spectrum of A with respect to the Hausdorff distance if $N \to \infty$ and $\epsilon \to 0$. Moreover,

$$\lim_{N\to\infty} \limsup_{m\to\infty} \limsup_{n\to\infty} d_H(B_\epsilon(\operatorname{sp} A), \Gamma_{N,\epsilon}^{m,n}) = 0,$$

that is, the spectrum of A can be approximated w.r.t. the Hausdorff metric even by the sets $\Gamma_{N,\epsilon}^{m,n}$.

Proof. Firstly, note that for compact sets S, T, U with $S \subset T \subset U$ we have

$$d_H(S,T) = \max_{t\in T} \operatorname{dist}(t,S) \leq \max_{u\in U} \operatorname{dist}(u,S) = d_H(S,U).$$

Clearly, $d_H(\operatorname{sp} A, B_\epsilon(\operatorname{sp} A)) \leq \epsilon$ which proves the first assertion. For the second one apply Theorem 2 and the estimate

$$d_H(B_\epsilon(\operatorname{sp} A), \operatorname{sp}_{\mathrm{N},\epsilon}(A)) \leq d_H(B_\epsilon(\operatorname{sp} A), B_\delta(\operatorname{sp} A)) \leq \delta - \epsilon,$$

which holds for every fixed $\delta > \epsilon > 0$ and sufficiently large N. For the last assertion we employ Theorem 3 in the same way. □

Remark 9. We want to point out that the ϵ-pseudospectrum does not behave continuously with respect to the parameter ϵ in general, that is the inclusion

$$\operatorname{clos}\{z \in \mathbb{C} : \|(A - zI)^{-1}\| > 1/\epsilon\} \subset \operatorname{sp}_\epsilon A$$

can be proper. Shargorodsky addressed a paper [24] to this fact, where an explanation and appropriate examples are given in great detail. The point is that the resolvent norm can take constant values on open sets. Actually, the history of the investigation of this phenomenon is much longer. Globevnik [9] posed the question "Can $\|(\lambda e - a)^{-1}\|$ be constant on an open subset of the resolvent set" for an element a of a complex Banach algebra. He could only derive a partial answer, but he was able to show that the answer is "No" for the resolvent norm of a bounded linear operator on a complex uniformly convex Banach space.[7] Unfortunately, it remained rather unnoticed, and so this question emerged again in the 90's where, independently from the earlier, Böttcher asked this and, together with Daniluk, he tackled the case of bounded linear operators on Hilbert spaces in [3], Proposition 6.1 (see also [8]), and later on the L^p-spaces with $1 < p < \infty$ as well. Shargorodsky combined all these observations and completed the picture in [24], where he pointed out that Hilbert spaces and the L^p-spaces are complex uniformly convex by [7], and that the outcome even extends to Banach spaces which have a complex uniformly convex dual space. This particularly permits to cover all L^p-spaces with $p \in [1, \infty]$.

Of course, the phenomenon of jumping pseudospectra (if it exists in the underlying setting) is reflected in the behavior of the (N, ϵ)-pseudospectra as well but, as Theorem 2 and Corollary 8 show, it gets less significant in any case, since the difference of the respective sets

$$\operatorname{clos}\{z \in \mathbb{C} : \|(A - zI)^{-2^N}\|^{2^{-N}} > 1/\epsilon\} \subset \operatorname{sp}_{\mathrm{N},\epsilon} A$$

becomes small as N grows.

2 Quasi-diagonal operators

The above observation that we can approximate the spectrum with the help of "rectangular finite sections" raises hope that this may even be possible by the usual finite sections L_nAL_n. In fact, such a result is not true in general as a simple and well known example demonstrates: Let V denote the shift operator $(x_i)_{i\in\mathbb{Z}} \mapsto (x_{i+1})_{i\in\mathbb{Z}}$ on l^2. It is invertible, has norm 1 and its inverse has norm 1 as well. Hence $V - \alpha I$ is invertible whenever $|\alpha| \neq 1$ by a Neumann series argument. On the other hand, consider the projections L_n, $n \in \mathbb{N}$ given by the rule

$$L_n : (x_i) \mapsto (\ldots, 0, x_{-n}, \ldots, x_n, 0, \ldots).$$

The spectrum of every finite section L_nVL_n equals $\{0\}$ and also their (N, ϵ)-pseudospectra contain 0, whereas the (N, ϵ)-pseudospectra of V approximate the unit circle.

Actually, for the description of the limiting set of such pseudospectra further additional limit operators are needed. This is well understood for many classes of operators and their classical ϵ-pseudospectra (see [2], [5], [12], [19] and the references cited there). For the (N, ϵ)-pseudospectra this will be part of future work.

Here we turn our attention to a class of operators which accomplish our desire.

Definition 10. Let (L_n) be a sequence of compact projections on a Banach space $\mathbf{X}$ having norm 1 and tending $*$-strongly to the identity. An operator $A \in \mathcal{L}(\mathbf{X})$ is called quasi-diagonal with respect to (L_n) (or (L_n) is said to quasi-diagonalize A) if

$$\|[A, L_n]\| := \|AL_n - L_nA\| \to 0 \quad \text{as} \quad n \to \infty.$$

[7] for a definition see [10] or [24].

The notion of quasi-diagonality is due to Halmos [11] and was initially studied for bounded linear operators A on a separable Hilbert space $\mathbf{X}$, where the projections (L_n) were additionally supposed to be orthogonal. In such a setting, Halmos observed that A is quasi-diagonal w.r.t. a sequence (L_n) if and only if $A = B + K$ where B is a block diagonal operator with respect to some orthonormal basis of $\mathbf{X}$ and K is compact. This means that there exist compact and orthogonal projections $\tilde{L}_n$ tending strongly to I which asymptotically commute with A and even fulfill $\tilde{L}_{n+1}\tilde{L}_n = \tilde{L}_n\tilde{L}_{n+1} = \tilde{L}_n$ for every n.

Berg [1] pointed out that every normal operator is quasi-diagonal in this stricter sense, hence also every self-adjoint operator. So, this class contains a lot of interesting examples like almost Mathieu operators, discretized Hamiltonians, difference operators coming from PDE's with constant coefficients, certain weighted shifts, etc. We will not go into great detail here, but we refer to the paper of Brown [6] which extensively treats the Hilbert space case with a filtration (L_n) (i.e. compact orthogonal projections which are nested in the sense $L_nL_{n+1} = L_{n+1}L_n = L_n$ for all n) and includes the discussion of convergence rates for certain classes of quasi-diagonal operators as well as the definition of some explicit algorithms for two special cases. This paper is heavily based on the results of the Standard Algebra approach in [12], which provides the machinery to describe stability [8] and several further asymptotic properties of operator sequences, such as convergence of norms and condition numbers, in a Hilbert space setting. In particular, the convergence of the classical ϵ-pseudospectra of the finite sections L_nAL_n to the ϵ-pseudospectrum of A have been proved. Also Hansen [14] considered the spectral approximation of self-adjoint operators, taking their quasi-diagonality as well as their (N, ϵ)-pseudospectra into account.

Of course, the approach of the present text suggests now to work with the (N, ϵ)-pseudospectra and to do this for Banach spaces $\mathbf{X}$, under weaker assumptions on (L_n) than before, and for all quasi-diagonal operators. This is what we obtain in the following corollaries.

Corollary 11. *Let A be quasi-diagonal with respect to (L_n). Then*

$$\begin{aligned} 0 &= \lim_{\epsilon\to 0} d_H(\operatorname{sp} A, B_\epsilon(\operatorname{sp} A)) \\ &= \lim_{\epsilon\to 0}\lim_{N\to\infty} d_H(\operatorname{sp} A, \operatorname{sp}_{\mathrm{N},\epsilon} A) \\ &= \lim_{\epsilon\to 0}\lim_{N\to\infty}\limsup_{m\to\infty} d_H(\operatorname{sp} A, \operatorname{sp}_{\mathrm{N},\epsilon}(L_mAL_m)). \end{aligned}$$

Proof. The first and second equation are done by Corollary 8. For the last one note that $\Gamma^{m,m}_{N,\epsilon}$ equals $\operatorname{sp}_{\mathrm{N},\epsilon}(L_mAL_m)$ and that $\gamma_N^m - \gamma_N^{m,m}$ already converges uniformly to zero on every compact subset of $\mathbb{C}$ as $m \to \infty$ since A is quasi-diagonal. □

Corollary 12. *Let (L_n) quasi-diagonalize A and set $A_n := L_nAL_n \in \mathcal{L}(\operatorname{im} L_n)$. Then*

- $\|A_n\| \to \|A\|$ *and* $\|A_n^{-1}\| \to \|A^{-1}\|$ *as* $n \to \infty$.
- (A_n) *ist stable if and only if A is invertible. In this case* $\operatorname{cond} A_n \to \operatorname{cond} A$ *as* $n \to \infty$.

Proof. Recall that

$$\gamma_0(0) = \begin{cases} \|A^{-1}\|^{-1} & : A \text{ invertible} \\ 0 & : \text{ otherwise} \end{cases}, \quad \gamma_0^{n,n}(0) = \begin{cases} \|A_n^{-1}\|^{-1} & : A_n \text{ invertible} \\ 0 & : \text{ otherwise} \end{cases}.$$

[8] We say that the sequence (A_n) of the operators A_n is stable if there is an n_0 such that all A_n with $n \geq n_0$ are invertible and $\sup_{n\geq n_0} \|A_n^{-1}\| < \infty$.

Since Proposition 6 and the quasi-diagonality of A provide

$$\gamma_0(0) - \gamma_0^{n,n}(0) = (\gamma_0(0) - \gamma_0^n(0)) + (\gamma_0^n(0) - \gamma_0^{n,n}(0)) \to 0 \quad \text{as} \quad n \to \infty,$$

we see that $\|A_n^{-1}\| \to \|A^{-1}\|$ if A is invertible and $\|A_n^{-1}\| \to \infty$ otherwise. This particularly yields the stability criterion and it remains to consider $\|A_n\|$. To avoid confusion we explicitly specify the spaces for all the norms under consideration. Fix $\epsilon > 0$ and choose $x \in \mathbf{X}$, $\|x\|_{\mathbf{X}} = 1$ and n_0 such that $\|Ax\|_{\mathbf{X}} \geq \|A\|_{\mathcal{L}(\mathbf{X})} - \epsilon$ as well as $\|[L_n, A]\|_{\mathcal{L}(\mathbf{X})} \leq \epsilon$, $\|(I - L_n)Ax\|_{\mathbf{X}} \leq \epsilon$ and $\|(I - L_n)x\|_{\mathbf{X}} \leq \epsilon \|A\|_{\mathcal{L}(\mathbf{X})}^{-1}$ for all $n \geq n_0$. Then

$$\begin{aligned}\|A\|_{\mathcal{L}(\mathbf{X})} &\leq \|Ax\|_{\mathbf{X}} + \epsilon \leq \|L_nAx\|_{\mathbf{X}} + 2\epsilon \leq \|L_nAL_nx\|_{\mathbf{X}} + 3\epsilon \\ &\leq \|L_nAL_n\|_{\mathcal{L}(\operatorname{im} L_n)} \|L_nx\|_{\mathbf{X}} + 3\epsilon \leq \|L_nAL_n\|_{\mathcal{L}(\operatorname{im} L_n)} + 4\epsilon.\end{aligned}$$

On the other hand

$$\|L_nAL_n\|_{\mathcal{L}(\operatorname{im} L_n)} \leq \|AL_n\|_{\mathcal{L}(\operatorname{im} L_n, \mathbf{X})} + \epsilon = \|A\|_{\mathcal{L}(\operatorname{im} L_n, \mathbf{X})} + \epsilon \leq \|A\|_{\mathcal{L}(\mathbf{X})} + \epsilon.$$

Since ϵ is arbitrary, we see that $\|A_n\| \to \|A\|$ as $n \to \infty$ and easily finish the proof. □

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