Parameter estimation for multivariate exponential sums

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The recovery of signal parameters from noisy sampled data is an essential problem in digital signal processing. In this paper, we discuss the numerical solution of the following parameter estimation problem. Let h be a multivariate exponential sum, i.e., h is a finite linear combination of complex exponentials with pairwise different frequency vectors. Determine all parameters of h, i.e., all frequency vectors, all coefficients, and the number of exponentials, if finitely many equispaced sampled data of h are given. Using Ingham–type inequalities, the stability of the reconstructed exponential sum $\tilde{h}$ is discussed both in the square and uniform norm. Further we show that a rectangular Fourier–type matrix has a bounded condition number, if the frequency vectors are well–separated and if the number of samples is sufficiently large. Then we reconstruct the parameters of an exponential sum h by a novel algorithm, the sparse approximate Prony method (SAPM), where we use only some data sampled along few lines. The first part of SAPM estimates the frequency vectors by using the approximate Prony method in the univariate case. The second part of SAPM computes all coefficients by solving an overdetermined linear Vandermonde–type system. Numerical experiments show the performance of our method.

Key words and phrases: Parameter estimation, multivariate exponential sum, multivariate exponential fitting problem, harmonic retrieval, sparse approximate Prony method, sparse approximate representation of signals.

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1 Parameter estimation problem

Let the dimension $d \in \mathbb{N}$ and a positive integer $M \in \mathbb{N} \setminus \{1\}$ be given. We consider a *d-variate exponential sum of order* M that is a linear combination

$$h(\boldsymbol{x}) := \sum_{j=1}^{M} c_j\, \mathrm{e}^{\mathrm{i}\boldsymbol{f}_j \cdot \boldsymbol{x}} \quad (\boldsymbol{x} = (x_l)_{l=1}^d \in \mathbb{R}^d) \tag{1.1}$$

of M complex exponentials with complex coefficients $c_j \neq 0$ and pairwise different frequency vectors $\boldsymbol{f}_j = (f_{j,l})_{l=1}^d \in \mathbb{T}^d \cong [-\pi,\, \pi)^d$. Here the torus $\mathbb{T}$ is identified with the interval $[-\pi,\, \pi)$. Further the dot in the exponent denotes the usual scalar product in $\mathbb{R}^d$. Then the d-variate function (1.1) is infinitely differentiable and bounded on $\mathbb{R}^d$. If h is real–valued, then we can represent (1.1) as linear combination of ridge functions

$$h(\boldsymbol{x}) = \sum_{j=1}^{M} |c_j| \cos\big(\boldsymbol{f}_j \cdot \boldsymbol{x} + \varphi_j\big)$$

with $c_j = |c_j|\, \mathrm{e}^{\mathrm{i}\varphi_j}$. Assume that the frequency vectors $\boldsymbol{f}_j \in \mathbb{T}^d$ $(j = 1, \ldots, M)$ fulfill the *gap condition on* $\mathbb{T}^d$

$$\mathrm{dist}(\boldsymbol{f}_j, \boldsymbol{f}_l) := \min\{\|(\boldsymbol{f}_j + 2\pi\boldsymbol{k}) - \boldsymbol{f}_l\|_\infty : \boldsymbol{k} \in \mathbb{Z}^d\} \geq q > 0$$

for all $j, l = 1, \ldots, M$ with $j \neq l$. Let $N \in \mathbb{N}$ with $N \geq 2M + 1$ be given. Suppose that perturbed sampled data

$$\tilde{h}(\boldsymbol{n}) := h(\boldsymbol{n}) + e(\boldsymbol{n}), \quad \|e(\boldsymbol{n})\|_2 \leq \varepsilon_1$$

of (1.1) for all $\boldsymbol{n} \in K \subseteq \mathbb{Z}_N^d := [-N, N]^d \cap \mathbb{Z}^d$ are given, where the error terms $e(\boldsymbol{n}) \in \mathbb{C}$ are bounded by certain accuracy $\varepsilon_1 > 0$.

Then we consider the following *parameter estimation problem* for the d–variate exponential sum (1.1): Recover the pairwise different frequency vectors $\boldsymbol{f}_j \in [-\pi,\, \pi)^d$ and the complex coefficients c_j in such a way that

$$|\tilde{h}(\boldsymbol{n}) - \sum_{j=1}^{M} c_j\, \mathrm{e}^{\mathrm{i}\boldsymbol{f}_j \cdot \boldsymbol{n}}| \leq \varepsilon \quad (\boldsymbol{n} \in K) \tag{1.2}$$

for very small accuracy $\varepsilon > 0$ and for minimal order M. With other words, we are interested in sparse approximate representations of the given noisy data $\tilde{h}(\boldsymbol{n}) \in \mathbb{C}$ by sampled data $h(\boldsymbol{n}) \in \mathbb{C}$ $(\boldsymbol{n} \in K)$ of the exponential sum (1.1), where the condition (1.2) is fulfilled.

The approximation of data by finite linear combinations of complex exponentials has a long history, see [17, 18]. There exists a variety of applications, such as fitting nuclear magnetic resonance (NMR) spectroscopic data [16] or the annihilating filter method [24, 6, 23]. Recently, the reconstruction method of [3] was generalized to bivariate exponential sums in [1]. In contrast to [3] we introduce a sparse approximate Prony method, where we use only some data sampled along few lines. Further we remark the relation to a reconstruction method for sparse multivariate trigonometric polynomials, see Remark 5.4 and [12, 10, 25].

In this paper, we extend the approximate Prony method (see [21]) to multivariate exponential sums. First we discuss the stability of the multivariate exponentials. Based on Ingham–type inequalities (see [13, 14]), we prove the Riesz stability of finitely many multivariate exponentials under some mild conditions in the squared norm (see Lemma 2.1) and more important for the applications in the uniform norm (see Corollary 2.3). Furthermore we present a result for the converse assertion, i.e., if finitely many d–variate exponentials are Riesz stable, then the corresponding frequency vectors are well–separated (see Lemma 2.2). In Section 3, we extend these stability results to draw conclusions for discrete norms. Further we prove that the condition number of a rectangular Fourier–type matrix is bounded. In Section 4 we give a short description of the approximative Prony method in the one–dimensional setting, and we extend this method to the multivariate case in Section 5. Here we suggest a new sparse approximative Prony method (SAPM). The main idea is to project the multivariate reconstruction problem to several one–dimensional problems and combine finally the one–dimensional results. We use only few data sampled along some lines in order to reconstruct a multivariate exponential sum. Finally, various numerical examples for the reconstruction of d–variate exponential sums with $d \in \{2, 3, 4\}$ are presented in Section 6.

2 Stability of exponentials

In this section, we discuss the stability of finitely many multivariate exponentials. We start with a generalization of the known Ingham inequalities (see [9]):

Lemma 2.1 *(see [13, pp. 153 − 156]). Let $d \in \mathbb{N}$, $M \in \mathbb{N} \setminus \{1\}$ and $T > 0$ be given. If the frequency vectors $\boldsymbol{f}_j \in \mathbb{R}^d$ $(j = 1, \ldots, M)$ fulfill the gap condition on $\mathbb{R}^d$*

$$\|\boldsymbol{f}_j - \boldsymbol{f}_k\|_\infty \ge q > \frac{\sqrt{d}\,\pi}{T} \quad (j,\, k = 1, \ldots, M;\, j \ne k),$$

then the exponentials $\mathrm{e}^{\mathrm{i}\boldsymbol{f}_j \cdot (\cdot)}$ $(j = 1, \ldots, M)$ are Riesz stable in $L^2([-T, T]^d)$, i.e., for all complex vectors $\boldsymbol{c} = (c_j)_{j=1}^M$

$$\alpha\, \|\boldsymbol{c}\|_2^2 \le \|\sum_{j=1}^{M} c_j\, \mathrm{e}^{\mathrm{i}\boldsymbol{f}_j \cdot (\cdot)}\|_2^2 \le \beta\, \|\boldsymbol{c}\|_2^2 \tag{2.1}$$

with some positive constants α, β, independent of the particular choice of the coefficients c_j. Here $\|\boldsymbol{c}\|_2$ denotes the Euclidean norm of $\boldsymbol{c} \in \mathbb{C}^M$ and

$$\|f\|_2 := \Big(\frac{1}{(2T)^d} \int_{[-T,T]^d} |f(\boldsymbol{x})|^2\, \mathrm{d}\boldsymbol{x}\Big)^{1/2} \quad (f \in L^2([-T, T]^d))\,.$$

For a proof see [13, pp. 153 – 156]. Note that for $d = 1$, we obtain exactly the classical Ingham inequalities (see [9]).
For pairwise different frequency vectors $\boldsymbol{f}_j \in \mathbb{R}^d$ $(j = 1, \ldots, M)$, the existence of a lower Riesz bound for the exponentials $\mathrm{e}^{\mathrm{i}\boldsymbol{f}_j \cdot (\cdot)}$ $(j = 1, \ldots, M)$ in $L^2([-T, T]^d)$ implies that the frequency vectors are well–separated and that this system of exponentials is Riesz stable in $L^2([-T, T]^d)$. The following lemma generalizes a former result [15] for univariate exponentials.

Lemma 2.2 *Let $d \in \mathbb{N}$, $M \in \mathbb{N} \setminus \{1\}$ and $T > 0$. Further let $\boldsymbol{f}_j \in \mathbb{R}^d$ $(j = 1, \ldots, M)$ be given. If there exists a constant $\alpha > 0$ such that*

$$\alpha \, \|\boldsymbol{c}\|_2^2 \le \| \sum_{j=1}^{M} c_j \, \mathrm{e}^{\mathrm{i}\boldsymbol{f}_j \cdot (\cdot)} \|_2^2$$

for all complex vectors $\boldsymbol{c} = (c_j)_{j=1}^M$, then the frequency vectors $\boldsymbol{f}_j$ are well–separated by

$$\|\boldsymbol{f}_j - \boldsymbol{f}_k\|_\infty \ge \frac{1}{d\,T} \, \ln(\sqrt{2\alpha} + 1)$$

for all $j, k = 1, \ldots, M$ $(j \ne k)$. Moreover the exponentials $\mathrm{e}^{\mathrm{i}\boldsymbol{f}_j \cdot (\cdot)}$ $(j = 1, \ldots, M)$ are Riesz stable in $L^2([-T, T]^d)$.

Proof. 1. The frequency vectors $\boldsymbol{f}_j \in \mathbb{R}^d$ $(j = 1, \ldots, M)$ are pairwise different, because from $\boldsymbol{f}_j = \boldsymbol{f}_k$ for certain $j \ne k$ it follows $\alpha = 0$ by

$$2\,\alpha = \alpha\,(|1|^2 + |-1|^2) \le \|\mathrm{e}^{\mathrm{i}\boldsymbol{f}_j \cdot (\cdot)} - \mathrm{e}^{\mathrm{i}\boldsymbol{f}_k \cdot (\cdot)}\|_2^2 = 0\,,$$

which contradicts our assumption.
2. For the following proof we use similar arguments as in [5, Theorem 7.6.5]. We choose $c_j = -c_k = 1$ for $j \ne k$. All the other coefficients are equal to 0. Then by the assumption, we obtain

$$\begin{aligned} 2\,\alpha &= \alpha\,(|1|^2 + |-1|^2) \le \|\mathrm{e}^{\mathrm{i}\boldsymbol{f}_j \cdot (\cdot)} - \mathrm{e}^{\mathrm{i}\boldsymbol{f}_k \cdot (\cdot)}\|_2^2 \\ &= \frac{1}{(2\,T)^d} \int_{[-T,T]^d} |1 - \mathrm{e}^{\mathrm{i}(\boldsymbol{f}_k - \boldsymbol{f}_j)\cdot \boldsymbol{x}}|^2 \, \mathrm{d}\boldsymbol{x}\,. \end{aligned} \tag{2.2}$$

Using the Taylor expansion of the exponential function and the triangle inequality, it follows that for $\boldsymbol{x} \in [-T,\, T]^d$

$$\begin{aligned} |1 - \mathrm{e}^{\mathrm{i}\,(\boldsymbol{f}_k - \boldsymbol{f}_j)\cdot \boldsymbol{x}}| &= |\sum_{n=1}^{\infty} \frac{1}{n!} \, (\mathrm{i}\,(\boldsymbol{f}_k - \boldsymbol{f}_j) \cdot \boldsymbol{x})^n| \\ &\le \sum_{n=1}^{\infty} \frac{1}{n!} \, |(\boldsymbol{f}_k - \boldsymbol{f}_j) \cdot \boldsymbol{x}|^n \le \sum_{n=1}^{\infty} \frac{1}{n!} \, (T\, \|\boldsymbol{f}_k - \boldsymbol{f}_j\|_1)^n \\ &= \mathrm{e}^{T\, \|\boldsymbol{f}_k - \boldsymbol{f}_j\|_1} - 1\,. \end{aligned}$$

Here we have used the estimate

$$|(\boldsymbol{f}_k - \boldsymbol{f}_j) \cdot \boldsymbol{x}| \le \|\boldsymbol{f}_k - \boldsymbol{f}_j\|_1 \, \|\boldsymbol{x}\|_\infty \le T \, \|\boldsymbol{f}_k - \boldsymbol{f}_j\|_1$$

for all $\boldsymbol{x} \in [-T,\, T]^d$. Therefore (2.2) shows that

$$2\,\alpha \le (\mathrm{e}^{T\,\|\boldsymbol{f}_k - \boldsymbol{f}_j\|_1} - 1)^2\,.$$

Thus the frequency vectors $\boldsymbol{f}_j$ are separated by

$$d\,\|\boldsymbol{f}_j - \boldsymbol{f}_k\|_\infty \, \|\boldsymbol{f}_j - \boldsymbol{f}_k\|_1 \ge \frac{1}{T}\, \ln(\sqrt{2\alpha} + 1)$$

for all $j, k = 1, \ldots, M$ $(j \ne k)$.
3. Immediately we see that M is an upper Riesz bound for the exponentials $\mathrm{e}^{\mathrm{i}\boldsymbol{f}_j \cdot (\cdot)}$ $(j = 1, \ldots, M)$ in $L^2([-T,T]^d)$. By the Cauchy–Schwarz inequality we obtain

$$|\sum_{k=1}^{M} c_k\, \mathrm{e}^{\mathrm{i}\boldsymbol{f}_j \cdot \boldsymbol{x}}|^2 \le M\, \|\boldsymbol{c}\|_2^2$$

for all $\boldsymbol{c} = (c_j)_{j=1}^M \in \mathbb{C}^M$ and all $\boldsymbol{x} \in [-T,\, T]^d$ such that

$$\|\sum_{k=1}^{M} c_k\, \mathrm{e}^{\mathrm{i}\boldsymbol{f}_j \cdot (\cdot)}\|_2^2 \le M\, \|\boldsymbol{c}\|_2^2\,.$$

This completes the proof. ■

Corollary 2.3 *If the assumptions of Lemma 2.1 are fulfilled, then the exponentials* $\mathrm{e}^{\mathrm{i}\boldsymbol{f}_j \cdot (\cdot)}$ $(j = 1, \ldots, M)$ *are Riesz stable in* $C([-T,T]^d)$*, i.e., for all complex vectors* $\boldsymbol{c} = (c_j)_{j=1}^M$

$$\sqrt{\frac{\alpha}{M}}\, \|\boldsymbol{c}\|_1 \le \|\sum_{j=1}^{M} c_j\, \mathrm{e}^{\mathrm{i}\boldsymbol{f}_j \cdot (\cdot)}\|_\infty \le \|\boldsymbol{c}\|_1$$

with the uniform norm

$$\|f\|_\infty := \max_{\boldsymbol{x} \in [-T,T]^d} |f(\boldsymbol{x})| \quad (f \in C([-T,T]^d)).$$

Proof. Let $h \in C([-T,T]^d)$ be defined by (1.1). Then $\|h\|_2 \le \|h\|_\infty < \infty$. Using the triangle inequality, we obtain that

$$\|h\|_\infty \le \sum_{j=1}^{M} |c_j| \cdot 1 = \|\boldsymbol{c}\|_1\,.$$

From Lemma 2.1 and $\|\boldsymbol{c}\|_1 \le \sqrt{M}\, \|\boldsymbol{c}\|_2$, it follows that

$$\sqrt{\frac{\alpha}{M}}\, \|\boldsymbol{c}\|_1 \le \sqrt{\alpha}\, \|\boldsymbol{c}\|_2 \le \|h\|_2\,.$$

This completes the proof. ■

Now we use the uniform norm of $C([-T,\,T]^d)$ and estimate the error $\|h-\tilde{h}\|_\infty$ between the original exponential sum (1.1) and its reconstruction

$$\tilde{h}(\boldsymbol{x}) := \sum_{j=1}^{M} \tilde{c}_j\, \mathrm{e}^{\mathrm{i}\tilde{\boldsymbol{f}}_j\cdot\boldsymbol{x}} \quad (\boldsymbol{x}\in[-T,\,T]^d).$$

We obtain a small error $\|h-\tilde{h}\|_\infty$ in the case $\sum_{j=1}^{M}|c_j-\tilde{c}_j| \ll 1$ and $\|\boldsymbol{f}_j-\tilde{\boldsymbol{f}}_j\|_\infty \le \delta \ll 1$ $(j=1,\ldots,M)$.

Theorem 2.4 *Let $M\in\mathbb{N}\setminus\{1\}$ and $T>0$ be given. Let $\boldsymbol{c}=(c_j)_{j=1}^M$ and $\tilde{\boldsymbol{c}}=(\tilde{c}_j)_{j=1}^M$ be arbitrary complex vectors. If $\boldsymbol{f}_j,\,\tilde{\boldsymbol{f}}_j\in\mathbb{R}^d$ $(j=1,\ldots,M)$ fulfill the conditions*

$$\begin{aligned}
\|\boldsymbol{f}_j-\boldsymbol{f}_k\|_\infty &\ \ge\ q > \frac{3\sqrt{d}\pi}{2T} \quad (j,\,k=1,\,\ldots,M;\,j\neq k),\\
\|\tilde{\boldsymbol{f}}_j-\boldsymbol{f}_j\|_\infty &\ \le\ \delta < \frac{\sqrt{d}\pi}{4T} \quad (j=1,\,\ldots,M),
\end{aligned}$$

then both

$$\mathrm{e}^{\mathrm{i}\boldsymbol{f}_j\cdot(\cdot)} \quad (j=1,\ldots,M)$$

and

$$\mathrm{e}^{\mathrm{i}\tilde{\boldsymbol{f}}_j\cdot(\cdot)} \quad (j=1,\ldots,M)$$

are Riesz stable in $C([-T,\,T]^d)$. Further

$$\|h-\tilde{h}\|_\infty \le \|\boldsymbol{c}-\tilde{\boldsymbol{c}}\|_1 + d\delta T\,\|\boldsymbol{c}\|_1\,.$$

Proof. 1. By the gap condition on $\mathbb{R}^d$ we know that

$$\|\boldsymbol{f}_j-\boldsymbol{f}_k\|_\infty \ge q > \frac{3\sqrt{d}\pi}{2T} > \frac{\sqrt{d}\pi}{T} \quad (j,\,k=1,\,\ldots,M;\,j\neq k).$$

Hence the original exponentials $\mathrm{e}^{\mathrm{i}\boldsymbol{f}_j\cdot(\cdot)}$ $(j=1,\ldots,M)$ are Riesz stable in $C([-T,\,T]^d)$ by Corollary 2.3. Using the assumptions, we conclude that

$$\begin{aligned}
\|\tilde{\boldsymbol{f}}_j-\tilde{\boldsymbol{f}}_k\|_\infty &\ \ge\ \|\boldsymbol{f}_j-\boldsymbol{f}_k\|_\infty - \|\tilde{\boldsymbol{f}}_j-\boldsymbol{f}_j\|_\infty - \|\boldsymbol{f}_k-\tilde{\boldsymbol{f}}_k\|_\infty\\
&\ \ge\ q-2\,\frac{\sqrt{d}\pi}{4T} > \frac{\sqrt{d}\pi}{T}\,.
\end{aligned}$$

Thus the reconstructed exponentials

$$\mathrm{e}^{\mathrm{i}\tilde{\boldsymbol{f}}_j\cdot(\cdot)} \quad (j=1,\ldots,M)$$

are Riesz stable in $C([-T,\,T]^d)$ by Corollary 2.3 too.
2. Now we estimate the normwise error $\|h-\tilde h\|_\infty$ by the triangle inequality. Then we obtain

$$\begin{aligned}\|h-\tilde h\|_\infty &\le \|\sum_{j=1}^M (c_j-\tilde c_j)\,\mathrm{e}^{\mathrm{i}\tilde{\boldsymbol f}_j\cdot(\cdot)}\|_\infty + \|\sum_{j=1}^M c_j\,(\mathrm{e}^{\mathrm{i}\boldsymbol f_j\cdot(\cdot)}-\mathrm{e}^{\mathrm{i}\tilde{\boldsymbol f}_j\cdot(\cdot)})\|_\infty \\ &\le \sum_{j=1}^M |c_j-\tilde c_j| + \sum_{j=1}^M |c_j|\,\max_{\boldsymbol x\in[-T,T]^d}|\mathrm{e}^{\mathrm{i}\boldsymbol f_j\cdot\boldsymbol x}-\mathrm{e}^{\mathrm{i}\tilde{\boldsymbol f}_j\cdot\boldsymbol x}|\,.\end{aligned}$$

Since for $\boldsymbol d_j := \tilde{\boldsymbol f}_j - \boldsymbol f_j$ $(j=1,\ldots,M)$ and arbitrary $\boldsymbol x\in[-T,\,T]^d$, we can estimate

$$\begin{aligned}|\mathrm{e}^{\mathrm{i}\boldsymbol f_j\cdot\boldsymbol x}-\mathrm{e}^{\mathrm{i}\tilde{\boldsymbol f}_j\cdot\boldsymbol x}| &= |1-\mathrm{e}^{\mathrm{i}\boldsymbol d_j\cdot\boldsymbol x}| = \sqrt{2-2\,\cos(\boldsymbol d_j\cdot\boldsymbol x)} \\ &= 2\,|\sin\frac{\boldsymbol d_j\cdot\boldsymbol x}{2}| \le |\boldsymbol d_j\cdot\boldsymbol x| \le \|\boldsymbol d_j\|_\infty\,\|\boldsymbol x\|_1 \le d\delta\,T\end{aligned}$$

such that we receive

$$\|h-\tilde h\|_\infty \le \|\boldsymbol c-\tilde{\boldsymbol c}\|_1 + d\delta T\,\|\boldsymbol c\|_1\,.$$

This completes the proof. ■

3 Stability of exponentials on a grid

In the former section we have studied the stability of d–variate exponentials defined on $[-T,\,T]^d$. Now we investigate the stability of d–variate exponentials restricted on a uniform grid $\mathbb Z_N^d$. First we will show that a discrete version of Lemma 2.1 is also true for d–variate exponential sums (1.1). If we sample an exponential sum (1.1) on the uniform grid $\mathbb Z_N^d$, then it is impossible to distinct between the frequency vectors $\boldsymbol f_j$ and $\boldsymbol f_j+2\pi\boldsymbol k$ with certain $\boldsymbol k\in\mathbb Z^d$, since by the periodicity of the complex exponential

$$\mathrm{e}^{\mathrm{i}\tilde{\boldsymbol f}_j\cdot\boldsymbol n} = \mathrm{e}^{\mathrm{i}\,(\tilde{\boldsymbol f}_j+2\pi\boldsymbol k)\cdot\boldsymbol n}\quad(\boldsymbol n\in\mathbb Z_N^d)\,.$$

Therefore we assume in the following that $\boldsymbol f_j\in[-\pi,\,\pi)^d$ $(j=1,\ldots,M)$ and we measure the distance between two different frequency vectors $\boldsymbol f_j,\ \boldsymbol f_l\in[-\pi,\,\pi)^d$ $(j,l=1,\ldots,M;\ j\ne l)$ by

$$\mathrm{dist}(\boldsymbol f_j,\boldsymbol f_l) := \min\{\|(\boldsymbol f_j+2\pi\boldsymbol k)-\boldsymbol f_l\|_\infty : \boldsymbol k\in\mathbb Z^d\}\,.$$

Then the *separation distance* of the set $\{\boldsymbol f_j\in[-\pi,\,\pi)^d : j=1,\ldots,M\}$ is defined by

$$\min\{\mathrm{dist}(\boldsymbol f_j,\boldsymbol f_l) : j,l=1,\ldots,M;\ j\ne l\}\in(0,\pi].$$

The separation distance can be interpreted as the smallest gap between two different frequency vectors in the d–dimensional torus $\mathbb T^d$.
Since we restrict an exponential sum h on the grid $\mathbb Z_N^d$, we use the norm

$$\frac{1}{(2N+1)^{d/2}}\Big(\sum_{\boldsymbol k\in\mathbb Z_N^d}|h(\boldsymbol k)|^2\Big)^{1/2}$$

in the Hilbert space $l^2(\mathbb Z_N^d)$.

Lemma 3.1 *(see [14]). Let $q \in (0, \pi]$ and $M \in \mathbb{N}\setminus\{1\}$ be given. If the frequency vectors $\boldsymbol{f}_m \in (-\pi + \frac{q}{2},\, \pi - \frac{q}{2})^d$ $(m = 1, \ldots, M)$ satisfy*

$$\|\boldsymbol{f}_m - \boldsymbol{f}_n\|_\infty \ge q > \frac{\sqrt{d}\pi}{N} \quad (m,\, n = 1, \ldots, M;\, m \ne n)\,, \tag{3.1}$$

then the exponentials $\mathrm{e}^{\mathrm{i}\boldsymbol{f}_j \cdot (\cdot)}$ $(j = 1, \ldots, M)$ are Riesz stable in $l^2(\mathbb{Z}_N^d)$, i.e., all complex vectors $\boldsymbol{c} = (c_j)_{j=1}^M$ satisfy the following Ingham–type inequalities

$$\alpha\, \|\boldsymbol{c}\|_2^2 \le \frac{1}{(2N+1)^d} \sum_{\boldsymbol{k} \in \mathbb{Z}_N^d} |\sum_{j=1}^M c_j\, \mathrm{e}^{\mathrm{i}\,\boldsymbol{f}_j \cdot \boldsymbol{k}}\,|^2 \le \beta\, \|\boldsymbol{c}\|_2^2$$

with some positive constants α and β, independent of the particular choice of $\boldsymbol{c}$.

Proof. 1. Note that $\boldsymbol{f}_m \in (-\pi + \frac{q}{2},\, \pi - \frac{q}{2})^d$ $(m = 1, \ldots, M)$ with (3.1) implies

$$\mathrm{dist}(\boldsymbol{f}_m, \boldsymbol{f}_n) > q$$

for all $m, n = 1, \ldots, M$ with $m \ne n$. Thus the separation distance of the frequency vectors is greater than q. Immediately we see that

$$\frac{1}{(2N+1)^d} \sum_{\boldsymbol{k} \in \mathbb{Z}_N^d} |\sum_{j=1}^M c_j\, \mathrm{e}^{\mathrm{i}\,\boldsymbol{f}_j \cdot \boldsymbol{k}}|^2 \le \beta\, \|\boldsymbol{c}\|_2^2$$

is valid for $\beta = M$, because by Cauchy–Schwarz inequality we obtain

$$|\sum_{j=1}^M c_j\, \mathrm{e}^{\mathrm{i}\,\boldsymbol{f}_j \cdot \boldsymbol{k}}|^2 \le M\, \|\boldsymbol{c}\|_2^2\,.$$

Thus we have to show only the existence of a constant $\alpha > 0$.
2. As usual, the space $H_0^1((-q,\, q)^d)$ consists of all functions (defined on $(-q,\, q)^d$) in the Sobolev space $H^1((-q,\, q)^d)$ of order 1, whose trace is zero. Let $\psi \in H_0^1((-q,\, q)^d)$ be a given function which we extend by 0 continuously on $\mathbb{R}^d \setminus (-q,\, q)^d$. Assume that ψ is infinitely differentiable outside the boundary of $[-q,\, q]^d$. By $\pi \ge q$, the Fourier transform of ψ reads as follows

$$\hat{\psi}(\boldsymbol{t}) = \int_{\mathbb{R}^d} \psi(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\,\boldsymbol{t} \cdot \boldsymbol{x}}\, \mathrm{d}\boldsymbol{x} = \int_{[-\pi,\, \pi]^d} \psi(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\,\boldsymbol{t} \cdot \boldsymbol{x}}\, \mathrm{d}\boldsymbol{x} \quad (\boldsymbol{t} \in \mathbb{R}^d).$$

Then we receive

$$\hat{\psi}(\boldsymbol{j}) = \int_{[-\pi,\, \pi]^d} \psi(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\,\boldsymbol{j} \cdot \boldsymbol{x}}\, \mathrm{d}\boldsymbol{x} = \int_{[-q,\, q]^d} \psi(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\,\boldsymbol{j} \cdot \boldsymbol{x}}\, \mathrm{d}\boldsymbol{x} \quad (\boldsymbol{j} \in \mathbb{Z}).$$

Since ψ is infinitely differentiable in the cube $[-q,\, q]^d$, it follows by repeated partial integration that

$$\lim_{N \to \infty} \sum_{\boldsymbol{j} \in \mathbb{Z}_N^d} |\hat{\psi}(\boldsymbol{j})| < \infty\,.$$

The exponential functions

$$\sqrt[2d]{\frac{1}{2\pi}}\,\mathrm{e}^{\mathrm{i}\,\boldsymbol{j}\cdot\boldsymbol{x}} \quad (\boldsymbol{j} \in \mathbb{Z})$$

form an orthonormal basis in $L^2([-\pi,\,\pi]^d)$. If

$$\psi_{2\pi}(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \psi(\boldsymbol{x} + 2\pi\,\boldsymbol{k}) \quad (\boldsymbol{x} \in \mathbb{R}^d)$$

is the 2π–periodization of ψ, then the d–variate Fourier series of $\psi_{2\pi}$ (with square partial sums) converges uniformly in $\mathbb{R}^d$ such that for all $\boldsymbol{x} \in \mathbb{R}^d$

$$\sum_{\boldsymbol{j}\in\mathbb{Z}^d} \hat{\psi}(\boldsymbol{j})\,\mathrm{e}^{\mathrm{i}\,\boldsymbol{j}\cdot\boldsymbol{x}} := \lim_{N\to\infty} \sum_{\boldsymbol{j}\in\mathbb{Z}_N^d} \hat{\psi}(\boldsymbol{j})\,\mathrm{e}^{\mathrm{i}\,\boldsymbol{j}\cdot\boldsymbol{x}} = (2\pi)^d\,\psi_{2\pi}(\boldsymbol{x}). \tag{3.2}$$

Note that $\psi_{2\pi}(\boldsymbol{x}) = \psi(\boldsymbol{x})$ for $\boldsymbol{x} \in [-\pi,\,\pi]^d$ and $\psi_{2\pi}(\boldsymbol{x}) = 0$ if $q < \|\boldsymbol{x}\|_\infty < 2\pi - q$. By our assumptions concerning the frequencies $\boldsymbol{f}_m$, we see that in the case $m \neq n$

$$q < \|\boldsymbol{f}_m - \boldsymbol{f}_n\|_\infty \le \|\boldsymbol{f}_m\|_\infty + \|\boldsymbol{f}_n\|_\infty < 2\left(\pi - \frac{q}{2}\right) = 2\pi - q$$

and hence $\psi_{2\pi}(\boldsymbol{f}_m - \boldsymbol{f}_n) = 0$. Thus for all $\boldsymbol{x} = \boldsymbol{f}_m - \boldsymbol{f}_n$ $(m,\,n = 1,\ldots,M)$, the function $\psi_{2\pi}$ is infinitely differentiable and the Fourier series (3.2) converges in these points. Then from (1.1) and (3.2) it follows that

$$\begin{aligned}
\sum_{\boldsymbol{j}\in\mathbb{Z}^d} \hat{\psi}(\boldsymbol{j})\,|h(\boldsymbol{j})|^2 &= \sum_{m,n=1}^{M} c_m\,\bar{c}_n \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \hat{\psi}(\boldsymbol{j})\,\mathrm{e}^{\mathrm{i}\,(\boldsymbol{f}_m - \boldsymbol{f}_n)\cdot\boldsymbol{j}} \\
&= (2\pi)^d \sum_{m,n=1}^{M} c_m\,\bar{c}_n\,\psi_{2\pi}(\boldsymbol{f}_m - \boldsymbol{f}_n) \\
&= (2\pi)^d\,\psi(\boldsymbol{0})\,\|\boldsymbol{c}\|_2^2\,.
\end{aligned} \tag{3.3}$$

3. By φ we denote the following eigenfunction of the Laplacian operator $-\Delta$ in the Sobolev space $H_0^1((-\frac{q}{2},\,\frac{q}{2})^d)$ corresponding to the first eigenvalue $\frac{d\pi^2}{q^2}$:

$$\varphi(\boldsymbol{x}) := \prod_{j=1}^{d} \cos\frac{\pi\,x_j}{q} \quad (\boldsymbol{x} \in (-\frac{q}{2},\,\frac{q}{2})^d).$$

Extending φ by zero on $\mathbb{R}^d \setminus (-\frac{q}{2},\,\frac{q}{2})^d$, we obtain a continuous function on $\mathbb{R}^d$, still denoted by φ. We compute the Fourier transform

$$\hat{\varphi}(\boldsymbol{t}) := \int_{\mathbb{R}^d} \varphi(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{t}}\,\mathrm{d}\boldsymbol{x} \quad (\boldsymbol{t} \in \mathbb{R}^d)$$

by

$$\int_{-q/2}^{q/2} \cos\frac{\pi x_j}{q}\,\mathrm{e}^{-\mathrm{i}x_j t_j}\,\mathrm{d}x_j = \frac{2\pi}{q}\,\frac{\cos\frac{q\,t_j}{2}}{\frac{\pi^2}{q^2} - t_j^2} \quad (t_j \neq \pm\frac{\pi}{q}),$$

where the limits of the right–hand side for $t_j \to \pm\frac{\pi}{q}$ are equal to $\frac{q}{2}$. Thus we obtain

$$\hat{\varphi}(\boldsymbol{t}) = \Big(\frac{2\pi}{q}\Big)^d \prod_{j=1}^{d} \frac{\cos\frac{q\,t_j}{2}}{\frac{\pi^2}{q^2} - t_j^2} \quad (\boldsymbol{t} = (t_j)_{j=1}^d \in \mathbb{R}^d).$$

Let $\psi := N^2\,(\varphi * \varphi) + \Delta\,(\varphi * \varphi)$. Then ψ satisfies the conditions of step 2 with $\psi(\boldsymbol{0}) > 0$. Further ψ is infinitely differentiable outside the boundary of $[-q,\, q]^d$. By the properties of the Fourier transform, we obtain

$$\hat{\psi}(\boldsymbol{t}) = (N^2 - \|\boldsymbol{t}\|_2^2)\,\big(\hat{\varphi}(\boldsymbol{t})\big)^2 \quad (\boldsymbol{t} \in \mathbb{R}^d).$$

Thus $\hat{\psi}$ is bounded from above by some positive constant ν and $\hat{\psi} \le 0$ on $\mathbb{R}^d \setminus B_N$, where B_N is the open ball of center $\boldsymbol{0}$ and radius N. Thus from (3.3) it follows that

$$\frac{(2\pi)^d\,\psi(\boldsymbol{0})}{\nu}\,\|\boldsymbol{c}\|_2^2 \le \sum_{\|\boldsymbol{j}\|_2 \le N} |h(\boldsymbol{j})|^2 \le \sum_{\boldsymbol{j}\in\mathbb{Z}_N^d} |h(\boldsymbol{j})|^2\,.$$

This completes the proof. ■

Lemma 3.2 *Let $d \in \mathbb{N}$, $M \in \mathbb{N}\setminus\{1\}$ and $N \in \mathbb{N}$ with $N \ge 2M+1$ be given. Further let $\boldsymbol{f}_j \in [-\pi,\, \pi)^d$ $(j = 1,\ldots,M)$. If there exists a constant $\alpha > 0$ such that*

$$\alpha\,\|\boldsymbol{c}\|_2^2 \le \frac{1}{(2N+1)^d} \sum_{\boldsymbol{k}\in\mathbb{Z}_N^d} \Big|\sum_{j=1}^{M} c_j\,\mathrm{e}^{\mathrm{i}\boldsymbol{f}_j\cdot\boldsymbol{k}}\Big|^2$$

for all complex vectors $\boldsymbol{c} = (c_j)_{j=1}^M$, then the frequency vectors $\boldsymbol{f}_j$ are well–separated by

$$\mathrm{dist}(\boldsymbol{f}_j, \boldsymbol{f}_l) \ge \frac{1}{d\,N}\,\ln(\sqrt{2\alpha}+1)$$

for all $j, l = 1,\ldots,M$ with $j \ne l$. Moreover the exponentials $\mathrm{e}^{\mathrm{i}\boldsymbol{f}_j\cdot(\cdot)}$ $(j = 1,\ldots,M)$ are Riesz stable in $l^2(\mathbb{Z}_N^d)$.

Proof. 1. The frequency vectors $\boldsymbol{f}_j \in [-\pi,\, \pi)^d$ $(j = 1,\ldots,M)$ are pairwise different, because from $\boldsymbol{f}_j = \boldsymbol{f}_l$ for certain $j \ne l$ it follows $\alpha = 0$ by

$$2\,\alpha = \alpha\,(|1|^2 + |-1|^2) \le \frac{1}{(2N+1)^d} \sum_{\boldsymbol{k}\in\mathbb{Z}_N^d} |\mathrm{e}^{\mathrm{i}\boldsymbol{f}_j\cdot\boldsymbol{k}} - \mathrm{e}^{\mathrm{i}\boldsymbol{f}_l\cdot\boldsymbol{k}}|^2 = 0\,,$$

which contradicts our assumption.
2. We choose $c_j = -c_l = 1$ for $j \ne l$. All the other coefficients are equal to 0. Then by the assumption, we obtain

$$2\,\alpha = \alpha\,(|1|^2 + |-1|^2) \le \frac{1}{(2N+1)^d} \sum_{\boldsymbol{k}\in\mathbb{Z}_N^d} |1 - \mathrm{e}^{\mathrm{i}(\boldsymbol{f}_l - \boldsymbol{f}_j)\cdot\boldsymbol{k}}|^2\,. \tag{3.4}$$

Using the Taylor expansion of the exponential function and the triangle inequality, it follows that for $\boldsymbol{k} \in \mathbb{Z}_N^d$

$$\begin{aligned} |1 - \mathrm{e}^{\mathrm{i}\,(\boldsymbol{f}_l - \boldsymbol{f}_j)\cdot \boldsymbol{k}}| &= |\sum_{n=1}^{\infty} \frac{1}{n!}\,(\mathrm{i}\,(\boldsymbol{f}_l - \boldsymbol{f}_j)\cdot \boldsymbol{k})^n| \\ &\le \sum_{n=1}^{\infty} \frac{1}{n!}\,|(\boldsymbol{f}_l - \boldsymbol{f}_j)\cdot \boldsymbol{k}|^n \le \sum_{n=1}^{\infty} \frac{1}{n!}\,(N\,\|\boldsymbol{f}_l - \boldsymbol{f}_j\|_1)^n \\ &= \mathrm{e}^{N\,\|\boldsymbol{f}_l - \boldsymbol{f}_j\|_1} - 1\,. \end{aligned}$$

Here we have used the estimate

$$|(\boldsymbol{f}_l - \boldsymbol{f}_j)\cdot \boldsymbol{k}| \le \|\boldsymbol{f}_l - \boldsymbol{f}_j\|_1\,\|\boldsymbol{k}\|_\infty \le N\,\|\boldsymbol{f}_l - \boldsymbol{f}_j\|_1$$

for all $\boldsymbol{k} \in \mathbb{Z}_N^d$. Therefore (3.4) shows that

$$2\,\alpha \le (\mathrm{e}^{N\,\|\boldsymbol{f}_l - \boldsymbol{f}_j\|_1} - 1)^2\,.$$

Thus the frequency vectors $\boldsymbol{f}_j$ $(j = 1, \ldots, M)$ are separated by

$$d\,\|\boldsymbol{f}_j - \boldsymbol{f}_l\|_\infty \ge \|\boldsymbol{f}_j - \boldsymbol{f}_l\|_1 \ge \frac{1}{N}\,\ln(\sqrt{2\alpha} + 1)\,.$$

By the periodicity of the exponential function, we can replace $\boldsymbol{f}_j$ by $\boldsymbol{f}_j + 2\pi\,\boldsymbol{n}$ with arbitrary $\boldsymbol{n} \in \mathbb{Z}^d$. Then we obtain that

$$d\,\|(\boldsymbol{f}_j + 2\pi\,\boldsymbol{n}) - \boldsymbol{f}_l\|_\infty \ge \frac{1}{N}\,\ln(\sqrt{2\alpha} + 1)$$

such that

$$d\,\mathrm{dist}(\boldsymbol{f}_j, \boldsymbol{f}_l) \ge \frac{1}{N}\,\ln(\sqrt{2\alpha} + 1)$$

for all $j, l = 1, \ldots, M$ with $j \ne l$.

3. Immediately we see that M is an upper Riesz bound for the exponentials $\mathrm{e}^{\mathrm{i}\boldsymbol{f}_j\cdot(\cdot)}$ $(j = 1, \ldots, M)$ in $l^2(\mathbb{Z}_N^d)$, because by the Cauchy–Schwarz inequality we obtain

$$|\sum_{j=1}^{M} c_j\,\mathrm{e}^{\mathrm{i}\,\boldsymbol{f}_j\cdot \boldsymbol{k}}|^2 \le M\,\|\boldsymbol{c}\|_2^2$$

for all $\boldsymbol{c} = (c_j)_{j=1}^M \in \mathbb{C}^M$ and all $\boldsymbol{k} \in \mathbb{Z}_N^d$ and hence

$$\frac{1}{(2N+1)^d} \sum_{\boldsymbol{k}\in \mathbb{Z}_N^d} |\sum_{j=1}^{M} c_j\,\mathrm{e}^{\mathrm{i}\,\boldsymbol{f}_j\cdot \boldsymbol{k}}|^2 \le M\,\|\boldsymbol{c}\|_2^2\,.$$

This completes the proof. ■

Introducing the *rectangular Fourier–type matrix*

$$\boldsymbol{F} := (2N+1)^{-d/2}\,\left(\mathrm{e}^{\mathrm{i}\,\boldsymbol{f}_j\cdot \boldsymbol{k}}\right)_{\boldsymbol{k}\in \mathbb{Z}_N^d,\, j=1,\ldots,M} \in \mathbb{C}^{(2N+1)^d \times M}\,,$$

we improve the result of [20, Theorem 4.3].

Corollary 3.3 *Under the assumptions of Lemma* 3.1, *the rectangular Fourier–type matrix* $\boldsymbol{F}$ *has a bounded condition number* $\mathrm{cond}_2(\boldsymbol{F})$ *for all integers* $N > \frac{\sqrt{d}\,\pi}{q}$.

Proof. By Lemma 3.1, we know that for all $\boldsymbol{c} \in \mathbb{C}^M$

$$\alpha\, \boldsymbol{c}^{\mathrm{H}} \boldsymbol{c} \le \boldsymbol{c}^{\mathrm{H}} \boldsymbol{F}^{\mathrm{H}} \boldsymbol{F}\, \boldsymbol{c} \le \beta\, \boldsymbol{c}^{\mathrm{H}} \boldsymbol{c} \tag{3.5}$$

with positive constants α, β. Let $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_M \ge 0$ be the ordered eigenvalues of $\boldsymbol{F}^{\mathrm{H}} \boldsymbol{F} \in \mathbb{C}^{M\times M}$. Using the Rayleigh–Ritz Theorem and (3.5), we receive that

$$\alpha\, \boldsymbol{c}^{\mathrm{H}} \boldsymbol{c} \le \lambda_M\, \boldsymbol{c}^{\mathrm{H}} \boldsymbol{c} \le \boldsymbol{c}^{\mathrm{H}} \boldsymbol{F}^{\mathrm{H}} \boldsymbol{F}\, \boldsymbol{c} \le \lambda_1\, \boldsymbol{c}^{\mathrm{H}} \boldsymbol{c} \le \beta\, \boldsymbol{c}^{\mathrm{H}} \boldsymbol{c}$$

and hence

$$0 < \alpha \le \lambda_M \le \lambda_1 \le \beta < \infty\,.$$

Thus $\boldsymbol{F}^{\mathrm{H}} \boldsymbol{F}$ is positive definite and

$$\mathrm{cond}_2(\boldsymbol{F}) = \sqrt{\frac{\lambda_1}{\lambda_M}} \le \sqrt{\frac{\beta}{\alpha}}\,.$$

This completes the proof. ■

4 Approximate Prony method for $d = 1$

In this section we give a short description of the *approximate Prony method* (APM) in the case $d = 1$. For details see [3, 21, 19]. Let $M \in \mathbb{N} \setminus \{1\}$ and $N \in \mathbb{N}$ with $N \ge 2M + 1$ be given. By $\mathbb{Z}_N$ we denote the finite set $[-N,\, N] \cap \mathbb{Z}$. We consider a univariate exponential sum

$$h(x) := \sum_{j=1}^{M} c_j\, \mathrm{e}^{\mathrm{i} f_j x} \quad (x \in \mathbb{R})$$

with pairwise different, ordered frequencies $-\pi \le f_1 < f_2 < \ldots < f_M < \pi$ and nontrivial complex coefficients c_j. Assume that these frequencies are well–separated in the sense that

$$\mathrm{dist}(f_j, f_l) := \min\{|(f_j + 2\pi k) - f_l| : k \in \mathbb{Z}\} > \frac{\pi}{N}$$

for all $j, l = 1, \ldots, M$ with $j \ne l$. Suppose that noisy sampled data $\tilde{h}(k) := h(k) + e(k) \in \mathbb{C}$ $(k \in \mathbb{Z}_N)$ are given, where the magnitudes of the error terms $e(k)$ are uniformly bounded by a certain accuracy $\varepsilon_1 > 0$. Then we consider the following *nonlinear approximation problem*: Recover the pairwise different frequencies $f_j \in [-\pi, \pi)$ and the complex coefficients c_j in such a way that

$$|\tilde{h}(k) - \sum_{j=1}^{M} c_j\, \mathrm{e}^{\mathrm{i} f_j k}| \le \varepsilon \quad (k \in \mathbb{Z}_N)$$

for very small accuracy $\varepsilon > 0$ and for minimal number M of nontrivial summands. This problem can be solved by the following algorithm.

Algorithm 4.1 (APM)

Input: $L,\, N \in \mathbb{N}$ ($3 \le L \le N$, L is upper bound of the number of exponentials), $\tilde{h}(k) = h(k) + e(k) \in \mathbb{C}$ ($k \in \mathbb{Z}_N$) with $|e(k)| \le \varepsilon_1$, accuracy bounds $\varepsilon_1,\, \varepsilon_2 > 0$.

1. Determine the smallest singular value $\tilde{\sigma}$ of the rectangular Hankel matrix

$$\tilde{\boldsymbol{H}} := (\tilde{h}(k+l))_{k=-N,\, l=0}^{N-L,\, L}$$

and related right singular vector $\tilde{\boldsymbol{u}} = (\tilde{u}_l)_{l=0}^{L}$ by singular value decomposition.
2. Compute all zeros $\tilde{z}_j$ ($j = 1, \dots, L$) of the polynomial $\sum_{l=0}^{L} \tilde{u}_l\, z^l$ and determine all that zeros $\tilde{z}_j$ ($j = 1, \dots, \tilde{M}$) with the property $|\,|\tilde{z}_j| - 1| \le \varepsilon_2$. Use here the QR decomposition of the corresponding companion matrix. Note that $L \ge \tilde{M}$.
3. For $\tilde{w}_j := \tilde{z}_j / |\tilde{z}_j|$ ($j = 1, \dots, \tilde{M}$), compute $\tilde{c}_j \in \mathbb{C}$ ($j = 1, \dots, \tilde{M}$) as least squares solution of the overdetermined linear Vandermonde–type system

$$\sum_{j=1}^{\tilde{M}} \tilde{c}_j\, \tilde{w}_j^k = \tilde{h}(k) \quad (k \in \mathbb{Z}_N)\,.$$

For large $\tilde{M}$ and N, we can apply the CGNR method (conjugate gradient on the normal equations), where the multiplication of the rectangular Fourier–type matrix $(\tilde{w}_j^k)_{k=-N,j=1}^{N,\tilde{M}}$ is realized in each iteration step by the nonequispaced fast Fourier transform (NFFT) (see [11]).
4. Delete all the $\tilde{w}_l$ ($l \in \{1, \dots, \tilde{M}\}$) with $|\tilde{c}_l| \le \varepsilon_1$ and denote the remaining entries by $\tilde{w}_j$ ($j = 1, \dots, M$) with $M \le \tilde{M}$.
5. Repeat step 3 and compute $\tilde{c}_j \in \mathbb{C}$ ($j = 1, \dots, \tilde{M}$) as least squares solution of the overdetermined linear Vandermonde–type system

$$\sum_{j=1}^{M} \tilde{c}_j\, \tilde{w}_j^k = \tilde{h}(k) \quad (k \in \mathbb{Z}_N)$$

with respect to the new set $\{\tilde{w}_j : j = 1, \dots, M\}$ again. Set $\tilde{f}_j := \mathrm{Im}\,(\log \tilde{w}_j)$ ($j = 1, \dots, M$).

Output: The reconstructed parameters of h are $M \in \mathbb{N}$, $\tilde{f}_j \in [-\pi,\, \pi)$, and $\tilde{c}_j \in \mathbb{C}$ ($j = 1, \dots, M$).

Remark 4.2 The convergence and stability properties of Algorithm 4.1 are discussed in [21]. In all numerical tests of Algorithm 4.1 (see Section 6 and [21, 19]), we have obtained very good reconstruction results. All frequencies and coefficients can be computed such that

$$\max_{j=1,\dots,M} |f_j - \tilde{f}_j| \ll 1, \quad \sum_{j=1}^{M} |c_j - \tilde{c}_j| \ll 1\,.$$

We have to assume that the frequencies f_j are well–separated, that $|c_j|$ are not too small, that the number $2N+1$ of samples is sufficiently large, that a convenient upper bound L of the number of exponentials is known, and that the error bound ε_1 of the sampled data is small. Up to now, useful error estimates of $\max |f_j - \tilde{f}_j|$ and $\sum_{j=1}^{M} |c_j - \tilde{c}_j|$ are unknown. □

Remark 4.3 The above algorithm has been tested for $M \le 100$ and $N \le 10^5$ in MATLAB with double precision. For fixed upper bound L and variable N, the arithmetic cost of this algorithm are very moderate with about $\mathcal{O}(N \log N)$ flops. In the step 1, the singular value decomposition needs $14\,(2N-L+1)(L+1)^2 + 8\,(L+1)^2$ flops. In the step 2, the QR decomposition of the companion matrix requires $\frac{4}{3}\,(L+1)^3$ flops (see [8], p. 337). For large values N and $\tilde{M}$, one can use the nonequispaced fast Fourier transform iteratively in steps 3 and 5. Since the condition number of the Fourier–type matrix $(\tilde{w}_j^k)_{k=-N,j=1}^{N,\tilde{M}}$ is uniformly bounded by Corollary 3.3, we need finitely many iterations of the CGNR method. In each iteration step, the product between this Fourier–type matrix and an arbitrary vector of length $\tilde{M}$ can be computed with the NFFT by $\mathcal{O}(N \log N + L\,|\log \varepsilon|)$ flops, where $\varepsilon > 0$ is the wanted accuracy (see [11]). □

Remark 4.4 By similar ideas, we can reconstruct also all parameters of an *extended exponential sum*

$$h(x) = \sum_{j=1}^{M} p_j(x)\, \mathrm{e}^{\mathrm{i}\, f_j x} \quad (x \in \mathbb{R})\,,$$

where p_j $(j = 1, \ldots, M)$ is an algebraic polynomial of degree $m_j \ge 0$ (see [4, p.169]). Then we can interpret the exactly sampled values

$$h(n) = \sum_{j=1}^{M} p_j(n)\, z_j^n \quad (n \in \mathbb{Z}_N)$$

with $z_j := \mathrm{e}^{\mathrm{i}\, f_j}$ as a solution of a homogeneous linear difference equation

$$\sum_{k=0}^{K} p_k\, h(j+k) = 0 \quad (j \in \mathbb{Z})\,, \tag{4.1}$$

where p_k $(k = 0, \ldots, K)$ are defined by

$$\prod_{j=1}^{M} (z - z_j)^{m_j+1} = \sum_{k=0}^{M_{\text{total}}} p_k\, z^k\,, \quad M_{\text{total}} := \sum_{j=1}^{M} (m_j + 1)\,.$$

Note that in this case z_j is a zero of order m_j of the polynomial and we can cover multiple zeros with this approach. Consequently, (4.1) has the general solution

$$h(k) = \sum_{j=1}^{M} \Big(\sum_{l=0}^{m_j} c_{j,l}\, k^l\Big)\, z_j^k \quad (k \in \mathbb{Z})\,.$$

Then we determine the coefficients $c_{j,l}$ $(j = 1, \ldots, M;\, l = 0, \ldots, m_j)$ in such a way that

$$\sum_{j=1}^{M} \Big(\sum_{l=0}^{m_j} c_{j,l}\, k^l\Big)\, z_j^k \approx h(k) \quad (k \in \mathbb{Z}_N)\,,$$

where we assume that $N \ge 2M_{\text{total}} + 1$. To this end, we compute the least squares solution of the above overdetermined linear system. □

5 Sparse approximative Prony method for $d > 1$

Let d, $M \in \mathbb{N} \setminus \{1\}$ and $N \in \mathbb{N}$ with $N \ge 2M + 1$ be given. The aim of this section is to present a new efficient parameter estimation method for a d–variate exponential sum of order M using only $\mathcal{O}(N)$ sampling points. The main idea is to project the multivariate reconstruction problem to several one–dimensional problems and to solve these problems by methods from the previous Section 4. Finally we combine the results from the one–dimensional problems. Note that it is not necessary to sample the d–variate function h on the whole grid $\mathbb{Z}_N^d$.

For simplicity, first we consider a bivariate exponential sum

$$h(x_1,\, x_2) = \sum_{j=1}^{M} c_j\, \mathrm{e}^{\mathrm{i}\,(f_{j,1}x_1 + f_{j,2}x_2)}\,.$$

Assume that the frequency vectors $\boldsymbol{f}_j = (f_{j,1},\, f_{j,2})^\top \in [-\pi,\, \pi)^2$ $(j = 1, \ldots, M)$ are *well–separated* by

$$\mathrm{dist}(\boldsymbol{f}_j, \boldsymbol{f}_k) > \frac{\sqrt{2}\,\pi}{N}\,.$$

Additionally we suppose that the components of the frequency vectors fulfill $\mathrm{dist}(f_{j,l},\, f_{k,l}) > \pi/N$ or $f_{j,l} = f_{k,l}$ for all $j, k = 1, \ldots, M$ and $l = 1,\, 2$. We solve the corresponding parameter estimation problem stepwise and denote this new procedure by *sparse approximate Prony method* (SAPM). Here we use only values $h(n, 0)$, $h(0, n)$, $h(n, \alpha n + \beta)$ $(n \in \mathbb{Z}_N)$ sampled along straight lines, where $\alpha \in \mathbb{Z} \setminus \{0\}$ and $\beta \in \mathbb{Z}$ are conveniently chosen.

First we consider the given noisy data $\tilde{h}(n, 0)$ $(n \in \mathbb{Z}_N)$ of

$$h(n, 0) = \sum_{j=1}^{M} c_j\, \mathrm{e}^{\mathrm{i} f_{j,1} n} = \sum_{j=1}^{M_1} c_{j,1}\, \mathrm{e}^{\mathrm{i} f'_{j,1} n}\,, \tag{5.1}$$

where $1 \le M_1 \le M$, $f'_{j,1} \in [-\pi,\, \pi)$ $(j = 1, \ldots, M_1)$ are the pairwise different values of $f_{j,1}$ $(j = 1, \ldots, M)$ and $c_{j,1} \in \mathbb{C}$ are certain linear combinations of the coefficients c_j. Assume that $c_{j,1} \neq 0$ (without cancellation). Using the Algorithm 4.1, compute the pairwise different frequencies $f'_{j,1} \in [-\pi,\, \pi)$ $(j = 1, \ldots, M_1)$.

Analogously, we consider the given noisy data $\tilde{h}(0, n)$ $(n = -N, \ldots, N)$ of

$$h(0, n) = \sum_{j=1}^{M} c_j\, \mathrm{e}^{\mathrm{i} f_{j,2} n} = \sum_{j=1}^{M_2} c_{j,2}\, \mathrm{e}^{\mathrm{i} f'_{j,2} n}, \tag{5.2}$$

where $1 \le M_2 \le M$, $f'_{j,2} \in [-\pi,\, \pi)$ $(j = 1, \ldots, M_2)$ are the pairwise different values of $f_{j,2}$ $(j = 1, \ldots, M)$ and $c_{j,2} \in \mathbb{C}$ are certain linear combinations of the coefficients c_j. Assume that $c_{j,2} \neq 0$ (without cancellation). Using the Algorithm 4.1, we compute the pairwise different frequencies $f'_{j,2} \in [-\pi,\, \pi)$ $(j = 1, \ldots, M_2)$.

Then we form the Cartesian product

$$F = \{(f'_{j_1,1}, f'_{j_2,2})^\top \in [-\pi,\, \pi)^2 : \, j_1 = 1, \ldots, M_1,\, j_2 = 1, \ldots, M_2\} \tag{5.3}$$

of the sets $\{f'_{1,1}, \ldots, f'_{M_1,1}\}$ and $\{f'_{1,2}, \ldots, f'_{M_2,2}\}$. Now the question arises which $(f'_{k,1}, f'_{k,2})^\top$ $(k = 1, \ldots, M_1 M_2)$ are approximations of the frequency vectors $\boldsymbol{f}_j = (f_{j,1}, f_{j,2})^\top$ $(j = 1, \ldots, M)$. Therefore we choose further parameters $\alpha \in \mathbb{Z} \setminus \{0\}$, $\beta \in \mathbb{Z}$ and consider the given noisy data $\tilde{h}(n, \alpha n + \beta)$ $(n \in \mathbb{Z}_N)$ of

$$h(n, \alpha n + \beta) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\mathrm{i}\beta f_{j,2}} \, \mathrm{e}^{\mathrm{i}(f_{j,1} + \alpha f_{j,2})n} = \sum_{j=1}^{M_3} c_{j,3} \, \mathrm{e}^{\mathrm{i} f_j(\alpha) n}, \tag{5.4}$$

where $1 \le M_3 \le M$, $f_j(\alpha) \in [-\pi,\, \pi)$ $(j = 1, \ldots, M_3)$ are the pairwise different values of $(f_{j,1} + \alpha f_{j,2})_{2\pi}$ $(j = 1, \ldots, M)$. Here $(f_{j,1} + \alpha f_{j,2})_{2\pi}$ is the symmetric residuum of $f_{j,1} + \alpha f_{j,2}$ modulo 2π, i.e. $f_{j,1} + \alpha f_{j,2} \in (f_{j,1} + \alpha f_{j,2})_{2\pi} + 2\pi\,\mathbb{Z}$ and $(f_{j,1} + \alpha f_{j,2})_{2\pi} \in [-\pi,\, \pi)$. Note that $f_j(\alpha) \in [-\pi,\, \pi)$ and that $f_{j,1} + \alpha f_{j,2}$ can be located outside of $[-\pi,\, \pi)$. The coefficients $c_{j,3} \in \mathbb{C}$ are certain linear combinations of the coefficients $c_j \, \mathrm{e}^{\mathrm{i}\beta f_{j,2}}$. Assume that $c_{j,3} \neq 0$ (without cancellation). Using the Algorithm 4.1, we compute the pairwise different frequencies $f_j(\alpha) \in [-\pi,\, \pi)$ $(j = 1, \ldots, M_3)$.

Then we form the subset $\tilde{F} \subset F$ of all those $(f'_{k,1}, f'_{k,2})^\top \in F$ $(k = 1, \ldots, M_1 M_2)$ such that there exists a frequency $f_j(\alpha)$ with

$$|f_j(\alpha) - (f'_{k,1} + \alpha f'_{k,2})_{2\pi}| < \varepsilon_1 \,,$$

where $\varepsilon_1 > 0$ is an accuracy bound. It depends on the problem how often we have to repeat the last step with different parameters α and β to obtain a small set $\tilde{F} := \{\tilde{\boldsymbol{f}}_j = (\tilde{f}_{j,1}, \tilde{f}_{j,2})^\top : \, j = 1, \ldots, |\tilde{F}|\}$. Finally we compute the coefficients $\tilde{c}_j$ $(j = 1, \ldots, |\tilde{F}|)$ as least squares solution of the overdetermined linear system

$$\sum_{j=1}^{|\tilde{F}|} \tilde{c}_j \, \mathrm{e}^{\mathrm{i}\tilde{\boldsymbol{f}}_j \cdot \boldsymbol{n}} = \tilde{h}(\boldsymbol{n}) \quad (\boldsymbol{n} \in I)\,. \tag{5.5}$$

With other words, this linear system (5.5) reads as follows

$$\begin{aligned}
\sum_{j=1}^{|\tilde{F}|} \tilde{c}_j \, \mathrm{e}^{\mathrm{i}\tilde{f}_{j,1} n} &= \tilde{h}(n, 0) \quad (n \in \mathbb{Z}_N)\,, \\
\sum_{j=1}^{|\tilde{F}|} \tilde{c}_j \, \mathrm{e}^{\mathrm{i}\tilde{f}_{j,2} n} &= \tilde{h}(0, n) \quad (n \in \mathbb{Z}_N)\,, \\
\sum_{j=1}^{|\tilde{F}|} \tilde{c}_j \, \mathrm{e}^{\mathrm{i}\beta\tilde{f}_{j,2}} \, \mathrm{e}^{\mathrm{i}(\tilde{f}_{j,1} + \alpha\tilde{f}_{j,2}) n} &= \tilde{h}(n, \alpha n + \beta) \quad (n \in \mathbb{Z}_N)\,.
\end{aligned}$$

Unfortunately, these three system matrices can possess equal columns. Therefore we represent these matrices as products $\boldsymbol{F}_l\, \boldsymbol{M}_l$ ($l = 1, 2, 3$), where $\boldsymbol{F}_l$ is a nonequispaced Fourier matrix with pairwise different columns and where all entries of $\boldsymbol{M}_l$ are equal to 0 or 1 and only one entry of each column is equal to 1. By [20, Theorem 4.3] the nonequispaced Fourier matrices $\boldsymbol{F}_l$ ($l = 1, 2, 3$) possess left inverses $\boldsymbol{L}_l$. If we introduce the vectors $\tilde{\boldsymbol{h}}_1 := (\tilde{h}(n, 0))_{n=-N}^N$, $\tilde{\boldsymbol{h}}_2 := (\tilde{h}(0, n))_{n=-N}^N$, $\tilde{\boldsymbol{h}}_3 := (\tilde{h}(n, \alpha n + \beta))_{n=-N}^N$, $\tilde{\boldsymbol{c}} := (\tilde{c}_j)_{j=1}^{|\tilde{F}|}$, and the diagonal matrix $\boldsymbol{D} := \operatorname{diag}\,(\exp(\mathrm{i}\beta \tilde{f}_{j,2}))_{j=1}^{|\tilde{F}|}$, we obtain the linear system

$$\begin{pmatrix} \boldsymbol{M}_1 \\ \boldsymbol{M}_2 \\ \boldsymbol{M}_3\, \boldsymbol{D} \end{pmatrix} \tilde{\boldsymbol{c}} = \begin{pmatrix} \boldsymbol{L}_1\, \tilde{\boldsymbol{h}}_1 \\ \boldsymbol{L}_2\, \tilde{\boldsymbol{h}}_2 \\ \boldsymbol{L}_3\, \tilde{\boldsymbol{h}}_3 \end{pmatrix}. \tag{5.6}$$

By a convenient choice of the parameters α, β, the rang of the above system matrix is equal to $|\tilde{F}|$. If this is not the case, we can use sampled values of h along another straight line. In our numerical experiments we have used only the values $\tilde{h}$ sampled on grid points of the straight lines. We summarize:

Algorithm 5.1 (SAPM for $d = 2$)
Input: $\tilde{h}(n, 0), \tilde{h}(0, n) \in \mathbb{C}$ ($n \in \mathbb{Z}_N$), accuracies $\varepsilon_1, \varepsilon_2 > 0$.
m number of additional straight lines, parameters $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$, $\tilde{h}(n, \alpha_l n + \beta_l) \in \mathbb{C}$ ($n \in \mathbb{Z}_N$; $l = 1, \ldots, m$)

Step 1. From the noisy data $\tilde{h}(n, 0)$ ($n \in \mathbb{Z}_N$) and $\tilde{h}(0, n)$ ($n \in \mathbb{Z}_N$) compute by Algorithm 4.1 the pairwise different frequencies $f'_{j,1} \in [-\pi, \pi)$ ($j = 1, \ldots, M_1$) in (5.1) and $f'_{j,2} \in [-\pi, \pi)$ ($j = 1, \ldots, M_2$) in (5.2), respectively. Set $I := \{(n, 0) : n \in \mathbb{Z}_N\} \cup \{(0, n) : n \in \mathbb{Z}_N\}$.
Step 2. Form the Cartesian product (5.3).
Step 3. For $l = 1, \ldots, m$ do:

> From the noisy data $\tilde{h}(n, \alpha_l n + \beta_l)$ ($n \in \mathbb{Z}_N$), compute the pairwise different frequencies $f_j(\alpha_l) \in [-\pi, \pi)$ ($j = 1, \ldots, M_3$) in (5.4) by Algorithm 4.1. Form the subset $F' := \{\boldsymbol{f}'_j : j = 1, \ldots, |F'|\}$ of F of all those $(f'_{k,1}, f'_{k,2})^\top \in F$ ($k = 1, \ldots, |F|$) such that there exists a frequency $f_j(\alpha_l)$ with
>
> $$|f_j(\alpha_l) - (f'_{k,1} + \alpha f'_{k,2})_{2\pi}| < \varepsilon_1\,.$$
>
> Set $I := I \cup \{(n, \alpha_l n + \beta_l) : n \in \mathbb{Z}_N\}$.

Step 4. Compute the least squares solution of the overdetermined linear system

$$\sum_{j=1}^{|F'|} c'_j\, \mathrm{e}^{\mathrm{i}\boldsymbol{f}'_j \cdot \boldsymbol{n}} = \tilde{h}(\boldsymbol{n}) \quad (\boldsymbol{n} \in I)$$

for the frequency set F'.

Step 5. Form the subset $\tilde{F} = \{\tilde{\boldsymbol{f}}_j : j = 1, \ldots, M\}$ of F' of all those $\boldsymbol{f}'_k \in F'$ $(k = 1, \ldots, |F'|)$ with $|c'_k| > \varepsilon_2$.
Step 6. Compute the least squares solution of the overdetermined linear system (5.5) corresponding to the new frequency set $\tilde{F}$.

Output: $M := |\tilde{F}| \in \mathbb{N}$, $\tilde{\boldsymbol{f}}_j \in [-\pi, \pi)^2$, $\tilde{c}_j \in \mathbb{C}$ $(j = 1, \ldots, M)$.

Note that it can be useful in some applications to choose the straight lines $\alpha_l n + \beta_l$ $(n \in \mathbb{Z}_N)$ at random.

In the following we extend the Algorithm 5.1 to the case $d > 2$. To this end, we add a dimension step by step. More precisely, in order to solve the parameter estimation problem for $d = 3$, we use Algorithm 5.1 with given values $\tilde{h}(n, 0, 0)$, $\tilde{h}(0, n, 0)$, $\tilde{h}(n, \alpha^{(1)} n + \beta^{(1)}, 0)$ $(n \in \mathbb{Z}_N)$. Then we compute from the noisy data $\tilde{h}(0, 0, n)$ $(n \in \mathbb{Z}_N)$ of

$$h(0, 0, n) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\mathrm{i} f_{j,3} n} = \sum_{j=1}^{M_3} c_{j,3} \, \mathrm{e}^{\mathrm{i} f'_{j,3} n},$$

where $1 \leq M_3 \leq M$, $f'_{j,3} \in [-\pi, \pi)$ $(j = 1, \ldots, M_3)$ the pairwise different values of $f_{j,3}$ $(j = 1, \ldots, M_3)$ and $c_{j,3} \in \mathbb{C}$ are certain linear combinations of the coefficients c_j. Assume that $c_{j,3} \neq 0$ (without cancellation). Using the Algorithm 4.1, we compute the pairwise different frequencies $f'_{j,3} \in [-\pi, \pi)$ $(j = 1, \ldots, M_3)$ and form the Cartesian product

$$F = \{(f'_{k,1}, f'_{k,2}, f'_{j,3})^\top \in [-\pi, \pi)^3 : k = 1, \ldots, K, j = 1, \ldots, M_3\}$$

of the sets $\{(f'_{1,1}, f'_{1,2})^\top, \ldots, (f'_{K,1}, f'_{K,2})^\top\}$ and $\{f'_{1,3}, \ldots, f'_{M_3,3}\}$, where the set

$$\{(f'_{k,1}, f'_{k,2})^\top \in [-\pi, \pi)^2 : k = 1, \ldots, K\}$$

is the corresponding set after Step 5 of Algorithm 5.1. Now we form a subset of F by using further straight lines. We denote by m_r the number of straight lines to restrict the set F for the dimension r $(r = 2, \ldots, d)$. We describe this lines by the parameters

$$\begin{aligned}
\boldsymbol{\alpha}^{(r)} &= \begin{pmatrix} \alpha^{(r)}_{1,1} & \cdots & \alpha^{(r)}_{1,r-1} \\ \vdots & \ddots & \vdots \\ \alpha^{(r)}_{m_r,1} & \cdots & \alpha^{(r)}_{m_r,r-1} \end{pmatrix} \in \mathbb{R}^{m_r \times (r-1)}, \\
\boldsymbol{\beta}^{(r)} &= \begin{pmatrix} \beta^{(r)}_{1,1} & \cdots & \beta^{(r)}_{1,r-1} \\ \vdots & \ddots & \vdots \\ \beta^{(r)}_{m_r,1} & \cdots & \beta^{(r)}_{m_r,r-1} \end{pmatrix} \in \mathbb{R}^{m_r \times (r-1)} \quad (r = 2, \ldots, d),
\end{aligned}$$

where $\alpha^{(r)}_{k,1}, \ldots, \alpha^{(r)}_{k,r-1}$ and $\beta^{(r)}_{k,1}, \ldots, \beta^{(r)}_{k,r-1}$ are the parameters for the k-th line $(k = 1, \ldots, m_r)$ for the restriction in dimension r $(r = 2, \ldots, d)$.

Thus we obtain the following algorithm:

Algorithm 5.2 (SAPM for $d > 2$)
Input: $\tilde{h}(n, 0, \ldots, 0), \tilde{h}(0, n, 0, \ldots, 0), \ldots, \tilde{h}(0, \ldots, 0, n)$ $(n \in \mathbb{Z}_N)$, accuracies ε_1, $\varepsilon_2 > 0$. m_r number of straight lines for dimension $r = 2, \ldots, d$, parameters of straight lines $\boldsymbol{\alpha}^{(r)}$, $\boldsymbol{\beta}^{(r)} \in \mathbb{R}^{m_r \times (r-1)}$.

Step 1. From the noisy data $\tilde{h}(n, 0, \ldots, 0)$, $\tilde{h}(0, n, 0, \ldots, 0)$, …, $\tilde{h}(0, \ldots, 0, n)$ $(n \in \mathbb{Z}_N)$ compute by Algorithm 4.1 the pairwise different frequencies $f'_{j,1} \in [-\pi,\, \pi)$ $(j = 1, \ldots, M_1)$, $f'_{j,2} \in [-\pi,\, \pi)$ $(j = 1, \ldots, M_2)$, …, $f'_{j,d} \in [-\pi,\, \pi)$ $(j = 1, \ldots, M_d)$.
Set $I := \{(n, 0, \ldots, 0) : n \in \mathbb{Z}_N\} \cup \cdots \cup \{(0, \ldots, n) : n \in \mathbb{Z}_N\}$.
Step 2. Set $F := \{f'_{j,1} : j = 1, \ldots, M_1\}$.
Step 3. For $r = 2, \ldots, d$ do:

Form the Cartesian product

$$F := F \times \{f'_{j,r} : j = 1, \ldots, M_r\} = \{(\boldsymbol{f}_l^\top, f'_{j,r})^\top : l = 1, \ldots |F|,\, j = 1, \ldots, M_r\}\,.$$

For $l = 1, \ldots, m_r$ do:

For the noisy data

$$\tilde{h}(n, \alpha_{n,1}^{(r)} n + \beta_{n,1}^{(r)}, \ldots, \alpha_{n,r-1}^{(r)} n + \beta_{n,r-1}^{(r)}, 0, \ldots, 0) \quad (n \in \mathbb{Z}_N)\,,$$

compute the pairwise different frequencies $G := \{f_j(\boldsymbol{\alpha}_n^{(r)})' \in [-\pi,\, \pi)\, (j = 1, \ldots, M_r)\}$ by Algorithm 4.1. Form the subset $\tilde{F}$ of F of all those $(f_{n_1,1}, f_{n_2,2}, \ldots, f'_{n_r,r})^\top \in F$ such that there exists a frequency $f_j(\boldsymbol{\alpha}_n^{(r)})' \in G$ with

$$|f_j(\boldsymbol{\alpha}_n^{(r)})' - (f_{n_1,1} + \alpha_{n,1}^{(r)} f_{n_2,2} + \cdots + \alpha_{n,r-1}^{(r)} f'_{n_r,r})_{2\pi}| < \varepsilon_1\,.$$

Set $F := \tilde{F}$ and $I := I \cup \{(n, \alpha_{n,1}^{(r)} n + \beta_{n,1}^{(r)}, \ldots, \alpha_{n,r-1}^{(r)} n + \beta_{n,r-1}^{(r)}, 0, \ldots, 0) : n \in \mathbb{Z}_N\}$.

Step 4. Compute the least squares solution of the overdetermined linear system

$$\sum_{j=1}^{|F|} \tilde{c}_j\, \mathrm{e}^{\mathrm{i} \boldsymbol{f}_j \cdot \boldsymbol{n}} = \tilde{h}(\boldsymbol{n}) \quad (\boldsymbol{n} \in I) \tag{5.7}$$

for the frequency set F.
Step 5. Form the set $\tilde{F} := \{\tilde{\boldsymbol{f}}_j : j = 1, \ldots, |\tilde{F}|\}$ of all those $\boldsymbol{f}_k \in F$ $(k = 1, \ldots, |F|)$ with $|\tilde{c}_k| > \varepsilon_2$.
Step 6. Compute the least squares solution of the overdetermined linear system

$$\sum_{j=1}^{|\tilde{F}|} \tilde{c}_j\, \mathrm{e}^{\mathrm{i} \tilde{\boldsymbol{f}}_j \cdot \boldsymbol{n}} = \tilde{h}(\boldsymbol{n}) \quad (\boldsymbol{n} \in I) \tag{5.8}$$

corresponding to the new frequency set $\tilde{F}$.

Output: $M := |\tilde{F}| \in \mathbb{N}$, $\tilde{\boldsymbol{f}}_j \in [-\pi, \pi)^d$, $\tilde{c}_j \in \mathbb{C}$ $(j = 1, \ldots, M)$.

Remark 5.3 Note that we solve the overdetermined linear systems (5.7) and (5.8) only by using the values $\tilde{h}(\boldsymbol{n})$ $(\boldsymbol{n} \in I)$, which we have used to determine the frequencies $\tilde{\boldsymbol{f}}_j$. If more values $\tilde{h}(\boldsymbol{n})$ available, clearly one can use further values as well in the final step to ensure a better least squares solvability of the linear systems, see (5.6) for the case $d = 2$ and Corollary 3.3. In addition we mention that there are variety of possibilities to combine the different dimensions, see e.g. Example 6.4. □

Remark 5.4 Our method can be interpreted as a reconstruction method for sparse multivariate trigonometric polynomials from few samples, see [12, 10, 25] and the references therein. More precisely, let Π_N^d denote the space of all d–variate trigonometric polynomials of maximal order N. An element $p \in \Pi_N^d$ can be represented in the form

$$p(\boldsymbol{y}) = \sum_{\boldsymbol{k} \in \mathbb{Z}_N^d} c_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \, \boldsymbol{k} \cdot \boldsymbol{y}} \quad (\boldsymbol{y} \in [-\frac{1}{2}, \frac{1}{2}]^d)$$

with $c_{\boldsymbol{k}} \in \mathbb{C}$. There exist completely different methods for the reconstruction of "sparse trigonometric polynomials", i.e., one assumes that the number M of the nonzero coefficients $c_{\boldsymbol{k}}$ is much smaller than the dimension of Π_N^d. Therefore our method can be used with

$$h(\boldsymbol{x}) := p(\frac{\boldsymbol{x}}{2N}) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\mathrm{i} \, \boldsymbol{f}_j \cdot \boldsymbol{x}} \quad (\boldsymbol{x} \in [-N, N]^d),$$

and $\boldsymbol{x} = 2N\boldsymbol{y}$ and $\boldsymbol{f}_j = \pi \boldsymbol{k}/N$ if $c_{\boldsymbol{k}} \neq 0$. Using Algorithm 5.2, we find the frequency vectors $\boldsymbol{f}_j$ and the coefficients c_j and finally set $\boldsymbol{k} := \mathrm{round}(N\boldsymbol{f}_j/\pi)$, $c_{\boldsymbol{k}} := c_j$. By [7] one knows sharp versions of L^2–norm equivalences for trigonometric polynomials under the assumption that the sampling set contains no holes larger than the inverse polynomial degree, see also [2]. □

6 Numerical experiments

Finally, we apply the algorithms suggested in Section 5 to various examples. We have implemented our algorithms in MATLAB with IEEE double precision arithmetic. We compute the relative error of the frequencies given by

$$e(\boldsymbol{f}) := \max_{l=1,\ldots,d} \frac{\max\limits_{j=1,\ldots,M} |f_{j,l} - \tilde{f}_{j,l}|}{\max\limits_{j=1,\ldots,M} |f_{j,l}|},$$

where $\tilde{f}_{j,l}$ are the frequency components computed by our algorithms. Analogously, the relative error of the coefficients is defined by

$$e(\boldsymbol{c}) := \frac{\max\limits_{j=1,\ldots,M} |c_j - \tilde{c}_j|}{\max\limits_{j=1,\ldots,M} |c_j|},$$

where $\tilde{c}_j$ are the coefficients computed by our algorithms. Further we determine the relative error of the exponential sum by

$$e(h) := \frac{\max |h(\boldsymbol{x}) - \tilde{h}(\boldsymbol{x})|}{\max |h(\boldsymbol{x})|},$$

where the maximum is built from approximately 10000 equispaced points from a grid of $[-N, N]^d$, and where

$$\tilde{h}(\boldsymbol{x}) := \sum_{j=1}^{M} \tilde{c}_j \, \mathrm{e}^{\tilde{\boldsymbol{f}}_j \cdot \boldsymbol{x}}$$

is the exponential sum recovered by our algorithms. We begin with an example previously considered in [22].

Example 6.1 The bivariate exponential sum (1.1) taken from [22, Example 1] possesses the following parameters

$$(\boldsymbol{f}_j^\top)_{j=1}^3 = \begin{pmatrix} 0.48\pi & 0.48\pi \\ 0.48\pi & -0.48\pi \\ -0.48\pi & 0.48\pi \end{pmatrix}, \quad (c_j)_{j=1}^8 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We sample this exponential sum (1.1) at the nodes $h(k, 0)$, $h(0, k)$ and $h(k, \alpha k + \beta)$, $(k \in \mathbb{Z}_N)$, where α, $\beta \in \mathbb{Z}$ are given in Table 6.1. Then we apply our Algorithm 5.1 for exact sampled data and for noisy sampled data $\tilde{h}(\boldsymbol{k}) = h(\boldsymbol{k}) + 10^{-\delta} e_{\boldsymbol{k}}$, where $e_{\boldsymbol{k}}$ is uniformly distributed in $[-1, 1]$. The notation $\delta = \infty$ means that exact data are given. We present the results in Table 6.1. It is remarkable that we obtain very precise results even in the case, where the unknown number $M = 3$ is estimated by L. □

L	N	ε_1	α	β	δ	$e(\boldsymbol{f})$	$e(\boldsymbol{c})$	$e(h)$
5	6	10^{-4}	1	0	∞	1.7e−15	5.9e−14	3.2e−13
10	20	10^{-4}	1	0	∞	5.4e−15	4.5e−14	4.5e−14
5	25	10^{-3}	1	0	6	5.6e−09	1.6e−07	2.5e−07
5	25	10^{-3}	1, 2	0, 0	6	1.0e−08	5.9e−07	7.4e−07
5	25	10^{-3}	1	0	5	1.7e−08	1.2e−06	1.3e−06

Table 6.1: Results of Example 6.1.

Example 6.2 We consider the bivariate exponential sum (1.1) with following parameters

$$(\boldsymbol{f}_j^\top)_{j=1}^8 = \begin{pmatrix} 0.1 & 1.2 \\ 0.19 & 1.3 \\ 0.3 & 1.5 \\ 0.35 & 0.3 \\ -0.1 & 1.2 \\ -0.19 & 0.35 \\ -0.3 & -1.5 \\ -0.3 & 0.3 \end{pmatrix}, \quad (c_j)_{j=1}^8 = \begin{pmatrix} 1+\mathrm{i} \\ 2+3\,\mathrm{i} \\ 5-6\,\mathrm{i} \\ 0.2-\mathrm{i} \\ 1+\mathrm{i} \\ 2+3\,\mathrm{i} \\ 5-6\,\mathrm{i} \\ 0.2-\mathrm{i} \end{pmatrix}.$$

For given exact data, the results are presented in Table 6.2. The dash $-$ means that we are not able to reconstruct the signal.
Then we use noisy sampled data $\tilde{h}(\boldsymbol{k}) = h(\boldsymbol{k}) + 10^{-\delta}\, e_{\boldsymbol{k}}$, where $e_{\boldsymbol{k}}$ is uniformly distributed in $[-1, 1]$. Instead of predeterminated values α and β, we choose these values randomly. We use only one additional line for sampling and present the results in Table 6.3, where $e(\boldsymbol{f})$, $e(\boldsymbol{c})$ and $e(h)$ are the averages of 100 runs. □

L	N	ε_1	α	β	$e(\boldsymbol{f})$	$e(\boldsymbol{c})$	$e(h)$
8	15	10^{-4}	1	0	2.7e−09	5.7e−09	3.4e−09
8	15	10^{-4}	1, 2, 3	0, 1, 2	2.7e−09	5.9e−09	3.3e−09
15	30	10^{-4}	1	0	1.4e−13	3.4e−13	6.5e−13
15	30	$2 \cdot 10^{-1}$	1	0	−	−	−
15	30	$2 \cdot 10^{-1}$	1, 2	0, 0	1.4e−13	4.0e−13	6.0e−13
15	80	$2 \cdot 10^{-1}$	1	0	3.5e−15	3.2e−14	7.5e−14

Table 6.2: Results of Example 6.2 with exact data.

L	N	ε_1	δ	$e(\boldsymbol{f})$	$e(\boldsymbol{c})$	$e(h)$
8	35	10^{-3}	6	1.4e−06	3.9e−06	5.5e−06
15	30	10^{-3}	6	1.2e−05	3.9e−05	5.3e−05
15	50	10^{-3}	5	4.0e−07	4.1e−06	3.8e−06
15	50	10^{-3}	6	3.8e−08	3.6e−07	3.3e−07

Table 6.3: Results of Example 6.2 with noisy data.

Example 6.3 We consider the trivariate exponential sum (1.1) with following parameters

$$(\boldsymbol{f}_j^\top)_{j=1}^8 = \begin{pmatrix} 0.1 & 1.2 & 0.1 \\ 0.19 & 1.3 & 0.2 \\ 0.4 & 1.5 & 1.5 \\ 0.45 & 0.3 & -0.3 \\ -0.1 & 1.2 & 0.1 \\ -0.19 & 0.35 & -0.5 \\ -0.4 & -1.5 & 0.25 \\ -0.4 & 0.3 & -0.3 \end{pmatrix}, \quad (c_j)_{j=1}^8 = \begin{pmatrix} 1+\mathrm{i} \\ 2+3\,\mathrm{i} \\ 5-6\,\mathrm{i} \\ 0.2-\mathrm{i} \\ 1+\mathrm{i} \\ 2+3\,\mathrm{i} \\ 5-6\,\mathrm{i} \\ 0.2-\mathrm{i} \end{pmatrix}.$$

and present the results in Table 6.4. □

L	N	ε_1	$\boldsymbol{\alpha}^{(1)}$	$\boldsymbol{\alpha}^{(2)}$	$\boldsymbol{\beta}^{(1)}$	$\boldsymbol{\beta}^{(2)}$	δ	$e(\boldsymbol{f})$	$e(\boldsymbol{c})$	$e(h)$
8	15	10^{-4}	(1)	$\begin{pmatrix}1 & 1\end{pmatrix}$	(0)	$\begin{pmatrix}0 & 0\end{pmatrix}$	∞	1.5e–10	1.7e–10	8.2e–11
8	15	10^{-4}	(1)	$\begin{pmatrix}1 & 1\end{pmatrix}$	(1)	$\begin{pmatrix}1 & 1\end{pmatrix}$	∞	1.5e–10	1.7e–10	8.1e–11
10	30	10^{-3}	(1)	$\begin{pmatrix}1 & 1\end{pmatrix}$	(0)	$\begin{pmatrix}0 & 0\end{pmatrix}$	6	8.7e–07	1.5e–06	2.9e–06
10	30	10^{-3}	(1)	$\begin{pmatrix}1 & 1 \\ 1 & 2\end{pmatrix}$	(0)	$\begin{pmatrix}0 & 0 \\ 0 & 0\end{pmatrix}$	6	7.8e–08	1.1e–06	1.5e–06
10	30	10^{-3}	(1)	$\begin{pmatrix}1 & 1 \\ 1 & 2\end{pmatrix}$	(0)	$\begin{pmatrix}0 & 0 \\ 0 & 0\end{pmatrix}$	5	4.5e–06	1.0e–05	1.6e–05
10	30	10^{-3}	(1)	$\begin{pmatrix}1 & 1 \\ 1 & 2\end{pmatrix}$	(0)	$\begin{pmatrix}0 & 0 \\ 0 & 0\end{pmatrix}$	4	1.2e–05	2.5e–05	5.2e–05

Table 6.4: Results of Example 6.3.

Example 6.4 Now we consider the 4–variate exponential sum (1.1) with following parameters

$$(\boldsymbol{f}_j^\top)_{j=1}^8 = \begin{pmatrix} 0.1 & 1.2 & 0.1 & 0.45 \\ 0.19 & 1.3 & 0.2 & 1.5 \\ 0.3 & 1.5 & 1.5 & -1.3 \\ 0.45 & 0.3 & -0.3 & 0.4 \\ -0.1 & 1.2 & 0.1 & -1.5 \\ -0.19 & 0.35 & -0.5 & -0.45 \\ -0.4 & -1.5 & 0.25 & 1.3 \\ -0.4 & 0.3 & -0.3 & 0.4 \end{pmatrix}, \quad (c_j)_{j=1}^8 = \begin{pmatrix} 1+\mathrm{i} \\ 2+3\,\mathrm{i} \\ 5-6\,\mathrm{i} \\ 0.2-\mathrm{i} \\ 1+\mathrm{i} \\ 2+3\,\mathrm{i} \\ 5-6\,\mathrm{i} \\ 0.2-\mathrm{i} \end{pmatrix}.$$

Instead of using Algorithm 5.2 directly, we apply the Algorithm 5.1 for the first two variables and then for the last variables with the parameters $\boldsymbol{\alpha}^{(2)}$ and $\boldsymbol{\beta}^{(2)}$. Then we take the tensor product of the obtained two parameter sets and use the additional parameters from $\boldsymbol{\alpha}^{(4)}$ and $\boldsymbol{\beta}^{(4)}$ in order to find a reduced set. Finally we solve the overdetermined linear system. The results are presented in Table 6.5. □

L	N	ϵ_1	$\boldsymbol{\alpha}^{(2)}$	$\boldsymbol{\alpha}^{(4)}$	$\boldsymbol{\beta}^{(2)}$	$\boldsymbol{\beta}^{(4)}$	δ	$e(\boldsymbol{f})$	$e(\boldsymbol{c})$	$e(h)$
8	15	10^{-4}	1	$\begin{pmatrix}1 & 1 & 1\end{pmatrix}$	0	$\begin{pmatrix}0 & 0 & 0\end{pmatrix}$	∞	1.7e-10	2.5e-11	1.6e-10
8	15	10^{-4}	1	$\begin{pmatrix}1 & 1 & 1\end{pmatrix}$	1	$\begin{pmatrix}1 & 1 & 1\end{pmatrix}$	∞	1.7e-10	2.4e-11	1.6e-10
15	30	10^{-4}	1	$\begin{pmatrix}1 & 1 & 1\end{pmatrix}$	0	$\begin{pmatrix}0 & 0 & 0\end{pmatrix}$	∞	1.3e-14	6.4e-15	8.8e-14
15	30	10^{-3}	1	$\begin{pmatrix}1 & 1 & 1\end{pmatrix}$	0	$\begin{pmatrix}0 & 0 & 0\end{pmatrix}$	6	1.0e-06	3.2e-07	3.0e-06
15	30	10^{-3}	1	$\begin{pmatrix}1 & 1 & 1\end{pmatrix}$	0	$\begin{pmatrix}0 & 0 & 0\end{pmatrix}$	5	1.3e-05	3.4e-06	4.2e-05
15	30	10^{-3}	$\begin{pmatrix}1 \\ -1\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 1 \\ -1 & 1 & -1\end{pmatrix}$	$\begin{pmatrix}0 \\ 0\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}$	6	1.1e-06	2.7e-07	3.9e-06
15	30	10^{-3}	$\begin{pmatrix}1 \\ -1\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 1 \\ -1 & 1 & -1\end{pmatrix}$	$\begin{pmatrix}0 \\ 0\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}$	5	8.8e-06	1.9e-06	3.3e-05
15	50	10^{-3}	$\begin{pmatrix}1 \\ -1\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 1 \\ -1 & 1 & -1\end{pmatrix}$	$\begin{pmatrix}0 \\ 0\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}$	5	4.5e-07	1.2e-07	1.6e-06
15	50	10^{-3}	$\begin{pmatrix}1 \\ -1\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 1 \\ -1 & 1 & -1\end{pmatrix}$	$\begin{pmatrix}0 \\ 0\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}$	4	8.0e-07	2.4e-07	1.1e-05

Table 6.5: Results of Example 6.4.

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