SOME NOTE ON THE MODULUS OF CONTINUITY FOR ILL-POSED PROBLEMS IN HILBERT SPACE

BERND HOFMANN AND PETER MATHÉ

ABSTRACT. The authors study linear ill-posed operator equations in Hilbert space. Such equations become conditionally well-posed by imposing certain smoothness assumptions, often given relative to the operator which governs the equation. Usually this is done in terms of general source conditions. Recently smoothness of an element was given in terms of properties of the distribution function of this element with respect to the self-adjoint associate of the underlying operator. In all cases the original ill-posed problem becomes well-posed, and properties of the corresponding modulus of continuity are of interest, specifically whether this is a concave function. The authors extend previous concavity results of a function related to the modulus of continuity, and obtained for compact operators in B. Hofmann, P. Mathé, and M. Schieck, *Modulus of continuity for conditionally stable ill-posed problems in Hilbert space*, J. Inverse Ill-Posed Probl. **16** (2008), no. 6, 567–585, to the general case of bounded operators in Hilbert space, and for recently introduced smoothness classes. This paper is dedicated to the 70th anniversary of the Corresponding-Member of the Russian Academy of Sciences VLADIMIR V. VASIN (Yekaterinburg).

1. INTRODUCTION

The focus of this note is on *linear ill-posed problems* that can be written as operator equations

$$Ax = y, \qquad x \in X,\ y \in Y, \tag{1}$$

where $A : X \to Y$ is a bounded injective linear mapping between infinite-dimensional separable Hilbert spaces X and Y endowed with inner products $\langle \cdot, \cdot \rangle$ and norms $\| \cdot \|$. We associate A with the positive self-adjoint operator

$$H := A^*A : X \to X \tag{2}$$

and set $a := \|H\| = \|A\|^2$ such that a is the maximum value of the spectrum $\sigma(H)$ of H and zero the corresponding minimum value which moreover represents an accumulation point of $\sigma(H)$ in the ill-posed case. Following the notation of [14] for the ill-posedness of (1) characterized by a non-closed range $\mathcal{R}(A)$ of A we distinguish between the ill-posedness of *type I* where A is *non-compact* and of *type II* where A is compact, for more details see also [3, 7].

The solution theory of ill-posed problems is preferably based on the fact that these problems become conditionally well-posed after imposing certain smoothness assumptions by restricting the admissible solutions to a set $\mathfrak{M}$. Then the severity of the ill-posedness phenomenon in solving a problem (1) depends on the interplay between the smoothing properties of the operator A and the smoothness of potential solutions $x \in \mathfrak{M} \subset X$. The solution theory may be considered element-wise and, as this is traditionally done, uniformly for smoothness classes.

For the analysis of ill-posed problems solution smoothness is most often measured *relative* to the operator A governing the equation (1), precisely its self-adjoint associate H. First, one can quantify the *individual smoothness* of an element $x \in X$ with respect to H by using the *point-wise* spectral information, i.e., the *distribution function*

$$F_x^2(t) := \|E_t x\|^2 := \langle \chi_{(0,t]}(H)\, x, x\rangle = \|\chi_{(0,t]}(H)x\|^2, \quad 0 < t < \infty, \tag{3}$$

where $\chi_{(0,t]}$ is the characteristic function on the interval $(0,t]$. This idea goes back to [15, 16] and it was further explored in [2], This non-decreasing and right-continuous function F_x^2, which satisfies the limit condition $\lim\limits_{t\to 0} F_x^2(t) = 0$ as a consequence of the ill-posedness, can be rewritten as $F_x^2(t) = \int\limits_0^t d\|E_s x\|^2$ for $t > 0$, where $E_t = E_t(H)$, $0 \le t \le a$, denotes the spectral resolution of the operator H. Note that

$$\|h(H)x\|^2 = \int\limits_0^a h^2(t)\; d\|E_t x\|^2 = \int\limits_0^a h^2(t)\; dF_x^2(t)$$

holds for any bounded measurable real function h. We refer to [1, Sect. 2.3] and [18, Chapt. 12] for details on spectral theory of bounded and self-adjoint linear operators in Hilbert space.

The most prominent, and traditional way of quantifying solution smoothness uses smoothness classes in terms of *source sets* $\mathfrak{M} = \mathcal{M}_{\varphi,R}$ defined as

$$\mathcal{M}_{\varphi,R} := \{x \in X :\; x = \varphi(H)\, v,\; v \in X,\; \|v\| \le R\}, \qquad R > 0. \tag{4}$$

Above, the functions $\varphi : (0,a] \to (0,\infty)$ are derived from variable Hilbert scales and called *index functions*; these are assumed to be increasing with $\lim\limits_{t\to+0} \varphi(t) = 0$. Here we follow the concept of [12, 13] or more recently [5, 11]. Source sets express the solution smoothness with respect to the spectrum of H in an *integral* manner, since we have that

$$x \in \mathcal{M}_{\varphi,R} \quad \text{if and only if} \quad \int_0^a \frac{1}{\varphi^2(t)}\, d\|E_t x\|^2 \le R^2.$$

Alternatively, and as this was recently suggested and roughly discussed in [4], one can assign smoothness classes by considering, in analogy to (4), the *level sets* $\mathfrak{M} = \mathcal{E}_{\psi,E}$ defined as

$$\mathcal{E}_{\psi,E} := \{x \in X :\ F_x^2(t) \le E^2\, \psi^2(t),\ 0 < t \le a\}, \qquad E > 0, \tag{5}$$

for index functions $\psi : (0,a] \to (0,\infty)$. It is easy to see and was established in [4, Prop. 9] that $\mathcal{M}_{\psi,E} \subset \mathcal{E}_{\psi,E}$ if φ and ψ in (4) and (5) coincide.

The following structural properties of both the source and the level sets are given next.

Proposition 1. *Let the operator A be as in (1) with associate H, see (2). Then the following properties hold true:*

1. *For arbitrary index functions φ and ψ the sets $\mathcal{M}_{\varphi,R}$ and $\mathcal{E}_{\psi,E}$ are centrally symmetric and convex.*
2. *If the operator A is compact then the sets $\mathcal{M}_{\varphi,R}$ and $\mathcal{E}_{\psi,E}$ are compact.*

Proof. The central symmetry is evident by definition. Also the convexity of $\mathcal{M}_{\varphi,R}$ is clear since this set is a linear transformation of a ball. For $x \in \mathcal{E}_{\psi,E}$ one has that $\|\chi_{(0,t]}(H)x\| = F_x(t) \le E\,\psi(t)$ for all $0 < t \le a$. Therefore the convexity follows from the fact that the inequalities under consideration remain valid for convex linear combinations of the elements x.

We turn to the second assertion. The compactness of the sets $\mathcal{M}_{\varphi,R}$ was established in [5, Lemma 2.8], and it follows from the fact that $\mathcal{M}_{\varphi,R}$ is the image of a closed convex set under a compact operator. Closedness of the sets $\mathcal{E}_{\psi,E}$ is immediate from (5). Also, it is clear that the set $\mathcal{E}_{\psi,E}$ is bounded, by letting $t := a$ in (5). To see the relative compactness, let us denote by $s_1 \ge s_2, \dots > 0$ the eigenvalues and by $u_1, u_2, \dots$ the corresponding eigenelements of the non-negative operator H. With this notation we can write

$$F_x^2(t) = \|\chi_{(0,t]}(H)x\|^2 = \sum_{s_j \le t} |\langle x, u_j\rangle|^2 \le E^2\psi^2(t)$$

for elements $x \in \mathcal{E}_{\psi,E}$ and for each $0 < t \le a$. But this yields that for any index function ψ the tails $\sum_{j=k}^{\infty} |\langle x, u_j\rangle|^2$ tend to zero as

$k \to \infty$ uniformly for all $x \in \mathcal{E}_{\psi,E}$. This gives the compactness, see [10, Chapt. II], and completes the proof. $\square$

For source sets and level sets representing $\mathfrak{M} \subset X$ the best possible error for reconstruction of the solution $x \in \mathfrak{M}$ based on noisy data y^δ, satisfying $\|y^\delta - y\| \le \delta$ instead of the exact right-hand side $y \in \mathcal{R}(A)$, is given by

$$(6)\quad \tilde{\omega}(A, \mathfrak{M}, \delta) := \sup\left\{\|x_1 - x_2\| : \ x_1, x_2 \in \mathfrak{M}, \ \|A(x_1 - x_2)\| \le \delta\right\}.$$

For *ellipsoidal sets* $\mathfrak{M} := \{x \in X : \ x = Gv, \ \|v\| \le 1\}$ with some bounded linear operator $G : X \to X$ this function coincides with the *modulus of continuity*

$$(7)\qquad \omega\left(A, \mathfrak{M}, \delta\right) := \sup\left\{\|x\| : \ x \in M, \ \|Ax\| \le \delta\right\}, \qquad \delta > 0,$$

of the inverse operator A^{-1} restricted to $\mathfrak{M}$. This modulus acts as a measure of ill-posedness pre-estimating the reconstruction error in solving (1) for given $\delta > 0$.

The mathematical school of Sverdlovsk/Yekaterinburg, see the monograph [9] by Ivanov, Vasin and Tanana, very early studied such moduli in connection with the development of the method of quasi-solutions. In the last decades Vladimir V. Vasin continued, extended and improved such studies on regularization methods for the stable approximate solution of ill-posed operator equations, see for example [20, 21, 22].

In Section 2 we collect some properties of the modulus of continuity $\omega\left(A, \mathfrak{M}, \delta\right)$, in particular with respect to the source sets and level sets. Our main result presented in Section 3 is to show the concavity of the associated function $\omega^2(A, \mathfrak{M}, \sqrt{\delta})$ for both classes. The paper will be completed with some remarks on upper and lower bounds for the modulus of continuity.

2. Basic properties of the modulus of continuity

We are going to study the modulus of continuity (7) with focus on the sets $\mathcal{M}_{\varphi,R}$ and $\mathcal{E}_{\psi,E}$. At the beginning we recall the following proposition formulated and proved in [6, Thm. 2.1] that characterizes the main properties of such modulus. Below we shall set

$$U\,\mathfrak{M} := \{z \in Z : \ z = U\,x, \ x \in \mathfrak{M}\},$$

for linear operators $U \colon X \to Z$ and some Hilbert space Z. In that sense, we use $K\,\mathfrak{M} := \{x \in X : \ x = K\,\tilde{x}, \ \tilde{x} \in \mathfrak{M}\}$ for constants $K > 0$ by identifying the constant K with the multiple KI of the unit operator.

We state the following useful results.

Proposition 2. *For centrally symmetric and convex sets $\mathfrak{M}$, which means that with $x_1, x_2 \in \mathfrak{M}$ also the elements $-x_2$ and $(x_1 - x_2)/2$ belong to $\mathfrak{M}$, the following properties hold for the moduli of continuity from (7):*

(a) If $\mathfrak{M}$ is bounded then $\omega(A, \mathfrak{M}, \delta)$ is a finite, positive and non-decreasing function for $\delta > 0$ and it is constant for $\delta \geq \bar{\delta} := \sup\limits_{x \in \mathfrak{M}} \|Ax\|$.
(b) If $\mathfrak{M}$ is relatively compact then $\lim\limits_{\delta \to 0} \omega(A, \mathfrak{M}, \delta) = 0$.
(c) $\omega(A, K\mathfrak{M}, \delta) = K\,\omega(A, \mathfrak{M}, \delta/K)$ for $K > 0$.
(d) $\omega(A, \mathfrak{M}, C\delta) \leq C\omega(A, \mathfrak{M}, \delta)$ for $C > 1$.
(e) $\omega(A, K\mathfrak{M}, C\delta) \leq \max\{C, K\}\,\omega(A, \mathfrak{M}, \delta)$ for $C, K > 0$.
(f) the decay rate of $\omega(A, \mathfrak{M}, \delta) \to 0$ as $\delta \to 0$ is at most linear.

We add a result on the behavior of the modulus of continuity with respect to unitary transformations.

Proposition 3.

(i) Let A and H be as in (1) and (2). Then we have that

$$\omega(A, \mathfrak{M}, \delta) = \omega\left(H^{1/2}, \mathfrak{M}, \delta\right), \quad \delta > 0.$$

(ii) If $B = UGU^* : Z \to Z$ for some unitary operator $U : X \to Z$ mapping into the Hilbert space Z with norm $\|\cdot\|_*$ and some bounded linear operator $G : X \to X$, then

$$\omega(G, \mathfrak{M}, \delta) = \omega(B, U\mathfrak{M}, \delta), \quad \delta > 0.$$

Proof. The first assertion (i) is an immediate consequence of $\|Ax\| = \|H^{1/2}x\|$ for $x \in X$. To prove (ii), let $B = UGU^*$. Then we have with $v := Ux$ and $\|v\|_* = \|x\|$

$$\begin{aligned}
\omega(G, \mathfrak{M}, \delta) &= \sup\{\|x\| : \ x \in \mathfrak{M}, \ \|Gx\| \leq \delta\} \\
&= \sup\{\|U^*v\| : \ \ U^*v \in \mathfrak{M}, \ \|GU^*v\| \leq \delta\} \\
&= \sup\{\|v\|_* : \ v \in U\mathfrak{M}, \ \|Bv\|_* \leq \delta\} \\
&= \omega(B, U\mathfrak{M}, \delta).
\end{aligned}$$

This completes the proof. ☐

We established in Proposition 1 that the sets $\mathcal{M}_{\varphi,R}$ and $\mathcal{E}_{\psi,E}$ are centrally symmetric and convex. Therefore Proposition 2 applies. Within the traditional setup when smoothness is given in terms of source sets with power type index function φ then it is known that the modulus of continuity is *concave* and that $\omega(A, \mathfrak{M}, \delta) \sim \delta^\kappa$ with $0 < \kappa < 1$. This also holds for the logarithmic case $\omega(A, \mathfrak{M}, \delta) \sim (\log(1/\delta))^{-\kappa}$ with $\kappa > 0$ if $\delta > 0$, and the concavity of the modulus of continuity seems to be typical. However, not necessarily the functions $\omega\left(A, \mathfrak{M}, \delta\right)$ are convex for any classes $\mathfrak{M} = M_{\varphi,R}$ and $\mathfrak{M} = E_{\psi,E}$. Nevertheless, we can show that the associated function $\omega^2(A, \mathfrak{M}, \sqrt{\delta})$ is convex in any case for both of the classes and for all $\delta > 0$. Because of Proposition 2(c) and because $\mathcal{M}_{\varphi,R} = R\,\mathcal{M}_{\varphi,1}$ it is sufficient to consider the case $R = 1$ and the set $\mathcal{M}_\varphi := \mathcal{M}_{\varphi,1}$. Similar holds for $\mathcal{E}_{\psi,E}$ and we let $\mathcal{E}_\psi := \mathcal{E}_{\psi,1}$.

3. Concavity of the modulus on smoothness classes

Based on results from [8], see also [13, Thm. 1], it was proved in [6, Rem. 3.6] that for compact operators A, and under a rather weak additional condition on φ, the function

$$\tau(A, \mathfrak{M}, \delta) := \omega^2(A, \mathfrak{M}, \sqrt{\delta}), \quad \delta > 0, \tag{8}$$

is for $\mathfrak{M} = \mathcal{M}_\varphi$ a concave linear spline, or more precisely the smallest concave index function that interpolates points defined by spectral properties of A and their interplay with the function φ. Later in [6, Prop. 3.5] the authors have proved the concavity of τ with $\mathfrak{M} = \mathcal{M}_\varphi$ for compact A, i.e., for ill-posedness of type II. The following Theorem 3 extends this result to ill-posedness of type I, thus covering the case of multiplication operators with multiplier functions having an essential zero. Moreover we can prove concavity of τ for all $\delta > 0$ also in the case $\mathfrak{M} = \mathcal{E}_\psi$.

We start with the following preliminary discussion, and we recall the spectral theorem for bounded self-adjoint linear operators in Hilbert space, see [23, Chapt. VII.1] and [17, Chapt. VII].

Proposition 4. *For every bounded self-adjoint linear operator $H : X \to X$ mapping in the the separable Hilbert space X there exist a measurable space $(\Omega, \mathcal{A}, \mu)$, a unitary transformation $U : X \to Z := L^2(\Omega, \mathcal{A}, \mu)$, and a measurable function $f : \Omega \to \sigma(H) \setminus \{0\} \subseteq (0, \|H\|] \subset \mathbb{R}$ such that $M_f := UHU^*$ is a multiplication operator defined as*

$$[M_f\, h](\omega) := f(\omega)\, h(\omega), \quad \omega \in \Omega,$$

and mapping Z into itself. Moreover we have $\eta(H) = U^* M_{\eta(f)} U$ for bounded measurable functions η.

We can apply this result for the non-negative operator $H = A^*A$, and thus find a non-negative function f together with a unitary mapping U such that $H = U^* M_f U$, where we shall abbreviate $\|g\|_* := \|g\|_{L_2(\Omega,\mathcal{A},\mu)}$. By Proposition 3 we find that

$$\omega\left(A, \mathfrak{M}, \delta\right) = \omega\left(H^{1/2}, \mathfrak{M}, \delta\right) = \omega\left(M_{f^{1/2}}, U\mathfrak{M}, \delta\right), \quad \delta > 0. \tag{9}$$

It is thus interesting to determine the analogs of $\mathcal{M}_{\varphi,R}$ and $\mathcal{E}_{\psi,E}$ in the multiplication context, i.e., the images $U\mathcal{M}_{\varphi,R}$ and $U\mathcal{E}_{\psi,E}$, respectively. We state the following without proof.

Lemma 1. *We have that*

$$U\mathcal{M}_{\varphi,R} = \left\{g \in Z : \quad g = \varphi(f)h, \ \|h\|_* \le 1\right\}.$$

and

$$U\mathcal{E}_{\psi,E} = \left\{ g \in Z : \int\limits_{0<f(\omega)\le t} |g(\omega)|^2 \ d\mu(\omega) \le E^2\psi^2(t), \ 0 < t \le a \right\}.$$

The main result is the following.

Theorem. *For every bounded linear operator $A : X \to Y$ with non-closed range $\mathcal{R}(A)$ and arbitrary index functions φ and ψ defined on the interval $(0, \|A\|^2]$ the functions $\tau(A, \mathfrak{M}, \delta)$, $\delta > 0$, from (8) are concave for the classes $\mathfrak{M} := \mathcal{M}_\varphi$ and $\mathfrak{M} := \mathcal{E}_\psi$.*

Proof. We first carry out the proof for $\mathfrak{M} := \mathcal{M}_\varphi$, and we use (9) together with Lemma 1. By introducing the function $\Theta(t) := \sqrt{t}\,\varphi(t)$, $0 < t \le a$, we find that

$$\begin{aligned}\tau(A, \mathcal{M}_\varphi, \delta) &= \sup\left\{\|g\|_*^2 : \ g = \varphi(f)h, \ \|h\|_* \le 1, \ \|\sqrt{f}g\|_*^2 \le \delta\right\} \\ &= \sup\left\{\|\varphi(f)h\|_*^2 : \ \|h\|_* \le 1, \ \|\Theta(f)h\|_*^2 \le \delta\right\}.\end{aligned}$$

Consider arbitrarily chosen $0 < \delta_1 < \delta < \delta_2$ and $\delta = \lambda\delta_1 + (1-\lambda)\delta_2$ for some appropriate $0 < \lambda < 1$. With given $\varepsilon > 0$ we can find elements $h_1, h_2 \in L^2(\Omega, \mathcal{A}, \mu)$, $\|h_1\|_* \le 1, \|h_2\|_* \le 1$, satisfying the conditions

$$\int_0^a \Theta^2(f(\omega))h_1^2(\omega)\, d\mu(\omega) \le \delta_1, \ \int_0^a \varphi^2(f(\omega))h_1^2(\omega)\, d\mu(\omega) \ge \tau(A, \mathcal{M}_\varphi, \delta_1) - \varepsilon$$

and

$$\int_0^a \Theta^2(f(\omega))h_2^2(\omega)\, d\mu(\omega) \le \delta_1, \ \int_0^a \varphi^2(f(\omega))h_2^2(\omega)\, d\mu(\omega) \ge \tau(A, \mathcal{M}_\varphi, \delta_2) - \varepsilon.$$

We let h be chosen such that

$$h^2(\omega) := \lambda h_1^2(\omega) + (1-\lambda)h_2^2(\omega), \quad \omega \in \Omega. \tag{10}$$

Plainly, $\|h\|_* \leq 1$. Also we have that

$$\begin{aligned}
&\int_0^a \Theta^2(f(\omega))h^2(\omega)\; d\mu(\omega)\\
&= \lambda \int_0^a \Theta^2(f(\omega))h_1^2(\omega)\; d\mu(\omega) + (1-\lambda)\int_0^a \Theta^2(f(\omega))h_2^2(\omega)\; d\mu(\omega)\\
&\leq \lambda\delta_1 + (1-\lambda)\delta_2 = \delta.
\end{aligned}$$

Therefore we conclude that

$$\tau(A, \mathcal{M}_\varphi, \delta) \geq \int_0^a \varphi^2(f(\omega))h^2(\omega)\; d\mu(\omega)$$

$$= \lambda \int_0^a \varphi^2(f(\omega))h_1^2(\omega)\; d\mu(\omega) + (1-\lambda)\int_0^a \varphi^2(f(\omega))h_2^2(\omega)\; d\mu(\omega)$$

$$\geq \lambda\, \tau(A, \mathcal{M}_\varphi, \delta_1) + (1-\lambda)\, \tau(A, \mathcal{M}_\varphi, \delta_2) - \varepsilon.$$

Letting $\varepsilon \to 0$ this proves the required concavity assertion for the source set $\mathcal{M}_\varphi$.

For the level set $\mathcal{E}_\psi$ the proof is similar. We start from

$$\tau(A, \mathcal{E}_\psi, \delta) = \sup\left\{ \|g\|_*^2 : \int\limits_{0<f(\omega)\leq t} |g(\omega)|^2\; d\mu(\omega) \leq \psi^2(t),\; 0 < t \leq a \right\}.$$

Again we choose h_1, h_2 such that

$$\|h_1\|_*^2 \geq \tau(A, \mathcal{E}_\psi, \delta_1) - \varepsilon \quad \text{and} \quad \|h_2\|_*^2 \geq \tau(A, \mathcal{E}_\psi, \delta_2) - \varepsilon,$$

together with

$$\int\limits_{0<f(\omega)\leq t} |h_1(\omega)|^2\; d\mu(\omega) \leq \psi^2(t), \quad 0 < t \leq a,$$

and

$$\int\limits_{0<f(\omega)\leq t} |h_2(\omega)|^2\; d\mu(\omega) \leq \psi^2(t), \quad 0 < t \leq a.$$

The same choice of h as in (10) allows us to complete the proof, and we leave the details to the reader. $\square$

Note that due to the identity (c) in Proposition 2 the proven concavity carries over to the functions $\omega^2(A, \mathcal{M}_{\varphi,R}, \sqrt{\delta})$ and $\omega^2(A, \mathcal{E}_{\psi,E}, \sqrt{\delta})$, respectively, for all $\delta > 0$ and $R > 0$.

4. Rates on smoothness classes

The modulus of continuity is a benchmark for the reconstruction error of regularization schemes, see the discussion in Section 1. Therefore its decay rate to zero as $\delta \to 0$ is of interest. In the compact case such rates for the classes $\mathcal{M}_{\varphi,R}$ and $\mathcal{E}_{\psi,E}$ are well studied. In particular, sharp bounds for smoothness given in terms of source sets $\mathcal{M}_{\varphi,R}$ are obtained by interpolation techniques. However, up to a factor 2 such upper bounds can be obtained by analyzing specific regularization techniques. It was mentioned in Section 2 that $\mathcal{M}_{\psi,E} \subset \mathcal{E}_{\psi,E}$, and upper bounds for the level sets provide also upper bounds for the source sets. But for level sets $\mathcal{E}_{\psi,E}$ upper bounds for the regularization error are easily obtained by noticing that the distribution function $F_x^2(t)$ is the square of the profile function (regularization error in the noise-free case) for *spectral cut-off*, we refer to [2]. This gives:

Proposition 5. *Let the operator A be as in (1), and let ψ be an index function, with associated function $\Theta(t) := \sqrt{t}\psi(t),\ 0 < t \le a$. Then*

$$\omega\left(A, \mathcal{E}_{\psi,E}, \delta\right) \le 2\, E\, \psi(\Theta^{-1}(\delta/E)), \quad 0 < \delta \le \Theta(a), \tag{11}$$

and

$$\omega\left(A, \mathcal{E}_{\psi,E}, \delta\right) \ge E\, \psi(\Theta^{-1}(\delta/E)), \quad \delta^2/E^2 \in \sigma(H\psi^2(H)). \tag{12}$$

Proof. We use the spectral cut-off regularization, i.e., when

$$x_\alpha^\delta := g_\alpha(H)A^*y^\delta \quad \text{with} \quad g_\alpha(t) := \begin{cases} 1/t, & t \ge \alpha, \\ 0, & t < \alpha, \end{cases}$$

determines the regularized solutions x_α^δ. As already mentioned we have in this case that $\|x - g_\alpha(H)Hx\|^2 = F_x^2(\alpha)$. For $x \in \mathcal{E}_{\psi,E}$ we have that

$$\|x - x_\alpha^\delta\| \le \|x - g_\alpha(H)Hx\| + \|g_\alpha(H)Hx - g_\alpha(H)A^*y^\delta\|$$

$$\le F_x(\alpha) + \|g_\alpha(H)A^*\|\|Ax - y^\delta\| \le F_x(\alpha) + \frac{\delta}{\sqrt{\alpha}} = E\psi(\alpha) + \frac{\delta}{\sqrt{\alpha}}.$$

The choice of $\alpha = \alpha(\delta)$ as solution to $\Theta(\alpha) = \delta/E$ allows to complete the proof of inequality (11).

By $\mathcal{M}_{\psi,E} \subset \mathcal{E}_{\psi,E}$ we have that $\omega\left(A, \mathcal{E}_{\psi,E}, \delta\right) \ge \omega\left(A, \mathcal{M}_{\psi,E}, \delta\right)$. On the other hand, $\omega\left(A, \mathcal{M}_{\psi,E}, \delta\right) \ge E\, \psi(\Theta^{-1}(\delta/E))$ is valid for $\delta^2/E^2 \in \sigma(H\psi^2(H))$, see [19, Theorem 2.5], which yields (12) and completes the proof of the proposition. ∎

References

1. H. W. Engl, M. Hanke, and A. Neubauer, *Regularization of inverse problems*, Mathematics and its Applications, vol. 375, Kluwer Academic Publishers Group, Dordrecht, 1996.
2. J. Flemming, B. Hofmann, and P. Mathé, *Sharp converse results for the regularization error using distance functions*, Inverse Problems **27** (2011), no. 11, 025006, 18.
3. B. Hofmann and G. Fleischer, *Stability rates for linear ill-posed problems with compact and non-compact operators*, Z. Anal. Anwendungen **18** (1999), no. 2, 267–286.
4. B. Hofmann and S. Kindermann, *On the degree of ill-posedness for linear problems with non-compact operators*, Methods and Applications of Analysis **17** (2010), no. 4, To appear.
5. B. Hofmann and P. Mathé, *Analysis of profile functions for general linear regularization methods*, SIAM J. Numer. Anal. **45** (2007), no. 3, 1122–1141.
6. B. Hofmann, P. Mathé, and M. Schieck, *Modulus of continuity for conditionally stable ill-posed problems in Hilbert space*, J. Inverse Ill-Posed Probl. **16** (2008), no. 6, 567–585.
7. B. Hofmann and L. von Wolfersdorf, *Some results and a conjecture on the degree of ill-posedness for integration operators with weights*, Inverse Problems **21** (2005), no. 2, 427–433.
8. V. K. Ivanov and T. Korolyuk, *Error estimates for solutions of incorrectly posed linear problems*, USSR Comput. Math. Math. Phys. (1969), no. 9, 35–49.
9. V. K. Ivanov, V. V. Vasin, and V. P. Tanana, *Theory of linear ill-posed problems and its applications*, second ed., Inverse and Ill-posed Problems Series, VSP, Utrecht, 2002, Translated and revised from the 1978 Russian original ["Nauka", Moscow].
10. L. A. Lyusternik and V. I. Sobolev, *Kratkii kurs funktsionalnogo analiza*, "Vyssh. Shkola", Moscow, 1982.
11. P. Mathé and B. Hofmann, *Direct and inverse results in variable Hilbert scales*, J. Approx. Theory **154** (2008), no. 2, 77–89.
12. P. Mathé and S. V. Pereverzev, *Discretization strategy for linear ill-posed problems in variable Hilbert scales*, Inverse Problems **19** (2003), no. 6, 1263–1277.
13. ———, *Geometry of linear ill-posed problems in variable Hilbert scales*, Inverse Problems **19** (2003), no. 3, 789–803.
14. M. Z. Nashed, *A new approach to classification and regularization of ill-posed operator equations*, Inverse and Ill-posed Problems (Sankt Wolfgang, 1986), volume 4 of Notes Rep. Math. Sci. Engrg., Academic Press, Boston, MA, 1987, pp. 53–75.
15. A. Neubauer, *On converse and saturation results for regularization methods*, Beiträge zur angewandten Analysis und Informatik, Shaker, Aachen, 1994, pp. 262–270.
16. ———, *On converse and saturation results for Tikhonov regularization of linear ill-posed problems*, SIAM J. Numer. Anal. **34** (1997), no. 2, 517–527.
17. M. Reed and B. Simon, *Methods of modern mathematical physics. I*, 1980, Functional analysis.
18. W. Rudin, *Functional analysis*, McGraw-Hill Book Co., New York, 1973, McGraw-Hill Series in Higher Mathematics.

19. U. Tautenhahn, U. Hämarik, B. Hofmann, and Y. Shao, *Conditional stability estimates for ill-posed PDE problems by using interpolation*, Paper in preparation, 2011.
20. V. V. Vasin, *The method of quasi-solutions by Ivanov is the effective method of solving ill-posed problems*, J. Inverse Ill-Posed Probl. **16** (2008), no. 6, 537–552.
21. V. V. Vasin and A. L. Ageev, *Ill-posed problems with a priori information*, Inverse and Ill-posed Problems Series, VSP, Utrecht, 1995.
22. V. V. Vasin and M. A. Korotkii, *Tikhonov regularization with nondifferentiable stabilizing functionals*, J. Inverse Ill-Posed Probl. **15** (2007), no. 8, 853–865.
23. D. Werner, *Funktionalanalysis*, Springer-Verlag, Berlin, 2007, 6. Auflage.

Department of Mathematics, Chemnitz University of Technology, 09107 Chemnitz, Germany

E-mail address: `hofmannb@mathematik.tu-chemnitz.de`

Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, 10117 Berlin, Germany

E-mail address: `mathe@wias-berlin.de`