THE ALGEBRAIC RICCATI EQUATION WITH TOEPLITZ MATRICES AS COEFFICIENTS

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Abstract. It is shown that, under appropriate assumptions, the continuous algebraic Riccati equation with Toeplitz matrices as coefficients has Toeplitz-like solutions. Both infinite and sequences of finite Toeplitz matrices are considered, and also studied is the finite section method, which consists in approximating infinite systems by large finite truncations. The results are proved by translating the problem into C^*-algebraic language and by using theorems on the Riccati equation in general C^*-algebras. The paper may serve as another illustration of the usefulness of C^*-algebra techniques in matrix theory.

Key words. Algebraic Riccati equation, Toeplitz matrix, C^*-algebra.

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1. Introduction. We consider the (continuous) algebraic Riccati equation in the form

$$XDX - XA - A^*X - C = 0. \tag{1.1}$$

Suppose the coefficients are infinite Toeplitz matrices generated by continuous functions,

$$XT(d)X - XT(a) - T(\overline{a})X - T(c) = 0. \tag{1.2}$$

Does equation (1.2) have a solution X which is Toeplitz-like, say $X = T(\varphi) + K$ with a continuous function φ and a compact operator K? A perhaps more important situation is the one when the coefficients are $n \times n$ Toeplitz matrices,

$$X_nT_n(d)X_n - X_nT_n(a) - T_n(\overline{a})X_n - T_n(c) = 0. \tag{1.3}$$

Does this equation possess a Toeplitz-like solution X_n? Defining Toeplitz-likeness for an individual $n \times n$ matrix is a delicate matter, but the concept makes perfect sense for the sequence $\{X_n\}_{n=1}^{\infty}$. Namely, we say that $\{X_n\}_{n=1}^{\infty}$ is a Toeplitz-like sequence if there exist a continuous function φ, compact operators K and L, and a sequence of $n \times n$ matrices U_n whose spectral norms $\|U_n\|$ go to zero such that

$$X_n = T_n(\varphi) + P_nKP_n + W_nLW_n + U_n, \tag{1.4}$$

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where P_n is the projection onto the first n coordinates and W_n stands for P_n followed by reversal of the coordinates. As the main mass of the matrix of a compact operator is contained in its upper-left corner, the main mass of the $n \times n$ perturbation matrices $P_nKP_n + W_nLW_n$ in (1.4) is concentrated in the upper-left and lower-right corners.

Searching for Toeplitz-like solutions of equations (1.2) and (1.3) is a special case of the problem of considering equation (1.1) in a C^*-algebra $\mathcal{A}$. Thus, given $D, A, C \in \mathcal{A}$, is there a solution $X \in \mathcal{A}$? This general question has already been explored, and the purpose of this paper is to show how these general C^*-algebraic results yield answers in the case of Toeplitz coefficients in a very quick and elegant way.

The question whether (1.3) has Toeplitz-like solutions was studied in [15], and that paper was in fact the motivation for the present paper. Paper [15] cites a theorem from [21], [22], which states that if $\mathcal{A}$ is an operator algebra and $D, A, A^*, C \in \mathcal{A}$ then, under certain assumptions, equation (1.1) has a solution in $\mathcal{A}$. Accordingly, a special operator algebra $\mathcal{T}_{\delta,\varrho,\alpha}$ of so-called almost Toeplitz matrices is constructed in [15] and the theorem is then applied to $\mathcal{A} = \mathcal{T}_{\delta,\varrho,\alpha}$. However, it remains a critical issue what exactly an operator algebra in this context is [6], and the algebra $\mathcal{T}_{\delta,\varrho,\alpha}$ is quite complicated. In contrast to this, working with (1.4) as the definition of Toeplitz-likeness has proven extremely useful since Silbermann's paper [23]. In [2], it was observed (and significantly exploited) that the set of all sequences $\{X_n\}_{n=1}^{\infty}$ of the form (1.4) is a C^*-algebra, and this favorable circumstance has been taken advantage of since then in many instances; see, for example, [1], [3], [19]. In this light the idea to invoke results on the Riccati equation in C^*-algebras, which are the nicest operator algebras, in order to treat equations (1.2) and (1.3) emerges very naturally.

The paper is organized as follows. In Section 2 we cite two results on the Riccati equation in general C^*-algebras. These results are then applied to C^*-algebras of Toeplitz matrices in Sections 3 and 4. Theorems 3.2 and 4.2 as well as Corollary 4.3 are our main results. In Section 5 we discuss some additional issues related to Riccati equations in C^*-algebras and W^*-algebras.

2. The Riccati equation in C^*-algebras. Let $\mathcal{A}$ be a unital C^*-algebra and denote the unit by I. The reader is referred to [12] for an introduction to C^*-algebras. We let $\sigma(H)$ stand for the spectrum of $H \in \mathcal{A}$. An element $H \in \mathcal{A}$ is called Hermitian if $H = H^*$, it is said to be positive, $H \geq 0$, if $H = H^*$ and $\sigma(H) \subset [0, \infty)$, and it is referred to as a positive definite element, $H > 0$, if $H = H^*$ and $\sigma(H) \subset (0, \infty)$. We write $G \leq H$ or $H \geq G$ if G and H are Hermitian and $H - G$ is positive.

In the case where $\mathcal{A} = \mathbf{C}^{N \times N}$, the Riccati equation arises from optimal control as follows; see, e.g., [16], [20]. Let $A, B, C, R \in \mathbf{C}^{N \times N}$ and suppose $B \geq 0$ and $R > 0$.

The control $u(t)$ of the system $\dot{x} = Ax + Bu$ which minimizes the cost functional

$$\int_0^\infty (x^*Cx + u^*Ru)\,\mathrm{d}t \tag{2.1}$$

is given by $u(t) = -R^{-1}B^*X_+x(t)$ where X_+ is the maximal Hermitian solution of the equation

$$XBR^{-1}B^*X - XA - A^*X - C = 0. \tag{2.2}$$

A Hermitian solution X_+ of (2.2) is said to be maximal if $X_+ \geq X$ for every Hermitian solution X of (2.2). Clearly, $D := BR^{-1}B^*$ is positive. Notice also that the feedback input $u = -Fx + v$ transforms the system $\dot{x} = Ax + Bu$ into the new system $\dot{x} = (A - BF)x + Bv$.

Now let $\mathcal{A}$ be a general unital C^*-algebra, let $D, A, C \in \mathcal{A}$, and suppose $D \geq 0$, $C \geq 0$. The pair (A, D) is called stabilizable in $\mathcal{A}$ if there exists an $F \in \mathcal{A}$ such that $\sigma(A - DF)$ is contained in the open left half-plane $\mathbf{C}_-$. Given a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on $\mathcal{H}$. It is well known that every C^*-algebra $\mathcal{A}$ may be identified with a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some $\mathcal{H}$. Bunce [4] (see also [14, Corollary 3.3]) showed that if the pair (A, D) is stabilizable in $\mathcal{B}(\mathcal{H})$, then it is automatically stabilizable in $\mathcal{A}$. A solution $X_+ \in \mathcal{A}$ of equation (1.1) is called maximal in $\mathcal{A}$ if X_+ is Hermitian and $X_+ \geq X$ for every Hermitian solution $X \in \mathcal{A}$ of (1.1). Dobovišek [14, pp. 74–75] observed that if $X_+ \in \mathcal{A}$ is a maximal solution in $\mathcal{A}$, then in fact $X_+ \geq X$ for every Hermitian solution $X \in \mathcal{B}(\mathcal{H})$. Thus, stabilizability and maximality do not depend on the algebra $\mathcal{A}$ and we therefore omit the "in $\mathcal{A}$". Curtain and Rodman [9] proved that if the pair (A, D) is stabilizable, then equation (1.1) has a maximal solution $X_+ \in \mathcal{B}(\mathcal{H})$ and this solution is positive. The big problem is whether this solution X_+ is in $\mathcal{A}$ or not. The following two theorems provide us with partial answers.

THEOREM 2.1. **(Bunce [4])** *If $D \geq 0$, $C = I$, and (A, D) is stabilizable, then equation (1.1) has a maximal solution $X_+ \in \mathcal{A}$ and this solution is positive.*

Note that the restriction to $C = I$ means that all state variables in (2.1) are considered to have equal rights.

THEOREM 2.2. **(Dobovišek [14, Theorem 3.7])** *If $D \geq 0$, $C \geq 0$, (A, D) is stabilizable, and $\sigma(A)$ does not intersect the imaginary axis, then equation (1.1) has a maximal solution $X_+ \in \mathcal{A}$ and this solution is positive.*

In [4], [14] it is shown that under the hypotheses of Theorems 2.1 or 2.2 the spectrum of $A - DX_+$ is a subset of $\mathbf{C}_-$, that is, $X_+ \in \mathcal{A}$ stabilizes (A, D).

3. Infinite Toeplitz matrices. Let $\mathbf{T}$ be the complex unit circle and let $C(\mathbf{T})$ be the C^*-algebra of all continuous complex-valued functions on $\mathbf{T}$. We abbreviate $C(\mathbf{T})$ to $\mathcal{C}$. Given $f \in \mathcal{C}$ with Fourier coefficients

$$f_k := \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}x})\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x, \quad k \in \mathbf{Z},$$

we consider the infinite Toeplitz matrix $T(f) := (f_{j-k})_{j,k=1}^{\infty}$. The function f is called the generating function or the symbol of $T(f)$. It is well known that $T(f)$ induces a bounded linear operator on $\ell^2 := \ell^2(\mathbf{N})$. Moreover, $\|T(f)\| = \|f\|_\infty$, where $\|\cdot\|_\infty$ is the L^∞ norm. A classical result by Gohberg says that the spectrum of $T(f)$ is the union of the range $f(\mathbf{T})$ and the points in $\mathbf{C}\setminus f(\mathbf{T})$ whose winding number with respect to $f(\mathbf{T})$ is nonzero; see, e.g., [3, Theorem 1.17]. In particular, if $f(\mathbf{T}) \subset \mathbf{C}_-$, then $T(f)$ is invertible.

We denote by $\mathcal{T}$ the smallest closed subalgebra of $\mathcal{B} := \mathcal{B}(\ell^2)$ which contains the set $T(\mathcal{C}) := \{T(f) : f \in \mathcal{C}\}$. The algebra $\mathcal{T}$ is a C^*-algebra and it turns out that the set $\mathcal{K} := \mathcal{K}(\ell^2)$ of all compact operators is a subset and thus a closed two-sided ideal of $\mathcal{T}$. Coburn [5] showed that $\mathcal{T}/\mathcal{K}$ is isometrically *-isomorphic to $\mathcal{C}$, the map $f \mapsto T(f)+\mathcal{K}$ being an isometric *-isomorphism of $\mathcal{C}$ onto $\mathcal{T}/\mathcal{K}$. Thus, an operator X belongs to $\mathcal{T}$ if and only if $X = T(f)+K$ with $f \in \mathcal{C}$ and $K \in \mathcal{K}$. This decomposition is unique, that is, $T(\mathcal{C}) \cap \mathcal{K} = \{0\}$.

Theorems 2.1 and 2.2 are applicable to $\mathcal{A} = \mathcal{T}$ and show that, under their hypotheses, equation (1.1) with $D, A, C \in \mathcal{T}$ has a maximal solution $X_+ \in \mathcal{T}$. To be more specific, we consider the equation

$$X\Big(T(d) + T(b)T^{-1}(r)T(\overline{b})\Big)X - XT(a) - T(\overline{a})X - T(c) = 0. \tag{3.1}$$

Note that the adjoint of $T(f)$ is just $T(\overline{f})$ where $\overline{f}(t) = \overline{f(t)}$ and the bar on the right stands for complex conjugation. We also write $T^{-1}(r) := [T(r)]^{-1}$. If $b = 0$, then (3.1) becomes (1.2), while if $d = 0$, equation (3.1) takes the form

$$XT(b)T^{-1}(r)T(\overline{b})X - XT(a) - T(\overline{a})X - T(c) = 0,$$

which is (2.2) in Toeplitz matrices. In what follows, if $f \in \mathcal{C}$ then $f > 0$ means that f is a positive definite element of $\mathcal{C}$, that is, f is real-valued and $f(t) > 0$ for all $t \in \mathbf{T}$. Analogously, we write $f \ge 0$ if f is real-valued and $f(t) \ge 0$ for all $t \in \mathbf{T}$.

LEMMA 3.1. *Let $b, r \in \mathcal{C}$ and suppose $r > 0$. Then*

$$T(b)T^{-1}(r)T(\overline{b}) = T(|b|^2 r^{-1}) + L$$

with some compact operator $L \ge 0$.

Proof. Suppose first that r is smooth. Then r has a Wiener-Hopf factorization $r = r_+\overline{r}_+$ with a function $r_+ \in \mathcal{C}$ such that $r_+^{-1} \in \mathcal{C}$ and all Fourier coefficients with negative indices of r_+ and r_+^{-1} vanish; see, e.g, [3, Theorem 1.14]. Standard computations with Toeplitz matrices, [3, Section 1.5], give

$$\begin{aligned} T(b)T^{-1}(r)T(\bar{b}) &= T(b)T(r_+^{-1})T(\overline{r}_+^{-1})T(\bar{b}) = T(br_+^{-1})T(\overline{r}_+^{-1}\bar{b}) \\ &= T(|b|^2r^{-1}) + H(g)H(\widetilde{\overline{g}}) =: T(|b|^2r^{-1}) + L, \end{aligned}$$

where $g := br_+^{-1}$, $\widetilde{f}$ is defined by $\widetilde{f}(t) := f(1/t)$ for $t \in \mathbf{T}$, and $H(f)$ is the Hankel operator induced by the matrix $(f_{j+k-1})_{j,k=1}^{\infty}$. Since $H(\widetilde{\overline{g}})$ is the adjoint operator of $H(g)$, the operator L is positive, and since g is continuous, the operator L is compact.

An arbitrary continuous function $r > 0$ may be approximated uniformly by smooth functions $r_n > 0$. Then $T^{-1}(r_n)$ and $T(|b|^2r_n^{-1})$ converge in the norm to $T^{-1}(r)$ and $T(|b|^2r^{-1})$, respectively. Since $T(b)T^{-1}(r_n)T(\bar{b}) - T(|b|^2r_n^{-1})$ was shown to be positive and compact, so also is $T(b)T^{-1}(r)T(\bar{b}) - T(|b|^2r^{-1})$. □

We denote the real part of a function f and an operator H by $\operatorname{Re} f$ and $\operatorname{Re} H$.

THEOREM 3.2. *Let $a, b, c, d, r \in \mathcal{C}$ with $d \geq 0$, $r > 0$ and suppose one of the following conditions is satisfied:*

(i) *$d + |b|^2r^{-1} > 0$ and $c = 1$;*

(ii) *$d + |b|^2r^{-1} > 0$, $c \geq 0$, and $\operatorname{Re} a > 0$;*

(iii) *$c \geq 0$ and $\operatorname{Re} a < 0$.*

Then equation (3.1) has a maximal solution $X_+ \in \mathcal{T}$, this solution is positive, and $X_+ = T(\varphi) + K$ where $\varphi \in \mathcal{C}$, $\varphi \geq 0$, and $K = K^*$ is compact. Furthermore, φ satisfies the equation

$$(d + |b|^2r^{-1})\varphi^2 - (a + \overline{a})\varphi - c = 0. \tag{3.2}$$

Proof. First of all, $C := T(c) \geq 0$ and $D := T(d) + T(b)T^{-1}(r)T(\bar{b}) \geq 0$. Put $A := T(a)$. If (iii) holds, then $\operatorname{Re} A < 0$ and hence (A, D) is trivially stabilizable. So assume (i) or (ii) is in force. Then $d + |b|^2r^{-1} > 0$. From Lemma 3.1 we infer that if $\gamma \in (0, \infty)$ is any constant, then

$$\operatorname{Re}(A - \gamma D) = T(\operatorname{Re} a - \gamma(d + |b|^2r^{-1})) - \gamma L,$$

and choosing γ large enough we can achieve that $\operatorname{Re}(A - \gamma D)$ is negative definite as the sum of a negative definite operator and a negative operator. Thus, (A, D) is stabilizable. The spectrum of $T(a)$ is in the left and right open half-planes if $\operatorname{Re} a < 0$

and $\operatorname{Re} a > 0$, respectively. Theorem 2.1 now yields a maximal and positive solution $X_+ \in \mathcal{T}$ under the hypothesis (i), while Theorem 2.2 does this if (ii) or (iii) is satisfied. Writing $X_+ = T(\varphi) + K$ and taking into account that passage from $\mathcal{T}$ to $\mathcal{T}/\mathcal{K} \cong \mathcal{C}$ preserves positivity, we conclude that $\varphi = \overline{\varphi} \geq 0$ and hence $K = K^*$. Inserting $X_+ = T(\varphi) + K$ in (3.1), and considering the resulting equation modulo compact operators we get

$$T(\varphi)\Big(T(d) + T(b)T^{-1}(r)T(\overline{b})\Big)T(\varphi) - T(\varphi)T(a) - T(\overline{a})T(\varphi) - T(c) \in \mathcal{K}.$$

Taking into account that $T^{-1}(r) - T(r^{-1})$ is compact (Lemma 3.1 with $b = 1$) and that $T(f)T(g) - T(fg)$ is compact for arbitrary $f, g \in \mathcal{C}$ (Coburn), we obtain that

$$T((d + |b|^2 r^{-1})\varphi^2 - (a + \overline{a})\varphi - c) \in \mathcal{K}.$$

Using that the only compact Toeplitz operator is the zero operator, we finally arrive at (3.2). □

4. Finite Toeplitz matrices. For $f \in \mathcal{C}$, we denote by $T_n(f)$ the $n \times n$ Toeplitz matrix $(f_{j-k})_{j,k=1}^n$. It is well known that if $\operatorname{Re} f < 0$ or $\operatorname{Re} f > 0$, then $T_n(f)$ is invertible for all $n \geq 1$,

$$\sup_{n \geq 1} \|T_n^{-1}(f)\| < \infty, \tag{4.1}$$

and $T_n^{-1}(f) \to T^{-1}(f)$ strongly as $n \to \infty$; see, e.g., [3, Proposition 2.17]. Here and throughout what follows, $\|\cdot\|$ is always the operator norm on ℓ^2 (= spectral norm in the case of matrices).

Let $\mathbf{B}$ denote the set of all sequences $F = \{F_n\}_{n=1}^\infty$ such that $F_n \in \mathbf{C}^{n \times n}$ and

$$\|F\| := \sup_{n \geq 1} \|F_n\| < \infty. \tag{4.2}$$

We henceforth abbreviate $\{F_n\}_{n=1}^\infty$ to $\{F_n\}$. With termwise operations, $\{F_n\} + \{G_n\} := \{F_n + G_n\}$, $\alpha\{F_n\} := \{\alpha F_n\}$, $\{F_n\}\{G_n\} := \{F_n G_n\}$, $\{F_n\}^* := \{F_n^*\}$, and with the norm (4.2), $\mathbf{B}$ is a C^*-algebra. We denote by $\mathbf{B}_0$ the elements $\{F_n\} \in \mathbf{B}$ for which $\|F_n\| \to 0$ as $n \to \infty$. Clearly, $\mathbf{B}_0$ is a closed two-sided ideal of $\mathbf{B}$. Let P_n and W_n be the operators

$$P_n : \ell^2 \to \ell^2, \;\; \{x_1, x_2, \ldots\} \mapsto \{x_1, \ldots, x_n, 0, 0, \ldots\},$$
$$W_n : \ell^2 \to \ell^2, \;\; \{x_1, x_2, \ldots\} \mapsto \{x_n, \ldots, x_1, 0, 0, \ldots\}.$$

Note that $W_n = W_n^* \to 0$ weakly, so that $W_n K W_n \to 0$ strongly whenever K is compact. For $H \in \mathcal{B}(\ell^2)$, we think of $P_n H P_n$ and $W_n H W_n$ as $n \times n$ matrices. Let $\mathbf{S}$ be the smallest closed subalgebra of $\mathbf{B}$ which contains the set $\{\{T_n(f)\} : f \in \mathcal{C}\}$.

One can show that $\mathbf{S}$ is a C^*-algebra and that actually $\mathbf{S}$ coincides with the set of all sequences $\{F_n\}$ of the form

$$F_n = T_n(f) + P_nKP_n + W_nLW_n + U_n \tag{4.3}$$

with $f \in \mathcal{C}$, $K \in \mathcal{K}$, $L \in \mathcal{K}$, $\{U_n\} \in \mathbf{B}_0$; see [2] or [3, Proposition 2.33]. Moreover, the set $\mathbf{J}$ of all sequences $\{F_n\}$ of the form $F_n = P_nKP_n + W_nLW_n + U_n$ with $K \in \mathcal{K}$, $L \in \mathcal{K}$, $\{U_n\} \in \mathbf{B}_0$ (that is those of the form (4.3) with $f = 0$) is closed two-sided ideal of $\mathbf{S}$ and the map $f \mapsto \{T_n(f)\} + \mathbf{J}$ is an isometric *-isomorphism of $\mathcal{C}$ onto $\mathbf{B}/\mathbf{J}$.

Using Theorems 2.1 and 2.2 with $\mathcal{A} = \mathbf{S}$, we get results for the Riccati equation (1.1) with $D = \{D_n\}$, $A = \{A_n\}$, $C = \{C_n\}$, $X = \{X_n\}$ is $\mathbf{S}$. We illustrate things for the equation

$$X_n\Big(T_n(d) + T_n(b)T_n^{-1}(r)T_n(\overline{b})\Big)X_n - X_nT_n(a) - T_n(\overline{a})X_n - T_n(c) = 0. \tag{4.4}$$

LEMMA 4.1. *If $b, r \in \mathcal{C}$ and $r > 0$ then*

$$T_n(b)T_n^{-1}(r)T_n(\overline{b}) = T_n(|b|^2r^{-1}) + P_nK'P_n + W_nL'W_n + U_n'$$

with compact operators $K' \geq 0$ and $L' \geq 0$ and Hermitian matrices U_n' such that $\|U_n'\| \to 0$ as $n \to \infty$.

Proof. By virtue of (4.1), $\{T_n(r)\}$ is invertible in $\mathbf{B}$ and thus also in $\mathbf{S}$. We therefore have

$$T_n(b)T_n^{-1}(r)T_n(\overline{b}) = T_n(s) + P_nK'P_n + W_nL'W_n + U_n' \tag{4.5}$$

with $s \in \mathcal{C}$, $K' \in \mathcal{K}$, $L' \in \mathcal{K}$, $\{U_n'\} \in \mathbf{B}_0$. Passing to the strong limit $n \to \infty$ in (4.5) we obtain $T(b)T^{-1}(r)T(b) = T(s) + K'$, and Lemma 3.1 now implies that $s = |b|^2r^{-1}$ and that K' is positive and compact. Multiplying (4.5) from the left and the right by W_n, taking into account that $W_nT_n(f)W_n = T_n(\widetilde{f})$, and then passing to the strong limit $n \to \infty$ we arrive at the equality $T(\widetilde{b})T^{-1}(\widetilde{r})T(\widetilde{\overline{b}}) = T(\widetilde{s}) + L'$. Lemma 3.1 shows again that L' is positive and compact. It is then immediate from (4.5) that the matrices U_n' are Hermitian. □.

THEOREM 4.2. *Let $a, b, c, d, r \in \mathcal{C}$ with $d \geq 0$, $r > 0$ and let at least one of the following conditions be satisfied:*

(i) *$d + |b|^2r^{-1} > 0$ and $c = 1$;*

(ii) *$d + |b|^2r^{-1} > 0$, $c \geq 0$, and $\operatorname{Re} a > 0$;*

(iii) *$c \geq 0$ and $\operatorname{Re} a < 0$.*

Then there is an n_0 such that for every $n \ge n_0$ equation (4.4) has a maximal solution $X_n^+ \in \mathbf{C}^{n\times n}$. This solution is positive and there exist $\varphi \in \mathcal{C}$, $\varphi \ge 0$, $K = K^* \in \mathcal{K}$, $L = L^* \in \mathcal{K}$, $U_n = U_n^* \in \mathbf{C}^{n\times n}$, $\|U_n\| \to 0$ such that

$$X_n^+ = T_n(\varphi) + P_nKP_n + W_nLW_n + U_n \tag{4.6}$$

for $n \ge n_0$. Moreover, φ satisfies the equation

$$(d + |b|^2 r^{-1})\varphi^2 - (a + \overline{a})\varphi - c = 0. \tag{4.7}$$

Proof. Put $D := \{T_n(d) + T_n(b)T_n^{-1}(r)T_n(\overline{b})\}$, $A := \{T_n(a)\}$, $C := \{T_n(c)\}$. Obviously, $D \ge 0$ and $C \ge 0$. It is clear that $\operatorname{Re} A < 0$ if (iii) holds. So consider the cases (ii) and (iii), where $d + |b|^2 r^{-1} > 0$. By Lemma 4.1,

$$\operatorname{Re}(A - \gamma D) = \{T_n(\operatorname{Re} a - \gamma(d + |b|^2 r^{-1})) - \gamma P_nK'P_n - \gamma W_nL'W_n - \gamma U_n'\}$$

for each constant $\gamma \in (0, \infty)$, and hence $\operatorname{Re}(A - \gamma D) \le -2I - \gamma\{U_n'\}$ for all $n \ge 1$ if only γ is sufficiently large. Since the matrices U_n' are Hermitian and $\|U_n'\| \to 0$, there is an n_0 such that $-2I - \gamma U_n' \le -I$ for all $n \ge n_0$. Thus, considering only sequences of the form $\{F_n\}_{n=n_0}^\infty$ and denoting the corresponding C^*-algebras by $\mathbf{B}(n_0)$, $\mathbf{B}_0(n_0)$, $\mathbf{S}(n_0)$, we see that (A, D) is stabilizable in $\mathbf{S}(n_0)$. Clearly, $\sigma(\{T_n(a)\})$ does not intersect the imaginary axis if $\operatorname{Re} a < 0$ or $\operatorname{Re} a > 0$. From Theorems 2.1 and 2.2 we now deduce that equation (1.1) has a positive maximal solution $X_+ = \{X_n^+\}_{n=n_0}^\infty$ in $\mathbf{S}(n_0)$.

The reasoning of the previous paragraph is applicable to each individual number $n \ge n_0$, that is, $D_n \ge 0$, $C_n \ge 0$, (A_n, D_n) is stabilizable, and $\sigma(T_n(a))$ has no points on the imaginary axis in the cases (ii) and (iii). Consequently, equation (4.4) has a positive maximal solution X_n for each $n \ge n_0$. We want to show that $X_n = X_n^+$. The matrix X_n can be obtained by an iterative procedure such as in [16]. This procedure yields Hermitian matrices $X_n^{(1)}, X_n^{(2)}, \ldots$ satisfying $X_n^{(1)} \ge X_n^{(2)} \ge \ldots \ge X_n$ and $X_n^{(k)} \to X_n$ as $k \to \infty$. In particular, $X_n \le X_n^{(1)}$. We have

$$X_n^{(1)} = \int_0^\infty \mathrm{e}^{t(A_n - \gamma D_n)^*}(\gamma^2 D_n + C_n)\mathrm{e}^{t(A_n - \gamma D_n)}\,\mathrm{d}t.$$

There is a constant $M < \infty$ such that $\|D_n\| \le M$ (due to (4.1)) and $\|C_n\| \le M$ (obvious). As $\operatorname{Re}(A_n - \gamma D_n) \le -I$, it follows that

$$\|X_n^{(1)}\| \le 2M \int_0^\infty \mathrm{e}^{-2t}\,\mathrm{d}t = M.$$

Consequently, $\|X_n\| \le M$ for $n \ge n_0$, which implies that $\{X_n\}_{n=n_0}^\infty$ is a Hermitian solution of equation (1.1) in $\mathbf{B}(n_0)$. As noted in Section 2, we then necessarily have

$\{X_n\}_{n=n_0}^{\infty} \le \{X_n^+\}_{n=n_0}^{\infty}$, that is $X_n \le X_n^+$ for all $n \ge n_0$. As X_n is the maximal solution of the nth equation and X_n^+ is a Hermitian solution of that equation, we also have $X_n^+ \le X_n$. Thus, $X_n = X_n^+$ for $n \ge n_0$.

As components of an element in $\mathbf{S}(n_0)$, the matrices X_n^+ are of the form (4.6) with $\varphi \in \mathcal{C}$, $K \in \mathcal{K}$, $L \in \mathcal{K}$, $\{U_n\} \in \mathbf{B}_0(n_0)$. We also know that $X_n^+ \ge 0$. As $n \to \infty$, the strong limit of (4.6) is $T(\varphi)+K$. This is positive operator if and only if $\varphi = \overline{\varphi} \ge 0$ and $K = K^*$ (recall the proof of Theorem 3.2). Multiplying (4.6) from the left and the right by W_n and passing to the strong limit $n \to \infty$, we obtain analogously that $L = L^*$. As $T_n(\varphi)$, K, L are Hermitian, so also is U_n.

Finally, passing in (4.4) to the strong limit $n \to \infty$, we arrive at (3.1). Theorem 3.2 therefore implies that φ must satisfy (4.7). □

The following result on the finite section method for equation (3.1) is immediate from Theorems 3.2 and 4.2. We silently already used it in the last paragraph of the previous proof to establish (4.7).

COROLLARY 4.3. *Under the hypotheses of Theorems 3.2 and 4.2, equations (4.4) have (unique) positive maximal solutions X_n^+ for all sufficiently large n, and $X_n^+ P_n$ converges strongly to the (unique) positive maximal solution X_+ of equation (3.1).*

5. Additional remarks.

5.1. Newton's method. Let $\mathcal{A}$ be a C^*-algebra and let D, A, C be arbitrary elements of $\mathcal{A}$. Newton's method for equation (1.1) consists in choosing an initial element $X_0 \in \mathcal{A}$ and in determining successive approximations as the solutions of the Sylvester equations

$$X_{k+1}(A - DX_k) + (A^* - X_k D)X_{k+1} = -X_k D X_k - C;$$

see, for instance, [13], [18], [24]. The separation of $-A$ and A^* is defined as the number $\operatorname{sep}(-A, A^*) := 1/\|S^{-1}\|$ where S is the Lyapunov operator

$$S : \mathcal{A} \to \mathcal{A}, \;\; X \mapsto XA + A^*X.$$

If S is not invertible, one puts $\operatorname{sep}(-A, A^*) = 0$. It follows from the results of Stewart [24] and Demmel [13] that if $X_0 = 0$ and

$$\frac{\|C\|\,\|D\|}{\operatorname{sep}(-A, A^*)^2} < \frac{1}{4}, \tag{5.1}$$

then X_k converges in the norm of $\mathcal{A}$ to some solution $X_\infty \in \mathcal{A}$ of equation (1.1) and that this equation has no other solution $X \in \mathcal{A}$ satisfying $\|X\| \le \|X_\infty\|$.

It is well known that if $\sigma(A)$ is contained in the open left half-plane, then

$$S^{-1}Y = \int_0^\infty \mathrm{e}^{tA^*} Y \mathrm{e}^{tA}\,\mathrm{d}t.$$

In the case where $\operatorname{Re} A \le -\varepsilon I$ for some $\varepsilon > 0$, this implies that $\operatorname{sep}(-A, A^*) \ge 2\varepsilon$. Thus, by (5.1), Newton's method delivers a solution $X_\infty \in \mathcal{A}$ whenever

$$\|C\|\,\|D\| < \varepsilon^2. \tag{5.2}$$

In the case of Toeplitz matrices as coefficients, we arrive at the following.

THEOREM 5.1. *Let $a, b, c, r \in \mathcal{C}$, $r \ge \delta > 0$, $\operatorname{Re} a \le -\varepsilon < 0$. If*

$$\|c\|_\infty(\|d\|_\infty + (2/\delta)\|b\|_\infty^2) < \varepsilon^2, \tag{5.3}$$

then equations (3.1) and (4.4) (for every $n \ge 1$) have a solution X_∞ and X_n^∞, respectively. These solutions are of the form $X_\infty = T(\varphi) + K$ and

$$X_n^\infty = T_n(\varphi) + P_nKP_n + W_nLW_n + U_n$$

where $\varphi \in \mathcal{C}$ satisfies equation (3.2), K and L are compact, and $\|U_n\| \to 0$, they are the limit in the norm of the iterates obtained by Newton's method with $X_0 = 0$, and the equations have no other solutions X and X_n with $\|X\| \le \|X_\infty\|$ and $\|X_n\| \le \|X_n^\infty\|$. If, in addition, $c \ge 0$ and $d \ge 0$, then X_∞ and X_n^∞ are positive and the maximal solutions of the equations.

Proof. We use what was said above with $\mathcal{A} = \mathcal{T}$ and $\mathcal{A} = \mathbf{S}$. We have $\|T_n(c)\| \le \|T(c)\| \le \|c\|_\infty$, and by the Brown–Halmos theorem (see, e.g., [3, Proposition 2.17]), $\|T^{-1}(r)\| < 2/\delta$ and $\|T_n^{-1}(r)\| < 2/\delta$. Consequently.

$$\|C\|\,\|D\| \le \|c\|_\infty(\|d\|_\infty + (2/\delta)\|b\|_\infty^2).$$

If $\operatorname{Re} a \le -\varepsilon$, then $\operatorname{Re} T(a) \le -\varepsilon I$ and $\operatorname{Re} T_n(a) \le -\varepsilon I$. Thus, (5.3) implies (5.2), and it follows that Newton's method converges to a solution in $\mathcal{A}$.

If $c \ge 0$ and $d \ge 0$, we have case (iii) of Theorems 3.2 and 4.2. Thus, the equation has a positive maximal solution. In [16], this solution was shown to be the limit of Newton's method starting with $X_0 = 0$. Therefore the maximal solution coincides with X_∞. □

5.2. The Riccati equation in W^*-algebras. By the Gelfand–Naimark theorem, every C^*-algebra $\mathcal{A}$ is isometrically *-isomorphic to a closed and selfadjoint subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. If that subalgebra is closed in the strong (= pointwise) operator topology for some $\mathcal{H}$, then $\mathcal{A}$ is called a W^*-algebra

or a von Neumann algebra. Known constructive procedures for solving Riccati equations [9], [16] yield the solution as the strong limit of iterates belonging to $\mathcal{A}$. This makes W^*-algebras a convenient terrain for Riccati equations. We remark that finite-dimensional C^*-algebras, $L^\infty(\mathbf{T})$, $\mathcal{B}(\mathcal{H})$, $\mathbf{B}$ are W^*-algebras, whereas $C(\mathbf{T})$, $\mathcal{T}$, $\mathcal{K}(\mathcal{H})$ with $\dim \mathcal{H} = \infty$, and $\mathbf{S}$ are C^*-algebras but not W^*-algebras.

THEOREM 5.2. **(Curtain and Rodman [9])** *Let $\mathcal{A}$ be a unital W^*-algebra and $D, A, C \in \mathcal{A}$. If $D \geq 0$, $C \geq 0$, and (A, D) is stabilizable, then equation (1.1) has a maximal solution $X_+ \in \mathcal{A}$ and this solution is positive.*

Unfortunately, this is of limited use for our purposes because, as said, $\mathcal{T}$ and $\mathbf{S}$ are not W^*-algebras.

If $\mathcal{J}$ is a closed two-sided ideal of some unital C^*-algebra $\mathcal{A}$, then

$$\mathbf{C}I + \mathcal{J} := \{\alpha I + J : \alpha \in \mathbf{C}, J \in \mathcal{J}\}$$

is a unital C^*-subalgebra of $\mathcal{A}$. For example, $\mathbf{C}I + \mathcal{K}(\mathcal{H})$ is a unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. Here is a result on the Riccati equation in such C^*-subalgebras.

THEOREM 5.3. *Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{J} \subset \mathcal{A}$ be a proper closed two-sided ideal. Suppose D, A, C are in $\mathbf{C}I + \mathcal{J}$, $D \geq 0$, $C > 0$ (sic!), and (A, D) is stabilizable. If $X \in \mathcal{A}$ is a positive solution of equation (1.1), then necessarily $X \in \mathbf{C}I + \mathcal{J}$.*

Proof. We abbreviate αI to α. Thus, let $D = d + K_D$, $A = a + K_A$, $C = c + K_C$ with $d, a, c \in \mathbf{C}$ and $K_D, K_A, K_C \in \mathcal{J}$. We claim that $c > 0$. To see this, assume $c \leq 0$. Then $0 \leq -cI < K_C$, and since ideals of C^*-algebras are hereditary (see, e.g., [12, Theorem I.5.3]), it follows that $-cI \in \mathcal{J}$. If $c < 0$, this implies $I \in \mathcal{J}$ and thus $\mathcal{J} = \mathcal{A}$, which contradicts our assumption that $\mathcal{J}$ is proper. Hence $c = 0$ and thus $C = K_C > 0$, which means that K_C is invertible and therefore again leads to the contradiction $\mathcal{J} = \mathcal{A}$. Consequently, $c > 0$. Analogously one can show that $d \geq 0$.

For $Z \in \mathcal{A}$, we denote by Z^π the coset $Z + \mathcal{J} \in \mathcal{A}/\mathcal{J}$. Since (A, D) is stabilizable, there is an $F = f + K_F \in \mathbf{C}I + J$ such that $\sigma(A - DF) \subset \mathbf{C}_-$, which implies that $\sigma(A^\pi - D^\pi F^\pi) \subset \mathbf{C}_-$. As $A^\pi - D^\pi F^\pi = a - df + \mathcal{J}$, it follows that $a - df \in \mathbf{C}_-$. This is impossible if d and $\operatorname{Re} a$ are both zero. Therefore one of d and $\operatorname{Re} a$ is nonzero. We put $\xi := c/(a + \overline{a})$ if $d = 0$, and

$$\xi := \frac{a + \overline{a}}{2d} + \sqrt{\left(\frac{a + \overline{a}}{2d}\right)^2 + \frac{c}{d}} \tag{5.4}$$

if $d > 0$. Then

$$d\xi^2 - (a + \overline{a})\xi - c = 0. \tag{5.5}$$

We may write $X = \xi + V$ with $V \in \mathcal{A}$. Our aim is to show that $V \in \mathcal{J}$. Equivalently, we want to prove that $v := V^\pi$ is the zero in $\mathcal{A}/\mathcal{J}$.

Inserting $X = \xi + V$ in (1.1) and passing to the quotient algebra $\mathcal{A}/\mathcal{J}$, we obtain

$$d(\xi + v)^2 - (a + \overline{a})(\xi + v) - c = 0,$$

and taking into account (5.5) we get

$$dv^2 + (2d\xi - a - \overline{a})v = 0. \tag{5.6}$$

If $d = 0$ and hence $a + \overline{a} \neq 0$, this implies $v = 0$, as desired. So assume $d > 0$. Then (5.6) reads

$$dv\left(v + 2\xi - \frac{a + \overline{a}}{d}\right) = 0.$$

Due to (5.4),

$$2\xi - \frac{a + \overline{a}}{d} = 2\sqrt{\left(\frac{a + \overline{a}}{2d}\right)^2 + \frac{c}{d}} =: 2\varrho > 0$$

and thus, $v(v + 2\varrho) = 0$. We can decompose v as $v = v_+ - v_-$ with $v_+ \geq 0$, $v_- \geq 0$, $v_+v_- = v_-v_+ = 0$. It follows that $(v_+ - v_-)(v_+ - v_- + 2\varrho) = 0$, whence

$$v_+^2 + v_-^2 + 2(v_+ - v_-) = 0. \tag{5.7}$$

Since $v_+^2 \geq 0$, $v_-^2 \geq 0$, $\varrho > 0$, we conclude from (5.7) that $v_+ - v_- \leq 0$. Consequently, $0 \leq v_+ \leq v_-$. Multiplying this from the left and the right by v_+, we see that $0 \leq v_+^3 \leq v_+v_-v_+ = 0$, that is, $v_+^3 = 0$. We may think of v_+ as a positive operator and thus of multiplication by a function $\psi \geq 0$. The equality $v_+^3 = 0$ shows that $\psi^3 = 0$, which gives $\psi = 0$ and thus $v_+ = 0$. Now (5.7) becomes $v_-(v_- - 2\varrho) = 0$. We have $0 \leq X^\pi = \xi + v = \xi - v_-$ and therefore $v_- \leq \xi$. Thus,

$$\begin{aligned} v_- - 2\varrho \leq \xi - 2\varrho &= \frac{a + \overline{a}}{2d} + \varrho - 2\varrho \\ &= \frac{a + \overline{a}}{2d} - \sqrt{\left(\frac{a + \overline{a}}{2d}\right)^2 + \frac{c}{d}} < 0, \end{aligned}$$

which shows that $v_- - 2\varrho$ is invertible. As $v_-(v_- - 2\varrho) = 0$, we arrive at the desired equality $v_- = 0$. □

We remark that Theorem 5.3 is no longer true if instead of $C > 0$ we require only that $C \geq 0$. To see this, consider $\mathcal{A} = \ell^\infty$ (bounded sequences) and $\mathcal{J} = c_0$ (sequences converging to zero) with termwise operations. Clearly, $\mathbf{C}I + \mathcal{J}$ is the

algebra of all convergent sequences. Take $D = A = \{1, 1, \ldots\}$ and $C = \{0, 0, \ldots\}$. Then $X = \{0, 2, 0, 2, \ldots\}$ is a positive solution of equation (1.1), but X is obviously not in $\mathbf{C}I + \mathcal{J}$. Note that the set of all solutions is the set of all sequences $\{X_n\}$ with $X_n \in \{0, 2\}$. Consequently, all solutions are positive. A solution is in $\mathbf{C}I + \mathcal{J}$ if and only if X_n eventually stabilizes. Thus, there are countably many solutions in $\mathbf{C}I + \mathcal{J}$ but uncountably many solutions in $\mathcal{A} \setminus (\mathbf{C}I + \mathcal{J})$. The maximal solution $X_+ = \{2, 2, \ldots\}$ is of course in $\mathbf{C}I + \mathcal{J}$, as it should be by virtue of Theorem 2.2.

COROLLARY 5.4. *Let $\mathcal{A}$ be a unital W^*-algebra and $\mathcal{J} \subset \mathcal{A}$ be a proper closed two-sided ideal. Suppose D, A, C are in $\mathbf{C}I + \mathcal{J}$, $D \geq 0$, $C > 0$, and (A, D) is stabilizable. The equation (1.1) has a maximal solution X_+ in $\mathbf{C}I + \mathcal{J}$ and $X_+ \geq 0$.*

Proof. The existence of $X_+ \in \mathcal{A}$ and the positivity of X_+ are ensured by Theorem 5.2, while Theorem 5.3 shows that actually $X_+ \in \mathbf{C}I + \mathcal{J}$. □

Corollary 5.4 is in particular applicable to $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $\mathcal{J} = \mathcal{K}(\mathcal{H})$. The corollary is not applicable to $\mathcal{A} = \mathbf{B}$ (which is a W^*-algebra) and the set $\mathbf{J}$ introduced in Section 4, because $\mathbf{J}$ is not an ideal of $\mathbf{B}$. Let $\mathbf{B}_c \subset \mathbf{B}$ denote the set of all $\{F_n\} \in \mathbf{B}$ for which F_n, F_n^*, $W_n F_n W_n$, $W_n F_n^* W_n$ have strong limits. The set $\mathbf{J}$ is a closed two-sided ideal of $\mathbf{B}_c$; this was discovered by Silbermann [23], and a proof is also in [3, Lemma 2.21]. However, $\mathbf{B}_c$ is not a W^*-algebra, its strong closure being all of $\mathbf{B}$, so that Corollary 5.4 is again not applicable. But using Theorem 5.3 with $\mathcal{A} = \mathbf{B}_c$ and $\mathcal{J} = \mathbf{J}$, we arrive at the following conclusion: if, under the hypothesis of the theorem, $X = \{X_n\} = \{X_n^*\}$ is a positive solution of equation (1.1), then either $X \in \mathbf{C}I + \mathbf{J}$ or $X \notin \mathbf{B}_c$. The latter means that X_n or $W_n X_n W_n$ (or both) do not have a strong limit as $n \to \infty$.

5.3. The Riccati equation with Laurent matrices as coefficients. The Laurent matrix $L(f)$ generated by a function $f \in \mathcal{C}$ is the doubly-infinite Toeplitz matrix $(f_{j-k})_{j,k=-\infty}^{\infty}$. This matrix induces a bounded operator on $\ell^2(\mathbf{Z})$ and is unitarily equivalent to the operator of multiplication by f on $L^2(\mathbf{T})$. The Riccati equation (1.1) in Laurent matrices reads

$$XL(d)X - XL(a) - L(\overline{a})X - L(c) = 0. \tag{5.8}$$

The smallest closed subalgebra of $\mathcal{B}(\ell^2(\mathbf{Z}))$ which contains the set $L(\mathcal{C}) := \{L(f) : f \in \mathcal{C}\}$ is in fact $L(\mathcal{C})$ itself, it is a (commutative) unital C^*-algebra, and the map $f \mapsto L(f)$ is an isometric *-isomorphism of $\mathcal{C}$ onto $L(\mathcal{C})$. Using Theorems 2.1 and 2.2, we see that if $d, a, c \in \mathcal{C}$ and at least one of the conditions (i) $d > 0$ and $c = 1$, (ii) $d > 0$, $c \geq 0$, and $\operatorname{Re} a > 0$, (iii) $c \geq 0$ and $\operatorname{Re} a < 0$ is satisfied, then equation (5.8) has a maximal solution $X_+ \in L(\mathcal{C})$, and this solution is actually $X_+ = L(\varphi)$ where $\varphi \in \mathcal{C}$, $\varphi \geq 0$, and $d\varphi^2 - (a + \overline{a})\varphi - c = 0$. The finite section method for equation

(5.8) consists in passing to the equations

$$Y_n L_n(d) Y_n - Y_n L_n(a) - L_n(\overline{a}) Y_n - L_n(c) = 0. \tag{5.9}$$

where $L_n(f) := (f_{j-k})_{j,k=-n}^n$ and $Y_n = (y_{jk}^{(n)})_{j,k=-n}^n$. Define the projection $\widetilde{P}_n$ on $\ell^2(\mathbf{Z})$ by $(\widetilde{P}_n x)_j = x_j$ if $|j| \le n$ and $(\widetilde{P}_n x)_j = 0$ if $|j| > n$.

THEOREM 5.5. *Under the hypotheses of the preceding paragraph, equations (5.9) have (unique) positive maximal solutions Y_n^+ for all sufficiently large n, and $Y_n^+ \widetilde{P}_n$ converges strongly to the (unique) positive maximal solution $X_+ = L(\varphi)$ of equation (5.8).*

Proof. Clearly, equation (5.9) is nothing but equation (1.3) with n replaced by $2n+1$, that is,

$$X_{2n+1} T_{2n+1}(d) X_{2n+1} - X_{2n+1} T_{2n+1}(a) - T_{2n+1}(\overline{a}) X_{2n+1} - T_{2n+1}(c) = 0. \tag{5.10}$$

Theorem 4.2 shows that, under the hypotheses of the preceding paragraph, the maximal solutions X_{2n+1}^+ of (5.10) are of the form

$$X_{2n+1}^+ = T_{2n+1}(\varphi) + P_{2n+1} K P_{2n+1} + W_{2n+1} L W_{2n+1} + C_{2n+1} \tag{5.11}$$

with compact operators K and L on $\ell^2(\mathbf{N})$ and $\|C_{2n+1}\| \to 0$. Let $U : \ell^2(\mathbf{Z}) \to \ell^2(\mathbf{Z})$ be the forward shift, $(Ux)_j = x_{j-1}$. If $M = (m_{jk})_{j,k=0}^{2n+1}$ is a $(2n+1) \times (2n+1)$ matrix which is thought of as an operator on $\ell^2(\mathbf{N})$, then $U^{-n} P_{2n+1} M P_{2n+1} U^n$ may be identified with the matrix $(m_{j+n,k+n})_{j,k=-n}^n$ regarded as an operator on $\ell^2(\mathbf{Z})$. This implies that the maximal solution of (5.9) is

$$Y_n^+ = U^{-n} P_{2n+1} X_{2n+1}^+ P_{2n+1} U^n. \tag{5.12}$$

Since $P_{2n+1} U^n$ and $W_{2n+1} U^n$ converge weakly to zero and K and L are compact, it follows that $U^{-n} P_{2n+1} K P_{2n+1} U^n \to 0$ and $U^{-n} W_{2n+1} L W_{2n+1} U^n \to 0$ strongly. We also have $U^{-n} P_{2n+1} T_{2n+1}(\varphi) P_{2n+1} U^n = L_n(\varphi)$. Thus, combining (5.11) and (5.12) we arrive at the conclusion that $Y_n^+ \widetilde{P}_n$ converges strongly to the maximal solution $X_+ = L(\varphi)$ of equation (5.8). □

Interesting problems for equation (5.8) arise when d, a, c are taken from a subset $\mathcal{W}$ of $\mathcal{C}$ and one is interested in the question whether special (e.g. maximal) solutions of the equation are of the form $X = L(\varphi)$ with $\varphi \in \mathcal{W}$. Usually $\mathcal{W}$ is characterized by decay properties of the Fourier coefficients, and an important case is where $\mathcal{W}$ is a weighted Wiener algebra (which is not a C^*-algebra). More generally, one takes d, a, c from $\mathcal{W}^{N \times N}$, the $N \times N$ matrix functions with entries from $\mathcal{W}$, and looks for solutions of equation (5.8) of the form $X = L(\varphi)$ with $\varphi \in \mathcal{W}^{N \times N}$. Note that in this case even the problem $\varphi d \varphi - \varphi a - a^* \varphi - c = 0$ is non-commutative. For results in this field of research, we refer to [7], [8], [10], [11], [17], for example.

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