LIPSCHITZ-CONTINUITY OF THE INTEGRATED DENSITY OF STATES FOR GAUSSIAN RANDOM POTENTIALS

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Abstract. The integrated density of states of a Schrödinger operator with random potential given by a homogeneous Gaussian field whose covariance function is continuous, compactly supported and has positive mean, is locally uniformly Lipschitz-continuous. This is proven using a Wegner estimate.

Let $d \in \mathbb{N}$, $\|x\| := (x_1^2 + \cdots + x_d^2)^{1/2}$ and $|x| := \max(|x_1|, \ldots, |x_d|)$ for any $x \in \mathbb{R}^d$, $\Lambda_R := \{x \in \mathbb{R}^d \mid |x| < R\}$, $A\colon \mathbb{R}^d \to \mathbb{R}^d$ be measurable with $x \mapsto |A(x)|$ in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$, $(\Omega, \mathcal{A}, \mathbb{P})$ a complete probability space with associated expectation $\mathbb{E}$, $V\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ a separable, jointly measurable version of a $\mathbb{R}^d$-homogeneous Gaussian stochastic field with zero mean and covariance function $x \mapsto \mathcal{C}(x) := \mathbb{E}\,(V(\cdot, x)V(\cdot, 0))$ which is continuous at $x = 0$ and satisfies $\mathcal{C}(0) \in (0, \infty)$. . For each $L \in (0, \infty)$ the restricted random operator $H_{\omega,L} := \sum_{j=1}^d (i\frac{\partial}{\partial x_j} + A_j)^2 + V(\omega, \cdot)$ on Λ_L with either Neumann or Dirichlet boundary conditions is almost surely selfadjoint, lower semibounded and has purely discrete spectrum. For a detailed dicussion of these facts see [2]. The hypotheses state so far will be referred to as (H). The indicator function of a set S is denoted by χ_S

Theorem 1. *Assume that (H) holds as well as*

$$\mathcal{C} \text{ is supported in } \Lambda_R \text{ and } \bar{\mathcal{C}} := \int dx\, \mathcal{C}(x) > 0. \tag{2}$$

Then there exists a isotone function $C_{WG} : \mathbb{R} \to \mathbb{R}$ depending only the covariance $\mathcal{C}$ such that for all $L \in [1, \infty)$, $E_1 \le E_2 \in \mathbb{R}$ and both choices of b.c. in (H) the Wegner estimate

$$N_L(E_2) - N_L(E_2) \le C_{WG}\,(2L)^d\,(E_2 - E_1) \tag{3}$$

holds, where $N_L(E) := \mathbb{E}\left\{\mathrm{Tr}\,\chi_{(-\infty,E]}(H_{\omega,L})\right\}$.

In [1, Thm. 1] and and [2, Cor. 4.3] the same statement as in the above Theorem is derived, in the case that assumption (2) is replaced by

(4) There exists a finite signed Borel-measure μ on $\mathbb{R}^d$, an open $\Gamma \subset \mathbb{R}^d$ and $\gamma \in (0, \infty)$ such that

$$\int \mu(dx) \int \mu(dy) \mathcal{C}(x - y) = \mathcal{C}(0) \text{ and } \mu * \mathcal{C} \ge \mathcal{C}(0)\,\gamma\,\chi_\Gamma.$$

In the references the reader may find explicit bounds on the function C_{WG}. To prove Theorem 1 it is sufficient to show that (2) implies (4). Before we do so in Lemma 6 below, let us state specific cases under which it was known before (cf. the references in [2]) that a measure μ as in (4) exists:

$$\mathcal{C}(x) \ge 0 \text{ for all } x \in \mathbb{R}^d \text{ and } \mathcal{C} \text{ not identically vanishing,} \tag{5a}$$

$$d = 1, \quad \mathcal{C}(x) = \int w(x - y) w(y) dy, \text{and } w = \chi_{[-3,3]} - \frac{5}{4}\chi_{[-1,1]}, \tag{5b}$$

$$\mathcal{C}(x) = \mathcal{C}(0) \exp\big(-\|x\|^2/(2t^2)\big)\big(1 - 7\|x\|^2/(16t^2) + \|x\|^4/(32t^4)\big), t > 0 \text{ arbitrary.} \tag{5c}$$

In all three listed cases the integral $\bar{\mathcal{C}}$ is positive. Note that while (4a) is an infinite family of conditions (one for each $x \in \mathbb{R}^d$), condition (2) is one-dimensional.

Lemma 6. *Assume (H) and (2). Choose b positive with $b \le \bar{\mathcal{C}}\,(2\mathrm{e}R\|\mathcal{C}\|_1)^{-1}$ and $f(x) = \mathrm{e}^{-b|x|}$. Then for all $x \in \mathbb{R}^d$ we have*

$$(f * \mathcal{C})(x) := \int dy\, f(y)\, \mathcal{C}(x-y) \ge \frac{\bar{\mathcal{C}}}{2} f(x). \tag{7}$$

In particular, condition (4) holds.

Proof. Since

$$(f * \mathcal{C})(x) = \bar{\mathcal{C}} f(x) + \int_{x+\Lambda_R} dy \big(f(y) - f(x)\big)\mathcal{C}(x-y),$$

(7) holds, if the absolute value of the second term is bounded by $f(x)\,\bar{\mathcal{C}}/2$. Note that

$$\Big| \int_{x+\Lambda_R} dy \big(\mathrm{e}^{-b|y|} - \mathrm{e}^{-b|x|}\big)\mathcal{C}(x-y)\Big| \le \mathrm{e}^{-b|x|} \int_{x+\Lambda_R} dy \big(\mathrm{e}^{b|x|-b|y|} - 1\big)|\mathcal{C}(x-y)|. \tag{8}$$

For $|x-y| \le R$ we have $\big|b|x| - b|y|\big| \le \frac{\bar{\mathcal{C}}}{2\mathrm{e}\|\mathcal{C}\|_1} \le \frac{1}{2\mathrm{e}}$ and thus $\mathrm{e}^{b|x|-b|y|} - 1 \le \frac{\bar{\mathcal{C}}}{2\|\mathcal{C}\|_1}$. Hence

$$(8) \le \mathrm{e}^{-b|x|} \frac{\bar{\mathcal{C}}}{2\|\mathcal{C}\|_1} \int_{x+\Lambda_R} dy\, |\mathcal{C}(x-y)| = \frac{\bar{\mathcal{C}}}{2} f(x).$$

Now we show that the measure $\mu(dx) := \alpha f(x) dx$ with an appropriate choice of $\alpha \in \mathbb{R}$ satisfies (4). Ineq. (7) implies $\int dx \int dy f(x) f(y) \mathcal{C}(x-y) \ge \frac{\alpha^2 \bar{\mathcal{C}}}{2} \|f\|_2^2 > 0$. Thus the choice $\alpha := \sqrt{\mathcal{C}(0)}\big(\int dx \int dy f(x) f(y) \mathcal{C}(x-y)\big)^{-1/2}$ is well defined and implies $\int \mu(dx) \int \mu(dy) \mathcal{C}(x-y) = \mathcal{C}(0)$. If we set $\gamma = \frac{\alpha \bar{\mathcal{C}}}{2\mathrm{e}\mathcal{C}(0)}$ and $\Gamma = \Lambda_{1/b} \subset \mathbb{R}^d$, then

$$(\mu * \mathcal{C})(x) := \alpha \int dy\, f(y)\, \mathcal{C}(x-y) \ge \alpha \frac{\bar{\mathcal{C}}}{2}\, \mathrm{e}^{-b|x|} \ge \mathcal{C}(0)\, \gamma\, \chi_\Gamma(x).$$

Thus condition (4) is satisfied. □

Under appropriate conditions on $A\colon \mathbb{R}^d \to \mathbb{R}^d$ (in particular for $A \equiv 0$) it is known that

(9) there exists an isotone, right-continuous function $N\colon \mathbb{R} \to \mathbb{R}$ such that $\lim_{L\to\infty} (2L)^{-d} N_L(E) = N(E)$ holds for every E where N is continuous.

Corollary 10. *Assume (H), (2), and (9). Then we have for all $E_1 \le E_2 \in \mathbb{R}$*

$$N(E_2) - N(E_1) \le C_{WG}(E_2)\,(E_2 - E_1). \tag{11}$$

Thus N is locally uniformly Lipschitz-continuous, hence differentiable a.e., and (11) implies an upper bound on its derivative, the density of states.

For more background information on this short note see the references. The author thanks P. Müller for enlightening discussions.

References

[1] W. Fischer, T. Hupfer, H. Leschke, and P. Müller. Existence of the density of states for multi-dimensional continuum Schrödinger operators with Gaussian random potentials. *Commun. Math. Phys.*, 190, No.1:133–141, 1997.

[2] T. Hupfer, H. Leschke, P. Müller, and S. Warzel. The absolute continuity of the integrated density of states for magnetic Schrödinger operators with certain unbounded random potentials. *Comm. Math. Phys.*, 221(2):229–254, 2001.

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