EXPONENTIAL DECAY OF GREEN'S FUNCTION FOR ANDERSON MODELS ON $\mathbb{Z}$ WITH SINGLE-SITE POTENTIALS OF FINITE SUPPORT

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ABSTRACT. One of the fundamental results in the theory of localisation for discrete Schrödinger operators with random potentials is the exponential decay of Green's function. In this note we provide a new variant of this result in the one-dimensional situation for sign-changing single-site potentials with arbitrary finite support using the fractional moment method.

1. INTRODUCTION

Anderson models are discrete Schrödinger operators with random potentials. Such models have been studied since a long time in computational and theoretical physics, as well as in mathematics. One of the fundamental results for these models is the physical phenomenon of localisation. There are various mathematical formulations of localisation: almost sure absence of continuous spectrum, non-spreading of wave packets, exponential decay of generalised eigensolutions or exponential decay of Green's function. Such properties have been established mainly by two different methods, the multiscale analysis and the fractional moment method. The multiscale analysis (MSA) was invented by Fröhlich and Spencer in [FS83]. The fractional moment method (FMM) was introduced by Aizenman and Molchanov [AM93], and further developed, e. g., in [Aiz94, Gra94, ASFH01].

Here we focus our attention on correlated Anderson models. More precisely, we consider models where the potential values at different sites need not be independent random variables. Assuming certain abstract regularity assumptions on the (possibly dependent) random potential localisation has been established using both methods, see e. g. [vDK91, AM93, Aiz94, ASFH01]. For continuous alloy-type models with sign-changing single-site potential localisation has been derived via MSA, e. g. in [Klo95, Ves02, KV06, Klo02], see also [Sto02]. To our knowledge, the FMM has not been applied to alloy-type models with sign-changing single-site potential so far (neither in the continuous nor the discrete setting).

In this paper we study a one-dimensional discrete alloy-type model using the FMM. In this model, the potential at the lattice site $x \in \mathbb{Z}$ is defined by a finite linear combination $V_\omega(x) = \sum_k \omega_k u(x-k)$ of i. i. d. random coupling constants ω_k. The function $u(\cdot - k)$ is called single-site potential and may be interpreted as a finite interaction range potential associated to the lattice site $k \in \mathbb{Z}$. In particular, the single-site potential is allowed to change sign. For such models we prove in one space dimension and at all energies a

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so called fractional moment bound, i. e. exponential off-diagonal decay of an averaged fractional power of Green's function. The restriction to the one-dimensional case allows an elegant and short proof in which the basic steps—decoupling and averaging—are particularly transparent. Currently we are working on the extension of our result to the multidimensional case.

2. Model and results

We consider a one-dimensional Anderson model. This is the random discrete Schrödinger operator

$$H_\omega := -\Delta + V_\omega, \quad \omega \in \Omega, \tag{1}$$

acting on $\ell^2(\mathbb{Z})$, the space of all square-summable sequences indexed by $\mathbb{Z}$ with an inner product $\langle \cdot, \cdot \rangle$. Here, $\Delta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ denotes the discrete Laplace operator and $V_\omega : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is a random multiplication operator. They are defined by

$$(\Delta\psi)(x) := \sum_{|e|=1} \psi(x+e) \quad \text{and} \quad (V_\omega\psi)(x) := V_\omega(x)\psi(x)$$

and represent the kinetic energy and the random potential energy, respectively. We assume that the probability space has a product structure $\Omega := \times_{k\in\mathbb{Z}} \mathbb{R}$ and is equipped with the probability measure $\mathrm{d}\mathbb{P}(\omega) := \prod_{k\in\mathbb{Z}} \rho(\omega_k)\mathrm{d}\omega_k$ where $\rho \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ with $\|\rho\|_{L^1} = 1$. Hence, each element ω of Ω may be represented as a collection $\{\omega_k\}_{k\in\mathbb{Z}}$ of independent identically distributed (i. i. d.) random variables, each distributed with the density ρ. The symbol $\mathbb{E}\{\cdot\}$ denotes the expectation with respect to the probability measure, i. e. $\mathbb{E}\{\cdot\} := \int_\Omega (\cdot)\mathrm{d}\mathbb{P}(\omega)$. For a set $\Gamma \subset \mathbb{Z}$, $\mathbb{E}_\Gamma\{\cdot\}$ denotes the expectation with respect to ω_k, $k \in \Gamma$. That is, $\mathbb{E}_\Gamma\{\cdot\} := \int_{\Omega_\Gamma} (\cdot) \prod_{k\in\Gamma} \rho(\omega_k)\mathrm{d}\omega_k$ where $\Omega_\Gamma := \times_{k\in\Gamma} \mathbb{R}$. Let the *single-site potential* $u : \mathbb{Z} \to \mathbb{R}$ be a function with finite and non-empty support $\Theta := \operatorname{supp} u = \{k \in \mathbb{Z} : u(k) \neq 0\}$. We assume that the random potential

$$V_\omega(x) := \sum_{k\in\mathbb{Z}} \omega_k u(x-k)$$

at a lattice site $x \in \mathbb{Z}$ is a linear combination of the i. i. d. random variables ω_k, $k \in \mathbb{Z}$, with coefficients provided by the single-site potential. The function $u(\cdot - k)$ may be interpreted as a finite range potential associated to the lattice site $k \in \mathbb{Z}$. The Hamiltonian (1) is possibly unbounded, but self-adjoint on a dense subspace of $\ell^2(\mathbb{Z})$, see e. g. [Kir07]. Finally, for the operator H_ω in (1) and $z \in \mathbb{C} \setminus \sigma(H_\omega)$ we define the corresponding *resolvent* by $G_\omega(z) := (H_\omega - z)^{-1}$. For the *Green's function*, which assigns to each $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ the corresponding matrix element of the resolvent, we use the notation

$$G_\omega(z; x, y) := \left\langle \delta_x, (H_\omega - z)^{-1}\delta_y \right\rangle. \tag{2}$$

For $\Gamma \subset \mathbb{Z}$, $\delta_k \in \ell^2(\Gamma)$ denotes the Dirac function given by $\delta_k(k) = 1$ for $k \in \Gamma$ and $\delta_k(j) = 0$ for $j \in \Gamma \setminus \{k\}$. The quantities $\|\rho\|_\infty^{-1}$ and (in the case that ρ is weakly differentiable) $\|\rho'\|_{L^1}^{-1}$ may be understood as a measure of the disorder present in the model. Our results in the case of strong disorder are the following two theorems.

Theorem 2.1. *Let $n \in \mathbb{N}$, $\Theta = \{0, \ldots, n-1\}$, $s \in (0,1)$, and $\|\rho\|_\infty$ be sufficiently small. Then there exist constants $C, m \in (0,\infty)$ such that for all $x, y \in \mathbb{Z}$ with $|x-y| \geq n$ and all $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\mathbb{E}\{|G_\omega(z; x, y)|^{s/n}\} \leq C\mathrm{e}^{-m|x-y|}. \tag{3}$$

Theorem 2.2. *Let $n \in \mathbb{N}$, $\Theta \subset \mathbb{Z}$ finite with* $\min \Theta = 0$ *and* $\max \Theta = n-1$, *r as in Eq.* (21) *(the width of the largest gap in Θ), and $s \in (0, n/(n+r))$. Assume*

(a) $\rho \in W^{1,1}(\mathbb{R})$ *with $\|\rho'\|_{L^1}$ sufficiently small,* ***or***
(b) $\operatorname{supp} \rho$ *compact with $\|\rho\|_\infty$ sufficiently small.*

Then there exist constants $C, m \in (0,\infty)$ such that the bound (3) *holds true for all $x, y \in \mathbb{Z}$ with $|x-y| \geq 2(n+r)$ and all $z \in \mathbb{C} \setminus \mathbb{R}$.*

The difference between the two theorems is the following: In Theorem 2.1 we assume that Θ is finite and *connected* (cf. §3). The latter condition can be dropped if ρ is sufficiently regular, cf. Theorem 2.2. Theorem 2.1 is proven in Section 3 and 4, compare also Theorem 4.3. An explicit formula for the constants m and C can be inferred from (20), and an explicit disorder requirement is given in Ineq. (19). A quantitative version of Theorem 2.2 is stated and proven in Section 5.

We can actually apply both theorems to arbitrary Θ with $\max \Theta - \min \Theta = n - 1$. In this situation a translation of the indices of the random variables $\{\omega_k\}_{k\in\mathbb{Z}}$ by $\min \Theta$ transforms the model to the case $\min \Theta = 0$ and $\max \Theta = n-1$. Note that $\min \Theta$ and $\max \Theta$ are well defined since $\Theta \subset \mathbb{R}$ is finite.

Remark 2.3. Our proof give estimates about fractional moments of *certain* matrix elements of the resolvent for somewhat more general models. Let us formulate this class of random potentials next. Assume that $V_\omega := V_\omega^{(1)} + V_\omega^{(2)}$ where $V_\omega^{(1)}, V_\omega^{(2)} : \mathbb{Z} \to \mathbb{R}$ are potentials indexed by the random parameter ω in some probability space Ω. Assume that $u : \mathbb{Z} \to \mathbb{R}$ has support equal to $\{0, \ldots, n-1\}$, and that there exists a sequence $\lambda_k : \Omega \to \mathbb{R}$ of i.i.d. random variables indexed by $k \in n\mathbb{Z}$, each being distributed according to a density $\rho \in L^\infty(\mathbb{R})$. Assume that $V_\omega^{(1)}(x) = \sum_{k\in n\mathbb{Z}} \lambda_k(\omega) u(x-k)$ and that $V_\omega^{(2)}$ is uniformly bounded on $\Omega \times \mathbb{Z}$, but otherwise arbitrary. If $F : \Omega \to [0,\infty)$ is a random variable we denote its average over all random variables $\lambda_k, k \in n\mathbb{Z}$ by $\mathbb{E}^{(1)}(F) := \int F(\omega) \prod_{k\in n\mathbb{Z}} \rho(\omega_k) \mathrm{d}\omega_k$, where the domain of integration is $\times_{k\in n\mathbb{Z}} \mathbb{R}$. It follows directly from the iterative application of Lemma 3.3 that for all $p \in \mathbb{N}$ and for the constant $C_{u,\rho}$ defined in (11) the following estimate holds

$$\mathbb{E}^{(2)}\{|G_\omega(z; x, x+np-1)|^{s/n}\} \leq C_{u,\rho}^p. \tag{4}$$

A decomposition of the type $V_\omega := V_\omega^{(1)} + V_\omega^{(2)}$ is implicitly used in the proof of Theorem 2.2, given in Section 5. Note, that in this particular situation the two stochstic processes $V_\omega^{(1)}, V_\omega^{(2)}$ are *not* independet from each other.

If $V_\omega^{(2)} \equiv 0$ then the full potential V_ω equals $\sum_{k\in n\mathbb{Z}} \lambda_k(\omega) u(x-k)$. Hence, in this case the bound (4) also holds true.

In the following remark we state two complementary results which are explained in detail in the appendix.

Remark 2.4. (i) The statements of Theorems 2.1 and 2.2 concern only off-diagonal elements. If we assume that ρ has compact support, $\mathbb{E}\{|G_\omega(E + i0; x, y)|^s\}$ is finite for any $x, y \in \mathbb{Z}$ and $s > 0$ sufficiently small. This implies in particular that $\sum_{y\in\mathbb{Z}} |G_\omega(E + i0; 0, y)|^2$ is finite almost surely for almost all $E \in \mathbb{R}$.

However, neither dynamical nor spectral localisation can be directly inferred from the behaviour of the Green's function using the existent methods ([SW86], [Aiz94]). The reason is that the random variables $V_\bullet(x)$, $x \in \mathbb{Z}$, are not independent, while the dependence of H_ω on the i. i. d. variables ω_x, $x \in \mathbb{Z}$, is not monotone.

(ii) If the polynomial $p_u(x) := \sum_{k=0}^{n-1} u(k)\, x^k$ does not vanish on $[0, \infty)$ it is possible to extract from V_ω a positive single-site potential with certain additional properties. In this situation the method of [AEN+06] applies and gives exponential decay of the fractional moments of the Green's function.

3. Fractional moment bounds for Green's function

In this section we present fractional moment bounds for Green's function. A very useful observation is that "important" matrix elements of the resolvent are given by the inverse of a determinant. The latter can be controlled using the following spectral averaging lemma for determinants.

Lemma 3.1. *Let $n \in \mathbb{N}$ and $A, V \in \mathbb{C}^{n\times n}$ be two matrices and assume that V is invertible. Let further $0 \le \rho \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $s \in (0, 1)$. Then we have for all $\lambda > 0$ the bound*

$$\int_{\mathbb{R}} |\det(A + rV)|^{-s/n}\, \rho(r)\mathrm{d}r \le |\det V|^{-s/n}\, \|\rho\|_{L^1}^{1-s} \|\rho\|_\infty^s \frac{2^s s^{-s}}{1 - s} \tag{5}$$

$$\le |\det V|^{-s/n} \Big(\lambda^{-s}\|\rho\|_{L^1} + \frac{2\lambda^{1-s}}{1 - s}\|\rho\|_\infty\Big). \tag{6}$$

Proof. Since V is invertible, the function $r \mapsto \det(A + rV)$ is a polynomial of order n and thus the set $\{r \in \mathbb{R} \colon A + rV \text{ is singular}\}$ is a discrete subset of $\mathbb{R}$ with Lebesgue measure zero. We denote the roots of the polynomial by $z_1, \ldots, z_n \in \mathbb{C}$. By multilinearity of the determinant we have

$$|\det(A + rV)| = |\det V| \prod_{j=1}^{n} |r - z_j| \ge |\det V| \prod_{j=1}^{n} |r - \operatorname{Re} z_j|.$$

The Hölder inequality implies for $s \in (0, 1)$ that

$$\int_{\mathbb{R}} |\det(A + rV)|^{-s/n}\, \rho(r)\mathrm{d}r \le |\det V|^{-s/n} \prod_{j=1}^{n} \left(\int_{\mathbb{R}} |r - \operatorname{Re} z_j|^{-s} \rho(r)\mathrm{d}r\right)^{1/n}.$$

For arbitrary $\lambda > 0$ and all $z \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} \frac{1}{|r - z|^s}\rho(r)\mathrm{d}r = \int\limits_{|r-z|\ge\lambda} \frac{1}{|r - z|^s}\rho(r)\mathrm{d}r + \int\limits_{|r-z|\le\lambda} \frac{1}{|r - z|^s}\rho(r)\mathrm{d}r$$

$$\le \lambda^{-s}\|\rho\|_{L^1} + \|\rho\|_\infty \frac{2\lambda^{1-s}}{1 - s}$$

which gives Ineq. (6). We now choose $\lambda = s\|\rho\|_{L^1}/(2\|\rho\|_\infty)$ (which minimises the right hand side of Ineq. (6)) and obtain Ineq. (5). ■

In order to use the estimate of Lemma 3.1 for our infinite-dimensional operator $G_\omega(z)$, we will use a special case of the Schur complement formula (also known as Feshbach formula or Grushin problem), see e. g. [BHS07, appendix]. Before providing such a formula, we will introduce some more notation. Let $\Gamma_1 \subset \Gamma_2 \subset \mathbb{Z}$. We define the operator $P_{\Gamma_1}^{\Gamma_2} : \ell^2(\Gamma_2) \to \ell^2(\Gamma_1)$ by

$$P_{\Gamma_1}^{\Gamma_2}\psi := \sum_{k\in\Gamma_1} \psi(k)\delta_k.$$

Note that the adjoint $(P_{\Gamma_1}^{\Gamma_2})^* : \ell^2(\Gamma_1) \to \ell^2(\Gamma_2)$ is given by $(P_{\Gamma_1}^{\Gamma_2})^*\phi = \sum_{k\in\Gamma_1} \phi(k)\delta_k$. If $\Gamma_2 = \mathbb{Z}$ we will drop the upper index and write P_{Γ_1} instead of $P_{\Gamma_1}^{\mathbb{Z}}$. For an arbitrary set $\Gamma \subset \mathbb{Z}$ we define the restricted operators $\Delta_\Gamma, V_\Gamma, H_\Gamma : \ell^2(\Gamma) \to \ell^2(\Gamma)$ by

$$\Delta_\Gamma := P_\Gamma \Delta P_\Gamma^*, \quad V_\Gamma := P_\Gamma V_\omega P_\Gamma^* \quad \text{and} \quad H_\Gamma := P_\Gamma H_\omega P_\Gamma^* = -\Delta_\Gamma + V_\Gamma.$$

Furthermore, we define $G_\Gamma(z) := (H_\Gamma - z)^{-1}$ and $G_\Gamma(z; x, y) := \langle \delta_x, G_\Gamma(z)\delta_y \rangle$ for $z \in \mathbb{C}\setminus\sigma(H_\Gamma)$ and $x, y \in \Gamma$. For an operator $T : \ell^2(\Gamma) \to \ell^2(\Gamma)$ the symbol $[T]$ denotes the matrix representation of T with respect to the basis $\{\delta_k\}_{k\in\Gamma}$. By $\partial\Gamma$ we denote the interior vertex boundary of the set Γ, i. e. $\partial\Gamma := \{k \in \Gamma : \#\{j \in \Gamma : |j - k| = 1\} < 2\}$. For finite sets $\Gamma \subset \mathbb{Z}$, $|\Gamma|$ denotes the number of elements of Γ. A set $\Gamma \subset \mathbb{Z}$ is called *connected* if $\partial\Gamma \subset \{\inf\Gamma, \sup\Gamma\}$. In particular, $\mathbb{Z}$ is a connected set.

Lemma 3.2. *Let $\Gamma \subset \mathbb{Z}$ and $\Lambda \subset \Gamma$ be finite and connected. Then we have the identity*

$$G_\Gamma(z; x, y) = \langle \delta_x, (H_\Lambda - B_\Gamma^\Lambda - z)^{-1}\delta_y \rangle$$

for all $z \in \mathbb{C} \setminus \sigma(H_\Gamma)$ and all $x, y \in \Lambda$, where $B_\Gamma^\Lambda : \ell^2(\Lambda) \to \ell^2(\Lambda)$ is specified in Eq. (7). Moreover, the operator B_Γ^Λ is diagonal and does not depend on $V_\omega(k)$, $k \in \Lambda$.

An analogous statement for arbitrary dimension was established in [EG].

Proof. Since Λ is finite, H_Λ is bounded and the Schur complement formula gives

$$P_\Lambda^\Gamma(H_\Gamma - z)^{-1}(P_\Lambda^\Gamma)^* = \Big[(H_\Lambda - z) - \underbrace{P_\Lambda^\Gamma \Delta_\Gamma (P_{\Gamma\setminus\Lambda}^\Gamma)^* \quad (H_{\Gamma\setminus\Lambda} - z)^{-1} \quad P_{\Gamma\setminus\Lambda}^\Gamma \Delta_\Gamma (P_\Lambda^\Gamma)^*}_{=:B_\Gamma^\Lambda}\Big]^{-1}.$$

It is straightforward to calculate that the matrix elements of B_Γ^Λ are given by

$$\Big\langle \delta_x, B_\Gamma^\Lambda \delta_y \Big\rangle = \begin{cases} \sum\limits_{\substack{k\in\Gamma\setminus\Lambda: \\ |k-x|=1}} \Big\langle \delta_k, (H_{\Gamma\setminus\Lambda} - z)^{-1}\delta_k \Big\rangle & \text{if } x = y \text{ and } x \in \partial\Lambda, \\ 0 & \text{else.} \end{cases} \tag{7}$$

Here we have used that Λ is connected. ■

Lemma 3.3. *Let $n \in \mathbb{N}$, $\Theta = \{0, \ldots, n - 1\}$, $s \in (0, 1)$, and $\Gamma \subset \mathbb{Z}$ be connected. Then,*

(i) *for every pair $x, x + n - 1 \in \Gamma$ and all $z \in \mathbb{C} \setminus \mathbb{R}$ we have*

$$\mathbb{E}_{\{x\}}\{|G_\Gamma(z; x, x + n - 1)|^{s/n}\} \le C_u C_\rho =: C_{u,\rho}. \tag{8}$$

(ii) *if* $1 \le |\Gamma| \le n$, *we have for all* $z \in \mathbb{C} \setminus \mathbb{R}$ *the bound*

$$\mathbb{E}_{\{\gamma_0\}}\{|G_\Gamma(z;\gamma_0,\gamma_1)|^{s/n}\} \le C_u^+ C_\rho^+ =: C_{u,\rho}^+ \tag{9}$$

where $\gamma_0 = \min\Gamma$ *and* $\gamma_1 = \max\Gamma$.

(iii) *if* $\Gamma = \{x, x+1, ...\}$ *and* $y \in \Gamma$ *with* $0 \le y - x \le n-1$, *we have for all* $z \in \mathbb{C} \setminus \mathbb{R}$ *the bound*

$$\mathbb{E}_{\{y-n+1\}}\{|G_\Gamma(z;x,y)|^{s/n}\} \le C_{u,+} C_\rho^+ =: C_{u,\rho,+}. \tag{10}$$

The constants C_u, C_ρ, C_u^+, C_ρ^+ *and* $C_{u,+}$ *are given in Eq.* (11), (12) *and* (13).

Proof. We start with the first statement of the lemma. By assumption $x, x+n-1 \in \Gamma$. We apply Lemma 3.2 with $\Lambda := \{x, x+1, \dots, x+n-1\} \subset \Gamma$ (since Γ is connected) and obtain for all $x, y \in \Lambda$

$$G_\Gamma(z;x,y) = \langle \delta_x, (H_\Lambda - B_\Gamma^\Lambda - z)^{-1}\delta_y \rangle,$$

where the operator B_Γ^Λ is given by Eq. (7). Set $D = H_\Lambda - B_\Gamma^\Lambda - z$. By Cramer's rule we have $G_\Gamma(z;x,y) = \det C_{y,x} / \det[D]$. Here, $C_{i,j} = (-1)^{i+j} M_{i,j}$ and $M_{i,j}$ is obtained from the tridiagonal matrix $[D]$ by deleting row i and column j. Thus $C_{x+n-1,x}$ is a lower triangular matrix with determinant ± 1. Hence,

$$|G_\Gamma(z;x,x+n-1)| = \frac{1}{|\det[D]|}.$$

Since $\Theta = \operatorname{supp} u = \{0, \dots, n-1\}$, every potential value $V_\omega(k)$, $k \in \Lambda$, depends on the random variable ω_x, while the operator B_Γ^Λ is independent of ω_x. Thus we may write $[D]$ as a sum of two matrices

$$[D] = A + \omega_x V,$$

where $V \in \mathbb{R}^{n\times n}$ is diagonal with the elements $u(k-x)$, $k = x, \dots, x+n-1$, and $A := [D] - \omega_x V$. Since A is independent of ω_x we may apply Lemma 3.1 and obtain for all $s \in (0,1)$ the estimate (8) with

$$C_u = \Bigl|\prod_{k\in\Theta} u(k)\Bigr|^{-s/n} \quad \text{and} \quad C_\rho = \|\rho\|_\infty^s \frac{2^s s^{-s}}{1-s}. \tag{11}$$

The proof of Ineq. (9) is similar but does not require Lemma 3.2. We have the decomposition $[H_\Gamma - z] = \tilde{A} + \omega_{\gamma_0}\tilde{V}$, where $d := \gamma_1 - \gamma_0$, $\tilde{V} \in \mathbb{R}^{(d+1)\times(d+1)}$ is diagonal with elements $u(k-\gamma_0)$, $k = \gamma_0, \dots, \gamma_1$, and $\tilde{A} := [H_\Gamma - z] - \omega_{\gamma_0}\tilde{V}$ is independent of ω_{γ_0}. By Cramer's rule and Lemma 3.1 we obtain

$$\mathbb{E}_{\{\gamma_0\}}\{|G_\Gamma(z;\gamma_0,\gamma_1)|^{t/(d+1)}\} \le \Bigl|\prod_{k=0}^{d} u(k)\Bigr|^{-t/(d+1)} \|\rho\|_\infty^t \frac{2^t t^{-t}}{1-t}$$

for all $t \in (0,1)$. We choose $t = s\frac{d+1}{n}$ and obtain Ineq. (9) with the constants

$$C_u^+ = \max_{i\in\Theta}\Bigl|\prod_{k=0}^{i} u(k)\Bigr|^{-s/n} \quad \text{and} \quad C_\rho^+ = \max\{\|\rho\|_\infty^s, \|\rho\|_\infty^{s/n}\}\frac{2^s s^{-s}}{1-s}. \tag{12}$$

In the final step we have used $s \ge t$ and the monotonicity of $(0,1) \ni x \mapsto 2^x x^{-x}/(1-x)$. For the proof of the third statement we apply Lemma 3.2 with $\Lambda = \{x, \dots, y\}$ and obtain using Cramer's rule $|G_\Gamma(z;x,y)| = |1/\det[H_\Lambda - B_\Gamma^\Lambda - z]|$. Set $d := y - x$. Notice that B_Γ^Λ is independent of ω_{y-n+1}, while every potential value $V_\omega(k)$, $k \in \Lambda$, depends on ω_{y-n+1}. Thus we have the decomposition $[H_\Lambda - B_\Gamma^\Lambda - z] = A + \omega_{y-n+1} V$, where $V \in \mathbb{R}^{(d+1)\times(d+1)}$ is diagonal with the elements $u(k)$, $k = n-1-d, \dots, n-1$, and $A := [H_\Lambda - B_\Gamma^\Lambda - z] - \omega_{y-n+1} V$. Since A is independent of ω_{y-n+1} we may apply Lemma 3.1 and obtain for all $t \in (0,1)$

$$\mathbb{E}_{\{y-n+1\}}\Big\{\Big|G_\Gamma(z;x,y)\Big|^{s/(d+1)}\Big\} \le \Big|\prod_{k=n-1-d}^{n-1} u(k)\Big|^{-t/(d+1)} \|\rho\|_\infty^t \, \frac{2^t t^{-t}}{1-t}.$$

We choose $t = s\frac{d+1}{n}$ and obtain Ineq (10) with

$$C_{u,+} := \max_{i\in\Theta}\Big|\prod_{k=n-1-i}^{n-1} u(k)\Big|^{-s/n}. \tag{13}$$

In the final step we have used $s \ge t$ and the monotonicity of $(0,1) \ni x \mapsto 2^x x^{-x}/(1-x)$. ■

4. Exponential decay of Green's function

In this section we use so called "depleted" Hamiltonians to formulate a geometric resolvent formula. Such Hamiltonians are obtained by setting to zero the "hopping terms" of the Laplacian along a collection of bonds. More precisely, let $\Lambda \subset \Gamma \subset \mathbb{Z}$ be arbitrary sets. We define the depleted Laplace operator $\Delta_\Gamma^\Lambda : \ell^2(\Gamma) \to \ell^2(\Gamma)$ by

$$\Big\langle \delta_x, \Delta_\Gamma^\Lambda \delta_y \Big\rangle := \begin{cases} 0 & \text{if } x \in \Lambda, y \in \Gamma\setminus\Lambda \text{ or } y \in \Lambda, x \in \Gamma\setminus\Lambda, \\ \langle \delta_x, \Delta_\Gamma \delta_y\rangle & \text{else.} \end{cases}$$

In other words, the hopping terms which connect Λ with $\Gamma \setminus \Lambda$ or vice versa are deleted. The depleted Hamiltonian $H_\Gamma^\Lambda : \ell^2(\Gamma) \to \ell^2(\Gamma)$ is then defined by

$$H_\Gamma^\Lambda := -\Delta_\Gamma^\Lambda + V_\Gamma.$$

Let further $T_\Gamma^\Lambda := \Delta_\Gamma - \Delta_\Gamma^\Lambda$ be the difference between the the "full" Laplace operator and the depleted Laplace operator. Analogously to Eq. (2) we use the notation $G_\Gamma^\Lambda(z) := (H_\Gamma^\Lambda - z)^{-1}$ and $G_\Gamma^\Lambda(z;x,y) := \langle \delta_x, G_\Gamma^\Lambda(z)\delta_y\rangle$. The second resolvent identity yields for arbitrary sets $\Lambda \subset \Gamma \subset \mathbb{Z}$

$$G_\Gamma(z) = G_\Gamma^\Lambda(z) + G_\Gamma(z) T_\Gamma^\Lambda G_\Gamma^\Lambda(z) \tag{14}$$

$$= G_\Gamma^\Lambda(z) + G_\Gamma^\Lambda(z) T_\Gamma^\Lambda G_\Gamma(z). \tag{15}$$

In the following we will use that $G_\Gamma^\Lambda(z;x,y) = G_\Lambda(z;x,y)$ for all $x,y \in \Lambda$, since H_Γ^Λ is block-diagonal, and that $G_\Gamma^\Lambda(z;x,y) = 0$ if $x \in \Lambda$ and $y \notin \Lambda$ or vice versa.

Lemma 4.1. *Let $n \in \mathbb{N}$, $\Theta = \{0,\dots,n-1\}$, $\Gamma \subset \mathbb{Z}$ be connected, and $s \in (0,1)$. Then we have for all $x,y \in \Gamma$ with $y - x \ge n$, $\Lambda := \{x+n, x+n+1, \dots\} \cap \Gamma$ and all $z \in \mathbb{C}\setminus\mathbb{R}$ the bound*

$$\mathbb{E}_{\{x\}}\{|G_\Gamma(z;x,y)|^{s/n}\} \le C_{u,\rho} \cdot |G_\Lambda(z;x+n,y)|^{s/n}.$$

In particular,

$$\mathbb{E}\{|G_\Gamma(z;x,y)|^{s/n}\} \le C_{u,\rho} \cdot \mathbb{E}\{|G_\Lambda(z;x+n,y)|^{s/n}\}. \tag{16}$$

Proof. Our starting point is Eq. (14). Taking the matrix element (x, y) yields

$$G_\Gamma(z; x, y) = G_\Gamma^\Lambda(z; x, y) + \langle \delta_x, G_\Gamma(z) T_\Gamma^\Lambda G_\Gamma^\Lambda(z) \delta_y \rangle.$$

Since $x \notin \Lambda$ and $y \in \Lambda$, the first summand on the right vanishes as the depleted Green's function $G_\Gamma^\Lambda(z; x, y)$ decouples x and y. For the second summand we calculate

$$G_\Gamma(z; x, y) = G_\Gamma(z; x, x + n - 1) G_\Lambda(z; x + n, y). \tag{17}$$

The second factor is independent of ω_x. Thus, taking expectation with respect to ω_x bounds the first factor using Ineq. (8) and the proof is complete. ∎

Lemma 4.2. *Let* $n \in \mathbb{N}$, $\Theta = \{0, \ldots, n - 1\}$, $\Gamma = \{x, x + 1, ...\}$, $y \in \Gamma$ *with* $n \leq y - x < 2n$, *and* $s \in (0, 1)$. *Then we have for all* $z \in \mathbb{C} \setminus \mathbb{R}$ *the bound*

$$\mathbb{E}_{\{y-n+1,x\}}\{|G_\Gamma(z; x, y)|^{s/n}\} \leq C_{u,\rho}^+ C_{u,\rho}. \tag{18}$$

Proof. The starting point is Eq. (15). Choosing $\Lambda = \{x, \ldots, y - n\}$ gives

$$G_\Gamma(z; x, y) = G_\Lambda(z; x, y - n) G_\Gamma(z; y - n + 1, y).$$

Since $G_\Lambda(z; x, y - n)$ depends only on the potential values at lattice sites in Λ it is independent of ω_{y-n+1}. We take expectation with respect to ω_{y-n+1} to bound the second factor of the above identity using Ineq. (8). Since $1 \leq |\Lambda| \leq n$ by assumption, we may apply Ineq. (9) to $G_\Lambda(z; x, y - n)$ which ends the proof. ∎

The proof of the following theorem will serve as a basis to complete the proof of

(i) Theorem 2.1 at the end of this section,
(ii) Theorem 2.2 in Section 5.

The difference between the proof of Theorem 2.1 and Theorem 4.3 is, that the latter is better suited for a generalisation to single-site potentials with disconnected support.

Theorem 4.3. *Let* $\Theta = \{0, \ldots, n - 1\}$ *and* $s \in (0, 1)$. *Assume*

$$\|\rho\|_\infty < \frac{(1 - s)^{1/s}}{2s^{-1}} \left| \prod_{k=0}^{n-1} u(k) \right|^{1/n}. \tag{19}$$

Then $m = -\ln C_{u,\rho}$ *is strictly positive and*

$$\mathbb{E}\{|G_\omega(z; x, y)|^{s/n}\} \leq C_{u,\rho}^+ \ \exp\left\{-m \left\lfloor \frac{|x - y|}{n} \right\rfloor\right\}$$

for all $x, y \in \mathbb{Z}$ *with* $|x - y| \geq 2n$ *and all* $z \in \mathbb{C} \setminus \mathbb{R}$. *Here,* $\lfloor \cdot \rfloor$ *is defined by* $\lfloor z \rfloor := \max\{k \in \mathbb{Z} | k \leq z\}$.

Proof. The constant m is larger than zero since $C_{u,\rho} < 1$ by assumption. By symmetry we assume without loss of generality $y - x \geq 2n$. In order to estimate $\mathbb{E}\{|G_\omega(z; x, y)|^{s/n}\}$, we iterate Eq. (16) of Lemma 4.1 and finally use Eq. (18) of Lemma 4.2 for the last step. Figure 1 shows this procedure schematically. We choose $p := \lfloor (y - x)/n \rfloor - 1 \in \mathbb{N}$ such that $y - 2n < x + pn \leq y - n$. We iterate Eq. (16) exactly p times, starting with $\Gamma = \mathbb{Z}$, and obtain

$$\mathbb{E}\{|G_\omega(z; x, y)|^{s/n}\} \leq C_{u,\rho}^p \cdot \mathbb{E}\{|G_{\Lambda_p}(z; x + pn, y)|^{s/n}\}$$

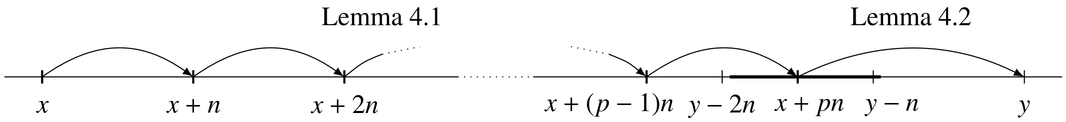

Figure 1. Illustration to the proof of Theorem 4.3

where $\Lambda_p = \{x + pn, x + pn + 1, \dots\}$. Now the first p jumps of Fig. 1 are done and it remains to estimate $\mathbb{E}\{|G_{\Lambda_p}(z; x + pn, y)|^{s/n}\}$. Since $n \le y - (x + pn) < 2n$ and $\Lambda_p = \{x + pn, x + pn + 1, \dots\}$ we may apply Lemma 4.2 and get

$$\mathbb{E}\{|G_\omega(z; x, y)|^{s/n}\} \le C_{u,\rho}^{p+1} C_{u,\rho}^{+} = C_{u,\rho}^{+}\ \mathrm{e}^{(p+1)\ln C_{u,\rho}}. \qquad \blacksquare$$

Proof of Theorem 2.1. Without loss of generality we assume $y - x \ge n$. We iterate Eq. (16) exactly $q := \lfloor (y - x)/n \rfloor \in \mathbb{N}$ times, starting with $\Gamma = \mathbb{Z}$, and obtain $\mathbb{E}\{|G_\omega(z; x, y)|^{s/n}\} \le C_{u,\rho}^{p} \cdot \mathbb{E}\{|G_{\Lambda_q}(z; x{+}qn, y)|^{s/n}\}$, where $\Lambda_q = \{x{+}pn, x{+}pn{+}1, \dots\}$. Since $0 \le y-(x{+}qn) \le n{-}1$ by construction, we may apply part (iii) of Lemma 3.3 and obtain

$$\mathbb{E}\{|G_\omega(z; x, y)|^{s/n}\} \le C_{u,\rho}^{q} C_{u,\rho,+} = C_{u,\rho,+}\ \exp\left\{-m\left\lfloor \frac{y - x}{n} \right\rfloor\right\} \tag{20}$$

where $m = -\ln C_{u,\rho}$. In particular, $m > 0$ if Ineq. (19) holds. $\blacksquare$

5. Single-site potentials with arbitrary finite support

In this section we consider the case in which the support Θ of the single-site potential is an arbitrary finite subset of $\mathbb{Z}$. By translation, we assume without loss of generality that $\min \Theta = 0$ and $\max \Theta = n - 1$ for some $n \in \mathbb{N}$. Furthermore, we define

$$r := \max\{\, b - a \mid [a, b] \subset \{0, \dots, n - 1\}, [a, b] \cap \Theta = \emptyset\}. \tag{21}$$

Thus r is the width of the largest gap in Θ. In order to handle arbitrary finite supports of the single-site potential, we need one of the following additional assumptions on the density $\rho \in L^\infty(\mathbb{R})$:

$$\mathcal{A}_1 : \rho \in W^{1,1}(\mathbb{R}) \qquad \mathcal{A}_2 : \operatorname{supp} \rho \subset [-R, R] \text{ for some } R > 0. \tag{22}$$

To illustrate the difficulties arising for non-connected supports Θ we consider an example. Suppose $\Theta = \{0, 2, 3, \dots, n - 1\}$ so that $r = 1$. If we set $\Lambda = \{0, \dots, n - 1\}$ there is no decomposition $H_\Lambda - B_\Gamma^\Lambda = A + \omega_0 V$ with an invertible V. If we set $\Lambda = \{0, \dots, n - 1 + r\} = \{0, \dots, n\}$ we observe that every diagonal element of H_Λ depends at least on one of the variables ω_0 and $\omega_1 = \omega_r$, while the elements of B_Γ^Λ (which appear after applying Lemma 3.2) are independent of ω_k, $k \in \{0, \dots, r\} = \{0, 1\}$. Thus we have a decomposition $H_\Lambda - B_\Gamma^\Lambda = A + \omega_0 V_0 + \omega_1 V_1$, where A is independent of ω_k, $k \in \{0, 1\}$, and for all $i \in \Lambda$ either $V_0(i)$ or $V_1(i)$ is not zero. As a consequence there is an $\alpha \in \mathbb{R}$ such that $V_0 + \alpha V_1$ is invertible on $\ell^2(\Lambda)$. Motivated by this observation, we prove the following lemma.

Lemma 5.1. *Let $N, d \in \mathbb{N}$ and $A, V_0, V_1, \dots, V_N \in \mathbb{C}^{d\times d}$ be matrices. Let $(\alpha_k)_{k=0}^N \in \mathbb{R}^{N+1}$ with $\alpha_0 \neq 0$. Assume that $\sum_{k=0}^N \alpha_k V_k$ is invertible. Let further $0 \le \rho \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with*

$\|\rho\|_{L^1} = 1$, $t \in (0,1)$*, and* $\mathcal{A}_1, \mathcal{A}_2$ *be as in* (22)*. Then, if the condition* $\mathcal{A}_1$ *is satisfied, we have the bound*

$$I := \int_{\mathbb{R}^{N+1}} \Bigl|\det\Bigl(A + \sum_{i=0}^N r_i V_i\Bigr)\Bigr|^{t/d} \prod_{i=0}^N \rho(r_i)\mathrm{d}r_i \leq \Bigl|\det\Bigl(\sum_{k=0}^N \alpha_k V_k\Bigr)\Bigr|^{-t/d} \Bigl(\sum_{k=0}^N |\alpha_k|\Bigr)^t \ \frac{t^{-t}}{1-t} \|\rho'\|_{L^1}^t .$$

If the condition $\mathcal{A}_2$ *is satisfied, we have the bound*

$$I \leq \Bigl|\det\Bigl(\sum_{k=0}^N \alpha_k V_k\Bigr)\Bigr|^{-t/d} |\alpha_0|^t \Bigl(1 + \max_{i\in\{1,\ldots,N\}} \frac{|\alpha_i|}{|\alpha_0|}\Bigr)^{Nt} \ \frac{2^t t^{-t}}{1-t} (2R)^{Nt} \|\rho\|_\infty^{(N+1)t} .$$

Proof. Substituting

$$\begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ \vdots \\ r_N \end{pmatrix} = T \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} \alpha_0 & 0 & \cdots & \cdots & 0 \\ \alpha_1 & \alpha_0 & 0 & & \vdots \\ \alpha_2 & 0 & \alpha_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \alpha_0 & 0 \\ \alpha_N & 0 & \ldots & 0 & \alpha_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} \alpha_0 x_0 \\ \alpha_1 x_0 + \alpha_0 x_1 \\ \alpha_2 x_0 + \alpha_0 x_2 \\ \vdots \\ \alpha_N x_0 + \alpha_0 x_N \end{pmatrix}$$

we get

$$I = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}} \Bigl|\det\Bigl(\tilde{A} + x_0 \sum_{i=0}^N \alpha_i V_i\Bigr)\Bigr|^{-t/d} g(x_0,\ldots,x_N)\mathrm{d}x_0 \right) |\alpha_0|^{N+1} \mathrm{d}x_1 \ldots \mathrm{d}x_N$$

where $\tilde{A} = A + \alpha_0 \sum_{i=1}^N x_i V_i$ and $g(x_0,\ldots,x_N) = \rho(\alpha_0 x_0) \prod_{i=1}^N \rho(\alpha_i x_0 + \alpha_0 x_i)$. Since $x_0 \mapsto g(x_0,\ldots,x_N)$ is an element of $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ we may apply Lemma 3.1 and obtain for all $\lambda > 0$

$$\begin{aligned} I &\leq \Bigl|\det\Bigl(\sum_{i=0}^N \alpha_i V_i\Bigr)\Bigr|^{-t/d} \int_{\mathbb{R}^N} \Bigl(\lambda^{-t} \int_{\mathbb{R}} g(x_0,\ldots,x_N)\mathrm{d}x_0 + \frac{2\lambda^{1-t}}{1-t} \sup_{x_0\in\mathbb{R}} g(x_0,\ldots,x_N)\Bigr) |\alpha_0|^{N+1}\mathrm{d}x \\ &= \Bigl|\det\Bigl(\sum_{i=0}^N \alpha_i V_i\Bigr)\Bigr|^{-t/d} \Bigl(\lambda^{-t} + \frac{2\lambda^{1-t}}{1-t} \int_{\mathbb{R}^N} \sup_{x_0\in\mathbb{R}} g(x_0,\ldots,x_N) |\alpha_0|^{N+1} \mathrm{d}x_1 \ldots \mathrm{d}x_N\Bigr) \end{aligned}$$

where $\mathrm{d}x = \mathrm{d}x_1 \ldots \mathrm{d}x_N$. In the case of $\mathcal{A}_1$ we use $\sup_{x_0\in\mathbb{R}} g \leq \frac{1}{2}\int_{\mathbb{R}} |\partial g/\partial x_0| \mathrm{d}x_0$, substitute back into the original coordinates and finally choose $\lambda = t/(\|\rho'\|_{L^1} \sum_{k=0}^N |\alpha_k|)$. To end the proof if the condition $\mathcal{A}_2$ is satisfied, we use $\operatorname{supp}\rho \subset [-R,R]$ and see that if $|x_j| > R\,\|T^{-1}\|_\infty$ for some $j = 0,\ldots,N$, then $g(x_0,\ldots,x_N) = 0$. Thus it is sufficient to integrate over the cube $[-R\|T^{-1}\|_\infty, R\|T^{-1}\|_\infty]^N$. We estimate $\sup_{x_0\in\mathbb{R}} g(x_0,\ldots,x_N) \leq \|\rho\|_\infty^{N+1}$ and choose $\lambda = t/(2\|\rho\|_\infty^{N+1} |\alpha_0^{N+1}| (2R\|T_\infty^{-1}\|)^N)$. The row-sum norm of T^{-1} equals $\|T^{-1}\|_\infty = \max_{i\in\{1,\ldots,N\}} (|\alpha_0|^{-1} + |\alpha_i/\alpha_0^2|) = (1 + \max_{i\in\{1,\ldots,N\}} |\alpha_i/\alpha_0|)/|\alpha_0|$. ∎

With the help of Lemma 5.1 we prove the following analogues of Lemma 3.3 and Theorem 4.3.

Lemma 5.2. *Let* $n \in \mathbb{N}$, $\Theta \subset \mathbb{Z}$ *with* $\min\Theta = 0$, $\max\Theta = n-1$*, and* $\Gamma \subset \mathbb{Z}$ *be connected. Let further* r *be as in Eq.* (21)*,* $\mathcal{A}_1, \mathcal{A}_2$ *as in* (22)*, and* $s \in (0,1)$*. Then there exists a* *constant D such that for all $x, x+n-1+r \in \Gamma$ and $z \in \mathbb{C} \setminus \mathbb{R}$*

$$\mathbb{E}_{\{x,\dots,x+r\}}\{|G_\Gamma(z; x, x+n-1+r)|^{s/(n+r)}\} \le D\,. \tag{23}$$

The constant D is characterised in Eq. (25) *and estimated in Ineq.* (27). *If $1 \le |\Gamma| \le n+r$ with $\gamma_0 = \min \Gamma$ and $\gamma_1 = \max \Gamma$ there exists a constant D^+ such that for all $z \in \mathbb{C} \setminus \mathbb{R}$*

$$\mathbb{E}_{\{\gamma_0,\dots,\gamma_0+r\}}\{|G_\Gamma(z; \gamma_0, \gamma_1)|^{s/(n+r)}\} \le D^+. \tag{24}$$

The constant D^+ is characterised in Eq. (28) *and estimated in Ineq.* (29).

Proof. The proof is similar to the proof of Lemma 3.3. Apply Lemma 3.2 with $\Lambda = \{x, x+1, \dots, x+n-1+r\}$ and Cramer's rule to get $|G_\Gamma(z; x, x+n-1+r)| = 1/\,|\det[D]|$ where $D = H_\Lambda - B_\Gamma^\Lambda - z$. Note that B_Γ^Λ is independent of ω_k, $k \in \{x, \dots, x+r\}$. We have the decomposition $[D] = A + \sum_{k=0}^r \omega_{x+k} V_k$ where the elements of the diagonal matrices $V_k \in \mathbb{R}^{(n+r)\times(n+r)}$, $k = 0, \dots, r$, are given by $V_k(i) = u(i-k)$, $i = 0, \dots, n-1+r$, and $A = D - \sum_{k=0}^r \omega_k V_k$ is independent of ω_k, $k \in \{x, \dots, x+r\}$. We apply Lemma 5.1 and obtain for all $\alpha = (\alpha_k)_{k=0}^r \in M := \{\alpha \in \mathbb{R}^{r+1} : \alpha_0 \ne 0,\ \sum_{k=0}^r \alpha_k V_k \text{ is invertible}\}$ the bound $\mathbb{E}_{\{x,\dots,x+r\}}\{|G_\Gamma(z; x, x+n-1+r)|^{s/(n+r)}\} \le D_\alpha$ where

$$D_\alpha = \|\rho'\|_{L^1}^s \frac{s^{-s}}{1-s} \Big(\sum_{k=0}^r |\alpha_k|\Big)^s \prod_{i=0}^{n-1+r} \Big|\sum_{k=0}^r \alpha_k u(i-k)\Big|^{-s/(n+r)}$$

if $\mathcal{A}_1$ is satisfied and

$$D_\alpha = \|\rho\|_\infty^{(r+1)s} (2R)^{rs} \frac{2^s s^{-s}}{1-s} |\alpha_0|^s \left(1 + \max_{i\in\{1,\dots,r\}} \frac{|\alpha_i|}{|\alpha_0|}\right)^{rs} \prod_{i=0}^{n-1+r} \Big|\sum_{k=0}^r \alpha_k u(i-k)\Big|^{-s/(n+r)}$$

if $\mathcal{A}_2$ is satisfied. The set M is non-empty and equal to the set $\{\alpha \in \mathbb{R}^{r+1} : \alpha_0 \ne 0, D_\alpha \text{ is finite}\}$. Thus Ineq. (23) holds with the constant

$$D := \inf_{\alpha\in M} D_\alpha. \tag{25}$$

In the following we establish an upper bound for D. Using a volume comparison criterion we can find a vector $\alpha' = (\alpha'_k)_{k=0}^r \in [0,1]^{r+1}$ which has to each hyperplane $\sum_{k=0}^r \alpha_k u(i-k) = 0$, $i = 0, \dots, n-1+r$, at least the Euclidean distance $(2(n+r)(r+1)^{r/2})^{-1}$, as outlined in Fig. 2. This implies $\alpha'_0 \ge (2(n+r)(r+1)^{r/2})^{-1}$ since the hyperplane for $i = 0$ is $\alpha_0 = 0$. With this choice of α and the notation $u_i = (u(i-k))_{k=0}^r$, $i \in \{0, \dots, n-1+r\}$, we have

$$\prod_{i=0}^{n-1+r} \Big|\sum_{k=0}^r \alpha'_k u(i-k)\Big|^{-\frac{s}{n+r}} = \prod_{i=0}^{n-1+r} \Big|\|u_i\|\, \langle \alpha', u_i/\|u_i\| \rangle_2\Big|^{-\frac{s}{n+r}} \le \frac{[2(n+r)(r+1)^{r/2}]^s}{\Big|\prod_{i=0}^{n-1+r}\Big(\sum_{k=0}^r u(i-k)^2\Big)\Big|^{\frac{s}{2(n+r)}}} \tag{26}$$

where $\langle\cdot,\cdot\rangle_2$ denotes the standard Euclidian scalar product. Now, in both cases $\mathcal{A}_1$ and $\mathcal{A}_2$ we choose $\alpha = \alpha'$ and obtain

$$D \le \|\rho'\|_{L^1}^s \frac{s^{-s}}{1-s} \frac{(r+1)^s [2(n+r)(r+1)^{r/2}]^s}{\Big|\prod_{i=0}^{n-1+r}\Big(\sum_{k=0}^r u(i-k)^2\Big)\Big|^{\frac{s}{2(n+r)}}} \tag{27a}$$

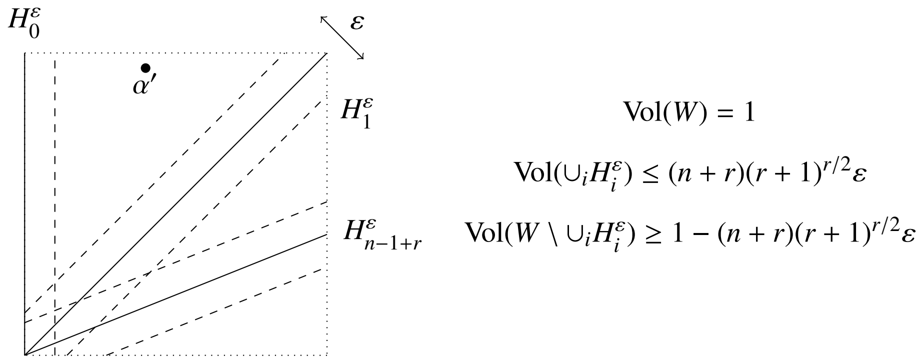

FIGURE 2. Sketch of the existence of a vector $\alpha' \in W = [0,1]^{r+1}$ with the desired properties: Let H_i^ε denote the ε-neighbourhood of the hyperplane $H_i := \{\alpha \in W \mid \sum_{k=0}^r \alpha_k u(i-k) = 0\}$ for $i \in \{0,\dots,n-1+r\}$. Since the volume of $W \setminus \cup_i H_i^\varepsilon$ is positive if ε is smaller than $(n+r)^{-1}(r+1)^{-r/2} = d_0$, we conclude (using continuity) that there is a vector α' whose distance to each hyperplane H_i, $i \in \{0,\dots,n-1+r\}$, is at least $d_0/2$.

if $\mathcal{A}_1$ is satisfied and

$$D \leq \lVert\rho\rVert_\infty^{(r+1)s}(2R)^{rs}\frac{2^s s^{-s}}{1-s}(1+2(n+r)(r+1)^{r/2})^{rs}\frac{[2(n+r)(r+1)^{r/2}]^s}{\left|\prod_{i=0}^{n-1+r}\left(\sum_{k=0}^r u(i-k)^2\right)\right|^{\frac{s}{2(n+r)}}} \tag{27b}$$

if $\mathcal{A}_2$ is satisfied.

The proof of the second statement is similar but without use of Lemma 3.2. By Cramer's rule we get $|G_\Gamma(z;\gamma_0,\gamma_1)| = 1/|\det[H_\Gamma - z]|$. Set $d = \gamma_1 - \gamma_0$. We have the decomposition $[H_\Gamma - z] = \tilde{A} + \sum_{k=0}^r \omega_{\gamma_0+k}\tilde{V}_k$, where the elements of the diagonal matrices $\tilde{V}_k \in \mathbb{R}^{(d+1)\times(d+1)}$, $k = 0,\dots,r$, are given by $\tilde{V}_k(i) = u(i-k)$, $i \in \{0,\dots,d\}$, and $\tilde{A} := [H_\Gamma - z] - \sum_{k=0}^r \omega_k \tilde{V}_k$ is independent of ω_k, $k \in \{x,\dots,x+r\}$. We apply Lemma 5.1 with $t = s\frac{d+1}{n+r}$ and obtain (using $s \geq t$) for all $\alpha = (\alpha_k)_{k=0}^r \in \tilde{M} := \{\alpha \in \mathbb{R}^{r+1} : \alpha_0 \neq 0,\ \sum_{k=0}^r \alpha_k \tilde{V}_k \text{ is invertible}\}$ that $\mathbb{E}_{\{\gamma_0,\dots,\gamma_0+r\}}\{|G_\Gamma(z;\gamma_0,\gamma_1)|^{s/(n+r)}\} \leq D_\alpha^+(d)$ where

$$D_\alpha^+(d) = \lVert\rho'\rVert_{L^1}^{s\frac{d+1}{n+r}}\frac{s^{-s}}{1-s}\Big(\sum_{k=0}^r |\alpha_k|\Big)^{s\frac{d+1}{n+r}}\prod_{i=0}^d\Big|\sum_{k=0}^r \alpha_k u(i-k)\Big|^{-s/(n+r)}$$

if $\mathcal{A}_1$ is satisfied and

$$D_\alpha^+(d) = \lVert\rho\rVert_\infty^{s\frac{(r+1)(d+1)}{n+r}}(2R)^{s\frac{r(d+1)}{n+r}}\frac{2^s s^{-s}}{1-s}|\alpha_0|^{s\frac{d+1}{n+r}}\left(1+\max_{i\in\{1,\dots,r\}}\frac{|\alpha_i|}{|\alpha_0|}\right)^{sr}\prod_{i=0}^d\Big|\sum_{k=0}^r \alpha_k u(i-k)\Big|^{\frac{s}{n+r}}$$

if $\mathcal{A}_2$ is satisfied. Since $\tilde{M} \supset M$ for each $d \in 0,\dots n-1+r$ the set $\tilde{M}$ is non-empty. Thus Ineq. (24) holds with the constant

$$D^+ := \max_{d\in\{0,\dots,n-1+r\}}\ \inf_{\alpha\in\tilde{M}} D_\alpha^+(d). \tag{28}$$

We again choose $\alpha = \alpha'$ as in Fig. 2, use $\alpha'_k \in [0, 1]$ and $\alpha'_0 \geq (2(n+r)(r+1)^{r/2})^{-1}$, estimate $D^+_{\alpha'}(d)$ similar to Ineq. (26), and obtain

$$D^+ \leq \max_{d\in\{0,\dots,n-1+r\}} \left\{ \|\rho'\|_{L^1}^{s\frac{d+1}{n+r}} \frac{s^{-s}}{1-s} \frac{(r+1)^s[2(d+1)(r+1)^{r/2}]^s}{\left|\prod_{i=0}^d \sum_{k=0}^r u(i-k)^2\right|^{s/(2(n+r))}} \right\} \tag{29a}$$

if $\mathcal{A}_1$ is satisfied and

$$D^+ \leq \max_{d\in\{0,\dots,n-1+r\}} \left\{ \|\rho\|_\infty^{s\frac{(r+1)(d+1)}{n+r}} (2R)^{s\frac{r(d+1)}{n+r}} \frac{[1+2(d+1)(r+1)^{r/2}]^{sr}[2(d+1)(r+1)^{r/2}]^s}{2^{-s}s^s(1-s)\left|\prod_{i=0}^d \sum_{k=0}^r u(i-k)\right|^{\frac{s}{2(n+r)}}} \right\} \tag{29b}$$

if $\mathcal{A}_2$ is satisfied. ■

Theorem 5.3. *Let $n \in \mathbb{N}$, $\Theta \subset \mathbb{Z}$, $\min\Theta = 0$, $\max\Theta = n-1$, $s \in (0,1)$, r as in Eq. (21), D the constant from Lemma 5.2, and let ρ satisfy one of the assumptions $\mathcal{A}_1$ or $\mathcal{A}_2$ from (22). Assume $D < 1$. Then $m = -\ln D$ is strictly positive and we have the bound*

$$\mathbb{E}\{|G_\omega(z; x, y)|^{s/(n+r)}\} \leq D^+ \ \mathrm{e}^{-m\left\lfloor \frac{|x-y|}{n+r} \right\rfloor}$$

for all $x, y \in \mathbb{Z}$ with $|x-y| \geq 2(n+r)$ and all $z \in \mathbb{C}\setminus\mathbb{R}$, where $\lfloor\cdot\rfloor$ is defined by $\lfloor z\rfloor := \max\{k \in \mathbb{Z} | k \leq z\}$.

Proof. The proof is similar to the proof of Theorem 4.3. We again assume $y > x$. Let $\Gamma_1 \subset \mathbb{Z}$ be connected. Using Eq. (14) with $\Lambda := \{x+n+r, \dots\} \cap \Gamma_1$ and Lemma 5.2 we have for all pairs $x, y \in \Gamma_1$ with $y - x \geq n + r$

$$\mathbb{E}\{|G_{\Gamma_1}(z; x, y)|^{s/(n+r)}\} \leq D\ \mathbb{E}\{|G_\Lambda(z; x+n+r, y)|^{s/(n+r)}\} \tag{30}$$

which is the analogue to Lemma 4.1. Now, let $\Gamma_2 = \{x, x+1, \dots\}$ and $y \in \Gamma_2$ with $n + r \leq y - x < 2(n+r)$. By Eq. (15) with $\Lambda = \{x, \dots, y-(n+r)\}$ and Lemma 5.2 we have

$$\mathbb{E}\{|G_{\Gamma_2}(z; x, y)|^{s/(n+r)}\} \leq DD^+ \tag{31}$$

which is the analogue of Lemma 4.2. Iterating Eq. (30) exactly $p = \lfloor (y-x)/(n+r)\rfloor - 1$ times, starting with $\Gamma_1 = \mathbb{Z}$, and finally using Eq. (31) once gives the statement of the theorem. ■

6. Appendix

Here we prove and discuss the two results which have been stated in Remark 2.4. In the appendix we assume throughout that assumption $\mathcal{A}_2$ holds, i. e. there is an $R \in (0, \infty)$ such that $\operatorname{supp}\rho \subset [-R, R]$. We use the notation $u_j(x) = u(x-j)$, for all $j, x \in \mathbb{Z}$, for the translated function as well as for the corresponding multiplication operator.

The following theorem concerns the first part of Remark 2.4. It gives a global uniform bound on $(x, y) \mapsto \mathbb{E}\{|G_\omega(z; x, y)|^s\}$ for $s > 0$ sufficiently small.

Theorem 6.1. *Let $s \in (0,1)$, $\Theta \subset \mathbb{Z}$ with $\min\Theta = 0$ and $\max\Theta = n-1$ for some $n \in \mathbb{N}$, and $\operatorname{supp}\rho$ be compact. Then there is a positive constant C such that for all $x, y \in \mathbb{Z}$ and all $z \in \mathbb{C}\setminus\mathbb{R}$ we have*

$$\mathbb{E}\{|G_\omega(z; x, y)|^{s/(4n)}\} \leq C.$$

For the proof we will need

Lemma 6.2. *Let $n \in \mathbb{N}$, $R \in \mathbb{R}$, $A \in \mathbb{C}^{n\times n}$ an arbitrary matrix, $V \in \mathbb{C}^{n\times n}$ an invertible matrix and $s \in (0,1)$. Then we have the bounds*

$$\|V^{-1}\| \leq \frac{\|V\|^{n-1}}{|\det V|} \tag{32}$$

and

$$\int_{-R}^{R} \left\|(A+rV)^{-1}\right\|^{s/n} \mathrm{d}r \leq \frac{2R^{1-s}(\|A\| + R\|V\|)^{s(n-1)/n}}{s^s(1-s)\,|\det V|^{s/n}}. \tag{33}$$

Proof. To prove Ineq. (32) let $0 < s_1 \leq s_2 \leq \ldots \leq s_n$ be the singular values of V. Then we have $\prod_{i=1}^n s_i \leq s_1 s_n^{n-1}$, that is,

$$\frac{1}{s_1} \leq \frac{s_n^{n-1}}{\prod_{i=1}^n s_i}. \tag{34}$$

For the norm we have $\|V^{-1}\| = 1/s_1$ and $\|V\| = s_n$. For the determinant of V there holds $|\det V| = \prod_{i=1}^n s_i$. Hence, Ineq. (32) follows from Ineq. (34). To prove Ineq. (33) recall that, since V is invertible, the set $\{r \in \mathbb{R} \colon A + rV \text{ is singular}\}$ is a discrete set. Thus, for almost all $r \in [-R, R]$ we may apply Ineq. (32) to the matrix $A + rV$ and obtain

$$\left\|(A+rV)^{-1}\right\|^{s/n} \leq \frac{(\|A\| + R\|V\|)^{s(n-1)/n}}{|\det(A+rV)|^{s/n}}.$$

Inequality (33) now follows from Lemma 3.1. ∎

Proof of Theorem 6.1. Since $\operatorname{supp}\rho \subset [-R,R]$, H_ω is a bounded operator. Set $m = \|H_\omega\|+1$. If $|z| \geq m$, we use $\|G_\omega(z)\| = \sup_{\lambda\in\sigma(H_\omega)} |\lambda - z|^{-1} \leq 1$ and obtain the statement of the theorem. Thus it is sufficient to consider $|z| \leq m$. If $|x-y| \geq 4n$ Theorem 5.2 applies, since $r \leq n$. We thus only consider the case $|x-y| \leq 4n-1$. By translation we assume $x = 0$ and by symmetry $y \geq 0$. Set $\Lambda_+ = \{-1, \ldots, 4n\}$ and $\Lambda = \{0, \ldots, 4n-1\}$. Lemma 3.2 gives

$$P_{\Lambda_+} G_\omega(z) P_{\Lambda_+}^* = (H_{\Lambda_+} - B_{\mathbb{Z}}^{\Lambda_+} - z)^{-1}$$

where $\langle \delta_x, B_\Gamma^{\Lambda_+} \delta_y\rangle = \sum_{k\in\Gamma\setminus\Lambda_+, |k-x|=1} \langle \delta_k, (H_{\Gamma\setminus\Lambda_+} - z)^{-1}\delta_k\rangle$ if $x = y$ and $x \in \partial\Lambda_+ = \{-1, 4n\}$, and zero else. Similarly, by another application of the Schur complement formula

$$P_\Lambda (H_{\Lambda_+} - B_{\mathbb{Z}}^{\Lambda_+} - z)^{-1} P_\Lambda^* = \left(H_\Lambda - z - P_\Lambda \Delta P_{\partial\Lambda_+}^* \left(P_{\partial\Lambda_+}^{\Lambda_+}(H_{\Lambda_+} - B_{\mathbb{Z}}^{\Lambda_+})(P_{\partial\Lambda_+}^{\Lambda})^* - z\right)^{-1} P_{\partial\Lambda_+} \Delta P_\Lambda^*\right)^{-1},$$

and consequently

$$P_\Lambda G_\omega(z) P_\Lambda^* = \left(H_\Lambda - z - P_\Lambda \Delta P_{\partial\Lambda_+}^* (K-z)^{-1} P_{\partial\Lambda_+} \Delta P_\Lambda^*\right)^{-1} \tag{35}$$

where

$$K = P_{\partial\Lambda_+}^{\Lambda_+}(H_{\Lambda_+} - B_{\mathbb{Z}}^{\Lambda_+})(P_{\partial\Lambda_+}^{\Lambda_+})^*.$$

Note that $B_{\mathbb{Z}}^{\Lambda_+}$ is independent of ω_k, $k \in \{-1, \ldots, 3n+1\}$, and K is independent of ω_k, $k \in \{0, \ldots, 3n\}$. Thus, in matrix representation with respect to the canonical basis, the operator $K : \ell^2(\partial\Lambda_+) \to \ell^2(\partial\Lambda_+)$ may be decomposed as

$$[K] = \begin{pmatrix} \omega_{-1} u(0) & 0 \\ 0 & \omega_{3n+1} u(n-1) \end{pmatrix} - \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$$

where $f_1 := \sum_{k\in\mathbb{Z}\setminus\{-1\}} \omega_k u(-1-k) - \langle \delta_{-1} B_{\mathbb{Z}}^{\Lambda_+} \delta_{-1}\rangle$ and $f_2 := \sum_{k\in\mathbb{Z}\setminus\{3n+1\}} \omega_k u(4n-k) - \langle \delta_{4n} B_{\mathbb{Z}}^{\Lambda_+} \delta_{4n}\rangle$ are independent of ω_{-1} and ω_{3n+1}. Standard spectral averaging or Lemma 3.1 gives for all $t \in (0,1)$

$$\mathbb{E}_{\{-1,3n+1\}}\Big\{\big\|(K-z)^{-1}\big\|^t\Big\} \leq (|u(0)|^{-t} + |u(n-1)|^{-t})\, \|\rho\|_\infty^t \, \frac{2^t t^{-t}}{1-t}. \tag{36}$$

Now, the operator H_Λ can be decomposed as $H_\Lambda = A + \sum_{k=0}^{3n} \omega_k u_k$ where $A := H_\Lambda - \sum_{k=0}^{3n} \omega_k u_k$ is independent of ω_k, $k \in \{0,\dots,3n\}$. Let $\alpha := (\alpha_k)_{k=0}^{3n} \in [0,1]^{3n+1}$ with $\alpha_0 \neq 0$. Similarly to the proof of Lemma 5.2, we use the substitution $\omega_0 = \alpha_0\zeta_0$ and $\omega_i = \alpha_i\zeta_0 + \alpha_0\zeta_i$ for $i \in \{1,\dots,3n\}$ and obtain from Eq. (35)

$$\begin{aligned}
E &:= \mathbb{E}_{\{0,\dots,3n\}}\Big\{\big\|P_\Lambda G_\omega(z) P_\Lambda^*\big\|^{s/(4n)}\Big\} \\
&\leq \|\rho\|_\infty^{3n+1} \int\limits_{[-R,R]^{3n+1}} \Big\|\Big(A + \sum_{k=0}^{3n} \omega_k u_k - z - P_\Lambda \Delta P_{\partial\Lambda_+}^* (K-z)^{-1} P_{\partial\Lambda_+}^{\Lambda} \Delta P_\Lambda^*\Big)^{-1}\Big\|^{s/(4n)} \mathrm{d}\omega_0 \dots \mathrm{d}\omega_{3n} \\
&\leq \|\rho\|_\infty^{3n+1} \int_{[-S,S]^{3n+1}} \Big\|\Big(A' + \zeta_0 \sum_{k=0}^{3n} \alpha_k u_k\Big)^{-1}\Big\|^{s/(4n)} |\alpha_0|^{3n+r} \mathrm{d}\zeta_0 \dots \mathrm{d}\zeta_{3n}
\end{aligned}$$

where $A' = A + \alpha_0 \sum_{k=1}^{3n} \zeta_k u_k - z - P_\Lambda \Delta P_{\partial\Lambda_+}^* (K-z)^{-1} P_{\partial\Lambda_+} \Delta P_\Lambda^*$ and $S = R(1 + \max_{i\in\{1,\dots,3n\}} |\alpha_i/\alpha_0|)/|\alpha_0|$. Since $\bigcup_{i=0}^{3n} \operatorname{supp} u_i = \Lambda$, there exists an $\alpha \in [0,1]^{3n+1}$ such that $\sum_{k=0}^{3n} \alpha_k u_k$ is invertible on $\ell^2(\Lambda)$, compare the proof of Lemma 5.2 and Figure 2. Thus we may apply Lemma 6.2 and obtain

$$E \leq \|\rho\|_\infty^{3n+1} \int_{[-S,S]^{3n}} \frac{2 s^{-s} S^{1-s}}{1-s} \frac{\Big(\|A'\| + S\, \big\|\sum_{k=0}^{3n} \alpha_k u_k\big\|\Big)^{s(4n-1)/(4n)}}{\big|\det(\sum_{k=0}^{3n} \alpha_k u_k)\big|^{s/(4n)}} \mathrm{d}\zeta_1 \dots \mathrm{d}\zeta_{3n} \tag{37}$$

Using $\zeta_k \in [-S,S]$ for $k \in \{1,\dots,3n\}$, $\omega_k \in [-R,R]$ for $k \in \mathbb{Z}\setminus\{0,\dots,3n\}$ and $\alpha_k \in [0,1]$ for $k \in \{0,\dots,3n\}$, the norm of A' can be estimated as

$$\begin{aligned}
\|A'\| &= \Big\|H_\Lambda - \sum_{k=0}^{3n} \omega_k u_k + \alpha_0 \sum_{k=1}^{3n} \zeta_k u_k - z - P_\Lambda \Delta P_{\partial\Lambda_+}^* (K-z)^{-1} P_{\partial\Lambda_+} \Delta P_\Lambda^*\Big\| \\
&\leq 2 + (n-1)R\, \|u\|_\infty + 3Sn\|u\|_\infty + m + 4\big\|(K-z)^{-1}\big\|.
\end{aligned} \tag{38}$$

All terms in the sum (38) are independent of ζ_k, $k \in \{0,\dots,3n\}$. Using $(\sum |a_i|)^t \leq \sum |a_i|^t$ for $t < 1$ we see from Ineq. (37) and (38) that there are constants C_1 and C_2 such that $E \leq C_1 + C_2\|(K-z)^{-1}\|^{s(4n-1)/(4n)}$. If we average over ω_{-1} and ω_{3n+1}, Ineq. (36) gives the desired result. ■

Next we turn to the second part of Remark 2.4. First we discuss a criterion which ensures that an appropriate one-parameter family of positive potentials can be extracted from the random potential V_ω.

Lemma 6.3. *Let $u = \sum_{k=0}^{n-1} u(k)\delta_k \colon \mathbb{Z} \to \mathbb{R}$. Then the following statements are equivalent.*

(A) *There exists an $N \in \mathbb{N}$ and real $\lambda_0,\dots,\lambda_N$ such that $w := u * \lambda := \lambda_0 u_0 + \dots + \lambda_N u_N$ is a non-negative function and $w(0) > 0$, $w(N+n-1) > 0$ hold.*

(B) *There exists an* $M \in \mathbb{N}$ *and real* $\gamma_0, \dots, \gamma_M$ *such that* $v := u * \gamma := \gamma_0 u_0 + \dots + \gamma_M u_M$ *is a non-negative function and* $\operatorname{supp} v = \{0, \dots, M + n - 1\}$ *holds.*

(C) *The polynomial* $\mathbb{C} \ni z \mapsto p_u(z) := \sum_{k=0}^{n-1} u(k) z^k$ *has no roots in* $[0, \infty)$.

Note, if $u(0) \neq 0$ and $u(n-1) \neq 0$, then $\{0, \dots, M + n - 1\}$ is the union of the supports of $u_0, \dots, u_M$. If (A) or (B) hold we may assume that $|\lambda_0|$, respectively $|\gamma_0|$, equals one.

Proof. If (A) holds, one may choose $v(x) = \sum_{j=0}^{N+n-2} w(x - j)$ to conclude (B). Thus it is sufficient to show (B)⇔(C). Using Fourier transform and the identity theorem for holomorphic functions one sees that (B) is equivalent to

(D) There exists an $M \in \mathbb{N}$ and real $\gamma_0, \dots, \gamma_M$ such that all coefficients of the polynomial $p_u(z) \cdot \sum_{j=0}^M \gamma_j z^j$ are strictly positive.

If (D) holds, $p_u(x) \cdot \sum_{j=0}^M \gamma_j x^j$ is strictly positive for $x \in [0, \infty)$. Thus its divisor p_u has no root in $[0, \infty)$ and one concludes (C). Assuming (C), one infers from Corrolary 2.7 of [MS69] that there exists a polynomial p such that $p_u \cdot p$ has strictly positive coefficients. Choosing $M = \deg(p)$ and $\gamma_0, \dots, \gamma_M$ to be the coefficients of p leads to (D). ∎

If the random potential V_ω contains a positive building block w as in (A) of the previous lemma, one obtains Theorems 2.2 with [AEN+06], as we outline now. The crucial tool is Proposition 3.2 of [AEN+06]. Here are two direct consequences of the latter:

Lemma 6.4. *Let* H *be bounded, selfadjoint on* $\ell^2(\mathbb{Z})$, $\phi, \psi : \mathbb{Z} \to [0, \infty)$ *bounded,* $z \in \mathbb{C}$ *with* $\operatorname{Im} z > 0$, *and* $t, S \in (0, \infty)$. *Then there is a universal constant* $C_W \in (0, \infty)$ *such that for all* $x, y \in \mathbb{Z}$

(i) $$\sqrt{\phi(x)\psi(y)}\, \mathcal{L}\{v_1, v_2 \in [-S, S] : |\langle \delta_x, (H + z - v_1\phi - v_2\psi)^{-1}\delta_y\rangle| > t\} \leq 4C_W \frac{S}{t}$$

where $\mathcal{L}$ *denotes Lebesgue measure.*

(ii) *If* $\phi(x)\psi(y) \neq 0$ *and* $s \in (0, 1)$*:*
$$\int_{[-S,S]^2} |\langle \delta_x, (H + z - v_1\phi - v_2\psi)^{-1}\delta_y\rangle|^s \mathrm{d}v_1 \mathrm{d}v_2 \leq \frac{4}{1-s}\left(\frac{C_W}{\sqrt{\phi(x)\psi(y)}}\right)^s S^{2-s}.$$

To obtain statement (ii) from (i) use the layer cake representation
$$\int_{[-S,S]^2} |f(v_1, v_2)|^s \mathrm{d}v_1 \mathrm{d}v_2 = \int_0^\infty \mathcal{L}\{|v_1|, |v_2| \leq S : |f(v_1, v_2)|^s > t\}\mathrm{d}t$$
and decompose the integration domain into $[0, \kappa]$ and $[\kappa, \infty)$ where $\kappa = (C_w / S\sqrt{\phi(x)\psi(y)})^s$.

Proposition 6.5. *Let* $\Gamma \subset \mathbb{N}$ *be connected,* $\Theta \subset \mathbb{N}$ *with* $\min \Theta = 0$ *and* $\max \Theta = n - 1$ *for some* $n \in \mathbb{N}$. *Assume that* u *satisfies condition (A) in Lemma 6.3 and that* $\operatorname{supp} \rho$ *is compact. Set* $\Lambda_x = \{x, \dots, x + N\}$ *and* $\Lambda_j = \{j - n + 1 - N, \dots, j - n + 1\}$. *Then we have for all* $x, j \in \Gamma$ *with* $|j - x| \geq 2(N + n) - 1$ *and all* $z \in \mathbb{C}$ *with* $\operatorname{Im} z > 0$
$$\mathbb{E}_\Lambda\{|G_\Gamma(z; x, j)|^s\} \leq C$$
where C *is defined in Eq.* (39) *and* $\Lambda = \Lambda_x \cup \Lambda_j$.

Proof. Without loss of generality we assume $j-x \geq 2(N+n)-1$ and $\lambda_0 = 1$. By assumption $\Gamma \supset \{x, x+1, \dots, j\}$. Note that the operator $A' := H_\Gamma - z - \sum_{k\in\Lambda_x}\omega_k u_k - \sum_{k\in\Lambda_j}\omega_k u_k$ is independent of ω_k, $k \in \Lambda$. To estimate the expectation

$$E := \mathbb{E}_\Lambda\Big\{\big|G_\Gamma(z;x,j)\big|^s\Big\} = \int_{[-R,R]^{|\Lambda|}} \Big|\Big\langle \delta_x, \Big(A' + \sum_{k\in\Lambda_x}\omega_k u_k + \sum_{k\in\Lambda_j}\omega_k u_k\Big)^{-1}\delta_j\Big\rangle\Big|^s \prod_{k\in\Lambda}\rho(\omega_k)\mathrm{d}\omega_k$$

we use the substitutions

$$\begin{pmatrix}\omega_x\\ \omega_{x+1}\\ \vdots\\ \vdots\\ \omega_{x+N}\end{pmatrix} = T\begin{pmatrix}\zeta_x\\ \zeta_{x+1}\\ \vdots\\ \vdots\\ \zeta_{x+N}\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}\omega_{j-n+1-N}\\ \omega_{j-n+2-N}\\ \vdots\\ \vdots\\ \omega_{j-n+1}\end{pmatrix} = T\begin{pmatrix}\zeta_{j-n+1-N}\\ \zeta_{j-n+2-N}\\ \vdots\\ \vdots\\ \zeta_{j-n+1}\end{pmatrix}$$

where the matrix T is the same as in Lemma 5.1 with α_k replaced by λ_k, $k = 0, \dots, N$. This gives the bound

$$E \leq \|\rho\|_\infty^{|\Lambda|}\int_{[-S,S]^{|\Lambda|}}\Big|\Big\langle\delta_x, \Big(A + \zeta_x\sum_{k\in\Lambda_x}\lambda_{k-x}u_k + \zeta_{j-n+1-N}\sum_{k\in\Lambda_j}\lambda_{k-(j-n+1-N)}u_k\Big)^{-1}\delta_j\Big\rangle\Big|^s \mathrm{d}\zeta_\Lambda.$$

where $\mathrm{d}\zeta_\Lambda = \prod_{k\in\Lambda}\mathrm{d}\zeta_k$, $S = R(1+\max_{i\in\{1,\dots,N\}}|\lambda_i|)$, and

$$A := A' + \sum_{k\in\Lambda_x\setminus\{x\}}\zeta_k u_k + \sum_{k\in\Lambda_j\setminus\{j-n+1-N\}}\zeta_k u_k$$

is independent of ζ_x and $\zeta_{j-n+1-N}$. By assumption the functions $\phi := \sum_{k\in\Lambda_x}\lambda_{k-x}u_k$ and $\psi := \sum_{k\in\Lambda_j}\lambda_{k-(j-n+1-N)}u_k$ are bounded and non-negative, with $\phi(x) = u(0) > 0$ and $\psi(j) = \lambda_N u(n-1) > 0$. Using Lemma 6.4 we obtain

$$\begin{aligned}E' &:= \int_{[-S,S]^2}\Big|\Big\langle\delta_x, (A + \zeta_x\phi + \zeta_{j-n+1-N}\psi)^{-1}\delta_j\Big\rangle\Big|^s \mathrm{d}\zeta_x\mathrm{d}\zeta_{j-n+1-N}\\ &\leq \frac{4}{1-s}\left(\frac{C_W}{\sqrt{\phi(x)\psi(j)}}\right)^s S^{2-s}.\end{aligned}$$

Thus the original integral is estimated by

$$\begin{aligned}E &\leq \|\rho\|_\infty^{|\Lambda|}(2S)^{|\Lambda|-2}\frac{4}{1-s}\left(\frac{C_W}{\sqrt{\phi(x)\psi(j)}}\right)^s S^{2-s}\\ &= \frac{4}{1-s}\left(\frac{C_W}{\sqrt{u(0)\lambda_N u(n-1)}}\right)^s (2S\,\|\rho\|_\infty)^{2(N+1)}\frac{1}{S^s} =: C\,. \qquad (39)\end{aligned}$$

■

The last proposition and a formula analogous to (17) now give for $j = x+2(N+n)-1$ and $x+2(N+n) \leq y$

$$\begin{aligned}\mathbb{E}_\Lambda\Big\{\big|G_{\Gamma_0}(z;x,y)\big|^s\Big\} &= \mathbb{E}_\Lambda\Big\{\big|G_{\Gamma_0}(z;x,x+2(N+n)-1)\big|^s\Big\}\big|G_{\Gamma_1}(z;x+2(N+n),y)\big|^s\\ &\leq C\big|G_{\Gamma_1}(z;x+2(N+n),y)\big|^s\end{aligned}$$

where $\Gamma_0 = \mathbb{Z}$ and $\Gamma_1 = \{x + 2(N + n), x + 2(N + n) + 1, \ldots\}$. In an appropriate large disorder regime, where the constant C in (39) is smaller than one, exponential decay now follows by iteration, similarly as in Theorem 4.3.

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References

[AEN+06] M. Aizenman, A. Elgart, S. Naboko, J. H. Schenker, and G. Stolz. Moment analysis for localization in random Schrödinger operators. *Invent. Math.*, 163:343–413, 2006.

[Aiz94] M. Aizenman. Localization at weak disorder: Some elementary bounds. *Rev. Math. Phys.*, 6:1163–1182, 1994.

[AM93] M. Aizenman and S. Molchanov. Localization at large disorder and at extreme energies: An elemantary derivation. *Commun. Math. Phys.*, 157:245–278, 1993.

[ASFH01] M. Aizenman, J. H. Schenker, R. M. Friedrich, and D. Hundertmark. Finite-volume fractional-moment criteria for Anderson localization. *Commun. Math. Phys.*, 224:219–253, 2001.

[BHS07] J. V. Bellissard, P. D. Hislop, and G. Stolz. Correlations estimates in the lattice Anderson model. *J. Stat. Phys.*, 129:649–662, 2007.

[EG] A. Elgart and G. M. Graf. Localization in the matrix valued Anderson model. in preperation.

[FS83] J. Fröhlich and T. Spencer. Absence of diffusion in the Anderson tight binding model for large disorder or low energy. *Commun. Math. Phys.*, 88:151–184, 1983.

[Gra94] G. M. Graf. Anderson localization and the space-time characteristic of continuum states. *J. Stat. Phys.*, 75:337–346, 1994.

[Kir07] W. Kirsch. An invitation to random Schrödinger operators. Preprint, arXiv:0709.3707v1 [math-ph], 2007.

[Klo95] F. Klopp. Localization for some continuous random Schrödinger operators. *Commun. Math. Phys.*, 167:553–569, 1995.

[Klo02] F. Klopp. Weak disorder localization and Lifshitz tails: continuous Hamiltonians. *Ann. Henri Poincaré*, 3:711–737, 2002.

[KV06] V. Kostrykin and I. Veselić. On the Lipschitz continuity of the integrated density of states for sign-indefinite potentials. *Math. Z.*, 252:367–392, 2006.

[MS69] T. S. Motzkin and E. G. Straus. Divisors of polynomials and power series with positive coefficients. *Pac. J. Math.*, 29:641–652, 1969.

[Sto02] G. Stolz. Strategies in localization proofs for one-dimensional random Schrödinger operators. *Proc. Indian Acad. Sci. Math. Sci.*, 112:229–243, 2002.

[SW86] B. Simon and T. Wolff. Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians. *Commun. Pur. Appl. Math.*, 39:75–90, 1986.

[vDK91] H. von Dreifus and A. Klein. Localization for random Sch?dinger operators with correlated potentials. *Commun. Math. Phys.*, 140:133–147, 1991.

[Ves02] I. Veselić. Wegner estimate and the density of states of some indefinite alloy type Schrödinger operators. *Lett. Math. Phys.*, 59:199–214, 2002.

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