Asymptotics of individual eigenvalues of a class of large Hessenberg Toeplitz matrices

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Abstract. We study the asymptotic behavior of individual eigenvalues of the n-by-n truncations of certain infinite Hessenberg Toeplitz matrices as n goes to infinity. The generating function of the Toeplitz matrices is supposed to be of the form $a(t) = t^{-1}(1-t)^{\alpha} f(t)$ $(t \in \mathbb{T})$, where α is a positive real number but not an integer and f is a smooth function in H^{∞}. The classes of generating functions considered here and in a recent paper by Dai, Geary, and Kadanoff are overlapping, and in the overlapping cases, our results imply in particular a rigorous justification of an asymptotic formula which was conjectured by Dai, Geary, and Kadanoff on the basis of numerical computations.

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1. Introduction and main results

The $n \times n$ Toeplitz matrix generated by a complex-valued function $a \in L^1$ on the unit circle $\mathbb{T}$ is the matrix $T_n(a) = \left(a_{j-k}\right)_{j,k=0}^{n-1}$, where a_k is the kth Fourier coefficient of the function a, that is, $a_k = \int_0^{2\pi} a\left(e^{i\theta}\right) e^{-ik\theta}\, d\theta/2\pi$, $k \in \mathbb{Z}$. The function a is referred to as the symbol of the matrices $T_n(a)$.

If a is real-valued, then the matrices $T_n(a)$ are all Hermitian, and in this case a number of results on the asymptotics of the eigenvalues of $T_n(a)$ is known; see, for example, [6], [7], [13], [16], [18], [20], [21], [23], [24], [26], [27], [29], [30]. We here consider genuinely complex-valued symbols, in which case the overall picture is less complete. Papers [12], [15], [19] describe the limiting behavior of the eigenvalues of $T_n(a)$ if a is a rational function, while papers [1] and [28] are devoted to the asymptotic eigenvalue distribution in the case of non-smooth symbols. In [25] and [28], it is in particular shown that if $a \in L^{\infty}$ and the essential range $\mathcal{R}(a)$ does

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not separate the plane, then the eigenvalues of $T_n(a)$ approximate $\mathcal{R}(a)$. Many of the results of the papers cited above can also be found in the books [5], [8], [9].

Throughout what follows we assume that a is a complex-valued continuous function on $\mathbb{T}$. In that case $\mathcal{R}(a) = a(\mathbb{T})$. When the eigenvalues of $T_n(a)$ approach $\mathcal{R}(a)$ asymptotically in the sense that

$$\lim_{n\to\infty} \frac{\operatorname{trace}\varphi(T_n(a))}{n} = \int_0^{2\pi} \varphi(a(e^{i\theta}))\,\frac{d\theta}{2\pi} \tag{1.1}$$

for a sufficiently rich supply of test functions φ, one says that they have canonical distribution. In 1990, Widom [28] showed that if $\mathcal{R}(a)$ is a Jordan curve and a is smooth on $\mathbb{T}$ minus a single point but not smooth on all of $\mathbb{T}$, then the spectrum of $T_n(a)$ has canonical distribution. He also raised the following intriguing conjecture, which is still an open problem:

The eigenvalues of $T_n(a)$ are canonically distributed except when
a extends analytically to an annulus $r < |z| < 1$ or $1 < |z| < R$.

Results like (1.1) or of the type that the spectrum of $T_n(a)$ converges to some limiting set in the Hausdorff metric do not provide us with information on the asymptotic behavior of individual eigenvalues. The asymptotic behavior of the extreme eigenvalues of Hermitian Toeplitz matrices is fairly well understood; see the references cited above. Paper [6] contains asymptotic expansions for individual inner eigenvalues of certain banded Hermitian Toeplitz matrices. The recent papers [11] and [17] concern asymptotic formulas for individual eigenvalues of Toeplitz matrices whose symbols are complex-valued and have a so-called Fisher-Hartwig singularity. These are special symbols that are smooth on $\mathbb{T}$ minus a single point but not smooth on the entire circle $\mathbb{T}$; see [8], [9].

To be more specific, Dai, Geary, and Kadanoff [11] considered symbols of the form

$$a(t) = \left(2 - t - \frac{1}{t}\right)^{\gamma} (-t)^{\beta}, \quad t \in \mathbb{T},$$

where $0 < \gamma < -\beta < 1$. They conjectured that the eigenvalues $\lambda_j = \lambda_{j,n}$ satisfy

$$\lambda_j \approx a\left(n^{(2\gamma+1)/n} \exp\left(-\frac{2\pi i}{n} j\right)\right), \quad j = 0, \dots, n-1, \tag{1.2}$$

and confirmed this conjecture numerically.

Let H^∞ be the usual Hardy space of (boundary values of) bounded analytic functions in the unit disk $\mathbb{D}$. Given $a \in C(\mathbb{T})$, we denote by $\operatorname{wind}_\lambda(a)$ the winding number of a about a point $\lambda \in \mathbb{C} \setminus \mathcal{R}(a)$ and by $\mathcal{D}(a)$ the set of all $\lambda \in \mathbb{C}$ for which $\operatorname{wind}_\lambda(a) \neq 0$. In this paper we study the eigenvalues of $T_n(a)$ for symbols $a(t) = t^{-1}h(t)$ with the following properties:

1. $h \in H^\infty$ and $h_0 \neq 0$;
2. $h(t) = (1-t)^\alpha f(t)$, where $\alpha \in [0,\infty) \setminus \mathbb{Z}$ and $f \in C^\infty(\mathbb{T})$;
3. h has an analytic extension to an open neighborhood W of $\mathbb{T}\setminus\{1\}$ not containing the point 1;
4. $\mathcal{R}(a)$ is a Jordan curve in $\mathbb{C}$ and $\operatorname{wind}_\lambda(a) = -1$ for each $\lambda \in \mathcal{D}(a)$.

According to [28], in our case the spectrum of $T_n(a)$ has canonical distribution. Note that when $\beta = \gamma - 1$ and $f \equiv 1$, our symbol coincides with the one of [11].

Let $D_n(a)$ denote the determinant of $T_n(a)$. Thus, the eigenvalues λ of $T_n(a)$ are the solutions of the equation $D_n(a-\lambda) = 0$. Our assumptions imply that $T_n(a)$ is a Hessenberg matrix, that is, it arises from a lower triangular matrix by adding the superdiagonal. This circumstance together with the Baxter-Schmidt formula for Toeplitz determinants allows us to express $D_n(a-\lambda)$ as a Fourier integral. The value of this integral mainly depends on λ and on the singularity of $(1-t)^\alpha$ at the point 1. Let W_0 be a small open neighborhood of zero in $\mathbb{C}$. We show that for every point $\lambda \in \mathcal{D}(a) \cap (a(W) \setminus W_0)$ there exists a unique point $t_\lambda \notin \overline{\mathbb{D}}$ such that $a(t_\lambda) = \lambda$. After exploring the contributions of λ and the singular point 1 to the Fourier integral, we get the following asymptotic expansion for $D_n(a-\lambda)$.

Theorem 1.1. *Let $a(t) = t^{-1}h(t)$ be a symbol with properties 1 to 4. Then for every small open neighborhood W_0 of zero in $\mathbb{C}$ and every $\lambda \in \mathcal{D}(a) \cap (a(W) \setminus W_0)$,*

$$D_n(a-\lambda) = (-h_0)^{n+1}\left[\frac{1}{t_\lambda^{n+2}a'(t_\lambda)} - \frac{f(1)\Gamma(\alpha+1)\sin(\alpha\pi)}{\pi\lambda^2 n^{\alpha+1}} + R_1(n,\lambda)\right], \quad (1.3)$$

where $R_1(n,\lambda) = \mathcal{O}\left(n^{-\alpha-\alpha_0-1}\right)$ as $n \to \infty$, uniformly in $\lambda \in a(W) \setminus W_0$. Here $\alpha_0 = \min\{\alpha, 1\}$ and h_0 is the zeroth Fourier coefficient of h.

The first term in brackets is the contribution of λ, while the second is the contribution of the point 1.

Here now are our main results. Let W_0 be a small open neighborhood of the origin in $\mathbb{C}$ and put $\omega_j := \exp(-2\pi i j/n)$. For each n there exists integers n_1 and n_2 such that $\omega_{n_1}, \omega_{n-n_2} \in a^{-1}(W_0)$ but $\omega_{n_1+1}, \omega_{n-n_2-1} \notin a^{-1}(W_0)$. Recall that $a(t_\lambda) = \lambda$.

Theorem 1.2. *Let $a(t) = t^{-1}h(t)$ be a symbol with properties 1 to 4. Then for every small open neighborhood W_0 of the origin in $\mathbb{C}$ and every j between n_1 and $n-n_2$,*

$$t_{\lambda_j} = n^{(\alpha+1)/n}\omega_j\left[1 + \frac{1}{n}\log\left(\frac{a^2(\omega_j)}{c_0(1)a'(\omega_j)\omega_j^2}\right) + R_2(n,j)\right], \quad (1.4)$$

where $R_2(n,j) = \mathcal{O}\left(n^{-\alpha_0-1}\right) + \mathcal{O}\left(n^{-2}\log n\right)$ as $n \to \infty$, uniformly with respect to j in $(n_1, n-n_2)$. Here $\alpha_0 = \min\{\alpha, 1\}$ and

$$c_0(1) = \frac{f(1)\Gamma(\alpha+1)\sin(\alpha\pi)}{\pi}.$$

Formula (1.4) proves conjecture (1.2) in the special case $\beta = \gamma - 1$. It shows that as n increases, the point t_{λ_j} is close to $n^{(\alpha+1)/n}\omega_j$. Finally, we take the value of a at the point (1.4) to obtain the following expression for λ_j.

Theorem 1.3. *Let $a(t) = t^{-1}h(t)$ be a symbol with properties 1 to 4. Then for every small neighborhood W_0 of zero in $\mathbb{C}$ and every j between n_1 and $n - n_2$,*

$$\lambda_j = a(\omega_j) + (\alpha+1)\,\omega_j a'(\omega_j)\frac{\log n}{n} + \frac{\omega_j a'(\omega_j)}{n}\log\left(\frac{a^2(\omega_j)}{c_0(1)a'(\omega_j)\omega_j^2}\right) + R_3(n,j), \tag{1.5}$$

where $c_0(1)$ is as in Theorem 1.2 and $R_3(n,j) = \mathcal{O}\left(n^{-\alpha_0-1}\right) + \mathcal{O}\left(n^{-2}(\log n)^2\right)$ as $n \to \infty$, uniformly with respect to j in $(n_1, n-n_2)$.

In the case where f is identically 1, the previous three theorems are essentially already in [3]. The idea to use just the singularity $(1-t)^\alpha$ in order to study phenomena connected with eigenvalue asymptotics was also employed in [4].

We remark that we wrote down only the first few terms in our asymptotic expansions but that our method is constructive and would allow us to get as many terms as we desire. Clearly, conjecture (1.2) corresponds to the first term in our asymptotic expansion (1.4). Figure 1 illustrates Theorem 1.3. In the last section, we present another simulation graphic and error tables made with *Matlab* software to show that incorporating the second term of our expansion (1.4) (= third term in (1.5)) reduces the error to nearly one tenth.

2. Toeplitz determinant

Lemma 2.1. *Let $a(t) = t^{-1}h(t)$ have properties 1 and 4. Then, for each $\lambda \in \mathcal{D}(a)$ and every $n \in \mathbb{N}$, and with $[\ ]_n$ denoting the nth Fourier coefficient,*

$$D_n(a-\lambda) = (-1)^n h_0^{n+1}\left[\frac{1}{h(t)-\lambda t}\right]_n. \tag{2.1}$$

Proof. This can be deduced from the Baxter-Schmidt formula [2], which is also in [5, p. 37]. For the reader's convenience, we include a direct proof of (2.1). Obviously,

$$T_{n+1}(h-\lambda t) = \left[\begin{array}{ccccc|c} h_0 & 0 & 0 & \cdots & 0 & 0 \\ \hline h_1-\lambda & h_0 & 0 & \cdots & 0 & 0 \\ h_2 & h_1-\lambda & h_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{n-1} & h_{n-2} & h_{n-3} & \cdots & h_0 & 0 \\ h_n & h_{n-1} & h_{n-2} & \cdots & h_1-\lambda & h_0 \end{array}\right] \tag{2.2}$$

and

$$T_n(a-\lambda) = \begin{bmatrix} h_1-\lambda & h_0 & 0 & \cdots & 0 \\ h_2 & h_1-\lambda & h_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1} & h_{n-2} & h_{n-3} & \cdots & h_0 \\ h_n & h_{n-1} & h_{n-2} & \cdots & h_1-\lambda \end{bmatrix}.$$

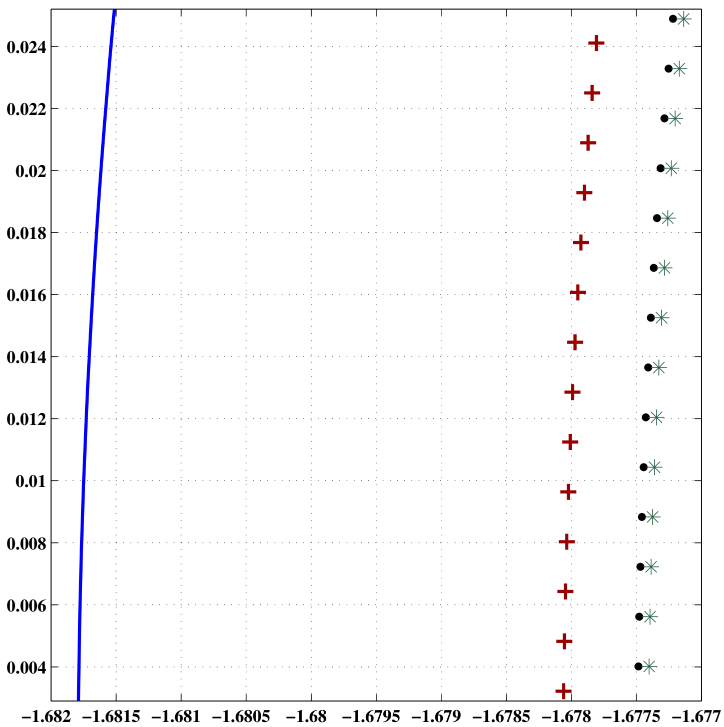

FIGURE 1. The picture shows a piece of $\mathcal{R}(a)$ for the symbol $a(t) = t^{-1}(1-t)^{3/4}$ (solid line) located "far" from zero. The dots are sp $T_{4096}(a)$ calculated by *Matlab*. The crosses and the stars are the approximations obtained by using 2 and 3 terms of (1.5), respectively.

Applying Cramer's rule to (2.2) we obtain

$$\left[T_{n+1}^{-1}(h-\lambda t)\right]_{(n+1,1)} = (-1)^{n+2}\frac{D_n(a-\lambda)}{D_{n+1}(h-\lambda t)}. \tag{2.3}$$

We claim that $h(t) - \lambda t$ is invertible in H^∞. To see this, we must show that $h(t) \neq \lambda t$ for all $t \in \overline{\mathbb{D}}$ and each $\lambda \in \mathcal{D}(a)$. Let λ be a point in $\mathcal{D}(a)$. For each $t \in \mathbb{T}$ we have $h(t) \neq \lambda t$ because $\lambda \notin \partial\mathcal{D}(a) = \mathcal{R}(a)$. By assumption, $\operatorname{wind}_\lambda(a) = -1$ and thus

$$\begin{aligned}-1 &= \operatorname{wind}_0(a-\lambda) = \operatorname{wind}_0(t^{-1}h(t)-\lambda) = \operatorname{wind}_0(t^{-1}(h(t)-\lambda t))\\ &= \operatorname{wind}_0(t^{-1}) + \operatorname{wind}_0(h(t)-\lambda t) = -1 + \operatorname{wind}_0(h(t)-\lambda t).\end{aligned}$$

It follows that $\text{wind}_0(h(t)-\lambda t)=0$, which means that the origin does not belong to the inside domain of the curve $\{h(t)-\lambda t : t\in\mathbb{T}\}$ (see [10, p. 204]). As $h\in H^\infty$, this shows that $h(t)\neq\lambda t$ for all $t\in\mathbb{D}$ and proves our claim.

If b is invertible in H^∞, then $T_{n+1}^{-1}(b)=T_{n+1}(1/b)$. Thus, the $(n+1,1)$ entry of the matrix $T_{n+1}^{-1}(h(t)-\lambda t)$ is in fact the nth Fourier coefficient of $(h(t)-\lambda t)^{-1}$,

$$\left[T_{n+1}^{-1}(h(t)-\lambda t)\right]_{(n+1,1)}=\left[\frac{1}{h(t)-\lambda t}\right]_n.$$

Inserting this in (2.3) we get

$$\begin{aligned}D_n(a-\lambda)&=(-1)^{n+2}D_{n+1}(h(t)-\lambda t)\left[\frac{1}{h(t)-\lambda t}\right]_n\\&=(-1)^n h_0^{n+1}\left[\frac{1}{h(t)-\lambda t}\right]_n,\end{aligned}$$

which completes the proof. □

Expression (2.1) says that the determinant $D_n(a-\lambda)$ can be expressed as the Fourier integral

$$D_n(a-\lambda)=(-1)^n h_0^{n+1}\int_{-\pi}^{\pi}\frac{e^{-in\theta}}{h\big(e^{i\theta}\big)-\lambda e^{i\theta}}\frac{d\theta}{2\pi},$$

which is our starting point to find an asymptotic expansion for the eigenvalues of $T_n(a)$. There are two major contributions to this integral. The first comes from λ, when it is close to $\mathcal{R}(a)$, and the second results from the singularity at the point 1. We will analyze them in separate sections.

3. Contribution of λ to the asymptotic behavior of D_n

Defining

$$b(z,\lambda):=\frac{1}{h(z)-\lambda z},$$

we have

$$b_n(\lambda)=\int_{-\pi}^{\pi}b\big(e^{i\theta},\lambda\big)e^{-in\theta}\frac{d\theta}{2\pi}. \tag{3.1}$$

From (2.1) we conclude that

$$D_n(a-\lambda)=(-1)^n h_0^{n+1}b_n(\lambda). \tag{3.2}$$

Lemma 3.1. *Let $a(t)=t^{-1}h(t)$ be a symbol such that $\mathcal{R}(a)$ is a Jordan curve in $\mathbb{C}$. Let W_0 be a small open neighborhood of zero in $\mathbb{C}$. Assume that h has an analytic extension to an open neighborhood W of $\mathbb{T}\setminus\{1\}$ in $\mathbb{C}$ not containing the point 1. Then, for each $\lambda\in\mathcal{D}(a)\setminus W_0$ sufficiently close to $\mathcal{R}(a)$, there exists a unique point t_λ in $W\setminus\overline{\mathbb{D}}$ such that $a(t_\lambda)=\lambda$. Moreover, the point t_λ is a simple pole for b.*

Proof. Without loss of generality, we may assume that the extension of a to W is bounded. As $h \in H^\infty$, this extension must map $W \setminus \overline{\mathbb{D}}$ to $\mathcal{D}(a) \cap a(W)$. As the range of a has no loops, we have $a'(t) \neq 0$ for all $t \in \mathbb{T}$. Consider the compact set $S := \{t \in \mathbb{T} : a(t) \notin W_0\}$. For every $t \in S$, there exists an open neighborhood V_t of t in $\mathbb{C}$ with $V_t \subset W$ such that $a'(t) \neq 0$ for each $t \in V_t$. Thus, there is an open set U_t such that $t \in U_t \subset V_t$ and a is a conformal map (and hence bijective) from U_t to $a(U_t)$. As S is compact, we can take a finite sub-cover from $\{U_t\}_{t \in S}$, say $U := \cup_{i=1}^{M} U_{t_i}$. It follows that a is a conformal map (and hence bijective) from $U \supset S$ to $a(U) \supset a(S)$; see Figure 2. The lemma then holds for every $\lambda \in a(U) \cap (\mathcal{D}(a) \setminus W_0)$. Finally, since $a'(t_\lambda) \neq 0$, the point t_λ must be a simple pole of b. □

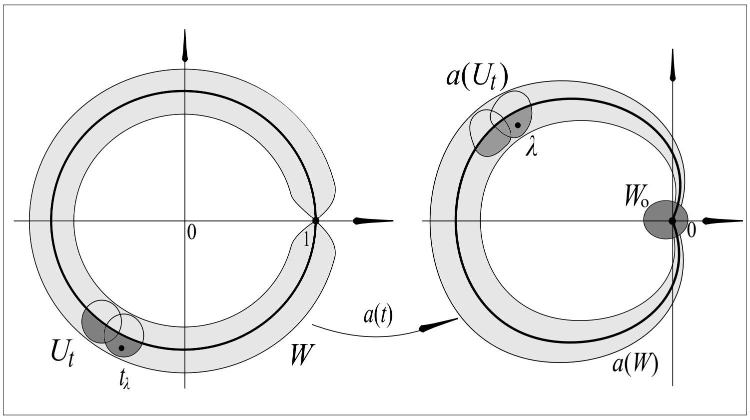

FIGURE 2. The map $a(t)$ over the unit circle.

Now using that t_λ is a simple pole of b, we split b as follows:

$$b(z,\lambda) = \frac{1}{z(a(z)-\lambda)} = \frac{1}{t_\lambda a'(t_\lambda)(z-t_\lambda)} + f_0(z,\lambda). \tag{3.3}$$

Here f_0 is analytic with respect to z in W and uniformly bounded with respect to λ in $a(W) \setminus W_0$. We calculate the Fourier coefficients of the first term in (3.3) directly and integrate the second term to get

$$b_n(\lambda) = \frac{-1}{t_\lambda^{n+2} a'(t_\lambda)} + \mathcal{I}, \tag{3.4}$$

where

$$\mathcal{I} := \int_{-\pi}^{\pi} f_0(e^{i\theta},\lambda) e^{-in\theta} \frac{d\theta}{2\pi}.$$

The first term in (3.4) times $(-1)^n h_0^{n+1}$ is the contribution of t_λ to the asymptotic expansion of $D_n(a-\lambda)$; see (3.2). The function f_0 has a singularity at $z = 1$ and we use this fact to expand $\mathcal{I}$ in the following section.

4. Contribution of 1 to the asymptotic behavior of D_n

In this section, we will show that the value of $\mathcal{I}$ in (3.4) depends mainly on the singularity at the point 1. Let us write $b(\theta,\lambda)$ and $f_0(\theta,\lambda)$ instead of $b(e^{i\theta},\lambda)$ and $f_0(e^{i\theta},\lambda)$, respectively. Let $\{\phi_1,\phi_2\}$ be a smooth partition of unity over the segment $[-\pi,\pi]$, which means that $\phi_1,\phi_2 \in C^\infty[-\pi,\pi]$, $\phi_1(\theta)+\phi_2(\theta)=1$ for all $\theta\in[-\pi,\pi]$, the support of ϕ_1 is contained in $[-\pi,-\varepsilon]\cup[\varepsilon,\pi]$, and the support of ϕ_2 is in $[-\delta,\delta]$, where $0<\varepsilon<\delta$ are small constants. By pasting segments $[-\pi,\pi]$ in both directions, we can continue ϕ_1 and ϕ_2 to the entire real line $\mathbb{R}$, and we will think of these two functions in that way.

Lemma 4.1. *For every sufficiently small positive δ, we have*

$$\mathcal{I} = \int_{-\delta}^{\delta} \phi_2(\theta) b(\theta,\lambda) e^{-in\theta}\frac{d\theta}{2\pi} + Q_1(n,\lambda), \tag{4.1}$$

where $Q_1(n,\lambda)=\mathcal{O}(n^{-\infty})$ as $n\to\infty$, uniformly with respect to λ in $a(W)\setminus W_0$.

Proof. Using the partition of unity $\{\phi_1,\phi_2\}$, we write $\mathcal{I}=\mathcal{I}_1+\mathcal{I}_2$ where

$$\mathcal{I}_1 := \int_{\varepsilon}^{2\pi-\varepsilon} \phi_1(\theta) f_0(\theta,\lambda) e^{-in\theta}\frac{d\theta}{2\pi}$$

and

$$\mathcal{I}_2 := \int_{-\delta}^{\delta} \phi_2(\theta) f_0(\theta,\lambda) e^{-in\theta}\frac{d\theta}{2\pi}.$$

The function $\phi_1(\theta)f_0(\theta,\lambda)$ belongs to $C^\infty[\varepsilon,2\pi-\varepsilon]$. Thus by [10, p. 22], we obtain that $\mathcal{I}_1=\mathcal{O}(n^{-\infty})$ as $n\to\infty$, uniformly with respect to λ in $a(W)\setminus W_0$.

Using (3.3) and writing $h(\theta)$ instead of $h(e^{i\theta})$, we arrive at $\mathcal{I}_2=\mathcal{I}_{21}+\mathcal{I}_{22}$ where

$$\mathcal{I}_{21} := \int_{-\delta}^{\delta} \frac{\phi_2(\theta)e^{-in\theta}}{h(\theta)-\lambda e^{i\theta}}\frac{d\theta}{2\pi} \tag{4.2}$$

and

$$\mathcal{I}_{22} := \frac{-1}{t_\lambda a'(t_\lambda)}\int_{-\delta}^{\delta} \frac{\phi_2(\theta)e^{-in\theta}}{e^{i\theta}-t_\lambda}\frac{d\theta}{2\pi}.$$

Once more, the function $\phi_2(\theta)/(e^{i\theta}-t_\lambda)$ belongs to $C^\infty[-\delta,\delta]$, we thus conclude that $\mathcal{I}_{22}=\mathcal{O}(n^{-\infty})$ as $n\to\infty$, uniformly with respect to λ in $a(W)\setminus W_0$. □

Expression (4.1) says that the value of $\mathcal{I}$ basically depends on the integrand $b(\theta,\lambda)e^{-in\theta}$ at $\theta=0$. As we can take δ as small as we desire, we can assume that θ is arbitrarily close to zero. Keeping this idea in mind, we will develop an asymptotic expansion for b. For future reference, we rewrite (4.1) as

$$\mathcal{I} = \mathcal{I}_{21} + Q_1(n,\lambda), \tag{4.3}$$

where $Q_1(n,\lambda)=\mathcal{O}(n^{-\infty})$ as $n\to\infty$, uniformly with respect to λ in $a(W)\setminus W_0$.

Lemma 4.2. *For every sufficiently small positive δ,*

$$\mathcal{I}_{21} = -\sum_{s=0}^{\infty} \frac{1}{\lambda^{s+1}} \int_{-\delta}^{\delta} \frac{\phi_2(\theta) h^s(\theta) e^{-in\theta}}{e^{i\theta(s+1)}} \frac{d\theta}{2\pi}. \tag{4.4}$$

Proof. From (4.2) we have

$$\mathcal{I}_{21} = \int_{-\delta}^{\delta} \phi_2(\theta) b(\theta, \lambda) e^{-in\theta} \frac{d\theta}{2\pi}. \tag{4.5}$$

Note that

$$b(\theta, \lambda) = \frac{1}{h(\theta) - \lambda e^{i\theta}} = \frac{-1}{\lambda e^{i\theta}} \cdot \frac{1}{1 - \lambda^{-1} e^{-i\theta} h(\theta)}.$$

As $|h(\theta)| \to 0$ when $\theta \to 0$, there exists a small positive constant δ such that

$$\left|\lambda^{-1} e^{-i\theta} h(\theta)\right| < 1$$

for every $|\theta| < \delta$. Thus,

$$b(\theta, \lambda) = \frac{-1}{\lambda e^{i\theta}} \sum_{s=0}^{\infty} \left(\lambda^{-1} e^{-i\theta} h(\theta)\right)^s = -\sum_{s=0}^{\infty} \frac{h^s(\theta)}{\lambda^{s+1} e^{i\theta(s+1)}} \tag{4.6}$$

for every $|\theta| < \delta$. Inserting (4.6) in (4.5) finishes the proof. □

We will use the notation

$$\mathcal{I}_{21s} := \frac{1}{\lambda^{s+1}} \int_{-\delta}^{\delta} \frac{\phi_2(\theta) h^s(\theta) e^{-in\theta}}{e^{i\theta(s+1)}} \frac{d\theta}{2\pi}.$$

Because $\phi_2(\theta) e^{-i\theta} \in C^{\infty}[-\delta, \delta]$, we have $\mathcal{I}_{21s}|_{s=0} = \mathcal{O}(n^{-\infty})$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$. With the previous notation, we can rewrite (4.4) as

$$\mathcal{I}_{21} = -\sum_{s=1}^{\infty} \mathcal{I}_{21s} + Q_2(n, \lambda), \tag{4.7}$$

where $Q_2(n, \lambda) = \mathcal{O}(n^{-\infty})$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$.

Finally we will work with $\mathcal{I}_{21s}$ and for this purpose we need the following well known result, which is, for example, in [14, p. 97].

Theorem 4.3. *Let $\beta > 0$, $\delta > 0$, $v(\theta) \in C^{\infty}[0, \delta]$, $v^{(s)}(\delta) = 0$ for all $s \geq 0$. Then, as $n \to \infty$,*

$$\int_0^{\delta} \theta^{\beta-1} v(\theta) e^{in\theta} d\theta \sim \sum_{s=0}^{\infty} \frac{a_s}{n^{s+\beta}},$$

where

$$a_s = \frac{v^{(s)}(0)}{s!} \Gamma(s+\beta) i^{s+\beta} \tag{4.8}$$

and $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$ is Euler's Gamma function.

Lemma 4.4. *Let $h(t) = (1-t)^\alpha f(t)$ with $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}$ and $f \in C^\infty(\mathbb{T})$. Then*

$$\mathcal{I}_{21} = \frac{f(1)\Gamma(\alpha+1)\sin(\alpha\pi)}{\pi\lambda^2 n^{\alpha+1}} + R_1(n,\lambda), \tag{4.9}$$

where $R_1(n,\lambda) = \mathcal{O}\left(n^{-\alpha-\alpha_0-1}\right)$ with $\alpha_0 = \min\{\alpha, 1\}$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$.

Proof. It is easy to verify that $h(\theta) = (-i\theta)^\alpha v(\theta) f\left(e^{i\theta}\right)$ as $\theta \to 0$, where $v(\theta) = \left(i\theta^{-1}\left(1-e^{i\theta}\right)\right)^\alpha$, the branch of the αth power being the one corresponding to the argument in $(-\pi, \pi]$; note that for every sufficiently small positive δ we have $v \in C^\infty[-\delta, \delta]$ and $v(0) = 1$. Thus,

$$\begin{aligned}\mathcal{I}_{21s} &= \frac{1}{\lambda^{s+1}} \int_{-\delta}^{\delta} \phi_2(\theta) h^s(\theta) e^{-i\theta(n+s+1)} \frac{d\theta}{2\pi} \\ &= \frac{(-i)^{\alpha s}}{\lambda^{s+1}} \int_{-\delta}^{\delta} \phi_2(\theta)\theta^{\alpha s} v^s(\theta) f^s\left(e^{i\theta}\right) e^{-i\theta(n+s+1)} \frac{d\theta}{2\pi}\end{aligned}$$

when $\theta \to 0$. The last integral can be written as

$$\begin{aligned}\mathcal{I}_{21s} &= \int_{-\delta}^{\delta} \theta^{\beta-1} w(\theta) e^{-in\theta} d\theta \\ &= \int_{-\delta}^{0} \theta^{\beta-1} w(\theta) e^{-in\theta} d\theta + \int_{0}^{\delta} \theta^{\beta-1} w(\theta) e^{-in\theta} d\theta \\ &= \int_{0}^{\delta} (-\tau)^{\beta-1} w(-\tau) e^{in\tau} d\tau + \int_{0}^{\delta} \theta^{\beta-1} w(\theta) e^{-in\theta} d\theta \\ &= \mathcal{I}_{21s1} + \mathcal{I}_{21s2},\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}\beta &:= \alpha s + 1, \\ w(\theta) &:= \frac{(-i)^{\alpha s}}{2\pi\lambda^{s+1}}\ \phi_2(\theta) v^s(\theta) f^s\left(e^{i\theta}\right) e^{-i\theta(s+1)}, \quad \theta \to 0,\end{aligned}$$

and

$$\mathcal{I}_{21s1} := (-1)^{\beta-1} \int_0^\delta \theta^{\beta-1} w(-\theta) e^{in\theta} d\theta, \quad \mathcal{I}_{21s2} := \int_0^\delta \theta^{\beta-1} w(\theta) e^{-in\theta} d\theta.$$

Note that $w(\pm\theta) \in C^\infty[0,\delta]$ and $w^{(s)}(\pm\delta) = 0$ for all $s \in \mathbb{N}$ because $\phi_2(\theta) \equiv 0$ in a small neighborhood of $\pm\delta$. Applying (4.8) to $\mathcal{I}_{21s1}$ and $\overline{\mathcal{I}_{21s2}}$, we obtain

$$\mathcal{I}_{21s1} = \frac{(-1)^{\alpha s} w(0)\Gamma(\alpha s+1) i^{\alpha s+1}}{n^{\alpha s+1}} + Q_3(s,n,\lambda)$$

and

$$\mathcal{I}_{21s2} = \frac{w(0)\Gamma(\alpha s+1) i^{-\alpha s-1}}{n^{\alpha s+1}} + Q_4(s,n,\lambda), \tag{4.11}$$

where $Q_3(s,n,\lambda)$ and $Q_4(s,n,\lambda)$ are $\mathcal{O}\left(n^{-\alpha s-2}\right)$ as $n\to\infty$, uniformly with respect to λ in $a(W)\setminus W_0$. Substitution of (4.11) in (4.10) yields

$$\begin{aligned}\mathcal{I}_{21s} =&\frac{w(0)\Gamma(\alpha s+1)}{n^{\alpha s+1}}\left[i^{-\alpha s-1}+(-1)^{\alpha s}i^{\alpha s+1}\right]+Q_5(s,n,\lambda)\\ =&\frac{-c_0(s)}{\lambda^{s+1}n^{\alpha s+1}}+Q_5(s,n,\lambda)\end{aligned}\tag{4.12}$$

where

$$c_0(s):=\frac{f^s(1)\Gamma(\alpha s+1)\sin(\alpha\pi s)}{\pi}\tag{4.13}$$

and $Q_5(s,n,\lambda)=\mathcal{O}\left(n^{-\alpha s-2}\right)$ as $n\to\infty$, uniformly in $\lambda\in a(W)\setminus W_0$. From (4.7) and (4.12) we obtain

$$\mathcal{I}_{21}=\frac{c_0(1)}{\lambda^2 n^{\alpha+1}}+R_1(n,\lambda),$$

where $R_1(n,\lambda)=\mathcal{O}\left(n^{-\alpha-\alpha_0-1}\right)$ as $n\to\infty$, uniformly in $\lambda\in a(W)\setminus W_0$. Here $\alpha_0:=\min\{\alpha,1\}$. □

The previous calculation gives us the main asymptotic term for $\mathcal{I}_{21}$. If more terms are needed, say m, we must expand $\mathcal{I}_{21}$ from $\mathcal{I}_{21s}|_{s=1}$ to $\mathcal{I}_{21s}|_{s=m}$ and expand each $\mathcal{I}_{21s}$ to m terms, after which, according to the value of α, we need to select the first m principal terms.

Finally we put all the lemmas together to prove Theorem 1.1.

Proof of Theorem 1.1. The proof of this theorem is a direct application of equations (3.2), (3.4), (4.3) and (4.9). □

5. Individual eigenvalues

In order to find the eigenvalues of the matrices $T_n(a)$, we need to solve the equations $D_n(a-\lambda)=0$. We start this section by locating the zeros of $D_n(a-\lambda)$.

Let W_0 be a small open neighborhood of zero in $\mathbb{C}$ and $\omega_j:=\exp(-2\pi ij/n)$. For each n there exists integers n_1 and n_2 such that $\omega_{n_1},\omega_{n-n_2}\in a^{-1}(W_0)$ but $\omega_{n_1+1},\omega_{n-n_2-1}\notin a^{-1}(W_0)$. Recall that $\lambda=a(t_\lambda)$. Take an integer j satisfying $n_1<j<n-n_2$. Using the relations

$$\frac{1}{t_\lambda^2 a'(t_\lambda)}=\frac{1}{\omega_j^2 a'(\omega_j)}+\mathcal{O}\left(|t_\lambda-\omega_j|\right)$$

and

$$\frac{1}{a^2(t_\lambda)}=\frac{1}{a^2(\omega_j)}+\mathcal{O}\left(|t_\lambda-\omega_j|\right),$$

where t_λ belongs to a small neighborhood of ω_j, we see that the determinant $D_n(a-\lambda)$ in (1.3) equals

$$\begin{aligned}&(-h_0)^{n+1}\left[T_1 - T_2 + \frac{1}{t_\lambda^n}\mathcal{O}\left(|t_\lambda-\omega_j|\right) + \frac{1}{n^{\alpha+1}}\mathcal{O}\left(|t_\lambda-\omega_j|\right) + Q_6(n,t_\lambda)\right]\\ &= (-h_0)^{n+1}\left[T_1 - T_2 + \mathcal{O}\left(\left|\frac{t_\lambda-\omega_j}{t_\lambda^n}\right|\right) + \mathcal{O}\left(\frac{|t_\lambda-\omega_j|}{n^{\alpha+1}}\right) + Q_6(n,t_\lambda)\right], \qquad (5.1)\end{aligned}$$

where $Q_6(n,t_\lambda) = \mathcal{O}\left(n^{-\alpha-\alpha_0-1}\right)$ as $n\to\infty$, uniformly with respect to t_λ in $W\setminus a^{-1}(W_0)$, and where t_λ belongs to a small neighborhood of ω_j. Here

$$T_1 := \frac{1}{t_\lambda^n\omega_j^2 a'(\omega_j)}, \quad T_2 := \frac{c_0(1)}{a^2(\omega_j)n^{\alpha+1}},$$

and $\alpha_0 := \min\{\alpha, 1\}$. Recall $c_0(1)$ from (4.13). Expression (5.1) makes sense only when t_λ is sufficiently "close" to ω_j and thus, it is necessary to know whether there exists a zero of $D_n(a-\lambda)$ "close" to ω_j. Let

$$t_\lambda = (1+\rho)\exp(i\theta).$$

It is easy to verify that $T_1 - T_2 = 0$ if and only if

$$\rho = \left(\frac{|a(\omega_j)|^2 n^{\alpha+1}}{|c_0(1)a'(\omega_j)|}\right)^{1/n} - 1 \qquad (5.2)$$

and

$$\theta = \theta_j = \frac{1}{n}\arg\left[\frac{a^2(\omega_j)}{c_0(1)\omega_j^2 a'(\omega_j)}\right] - \frac{2\pi}{n}$$

for some $j\in\{0,\dots,n-1\}$. When n tends to infinity, (5.2) shows that ρ remains positive and $\rho\to 0$. The function $T_1 - T_2$ has n zeros with respect to $\lambda\in\mathcal{D}(a)$ given by

$$a\left((1+\rho)e^{i\theta_0}\right), \quad \dots, \quad a\left((1+\rho)e^{i\theta_{n-1}}\right).$$

As Lemma 3.1 establishes a 1-1 correspondence between λ and t_λ, the function $D_n(a-\lambda)$ is analytic with respect to λ in $a(W)\setminus W_0$, that is, analytic with respect to t_λ in $W\setminus a^{-1}(W_0)$. We can therefore suppose that $T_1 - T_2$ has n zeros with respect to t_λ in the exterior of $\overline{\mathbb{D}}$ given by

$$t_0 := (1+\rho)e^{i\theta_0}, \quad \dots, \quad t_{n-1} := (1+\rho)e^{i\theta_{n-1}}.$$

We take the function "arg" in the interval $(-\pi,\pi]$. Thus, $t_j = (1+\rho)e^{i\theta_j}$ is the nearest zero to ω_j. Consider the neighborhood E_j of t_j sketched in Figure 3.

The boundary of E_j is $\Gamma := \Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$. We have chosen radial segments Γ_2 and Γ_4 so that their length is $1/n^\epsilon$ with $\epsilon\in(0,\alpha_0)$ and all the points in Γ_2 have the common argument $(\theta_{j+1}+\theta_j)/2$, while all the points in Γ_4 have the common argument $(\theta_{j-1}+\theta_j)/2$. As we can see in Figure 3, these points run from the unit circle $\mathbb{T}$ to $(1+1/n^\epsilon)\mathbb{T}$. Note also that $\Gamma_1\subset(1+1/n^\epsilon)\mathbb{T}$ and $\Gamma_3\subset\mathbb{T}$.

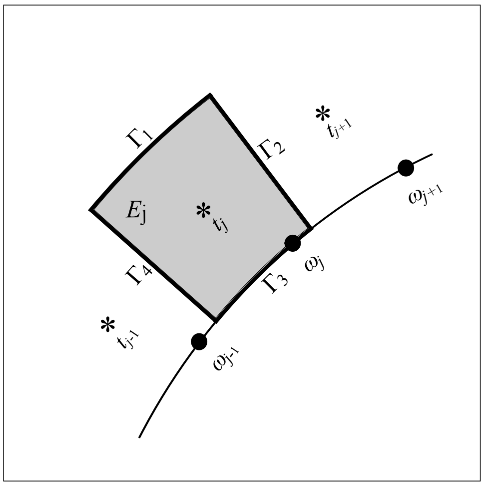

FIGURE 3. The neighborhood E_j of t_j in the complex plane.

Theorem 5.1. *Suppose $a(t) = t^{-1}h(t)$ is a symbol with properties 1 to 4. Let $\epsilon \in (0, \alpha_0)$ be a constant. Then there exists a family of sets $\{E_j\}_{j=n_1+1}^{n-n_2-1}$ in $\mathbb{C}$ such that*

1. *$\{E_j\}_{j=n_1+1}^{n-n_2-1}$ is a family of pairwise disjoint open sets,*
2. *$\operatorname{diam}(E_j) \leq \frac{2}{n^\epsilon}$,*
3. *$\omega_j \in \partial E_j$,*
4. *$D_n(a - a(t_\lambda)) = D_n(a - \lambda)$ has exactly one zero in each E_j.*

Here $\alpha_0 := \min\{\alpha, 1\}$ and $\operatorname{diam}(E_j) := \sup\{|z_1 - z_2| : z_1, z_2 \in E_j\}$.

Proof. Assertions 1, 2, and 3 can be deduced from the above construction. We prove assertion 4 by studying the behavior of $|D_n(a-\lambda)|$ in dependence on $t_\lambda \in \Gamma$. For $t_\lambda \in \Gamma_1$ we have, as $n \to \infty$,

$$|T_1|_{\Gamma_1} = \frac{1}{|a'(\omega_j)|} \cdot \left(1 + \frac{1}{n^\epsilon}\right)^{-n} = \frac{\exp(-n^{1-\epsilon})}{|a'(\omega_j)|} + \mathcal{O}\left(\frac{\exp(-n^{1-\epsilon})}{n^{2\epsilon-1}}\right),$$

$$|T_2|_{\Gamma_1} = \frac{1}{n^{\alpha+1}} \cdot \left|\frac{c_0(1)}{a^2(\omega_j)}\right|,$$

$$\left|\mathcal{O}\left(\left|\frac{t_\lambda - \omega_j}{t_\lambda^n}\right|\right)\right|_{\Gamma_1} = \mathcal{O}\left(\frac{\exp(-n^{1-\epsilon})}{n^\epsilon}\right),$$

$$\left|\mathcal{O}\left(\frac{|t_\lambda - \omega_j|}{n^{\alpha+1}}\right)\right|_{\Gamma_1} = \mathcal{O}\left(\frac{1}{n^{\alpha+\epsilon+1}}\right),$$

and

$$|Q_6(n, t_\lambda)|_{\Gamma_1} = \mathcal{O}\left(\frac{1}{n^{\alpha+\alpha_0+1}}\right).$$

When n goes to infinity, the absolute value of T_2 decreases at polynomial speed over Γ_1, while the absolute values of the remaining terms in (5.1) are smaller over Γ_1. Thus,

$$\left|\frac{D_n(a-\lambda)}{h_0^{n+1}}\right|_{\Gamma_1} = \frac{1}{n^{\alpha+1}} \cdot \left|\frac{c_0(1)}{a^2(\omega_j)}\right| + \mathcal{O}\left(\frac{1}{n^{\alpha+\epsilon+1}}\right) \text{ as } n \to \infty.$$

For $t_\lambda \in \Gamma_3$ we get, as $n \to \infty$,

$$|T_1|_{\Gamma_3} = \frac{1}{|a'(\omega_j)|}, \quad |T_2|_{\Gamma_3} = \frac{1}{n^{\alpha+1}} \cdot \left|\frac{c_0(1)}{a^2(\omega_j)}\right|,$$
$$\left|\mathcal{O}\left(\left|\frac{t_\lambda-\omega_j}{t_\lambda^n}\right|\right)\right|_{\Gamma_3} = \mathcal{O}\left(\frac{1}{n}\right),$$
$$\left|\mathcal{O}\left(\frac{|t_\lambda-\omega_j|}{n^{\alpha+1}}\right)\right|_{\Gamma_3} = \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right),$$

and

$$|Q_6(n,t_\lambda)|_{\Gamma_3} = \mathcal{O}\left(\frac{1}{n^{\alpha+\alpha_0+1}}\right).$$

When n goes to infinity, the modulus of T_1 remains constant over Γ_3, while the moduli of the remaining terms in (5.1) are smaller there. Consequently,

$$\left|\frac{D_n(a-\lambda)}{h_0^{n+1}}\right|_{\Gamma_3} = \frac{1}{|a'(\omega_j)|} + \mathcal{O}\left(\frac{1}{n}\right) \text{ as } n \to \infty.$$

As for the radial segments Γ_2 and Γ_4, we start by showing that T_1 and $-T_2$ have the same argument there. Since t_j is a zero of $T_1 - T_2$, we deduce that

$$\arg\left[\frac{1}{t_{j_0}^n \omega_j^2 a'(\omega_j)}\right] = \arg\left[\frac{c_0(1)}{a^2(\omega_j) n^{\alpha+1}}\right]$$

and thus

$$-n\theta_j + \arg\left[\frac{1}{\omega_j^2 a'(\omega_j)}\right] = \arg\left[\frac{c_0(1)}{a^2(\omega_j)}\right]. \tag{5.3}$$

For $t_\lambda \in \Gamma_2$ we have

$$\begin{aligned}
\arg(T_1) &= \arg\left[\frac{1}{t_\lambda^n \omega_j^2 a'(\omega_j)}\right] \\
&= -\frac{n}{2}(\theta_{j-1}+\theta_j) + \arg\left[\frac{1}{\omega_j^2 a'(\omega_j)}\right] \\
&= \frac{n}{2}(\theta_j - \theta_{j-1}) + \arg\left[\frac{c_0(1)}{a^2(\omega_j)}\right] \\
&= \pi + \arg\left[\frac{c_0(1)}{a^2(\omega_j)}\right] \\
&= \arg(-T_2).
\end{aligned}$$

Here, the third line is due to (5.3). In addition, as $n \to \infty$,

$$\left|\mathcal{O}\left(\left|\frac{t_\lambda - \omega_j}{t_\lambda^n}\right|\right)\right|_{\Gamma_2} = \mathcal{O}\left(\frac{1}{n^\epsilon |t_\lambda|^n}\right), \quad \left|\mathcal{O}\left(\frac{|t_\lambda - \omega_j|}{n^{\alpha+1}}\right)\right|_{\Gamma_2} = \mathcal{O}\left(\frac{1}{n^{\alpha+\epsilon+1}}\right),$$

and

$$|Q_6(n, t_\lambda)|_{\Gamma_2} = \mathcal{O}\left(\frac{1}{n^{\alpha+\alpha_0+1}}\right).$$

Furthermore,

$$\begin{aligned}\left|\frac{D_n(a-\lambda)}{h_0^{n+1}}\right|_{\Gamma_2} =& \frac{1}{|t_\lambda^n a'(\omega_j)|} + \mathcal{O}\left(\frac{1}{n^\epsilon |t_\lambda|^n}\right)\\ &+ \frac{1}{n^{\alpha+1}} \cdot \left|\frac{c_0(1)}{a^2(\omega_j)}\right| + \mathcal{O}\left(\frac{1}{n^{\alpha+\epsilon+1}}\right)\end{aligned}$$

over Γ_2 when $n \to \infty$. The situation is similar for the segment Γ_4.

From the previous analysis of $|D_n(a-\lambda)|$ over Γ we infer that for every sufficiently large n we have

$$|T_1 - T_2|_\Gamma \geq \frac{1}{2n^{\alpha+1}} \left|\frac{c_0(1)}{a^2(\omega_j)}\right|$$

and

$$\left|\mathcal{O}\left(\left|\frac{t_\lambda - \omega_j}{t_\lambda^n}\right|\right) + \mathcal{O}\left(\frac{|t_\lambda - \omega_j|}{n^{\alpha+1}}\right) + Q_6(n, t_\lambda)\right|_\Gamma \leq \mathcal{O}\left(\frac{1}{n^{\alpha+\epsilon+1}}\right).$$

Hence by Rouché's theorem, $D_n(a-\lambda)/(-h_0)^{n+1}$ and $T_1 - T_2$ have the same number of zeros in E_j, that is, a unique zero. □

As a consequence of Theorem 5.1, we can iterate the variable t_λ in the equation $D_n(a-\lambda) = 0$, where $D_n(a-\lambda)$ is given by (1.3). In this fashion we find the unique eigenvalue of $T_n(a)$ which is located "close" to each ω_j. We thus rewrite the equation $D_n(a-\lambda) = 0$ in a small neighborhood of ω_j as

$$t_{\lambda_j} = n^{(\alpha+1)/n} \omega_j \left[\frac{a^2(t_{\lambda_j})}{c_0(1) a'(t_{\lambda_j}) t_{\lambda_j}^2}\right]^{\frac{1}{n}} \cdot \Big[1 + Q_7(n, j)\Big]^{-\frac{1}{n}}; \tag{5.4}$$

recall $c_0(1)$ from (4.13). Here the function $z^{1/n}$ takes its principal branch, specified by the argument in $(-\pi, \pi]$. Also notice that $Q_7(n, j) = \mathcal{O}(n^{-\alpha_0})$ as $n \to \infty$, uniformly in $j \in (n_1, n - n_2)$, with n_1, n_2 as in Theorem 5.1.

Proof of Theorem 1.2. Equation (5.4) is an implicit expression for t_{λ_j}. We manipulate it to obtain two asymptotic terms for t_{λ_j}. Remember that λ belongs to $\mathcal{D}(a) \setminus W_0$; see Figure 2. We can choose W so thin that $\lambda_j = a(t_{\lambda_j})$, $a'(t_{\lambda_j})$, and t_{λ_j} are bounded and not too close to zero. After expanding and multiplying the terms in brackets in (5.4), we obtain

$$t_{\lambda_j} = n^{(\alpha+1)/n} \omega_j \left[1 + \frac{1}{n} \log\left(\frac{a^2(t_{\lambda_j})}{c_0(1) a'(t_{\lambda_j}) t_{\lambda_j}^2}\right) + Q_8(n, j)\right], \tag{5.5}$$

where $Q_8(n,j) = \mathcal{O}\left(n^{-\alpha_0-1}\right)$ as $n \to \infty$, uniformly with respect to j in $(n_1, n-n_2)$. Our first approximation for t_{λ_j} is

$$t_{\lambda_j} = n^{(\alpha+1)/n}\omega_j\Big[1 + Q_9(n,j)\Big],$$

where $Q_9(n,j) = \mathcal{O}\left(n^{-1}\right)$ as $n \to \infty$, uniformly in j from $(n_1, n-n_2)$. Replacing t_{λ_j} by this approximation in (5.5) shows that t_{λ_j} equals $n^{(\alpha+1)/n}\omega_j$ times

$$1 + \frac{1}{n}\log\left(\frac{a^2\big(n^{(\alpha+1)/n}\omega_j\left[1+Q_9(n,j)\right]\big)}{c_0(1)a'\big(n^{(\alpha+1)/n}\omega_j\left[1+Q_9(n,j)\right]\big)\big(n^{(\alpha+1)/n}\omega_j\left[1+Q_9(n,j)\right]\big)^2}\right),$$

plus $Q_{10}(n,j)$, where $Q_{10}(n,j) = \mathcal{O}\left(n^{-\alpha_0-1}\right)$ as $n \to \infty$, uniformly with respect to j in $(n_1, n-n_2)$. Now we use the analyticity of a and a' in W to obtain that t_{λ_j} is $n^{(\alpha+1)/n}\omega_j$ times

$$1 + \frac{1}{n}\log\left(\frac{a^2\big(n^{(\alpha+1)/n}\omega_j\big)}{c_0(1)a'\big(n^{(\alpha+1)/n}\omega_j\big)\big(n^{(\alpha+1)/n}\omega_j\big)^2}\right) + Q_{11}(n,j),$$

where $Q_{11}(n,j) = \mathcal{O}\left(n^{-\alpha_0-1}\right)$ as $n \to \infty$, uniformly in $j \in (n_1, n-n_2)$. Taking into account that

$$\frac{a^2\big(n^{(\alpha+1)/n}\omega_j\big)}{c_0(1)a'\big(n^{(\alpha+1)/n}\omega_j\big)\big(n^{(\alpha+1)/n}\omega_j\big)^2} = \frac{a^2(\omega_j)}{c_0(1)a'(\omega_j)\omega_j^2} + \mathcal{O}\left(\frac{\log n}{n}\right) \text{ as } n \to \infty,$$

we can simplify the expression for t_{λ_j} to

$$t_{\lambda_j} = n^{(\alpha+1)/n}\omega_j\left[1 + \frac{1}{n}\log\left(\frac{a^2(\omega_j)}{c_0(1)a'(\omega_j)\omega_j^2}\right) + R_2(n,j)\right],$$

where $R_2(n,j) = \mathcal{O}\left(n^{-\alpha_0-1}\right) + \mathcal{O}\left(n^{-2}\log n\right)$ as $n \to \infty$, uniformly with respect to j in $(n_1, n-n_2)$. □

Proof of Theorem 1.3. Note that

$$n^{(\alpha+1)/n} = \exp\left[(\alpha+1)\frac{\log n}{n}\right] = 1 + (\alpha+1)\frac{\log n}{n} + \mathcal{O}\left(\frac{\log n}{n}\right)^2 \text{ as } n \to \infty. \tag{5.6}$$

Inserting (5.6) in (1.4) we obtain

$$t_{\lambda_j} = \omega_j\left[1 + (\alpha+1)\frac{\log n}{n} + \frac{1}{n}\log\left(\frac{a^2(\omega_j)}{c_0(1)a'(\omega_j)\omega_j^2}\right) + Q_{12}(n,j)\right], \tag{5.7}$$

where $Q_{12}(n,j) = \mathcal{O}\left(n^{-\alpha_0-1}\right)$ as $n \to \infty$, uniformly in $j \in (n_1, n-n_2)$. Applying the symbol a to (5.7), we see that, as $n \to \infty$,

$$\lambda_j = a\big(\omega_j\big) + (\alpha+1)\omega_j a'(\omega_j)\frac{\log n}{n} + \frac{\omega_j a'(\omega_j)}{n}\log\left(\frac{a^2(\omega_j)}{c_0(1)a'(\omega_j)\omega_j^2}\right)$$
$$+ a'(\omega_j)Q_{12}(n,j) + \mathcal{O}\left(\frac{\log n}{n}\right)^2.$$

□

6. An example

The symbol studied by Dai, Geary, and Kadanoff [11] is

$$a(t) = \left(2 - t - \frac{1}{t}\right)^{\gamma} (-t)^{\beta} = (-1)^{3\gamma+\beta} t^{\beta-\gamma} (1-t)^{2\gamma},$$

where $0 < \gamma < -\beta < 1$. In the case $\beta = \gamma - 1$, this function a becomes our symbol with $h(t) = (-1)^{4\gamma-1}(1-t)^{2\gamma}$. We omit the constant $(-1)^{4\gamma-1}$, because it is just a rotation. The conjecture of [11] is that $t_{\lambda_j} \sim n^{(2\gamma+1)/n} \exp(-2\pi i j/n)$. Expansions (1.4) and (1.5) prove this result, giving us an error bound and a mathematical justification.

Our results are valid outside a small open neighborhood W_0 of the origin. Let $W_0 = B_{1/5}(0)$ be the disk of radius 1/5 centered at zero. Table 1 shows the data of numerical computations. It reveals that the maximum error of (1.4) with one term is reduced by nearly 10 times when considering the second term; see also Figure 1.

n	256	512	1024	2048	4096
(1.4) with one term	15.041	7.698	3.900	1.964	0.986
(1.4) with two terms	1.890	0.823	0.394	0.197	0.099
(1.5) with one term	49.374	27.373	14.983	8.126	4.376
(1.5) with two terms	14.655	7.574	3.861	1.952	0.983
(1.5) with three terms	2.633	1.052	0.479	0.226	0.109

TABLE 1. The table shows the maximum error $\times 10^3$, obtained with our different formulas for the eigenvalues of $T_n(t^{-1}(1-t)^{3/4})$ for different values of n. The data was obtained by comparison with the solutions given by *Matlab*, taking into account only the eigenvalues with absolute value greater than or equal to 1/5.

We also performed calculations with our expansions inside $W_0 = B_{1/5}(0)$, and although the error is nearly 8 times the one of outside, the approximation is still valid there because the distance between two consecutive eigenvalues is bigger than the one between an eigenvalue and the respective approximation given by (1.4) with two terms; compare Tables 1 and 2 and see Figure 4. Clearly, to describe the asymptotic behavior of the eigenvalues of $T_n(a)$ completely with mathematical rigor, we need an expression valid inside W_0. We hope to do this in future work.

We remark that if λ is an eigenvalue of $T_n(a)$ and $b_j(\lambda)$ is defined by (3.1), then $(b_j(\lambda))_{j=0}^{n-1}$ is an eigenvector for λ provided $b_{n-1}(\lambda) \neq 0$. In a forthcoming paper we will employ this observation to study the asymptotics of the eigenvectors.

We finally want to emphasize that the results of this paper can be easily translated to the case where the symbol is $\overline{a(t)} = t(1-t^{-1})^{\alpha}\overline{f(t)}$.

n	256	512	1024	2048	4096
(1.4) with one term	29.719	22.395	16.050	11.123	7.528
(1.4) with two terms	6.229	3.756	2.247	1.340	0.797
(1.5) with one term	37.328	22.423	13.406	7.993	4.760
(1.5) with two terms	35.062	25.994	18.573	12.914	8.796
(1.5) with three terms	3.178	1.867	1.107	0.658	0.391

Table 2. The same as in Table 1, only now considering eigenvalues with absolute value less than 1/5.

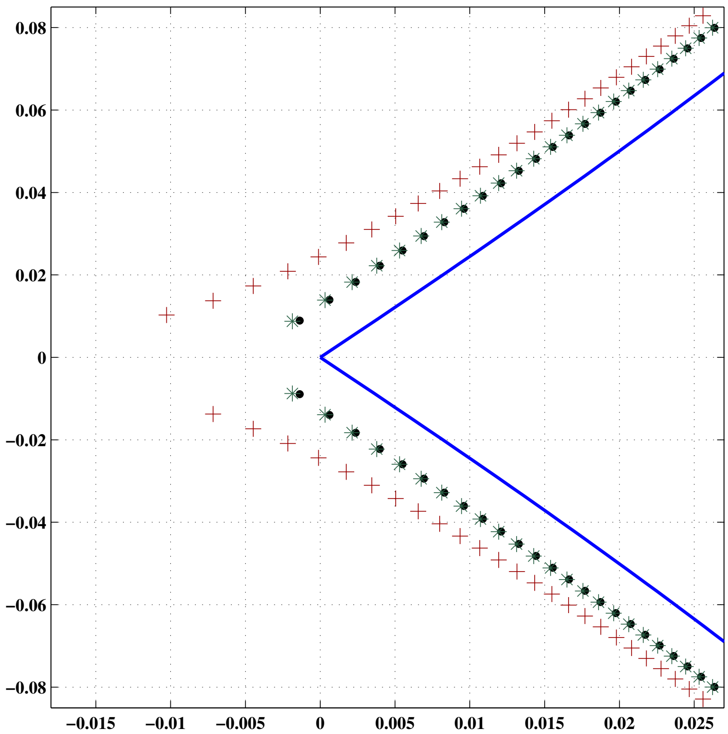

Figure 4. The picture shows a piece of $\mathcal{R}(a)$ for the symbol $a(t) = t^{-1}(1-t)^{3/4}$ (solid line), located "close" to zero. The dots are sp $T_{4096}(a)$ calculated by *Matlab*. The crosses and the stars are the approximations obtained by using 2 and 3 terms of (1.5), respectively.

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