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Eigenvalue problem meets Sierpinski triangle: computing the spectrum of a non-self-adjoint random operator

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Abstract. The purpose of this paper is to prove that the spectrum of the non-self-adjoint one-particle Hamiltonian proposed by J. Feinberg and A. Zee (Phys. Rev. E 59 (1999), 6433–6443) has interior points. We do this by first recalling that the spectrum of this random operator is the union of the set of ℓ^∞ eigenvalues of all infinite matrices with the same structure. We then construct an infinite matrix of this structure for which every point of the open unit disk is an ℓ^∞ eigenvalue, this following from the fact that the components of the eigenvector are polynomials in the spectral parameter whose non-zero coefficients are ± 1's, forming the pattern of an infinite discrete Sierpinski triangle.

Mathematics subject classification (2000): Primary 47B80; Secondary 47A10, 47B36.
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1 Introduction and Notations

In this paper we study infinite matrices of the form

$$\begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & 0 & 1 & & & \\ & b_{-1} & 0 & 1 & & \\ & & b_0 & 0 & 1 & \\ & & & b_1 & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix} \tag{1}$$

with $b_k \in \{\pm 1\} := \{-1, +1\}$ for all $k \in \mathbb{Z}$. We think of (1) as a linear operator acting via matrix-vector multiplication on $\ell^p(\mathbb{Z})$, the standard space of bi-infinite complex sequences with $p \in [1, \infty]$. By $\{\pm 1\}^{\mathbb{Z}}$ we denote the set of all sequences $b = (b_k)_{k\in\mathbb{Z}}$ with $b_k \in \{\pm 1\}$ for all $k \in \mathbb{Z}$, and we refer to the operator on $\ell^p(\mathbb{Z})$ that is induced by the matrix (1) as A^b. For convenience, we will also refer to the matrix (1) as A^b. For $p \in [1, \infty]$ and $b \in \{\pm 1\}^{\mathbb{Z}}$, we write

$$\begin{aligned} \operatorname{spec}^p A^b &:= \{\lambda \in \mathbb{C} : A^b - \lambda I \text{ is not invertible on } \ell^p(\mathbb{Z})\}, \\ \operatorname{spec}^p_{\mathrm{ess}} A^b &:= \{\lambda \in \mathbb{C} : A^b - \lambda I \text{ is not Fredholm on } \ell^p(\mathbb{Z})\}, \\ \operatorname{spec}^p_{\mathrm{point}} A^b &:= \{\lambda \in \mathbb{C} : A^b - \lambda I \text{ is not injective on } \ell^p(\mathbb{Z})\}. \end{aligned}$$

Because (1) is a band matrix, it holds (see [23] and [28]) that $\operatorname{spec}^p A^b$ and $\operatorname{spec}^p_{\mathrm{ess}} A^b$ do not depend on $p \in [1, \infty]$, so that it makes sense to abbreviate these as $\operatorname{spec} A^b$ and $\operatorname{spec}_{\mathrm{ess}} A^b$ in what follows. Note however that the set of eigenvalues, $\operatorname{spec}^p_{\mathrm{point}} A^b$, does depend on p.

Physicists have studied the operator A^b as the (non-self-adjoint) Hamiltonian of a particle hopping (asymmetrically) on a 1-dimensional lattice [15, 16, 9, 22]. Applications of such and related Hamiltonians, especially examples with random diagonals, include vortex line pinning in superconductors and growth models in population biology. The particular model (1) was proposed by Feinberg and Zee in [15], and some properties of its spectrum have been studied in [9, 22] (also see Paragraph 37, in particular Figure 37.7c, in [38]).

In all these studies the focus is on the case of a random sequence $b \in \{\pm 1\}^{\mathbb{Z}}$. A related but completely deterministic concept is that of a pseudo-ergodic sequence. In accordance with Davies [11], we call $b \in \{\pm 1\}^{\mathbb{Z}}$ *pseudo-ergodic* if every finite pattern of ± 1's can be found somewhere (as a string of consecutive entries) in b. If b is pseudo-ergodic (which is almost surely the case if all b_k, $k \in \mathbb{Z}$, are independent (or at least not fully correlated) samples from a random variable with values in $\{\pm 1\}$ and nonzero probability for both $+1$ and -1) then, as a consequence of [7] (also see [6, 8, 29, 30] and cf. [11]), it holds that

$$\operatorname{spec} A^b \;=\; \operatorname{spec}_{\mathrm{ess}} A^b \;=\; \bigcup_{c\in\{\pm1\}^{\mathbb{Z}}} \operatorname{spec} A^c \;=\; \bigcup_{c\in\{\pm1\}^{\mathbb{Z}}} \operatorname{spec}^{\infty}_{\mathrm{point}} A^c. \tag{2}$$

The contribution of [7] is the third "=" sign in (2) that enables, or at least simplifies, the explicit computation of the spectra of particular pseudo-ergodic operators in [6, 8, 29]. The first "=" sign in (2) follows immediately from the second; the second comes from the Fredholm theory of much more general operators and is typically expressed in the language of so-called limit operators [34, 35, 27, 8]. (A similar equality, often with the closure taken of the union of spectra, can be found in the literature on spectral properties of Schrödinger and more general Jacobi operators [32, 4, 10, 11, 21, 1, 31, 17, 18, 19, 20, 33, 26, 25, 36, 37]. The three last papers also shed some light on the role of limit operators in the study of the absolutely continuous spectrum.)

Note that, by (2), the spectrum of A^b does not depend on the actual sequence b – as long as it is pseudo-ergodic. In [6] we obtain information about the spectrum, pseudospectrum and numerical range of the bi-infinite matrix operator A^b, its contraction A^b_+ to the positive half axis (a semi-infinite matrix) and the finite sections A^b_n which, for $n \in \mathbb{N}$, are $n \times n$ submatrices of (1). Explicitly and precisely, these related matrices are

$$A^b_+ \;=\; \begin{pmatrix} 0 & 1 & & & \\ b_1 & 0 & 1 & & \\ & b_2 & 0 & 1 & \\ & & b_3 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \qquad \text{and} \qquad A^b_n \;=\; \begin{pmatrix} 0 & 1 & & & \\ b_1 & 0 & 1 & & \\ & b_2 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & b_{n-1} & 0 \end{pmatrix},$$

where in the case $n = 1$ we set $A^b_1 = (0)$. We explore in some detail in [6] the interrelations between the spectra and pseudospectra of A^b, A^b_+ and A^b_n. Here, for $\varepsilon > 0$ and a bounded operator A on $\ell^2(\mathbb{N})$ or $\ell^2(\mathbb{Z})$, or on $\mathbb{C}^n$ equipped with the 2-norm, we define the *ε-pseudospectrum* of A (see e.g. [2, 38]) by

$$\operatorname{spec}_\varepsilon A \;:=\; \{\lambda \in \mathbb{C} \,:\, \lambda \in \operatorname{spec} A \text{ or } \|(A-\lambda I)^{-1}\| > 1/\varepsilon\},$$

where $\|\cdot\|$ is the induced operator norm. It is convenient also to use the notation $\operatorname{spec}_0 A := \operatorname{spec} A$. Note that the finite matrix A^b_n only depends on the $n-1$ values $b_1, ..., b_{n-1} \in \{\pm 1\}$. Recognising this, we will use the notation $A^{b'}_n$, where $b' = (b_1, ..., b_{n-1}) \in \{\pm 1\}^{n-1}$, as an alternative notation for the same matrix A^b_n.

Here is a summary of our results from [6]:

Theorem 1.1 *[6] If $b \in \{\pm 1\}^{\mathbb{Z}}$ is pseudo-ergodic (which holds almost surely if b is random in the sense discussed above) then the following statements hold.*

a) $\operatorname{spec} A^b$ *is invariant under reflection about either axis as well as under a 90^o rotation around the origin.*

b) *Provided the "positive" part of the sequence b (by which we mean $(b_k)_{k\in\mathbb{N}}$) is itself pseudo-ergodic (contains every finite pattern of ± 1's), then, for all $\varepsilon \geq 0$ one has*

$$\operatorname{spec}_\varepsilon A^b \;=\; \operatorname{spec}_\varepsilon A^b_+.$$

c) *The numerical range of* A^b *(considered as an operator on* $\ell^2(\mathbb{Z})$*) is*

$$W(A^b) \;=\; \{x + \mathrm{i}y \,:\, x, y \in \mathbb{R},\, |x| + |y| < 2\},$$

and spec A^b *is a strict subset of the closure,* clos$\,(W(A^b))$*, of the numerical range, so that*

$$\mathrm{spec}\; A^b \;\subsetneqq\; \{x + \mathrm{i}y \,:\, x, y \in \mathbb{R},\, |x| + |y| \le 2\}.$$

d) *For every* $n \in \mathbb{N}$*, where* $\Pi_n := \big\{c \in \{\pm 1\}^{\mathbb{Z}} : c \text{ is } n\text{-periodic}\big\}$*, the set*

$$\pi_n \;:=\; \bigcup_{c \in \Pi_n} \mathrm{spec}\, A^c \;=\; \bigcup_{c \in \Pi_n} \mathrm{spec}^\infty_{\mathrm{point}}\, A^c \tag{3}$$

is contained in spec A^b*, by* (2)*. Each set* π_n *consists of* k *analytic arcs (see Figure 1.1) with* $\frac{2^n}{n} \le k \le 2^n$ *that can be computed explicitly (as unions of sets of eigenvalues of* $n \times n$ *matrices). In particular,*

$$\pi_1 \;=\; [-2, 2] \cup [-2\mathrm{i}, 2\mathrm{i}] \qquad \text{and} \qquad \pi_2 \;=\; \pi_1 \cup \{x + \mathrm{i}y : -1 \le x \le 1,\, y = \pm x\}.$$

e) *For all* $n \in \mathbb{N}$ *and* $\varepsilon \ge 0$*, the set*

$$\sigma_{n,\varepsilon} \;:=\; \bigcup_{c \in \{\pm 1\}^{n-1}} \mathrm{spec}_\varepsilon\, A_n^c \tag{4}$$

is contained in $\mathrm{spec}_\varepsilon\, A^b$ *(see Figure 1.2 for* $\varepsilon = 0$*).*

f) *In the case of spectra* ($\varepsilon = 0$)*, the finite matrix eigenvalues* $\sigma_n := \sigma_{n,0}$ *from* (4) *are connected with the periodic operator spectra* π_n *from* (3) *by*

$$\sigma_n \;\subset\; \pi_{2n+2} \;\subset\; \mathrm{spec}\, A^b \tag{5}$$

for all $n \in \mathbb{N}$ *(see Figure 2.1).*

g) *As a special case of a much more general spectral inclusion result from [5], we can complement the inclusion* $\sigma_{n,\varepsilon} \subset \mathrm{spec}_\varepsilon\, A^b$ *from e) by*

$$\sigma_{n,\varepsilon} \;\subset\; \mathrm{spec}_\varepsilon\, A^b \;\subset\; \sigma_{n,\,\varepsilon+\varepsilon_n} \qquad \text{and} \qquad \sigma_n \;\subset\; \mathrm{spec}\, A^b \;\subset\; \mathrm{clos}(\sigma_{n,\,\varepsilon_n})\,,$$

for $n \in \mathbb{N}$ *and* $\varepsilon > 0$*, where* $\varepsilon_n = 4 \sin\theta_n < 2\pi/(n+1)$*, with* θ_n *the unique solution in the interval* $\left(\dfrac{\pi}{2(n+3)}, \dfrac{\pi}{2(n+1)}\right)$ *of the equation*

$$2\cos\left((n+1)\theta\right) \;=\; \cos\left((n-1)\theta\right).$$

Remark 1.2 Note that the right "=" sign in (3) holds because $\mathrm{spec}\, A^c = \mathrm{spec}^\infty_{\mathrm{point}}\, A^c$ for all periodic sequences c, whereas the right "=" sign in (2) only holds as stated, with the union taken over all $c \in \{\pm 1\}^{\mathbb{Z}}$; the spectrum and point spectrum of A^c are different, in general, for specific $c \in \{\pm 1\}^{\mathbb{Z}}$. □

Remark 1.3 The inclusions in g) imply that $\mathrm{spec}\, A^b \subset \mathrm{clos}\,(\sigma_{n,\varepsilon_n}) \subset \mathrm{clos}\,(\mathrm{spec}_{\varepsilon_n} A^b)$. Since $\varepsilon_n \to 0$ so that $\mathrm{clos}\,(\mathrm{spec}_{\varepsilon_n} A^b) \to \mathrm{spec}\, A^b$ in the Hausdorff metric [38] as $n \to \infty$, it follows that $\mathrm{clos}\,(\sigma_{n,\varepsilon_n}) \to \mathrm{spec}\, A^b$ as $n \to \infty$. For small values of n the upper bound $\mathrm{clos}\,(\sigma_{n,\varepsilon_n})$ can be evaluated very explicitly. In particular, $\theta_1 = \pi/6$ so that $\varepsilon_1 = 2$ and, since $A_1^c = (0)$, we obtain that $\mathrm{spec}\, A^b \subset \mathrm{clos}\,(\sigma_{1,\varepsilon_1}) = \{\lambda \in \mathbb{C} : |\lambda| \le 2\}$. The result in c) above, that $\mathrm{spec}\, A^b$ is a strict subset of the closure of the numerical range, comes from the bound in g) applied with $n = 34$, when σ_{n,ε_n} is the union of the pseudospectra of $2^{33} \approx 8.6 \times 10^9$ matrices of size 34×34. □

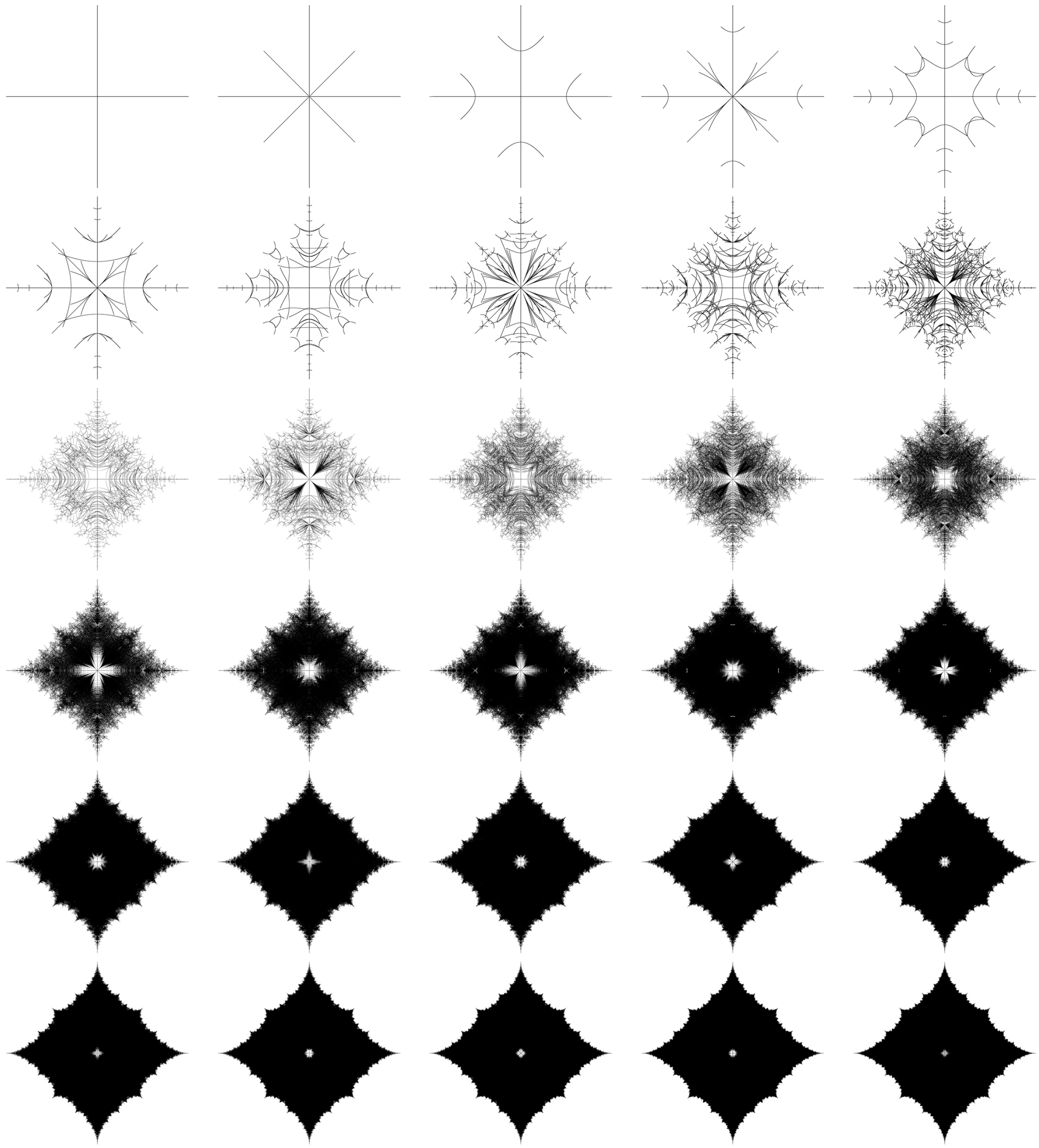


Figure 1.1: Our figure shows the sets π_n, as defined in (3), for $n = 1, ..., 30$. Recall that $\pi_1 = [-2, 2] \cup [-2\mathrm{i}, 2\mathrm{i}]$ and that, for each n, $\pi_1 \subset \pi_n \subset \{x + \mathrm{i}y : x, y \in \mathbb{R}, |x| + |y| \leq 2\}$. Note also that spectra of periodic infinite matrices can be expressed analytically (by Fourier transform techniques, see e.g. [3, 12]) and that each set π_n consists of k analytic arcs, where $2^n/n \leq k \leq 2^n$.

It follows from Theorem 1.1 d) and e) that both

$$\sigma_\infty := \bigcup_{n=1}^{\infty} \sigma_n \qquad \text{and} \qquad \pi_\infty := \bigcup_{n=1}^{\infty} \pi_n$$

are subsets of $\operatorname{spec} A^b$ (with $\sigma_\infty \subset \pi_\infty$, by (5)). These subsets consist of countably many points and countably many analytic arcs, respectively, and so both have zero (two-dimensional) Lebesgue measure. Indeed, it is not clear from any of the results in Theorem 1.1 (or other results in the literature) whether $\operatorname{spec} A^b$ has positive Lebesgue measure, in particular whether it has interior points. Related to this question, Holz et al. [22, Sections I, V, VI], conjecture that $\operatorname{clos}(\sigma_\infty) \subset \operatorname{spec} A^b$ has a fractal dimension in the range $(1, 2)$, and so has zero Lebesgue measure.

The purpose of the current paper is to shed light on these questions by constructing a sequence $c \in \{\pm 1\}^{\mathbb{Z}}$ for which $\operatorname{spec}^{\infty}_{\text{point}} A^c$ contains the open unit disk. As a consequence of formula (2) and the closedness of spectra, this shows that $\operatorname{spec} A^b$ contains the closed unit disk and therefore has dimension 2 and a positive Lebesgue measure. This is the main result of the next section. Intriguingly we will see that the sequence c constructed, while rather irregular, is such that each λ in the unit disk is an eigenvalue of A^c with an eigenvector $u \in \ell^\infty(\mathbb{Z})$ whose components are polynomials in λ with coefficients forming the regular self-similar pattern of a discrete Sierpinski triangle (7).

We will finish the paper with our own conjecture on the geometry of $\operatorname{clos}(\sigma_\infty)$ and $\operatorname{spec} A^b$.

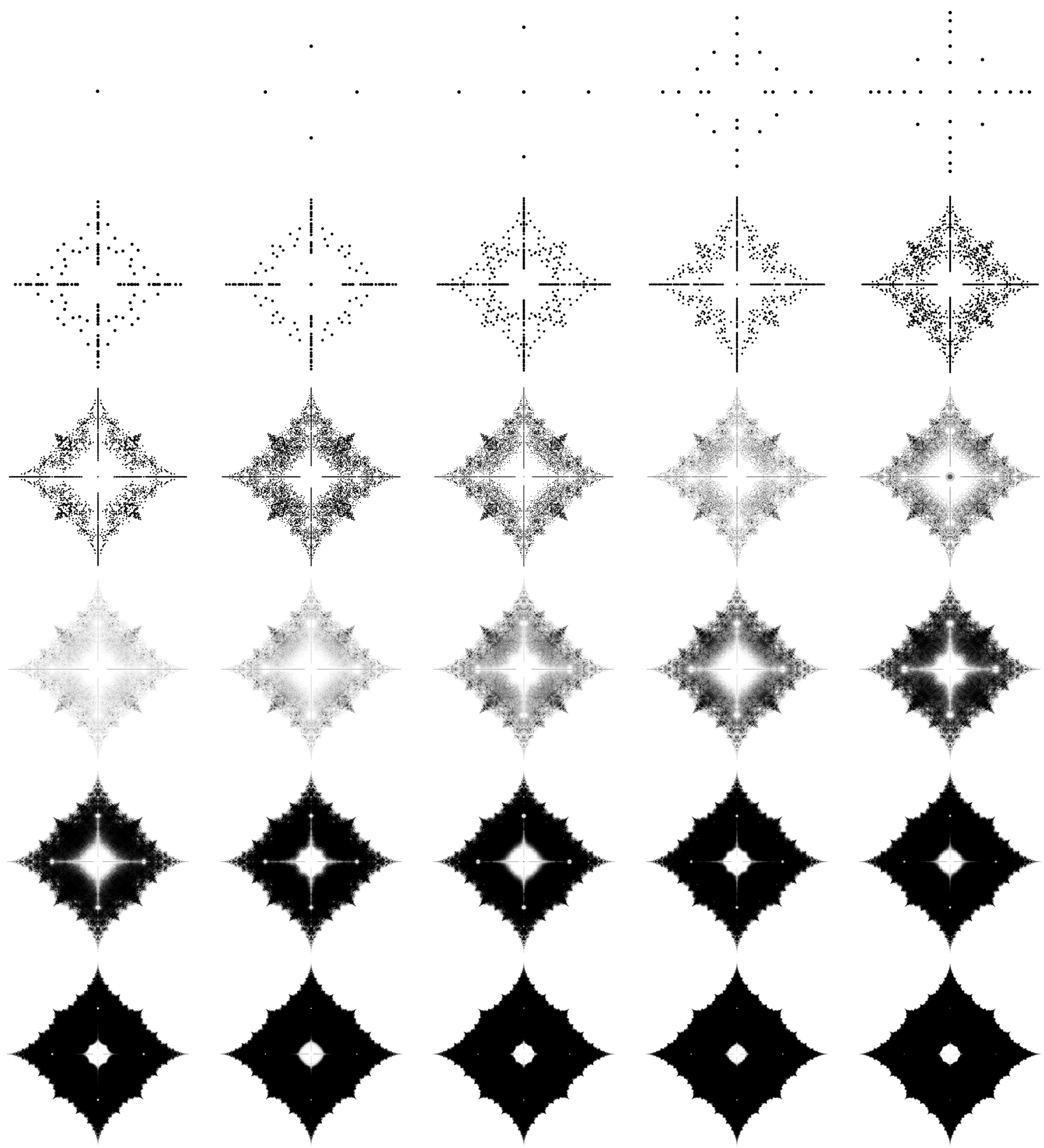


Figure 1.2: Our figure shows the sets $\sigma_n := \sigma_{n,0}$ of all $n \times n$ matrix eigenvalues, as defined in (4), for $n = 1, ..., 30$. Note that in the first pictures (with only a few eigenvalues), we have used heavier pixels for the sake of visibility. By (5), each of the sets with $n = 1, 2, ..., 14$ in this figure is contained, respectively, in the set number $2n + 2$ of Figure 1.1.

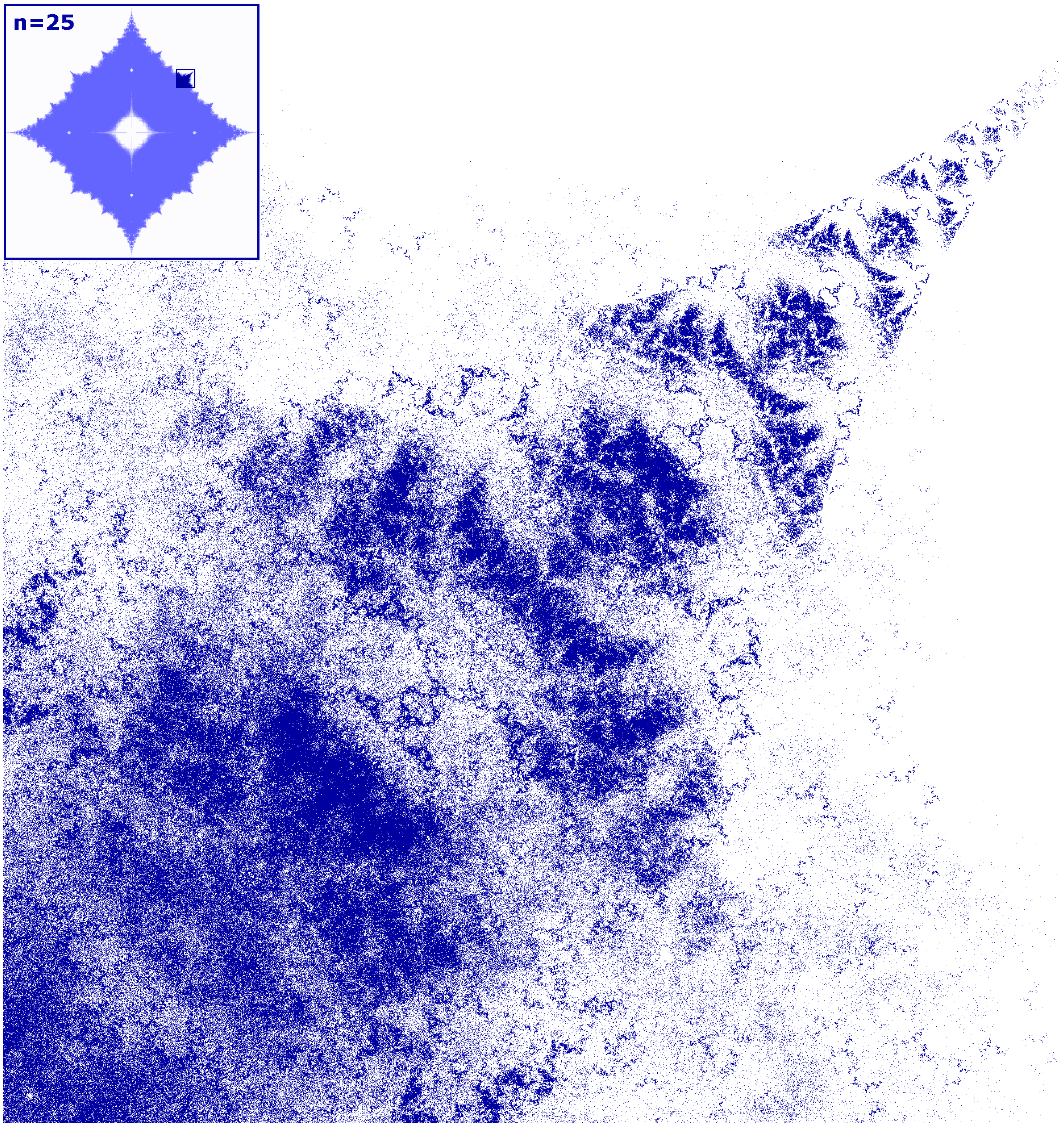


Figure 1.3: This is a zoom into σ_{25} – the 25th picture of Figure 1.2. The location of this zoom is near the point $1 + i$, which is the midpoint of the northeast edge of the square $\text{clos}\,(W(A^b)) = \text{conv}\{2, -2, 2\text{i}, -2\text{i}\}$. The picture clearly suggests self-similar features of the set σ_{25}.

2 A sequence c for which $\operatorname{spec} A^c$ contains the unit disk

The formula (2) for the spectrum of A^b when $b \in \{\pm 1\}^{\mathbb{Z}}$ is pseudo-ergodic motivates the following approach to decide whether a given point $\lambda \in \mathbb{C}$ is in $\operatorname{spec} A^b$ or not: look for a sequence $c \in \{\pm 1\}^{\mathbb{Z}}$ such that $\lambda \in \operatorname{spec}^{\infty}_{\text{point}} A^c$, in other words, such that there exists a non-zero $u \in \ell^{\infty}(\mathbb{Z})$ with $A^c u = \lambda u$, i.e.

$$u_{i+1} \; = \; \lambda u_i \; - \; c_i \, u_{i-1} \tag{6}$$

for $i \in \mathbb{Z}$. If such a sequence c exists then $\lambda \in \operatorname{spec} A^b$ – if not, then not.

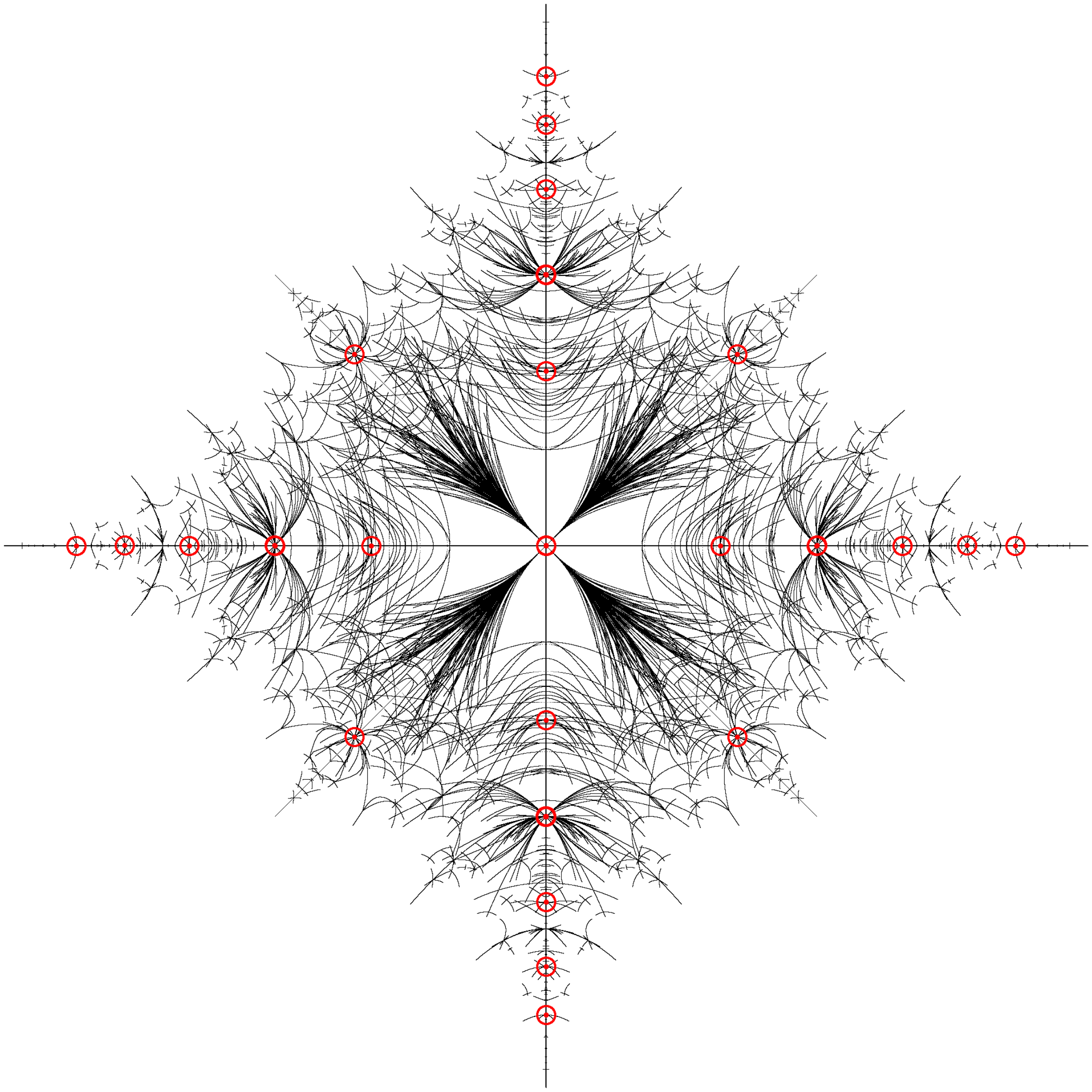


Figure 2.1: Here we see the inclusion $\sigma_5 \subset \pi_{12}$, which holds by (5) with $n = 5$. (The points in σ_5 are indicated by circled dots.)

Starting from $u_0 = 0$ and $u_1 = 1$, we will successively use (6) to compute u_i for $i = 2, 3, ...$ (an analogous procedure is possible for $i = -1, -2, -3, ...$) and see whether the sequence remains bounded. Doing so, we get

$$\begin{aligned} u_2 &= \lambda, \qquad u_3 = \lambda^2 - c_2, \qquad u_4 = \lambda^3 - (c_2 + c_3)\lambda, \\ u_5 &= \lambda^4 - (c_2 + c_3 + c_4)\lambda^2 + c_2 c_4, \end{aligned}$$

and so on. Explicitly, it is easy to check that, for $i \geq 3$, the solution of (6) with initial conditions $u_0 = 0$ and $u_1 = 1$ is given by the characteristic polynomial

$$u_i = \begin{vmatrix} \lambda & -1 & & \\ -c_2 & \lambda & \ddots & \\ & \ddots & \ddots & -1 \\ & & -c_{i-1} & \lambda \end{vmatrix}.$$

Thus, for $i \geq 3$, u_i is a polynomial of degree $i - 1$ in λ with coefficients depending on $c_2, ..., c_{i-1}$. We will aim to achieve that u be a bounded sequence at least for $|\lambda| < 1$.

With this in mind we should try to keep the coefficients of these polynomials small. Precisely, our strategy will be to try to choose $c_1, c_2, \ldots \in \{\pm 1\}$ such that each u_i is a polynomial in λ with coefficients in $\{-1, 0, 1\}$. The following table, where we abbreviate -1 by "$-$", $+1$ by "$+$", and 0 by a space, suggests that this seems to be possible.

		j $\rightarrow$							coefficients of λ^{j-1} in the polynomial u_i									
i	c_i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
1	$+$	$+$																
2	$+$		$+$															
3	$-$	$-$		$+$														
4	$-$				$+$													
5	$+$	$-$		$+$		$+$												
6	$-$		$-$				$+$											
7	$+$	$-$				$+$		$+$										
8	$-$								$+$									
9	$+$	$-$				$+$		$+$		$+$								
10	$+$		$-$				$+$				$+$							
11	$-$	$+$		$-$		$-$				$-$		$+$						
12	$+$				$-$								$+$					
13	$-$	$-$		$+$						$+$		$-$		$+$				
14	$-$		$-$								$+$				$+$			
15	$+$	$-$								$+$				$+$		$+$		
16	$-$																$+$	
$\vdots$	$\vdots$									$\vdots$								

(7)

For $i, j \in \mathbb{N}$, denote the coefficient of λ^{j-1} in the polynomial u_i by $p_{i,j}$. Table (7) shows the values $p_{i,j}$ for $i, j = 1, \ldots, 16$, given the specific choices indicated on the left hand side of the table for the coefficients c_i. From (6) it follows that

$$p_{i+1,j} \; = \; p_{i,j-1} \; - \; c_i \, p_{i-1,j}, \tag{8}$$

for $i \in \mathbb{N}$ and $j = 1, 2, \ldots, i+1$, where we have defined $p_{i,j} := 0$ if $j < 1$, $i < 1$, or $j > i$.

Let us explore more systematically whether it is possible to choose the coefficients c_i so as to ensure that all the coefficients $p_{i,j} \in \{-1, 0, 1\}$. Note first that, if this is possible, then if, for some i, j, one has that $p_{i,j-1} \neq 0$ and $p_{i-1,j} \neq 0$, then $p_{i,j-1}$, $p_{i-1,j} \in \{-1, 1\}$. Thus it follows from (8) that $p_{i+1,j} = 0$, i.e.

$$c_i \; = \; p_{i,j-1}/p_{i-1,j} \; = \; p_{i,j-1}\, p_{i-1,j}, \tag{9}$$

since otherwise $p_{i+1,j} \in \{-2, 2\}$. Illustrating this, look at $p_{15,1} = -1$ and $p_{14,2} = -1$ in the above table. If we chose $c_{15} = -1$, we would get from (8) that $p_{16,2} = -2 \notin \{-1, 0, 1\}$, so it is necessary to choose $c_{15} = 1 = p_{15,1}\, p_{14,2}$. Luckily, the same value $c_{15} = 1$ is required by the values of $p_{15,9}$ and $p_{14,10}$, as well as by $p_{15,13}$ and $p_{14,14}$. We will prove that this coincidence, i.e. that the right-hand side of (9) is (if non-zero) independent of j, is not a matter of fortune. As a result we will show that the pattern of coefficients in table (7) continues without end, only using values from $\{-1, 0, 1\}$ for $p_{i,j}$ and from $\{\pm 1\}$ for c_i. To prove this, we will make use of a particular self-similarity in the triangular pattern of (7); more precisely, we will show that the pattern of non-zero values of the coefficients $p_{i,j}$ forms a so-called infinite discrete Sierpinski triangle.

Proposition 2.1 *Define the sequence $c \in \{\pm 1\}^{\mathbb{Z}}$, for positive indices by $c_1 = 1$ and by the requirement that*

$$c_{2i} \; = \; c_{2i-1}\, c_i \qquad \textit{and} \qquad c_{2i+1} \; = \; -c_{2i}, \qquad i = 1, 2, \ldots,$$

and for non-positive indices by

$$c_{-i} = c_{i+1}, \quad i = 0, 1, \ldots .$$

Further, given $\lambda \in \mathbb{C}$, define the sequence $u = (u_i)_{i \in \mathbb{Z}}$, by the requirement that

$$u_{i+1} \; = \; \lambda u_i - c_i u_{i-1}, \quad i \in \mathbb{Z},$$

and by the initial conditions

$$u_0 = 0, \quad u_1 = 1.$$

Then, as a function of λ, for $i \in \mathbb{Z}$, u_i is a polynomial of degree $|i|-1$ with all its coefficients taking values in the set $\{-1,0,1\}$.

In more detail, denoting, for $i,j \in \mathbb{N}$, the coefficient of λ^{j-1} in the polynomial u_i by $p_{i,j}$, the following statements hold.

(i) $p_{i,j} = 0$ *for* $j > i$, *so that, for every* $i \in \mathbb{N}$,

$$u_i \;=\; \sum_{j=1}^{i} p_{i,j}\, \lambda^{j-1}.$$

(ii) *Defining, additionally,* $p_{i,j} := 0$ *if* $i,j \in \mathbb{N}\cup\{0\}$ *and* $i=0$ *or* $j=0$, *it holds that* $p_{1,1}=1$ *and that*

$$p_{i+1,j} \;=\; p_{i,j-1} \;-\; c_i\, p_{i-1,j}, \tag{10}$$

for $i \in \mathbb{N}$ *and* $j = 1,2,...,i+1$.

(iii) $p_{i,j} = 0$ *if* $i+j$ *is odd.*

(iv) *Writing the semi-infinite coefficient matrix* $P = (p_{i,j})_{i,j\in\mathbb{N}}$ *in block form as* $P = (\mathbf{p}_{i,j})_{i,j\in\mathbb{N}}$ *where, for* $i,j \in \mathbb{N}$,

$$\mathbf{p}_{i,j} := \begin{pmatrix} p_{2i-1,2j-1} & p_{2i-1,2j} \\ p_{2i,2j-1} & p_{2i,2j} \end{pmatrix},$$

it holds, for $i \in \mathbb{N}$, *that* $\mathbf{p}_{i.j} = 0$ *for* $j > i$ *while, for* $j = 1,...,i$,

$$\mathbf{p}_{i,j} = \begin{cases} p_{i,j}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } i+j \text{ is even,} \\ c_{2i-1}\, p_{i-1,j}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } i+j \text{ is odd.} \end{cases} \tag{11}$$

(v) $p_{i,j} \in \{-1,0,1\}$ *for* $i,j \in \mathbb{N}$.

(vi) *Let* $V := \{(0,0),(-1,-1),(1,-1)\}$ *and let* $S := \big\{(i,j) \in \mathbb{N}^2 : p_{i,j} \text{ is non-zero}\big\}$. *Let* $\Sigma := 2^{\mathbb{N}^2}$ *be the set of all subsets of* $\mathbb{N}^2$, *and define* $\mathbf{T} : \Sigma \to \Sigma$ *by*

$$\mathbf{T}(\sigma) \;:=\; 2\sigma + V \;=\; \{2a+b : a \in \sigma,\, b \in V\}, \qquad \text{for } \sigma \in \Sigma.$$

Then, where

$$S_1 \;:=\; \{(1,1)\}, \qquad \text{and} \qquad S_{n+1} \;:=\; \mathbf{T}(S_n), \quad n \in \mathbb{N},$$

it holds that

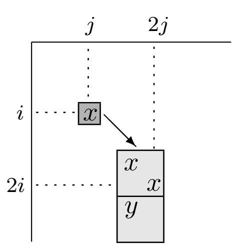
$$S \;=\; \bigcup_{n\in\mathbb{N}} S_n \qquad \text{and that} \qquad S \;=\; \mathbf{T}(S).$$

(vii) *For* $i \in \mathbb{N}\cup\{0\}$,

$$u_{-i} = d_i\, u_i,$$

where, for $j \in \mathbb{N}\cup\{0\}$,

$$d_{2j} := (-1)^j c_{2j}, \quad d_{2j+1} := (-1)^{j+1}.$$



Remark 2.2 Statements (*iv*) and (*vi*) reveal the self-similar nature of the pattern (7). With respect to a scaling of the pattern by the factor 2, an entry $p_{i,j}$, with $i+j$ even, replicates three times: as $p_{2i-1,2j-1}$, $p_{2i,2j}$ and, multiplied by c_{2i+1}, as $p_{2i+1,2j-1}$. So the "volume" of the pattern (7) triples under a scaling by 2, which is why (see [13]) its zeta dimension is $\log_2 3 \approx 1.585$ – exactly the fractal (Hausdorff or box-counting) dimension of its bounded version, the usual Sierpinski triangle or gasket [14]. □

As an immediate consequence of Proposition 2.1 and formula (2) we get our main result.

Theorem 2.3 *For the sequence $c \in \{\pm 1\}^{\mathbb{Z}}$ from Proposition 2.1, it holds that the closed unit disk $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$ is contained in* spec A^c. *Consequently, for a pseudo-ergodic $b \in \{\pm 1\}^{\mathbb{Z}}$, one has $\overline{\mathbb{D}} \subset \operatorname{spec} A^b$, so that* spec A^b *has dimension 2 and a positive Lebesgue measure.*

Proof. Let $\lambda \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, let c be the sequence from Proposition 2.1 and $u : \mathbb{Z} \to \mathbb{C}$ the corresponding eigenfunction from (6). Then, for every $i \in \mathbb{N}$,

$$|u_{-i}| \;=\; |u_i| \;=\; \left|\sum_{j=1}^{i} p_{i,j}\lambda^{j-1}\right| \;\le\; \sum_{j=1}^{i} |p_{i,j}|\,|\lambda|^{j-1} \;\le\; \sum_{j=1}^{\infty} |\lambda|^{j-1} \;=\; \frac{1}{1-|\lambda|}$$

since $p_{i,j} \in \{-1, 0, 1\}$ for all i, j, showing that $u \in \ell^\infty(\mathbb{Z})$, and, by our construction (6), $A^c u = \lambda u$. So $\mathbb{D} \subset \operatorname{spec}^\infty_{\text{point}} A^c \subset \operatorname{spec} A^c$. Since spec A^c is closed, it holds that $\overline{\mathbb{D}} \subset \operatorname{spec} A^c$. The claim for a pseudo-ergodic b now follows from $\operatorname{spec} A^c \subset \operatorname{spec} A^b$, by (2). Finally, from the monotonicity of (all notions of) dimension [14], it follows that $2 = \dim(\overline{\mathbb{D}}) \le \dim(\operatorname{spec} A^b) \le \dim(\mathbb{R}^2) = 2$. ■

Proof of Proposition 2.1. Statements (i) and (ii) are clear from the discussion preceding Proposition 2.1, and statement (iii) then follows easily by induction. Thus $\mathbf{p}_{i,j} = 0$ for $j > i$, and in every matrix $\mathbf{p}_{i,j}$ the off-diagonal entries are zero, i.e. $p_{2i-1,2j} = 0 = p_{2i,2j-1}$ for all i, j.

We will now prove (iv) by proving by induction that, for each $i \in \mathbb{N}$, (11) holds for $j = 1, ..., i$. It is easy to check that (11) holds for $i = j = 1$. Now suppose that, for some $k \in \mathbb{N}$, (11) holds for $i = 1, ..., k$, $j = 1, .., i$. We will show that this implies that (11) holds for $i = k + 1$ and $j = 1, ..., k + 1$.

We let $i = k + 1$ and start with the case when $i + j$ is even. By (10) we have that

$$\begin{aligned} p_{2i-1,2j-1} &= p_{2i-2,2j-2} - c_{2i-2}\, p_{2i-3,2j-1} \\ &= p_{2(i-1),2(j-1)} - c_{2i-2}\, p_{2(i-1)-1,2j-1}, \end{aligned} \tag{12}$$

with $p_{2(i-1),2(j-1)} = p_{i-1,j-1} = 0$ if $j = 1$ and, by the inductive hypothesis (and since $i-1+j-1$ is even), $p_{2(i-1),2(j-1)} = p_{i-1,j-1}$ if $j > 1$. Also, by the inductive hypothesis, $p_{2(i-1)-1,2j-1} = c_{2(i-1)-1}\, p_{i-2,j}$ since $i-1+j$ is odd, while, from the definition of the sequence c, $c_{2i-2} = c_{2i-3}\, c_{i-1}$. Inserting these results in (12), we get that

$$\begin{aligned} p_{2i-1,2j-1} &= p_{i-1,j-1} - c_{2i-3}\, c_{i-1}\, c_{2i-3}\, p_{i-2,j} \\ &= p_{i-1,j-1} - c_{i-1}\, p_{i-2,j} = p_{i,j}, \end{aligned} \tag{13}$$

by (10). We have observed already that $p_{2i-1,2j} = 0 = p_{2i,2j-1}$ for all i, j, so it remains to consider $p_{2i,2j}$. By (10), (13), and the inductive hypothesis which implies that $p_{2i-2,2j} = p_{2(i-1),2j} = 0$ as $i - 1 + j$ is odd, we have that

$$p_{2i,2j} \;=\; p_{2i-1,2j-1} - c_{2i-1}\, p_{2i-2,2j} \;=\; p_{i,j}.$$

Now suppose $i + j$ is odd. Then, by (10) and the inductive hypothesis,

$$\begin{aligned} p_{2i-1,2j-1} &= p_{2i-2,2j-2} - c_{2i-2}\, p_{2i-3,2j-1} \\ &= 0 - c_{2i-3}\, c_{i-1}\, p_{i-1,j} \;=\; c_{2i-1}\, p_{i-1,j}, \end{aligned}$$

since $c_{2i-1} = -c_{2i-2} = -c_{2i-3}\, c_{i-1}$. By (10) and the inductive hypothesis and noting that $i-1+j$ is even,

$$\begin{aligned} p_{2i,2j} &= p_{2i-1,2j-1} - c_{2i-1}\, p_{2i-2,2j} \\ &= c_{2i-1}\, p_{i-1,j} - c_{2i-1}\, p_{i-1,j} \;=\; 0. \end{aligned}$$

This completes the proof of (iv), and (v) follows from (iv) by a simple induction argument.

To see that (vi) is true, observe first that, from (i), (iii), and (iv) (and cf. Remark 2.2), it holds for $i', j' \in \mathbb{N}$ that $(i', j') \in S$ iff, for some $i, j \in \mathbb{N}$ either $(i', j') = (2i, 2j)$ and $(i, j) \in S$; or $(i', j') = (2i - 1, 2j - 1)$ and $(i, j) \in S$; or $(i', j') = (2i + 1, 2j - 1)$ and $(i, j) \in S$. From this it follows that $S = \mathbf{T}(S)$.

Define a metric d on Σ by

$$d(\sigma,\tau) \ := \sum_{(i,j)\in(\sigma\cup\tau)\setminus(\sigma\cap\tau)} 2^{-i-j}, \qquad \sigma,\tau\in\Sigma.$$

Then, since $\big(\mathbf{T}(\sigma)\cup\mathbf{T}(\tau)\big)\setminus\big(\mathbf{T}(\sigma)\cap\mathbf{T}(\tau)\big)\ \subset\ \mathbf{T}\big((\sigma\cup\tau)\setminus(\sigma\cap\tau)\big)$ for all $\sigma,\tau\in\Sigma$,

$$\begin{aligned} d\left(\mathbf{T}(\sigma),\mathbf{T}(\tau)\right) \ &\le \sum_{(i,j)\in(\sigma\cup\tau)\setminus(\sigma\cap\tau)} \left(2^{-2i-2j}+2^{-(2i-1)-(2j-1)}+2^{-(2i+1)-(2j-1)}\right) \\ &= \sum_{(i,j)\in(\sigma\cup\tau)\setminus(\sigma\cap\tau)} 2^{-i-j}\left(2^{-i-j}+2^{2-i-j}+2^{-i-j}\right)\ \le\ \frac{3}{4}\,d(\sigma,\tau), \qquad (14)\end{aligned}$$

if $(1,1)\notin(\sigma\cup\tau)\setminus(\sigma\cap\tau)$. Let $\Sigma_1 := \{\sigma\in\Sigma : (1,1)\in\sigma\}$. Then $\mathbf{T}(\Sigma_1)\subset\Sigma_1$ and, by (14), $\mathbf{T}$ is a contraction mapping on Σ_1. Thus, by the contraction mapping theorem, $\mathbf{T}$ has a unique fixed point in Σ_1, which is the set S, and, if $\sigma_1\in\Sigma_1$ and $\sigma_{n+1} := \mathbf{T}(\sigma_n)$, $n\in\mathbb{N}$, then $d(\sigma_n,S)\to 0$ as $n\to\infty$. In particular, $d(S_n,S)\to 0$ as $n\to\infty$. Since also (by an easy induction argument) $S_1\subset S_2\subset ...$, it follows that $S=\cup_{n\in\mathbb{N}}S_n$.

Define v_{-i} for $i=0,1,...$ by $v_{-i} := d_iu_i$, which implies that $v_0=0$, and set $v_1=1$. Then, since u_i is defined uniquely for $i\le 0$ by the requirement that it satisfy (6) for $i\le 0$ with the initial conditions that $u_0=0$ and $u_1=1$, to show (vii) it is enough to check that the sequence v_i satisfies (6) for $i\le 0$, i.e. that

$$v_{-i+1}=\lambda v_{-i}-c_{-i}v_{-i-1}, \quad i=0,1,\dots.$$

But $v_1-\lambda v_0+c_0v_{-1}=1+c_0d_1u_1=0$, so the equation holds for $i=0$, and, for $i\in\mathbb{N}$,

$$\begin{aligned} v_{-i+1}-\lambda v_{-i}+c_{-i}v_{-i-1} \ &= \ d_{i-1}u_{i-1}-\lambda d_iu_i+c_{i+1}d_{i+1}u_{i+1} \\ &= \ (d_{i-1}-c_ic_{i+1}d_{i+1})u_{i-1}-\lambda(d_i-c_{i+1}d_{i+1})u_i, \end{aligned}$$

since $u_{i+1}=\lambda u_i-c_iu_{i-1}$. Since $u_0=0$, the right hand side of this last equation is zero for $i\in\mathbb{N}$ provided that $d_i=c_{i+1}d_{i+1}$ for $i\in\mathbb{N}$. But this follows from the definitions of the sequences c and d. ■

Remark 2.4 The standard infinite discrete Sierpinski triangle (e.g. [24]) is the set $\tilde{S}\subset\mathbb{N}^2$ defined by $\tilde{S} := \cup_{n\in\mathbb{N}}\tilde{S}_n$, where $\tilde{S}_1 := \{(1,1)\}$ and the sets $\tilde{S}_n$, $n=2,3,...$, are defined recursively by $\tilde{S}_{n+1} := 2\tilde{S}_n+\tilde{V}$, where $\tilde{V} := \{(0,0),(-1,-1),(0,-1)\}$. One instance where $\tilde{S}$ arises is as the pattern of odd coefficients in Pascal's triangle: for $i\in\mathbb{N}$ and $j=1,...,i$ the coefficient of x^{j-1} in $(1+x)^{i-1}$ is odd iff $(i,j)\in\tilde{S}$, so that the discrete Sierpinski triangle is often referred to as Pascal's triangle modulo 2 (e.g. [13]). Proposition 2.1(vi) (cf. Remark 2.2) makes clear that the pattern $S\subset\mathbb{N}^2$ of the non-zero coefficients in table (7) is essentially that of the standard discrete Sierpinski triangle $\tilde{S}$; indeed, the sets $\tilde{S}$ and S are connected by a linear mapping: $(i,j)\in\tilde{S}$ iff $(2i-j,j)\in S$, for $i,j\in\mathbb{N}$. □

Remark 2.5 Note that the sequence c from Proposition 2.1 is not pseudo-ergodic since, by $c_{2i+1}=-c_{2i}$, the patterns "$+\,+\,+$" and "$-\,-\,-$" can never occur as consecutive entries in the sequence c. □

Based on Theorems 1.1 and 2.3 and the numerical results displayed in Figures 1.1 and 1.2, we make the following conjecture.

Conjecture. We conjecture that $\operatorname{clos}(\sigma_\infty)=\operatorname{clos}(\pi_\infty)=\operatorname{spec}A^b$, and that $\operatorname{spec}A^b$ is a simply connected set which is the closure of its interior and which has a fractal boundary.

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