EUROPEAN DOUBLE-BARRIER OPTIONS WITH A COMPOUND POISSON COMPONENT

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Abstract. We consider European double-barrier options for underlyings that are given by the superposition of a Gaussian and a compound Poisson process with discrete values. The determination of the price of such options leads to a Black-Scholes system that is perturbed by a Toeplitz matrix. On the basis of this observation, we design an effective algorithm for the computation of this price. Numerical examples are provided.

Key words. double barrier option, Lévy process, compound Poisson process, Toeplitz matrix

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1. Introduction. The problem of determining the price of a double barrier option when the stock price is modeled by geometric Brownian motion is considered in [10, 12, 13, 15, 19, 23, 25]. In [12, 13, 19, 25] the approach is to solve the Black-Scholes partial differential equation on a strip of finite width. However, for many situations geometric Brownian motion is not an adequate model for stock price, and in recent years Lévy processes have come to be used as models for logarithmic stock price. In this context European options [1, 8, 17, 18, 20, 21], perpetual American options [4, 5, 16], and single barrier options [4, 5, 6, 16] have been examined in detail. Recent papers concerning double barrier options under Lévy processes include [2, 3, 7, 9, 22].

In this article we consider European double-barrier options whose underlyings are Lévy processes formed by the superposition of a Gausssian and a compound Poisson process with discrete values. The determination of the price of such options leads to a Black-Scholes system that is perturbed by a Toeplitz matrix. On the basis of this observation, we design an effective algorithm for the computation of this price. Numerical examples are provided.

The mathematical setting will be a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_\tau\}, P)$ on which $\{X_\tau\}_{\tau\geq 0}$ is the Lévy process ([24], p. 202) specified by

$$E^P\left[e^{i\xi X_\tau}\right] = e^{-\tau\psi(\xi)},$$

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu_0\xi + \varepsilon\left(1 - \sum_{j=-\infty}^{\infty} q_j e^{ij\xi}\right), \tag{1.1}$$

E^P referring to the expected value taken with respect to the probability measure P. Here $\sigma, \mu_0, \varepsilon, q_j$ are real numbers subject to the constraints $\sigma > 0$, $\varepsilon \geq 0$, $q_j \geq 0$, $\sum q_j = 1$, and we also require that only finitely many of the numbers q_j are nonzero. We consider $\{X_\tau\}_{\tau\geq 0}$ under the assumption that we are given two absorbing barriers, one at 0 and one at a natural number $n \geq 2$. Let g be a function in $L^2(0,n)$. Our objective is, for a fixed $t > 0$, to compute the expected value on $(0,n)$ of $e^{-rt}g(X_t)$ with respect to a certain equivalent martingale measure (EMM) Q for P under the condition that X_0 is known to be a given value $x \in (0,n)$. Thus, we look for

$$u(x,t) := E^Q\left[e^{-rt}g(X_t)\mathbf{1}_{\eta>t}\Big|_{\mathcal{F}_0} X_0 = x\right], \tag{1.2}$$

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where the hitting time η is the random variable

$$\eta := \inf\{\tau > 0 : X_\tau \in (-\infty, 0] \text{ or } X_\tau \in [n, \infty)\}$$

and $\mathbf{1}_{(\cdot)}$ denotes the characteristic function of a set.

The value (1.2) may be interpreted as the price for a knock-out double-barrier option. Let $b^- < b^+$ and think of

$$S_\tau = S_0\, b^- e^{((1/n)\log(b^+/b^-))X_\tau} \tag{1.3}$$

as the market price of a stock at time τ. The market drift and volatility are μ_0 and σ, while the parameter r is the rate of the riskless bond. Fix $t > 0$ and let $X_0 = x$. The holder pays the premium $u(x,t)$ at time $\tau = 0$ to the writer and receives the amount $g(X_t) = h(S_t)$ at time $\tau = t$ from the writer provided that the condition $0 < X_\tau < n$, i.e., $b^- < S_\tau/S_0 < b^+$, is maintained for all $\tau \in [0,t]$.

Our assumptions say that

$$X_\tau = \sigma^2 B_\tau + \mu_0 \tau + \sum_{k=1}^{N_\tau} J_k$$

where $B_\tau \sim N(0, \sqrt{\tau})$ is normalized Brownian motion, μ_0 characterizes the drift, N_τ is the Poisson counting process at rate ε,

$$\text{Probability}\,(N_\tau = k) = \frac{(\varepsilon\tau)^k}{k!}\, e^{-\varepsilon\tau} \quad (k = 0, 1, 2, \ldots),$$

and $J_1, J_2, \ldots$ are independent identically distributed random variables with

$$\text{Probability}\,(J_k = j) = q_j \quad (j = 0, \pm 1, \pm 2, \ldots).$$

We remark that terminology is sometimes different, the holder paying the price to the writer at time t and receiving the amount $g(X_T)$ at an agreed point of time $T > t$. If one denotes the option price in this context by $U(x,\tau)$, then clearly this is just $u(x,t)$ with $t = T - \tau$.

It is well known that the problem of defining the option price $u(x,t)$ in the right way is delicate. Under our assumptions, we do not have a complete market. This implies that there is in general no unique EMM and hence the definition of $u(x,t)$ by (1.2) includes a high extent of arbitrariness. We will employ (1.2) with the EMM Q delivered by the Esscher transform, and we find that this is a reasonable starting point for the investigation of double-barrier options under processes with jumps. In Sections 2 and 3 we describe the EMM and give the existence result for the corresponding Black-Scholes system. Sections 4 and 5 contain the detailed numerical algorithm and the computational considerations. In Section 6 we describe the particular situation when $\sigma = \mu_0 = 0$, i.e., when X_τ is driven by pure jumps.

2. An equivalent martingale measure. We determine the EMM Q from the Esscher transform [4, pp. 98–99], that is, from the equation

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_\tau} = e^{\theta X_\tau - d(\theta,\tau)} \tag{2.1}$$

where θ is the real solution of the equation

$$\psi(-i(1+\theta)) - \psi(-i\theta) + r = 0, \tag{2.2}$$

and $d(\theta,\tau) = -\tau\psi(-i\theta)$.

PROPOSITION 1. *Equation (2.2) has a unique real solution* $\theta = \theta_\varepsilon$ *for every* $\varepsilon \in [0,\infty)$. *This solution depends continuously on* ε. *If* $\varepsilon = 0$, *the solution is* $\theta_0 = -(\sigma^2/2 + \mu_0 - r)/\sigma^2$.

Proof. By (1.1), equation (2.2) reads

$$\frac{\sigma^2}{2}(-i(1+\theta))^2 - i\mu_0(-i(1+\theta)) + \varepsilon\left(1 - \sum_j q_j e^{ij(-i(1+\theta))}\right)$$
$$-\frac{\sigma^2}{2}(-i\theta)^2 + i\mu_0(-i\theta) - \varepsilon\left(1 - \sum_j q_j e^{ij(-i\theta)}\right) + r = 0,$$

which can be simplified to

$$\varrho(\theta) := \sigma^2\theta + (\sigma^2/2 + \mu_0 - r) - \varepsilon \sum_{j\neq 0} q_j e^{j\theta}(1 - e^j) = 0. \tag{2.3}$$

If $\varepsilon = 0$, then (2.3) has the unique solution $\theta = \theta_0 = -(\sigma^2/2 + \mu_0 - r)/\sigma^2$. Clearly, $\varrho(-\infty) = -\infty$ and $\varrho(+\infty) = +\infty$, which shows that (2.3) has a solution. Since

$$\varrho'(\theta) = \sigma^2 - \varepsilon \sum_{j\neq 0} q_j j e^{j\theta}(1 - e^j) > 0$$

for all θ, the solution must be unique. The continuous dependence of θ_ε on the parameter ε is obvious. ∎

With Q given by (2.1),

$$E^Q\left[e^{i\xi X_t}\right] = e^{-t\psi^Q(\xi)}, \quad \psi^Q(\xi) := \psi(\xi - i\theta) - \psi(-i\theta),$$

and a simple computation yields

$$\psi^Q(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \delta\left(1 - \sum_{j=-\infty}^{\infty} p_j e^{ij\xi}\right) \tag{2.4}$$

where the EMM market parameters are given by

$$\mu := \mu_0 + \sigma^2\theta_\varepsilon, \quad \delta := \varepsilon S, \quad p_j := \frac{q_j e^{j\theta_\varepsilon}}{S}, \tag{2.5}$$

with

$$S := \sum_{j=-\infty}^{\infty} q_j e^{j\theta_\varepsilon}.$$

3. The generalized Black-Scholes equation. Let $\sigma > 0$, $r > 0$ and let μ, δ, p_j be the parameters (2.5). We consider the operator A defined by

$$(Af)(x) := -\frac{\sigma^2}{2}f''(x) - \mu f'(x) + rf(x) + \delta f(x) - \delta \sum_{j=-\infty}^{\infty} p_j f(x+j)\Big|_{(0,n)}, \tag{3.1}$$

where $f(x+j)\Big|_{(0,n)}$ is $f(x+j)$ for $x+j \in (0,n)$ and zero for $x+j \notin (0,n)$. We think of A as an operator on $L^2(0,n)$ with the (dense) domain $D(A) := C^2[0,n]$.

In [4] it is shown that the function given by (1.2) and (2.1) satisfies the generalized Black-Scholes equation[1]

$$u_t(x,t) + (Au)(x,t) = 0, \quad (x,t) \in (0,n) \times (0,\infty), \tag{3.2}$$

where A is taken in the variable x, along with the boundary conditions

$$u(x,0) = g(x), \quad x \in (0,n), \tag{3.3}$$

$$u(x,t) = 0, \quad (x,t) \in \Big((-\infty,0] \cup [n,\infty)\Big) \times (0,\infty). \tag{3.4}$$

Condition (3.4) is in fact superfluous because A is considered as acting on L^2 over $(0,n)$. We may also write (3.2), (3.3), (3.4) in the form

$$\begin{aligned} u_t(x,t) &= \frac{\sigma^2}{2}\, u_{xx}(x,t) + \mu\, u_x(x,t) - (r+\delta)u(x,t) \\ &\quad + \ \delta \sum_{j=-\infty}^{\infty} p_j\, u(x+j,t)\Big|_{(0,n)} \end{aligned} \tag{3.5}$$

on $(0,n) \times (0,\infty)$ with the boundary conditions

$$u(x,0) = g(x) \quad \text{for} \quad x \in (0,n), \tag{3.6}$$

$$u(0,t) = u(n,t) = 0 \quad \text{for} \quad t \in (0,\infty). \tag{3.7}$$

For $t \in [0,\infty)$, we define $\widetilde{u}(t) \in L^2(0,n)$ by $(\widetilde{u}(t))(x) := u(x,t)$. Then the problem (3.2), (3.3) can be interpreted as the Cauchy problem

$$\frac{d}{dt}\,\widetilde{u}(t) = -(A\widetilde{u})(t), \quad \widetilde{u}(0) = g. \tag{3.8}$$

Let $D_x := d/dx$. In the case $\delta = 0$, the solution of (3.8) (and hence of (3.2), (3.3)) is well known and can be found by separation of variables. Here it is.

THEOREM 2. *Let $A := -(\sigma^2/2)D_x^2 - \mu D_x + rI$. Then the problem (3.8) is well-posed in the sense that $-A$ generates a C_0 contraction semigroup on $L^2(0,n)$. The solution of (3.8) is*

$$u(x,t) = \sum_{k=1}^{\infty} B_k e^{-\lambda_k^0 t} e^{-(\mu/\sigma^2)x} \sin \frac{k\pi}{n} x$$

where

$$\lambda_k^0 := r + \frac{\mu^2}{2\sigma^2} + \frac{k^2\pi^2\sigma^2}{2n^2}, \quad \sum_{k=1}^{\infty} B_k \sin \frac{k\pi}{n} x = e^{(\mu/\sigma^2)x} g(x).$$

For $\delta > 0$, we have the following result. We denote by $\|\cdot\|_2$ the norm in L^2.

[1] More exactly, Theorem 2.13 of [4, p. 65] holds if the Lévy process satisfies the so-called (ACP) condition (see [4, p. 59]). Formally our case does not satisfy the (ACP) condition; however, a minor modification of the proof of their Theorem 2.13 allows one to apply it to our case (see Remarks 2.1 and 2.2 in [4, pp. 64, 66]).

THEOREM 3. *Let A be the operator (3.1). Problem (3.8) is well-posed in the sense that $-A$ generates a C_0 contraction semigroup and*

$$\|e^{-tA}g\|_2 \le e^{-rt}\|g\|_2. \tag{3.9}$$

The resolvent operator $(\lambda I + A)^{-1}$ is compact and hence the spectrum of $-A$ consists entirely of isolated eigenvalues of finite algebraic multiplicity.

Proof. We have

$$-A = \frac{\sigma^2}{2}D_x^2 + \mu D_x - rI - \delta(I - V) \tag{3.10}$$

where $(Vf)(x) := \sum_j p_j f(x+j)\big|_{(0,n)}$. Clearly, (3.9) will follow once we have shown that $(\sigma^2/2)D_x^2 + \mu D_x - \delta(I-V)$ generates a C_0 contraction semigroup. By Theorem 2 and [11, Theorem 2.6.1], it suffices to show that $-\delta(I-V)$ is bounded and dissipative. The boundedness of $-\delta(I-V)$ is obvious. To show that $-\delta(I-V)$ is dissipative, let F denote the Fourier transform, $(Ff)(\xi) := \int_{-\infty}^{\infty} e^{i\xi x} f(x)dx$ ($\xi \in \mathbf{R}$), and notice that $-\delta(I-V)$ can be written as $-\delta F^{-1}\varphi F$ with $\varphi(x) := 1 - \sum_j p_j e^{-ij\xi}$. Since

$$\begin{aligned}
&\operatorname{Re}((-\delta(I-V)f, f) = -\delta \operatorname{Re}(\mathbf{1}_{(0,n)}F^{-1}\varphi F f, f) \\
&= -\delta \operatorname{Re}(F^{-1}\varphi F f, f) = -\delta \operatorname{Re}(\varphi F f, F f) \le 0
\end{aligned}$$

(recall that $\operatorname{Re}\varphi \ge 0$), we see that $-\delta(I-V)$ is dissipative.

Finally, since $(\sigma^2/2)D_x^2 + \mu D_x - rI$ has compact resolvent ([14, p. 187]) and, by (3.10), $-A$ differs from $(\sigma^2/2)D_x^2 + \mu D_x - rI$ by a bounded operator, we deduce that $-A$ must also have a compact resolvent ([14, p. 214]). □

4. Algorithm. Let $\sigma, \mu, r, \delta, p_j$ be real numbers satisfying $\sigma > 0$, $r > 0$, $\delta \ge 0$, $p_j \ge 0$, $\sum p_j = 1$. For a natural number n, we consider the boundary value problem (3.5), (3.6), (3.7).

We divide $(0,n)$ into n pieces of length 1. Given a function f on $(0,n)$, we define functions $f_1, \dots, f_n$ on $(0,1)$ by

$$f_k(x) = f(x+k-1), \quad x \in (0,1),\ k = 1,2,\dots,n.$$

We now can write (3.5) as

$$\begin{pmatrix} u_{1,t} \\ \vdots \\ u_{n,t} \end{pmatrix} = \frac{\sigma^2}{2}\begin{pmatrix} u_{1,xx} \\ \vdots \\ u_{n,xx} \end{pmatrix} + \mu \begin{pmatrix} u_{1,x} \\ \vdots \\ u_{n,x} \end{pmatrix} + T_n(c)\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \tag{4.1}$$

where $T_n(c) = (c_{j-k})_{j,k=1}^n$ is the Toeplitz matrix

$$\begin{pmatrix}
-r-\delta+\delta p_0 & \delta p_1 & \dots & \delta p_{n-1} \\
\delta p_{-1} & -r-\delta+\delta p_0 & \dots & \delta p_{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
\delta p_{-(n-1)} & \delta p_{-(n-2)} & \dots & -r-\delta+\delta p_0
\end{pmatrix}.$$

The boundary conditions (3.6), (3.7) become

$$u_j(x,0) = g_j(x) \text{ for } x \in (0,1), \tag{4.2}$$

$$u_1(0,t) = u_n(1,t) = 0, \tag{4.3}$$

$$u_j(1,t) = u_{j+1}(0,t),\ u_j'(1,t) = u_{j+1}'(0,t) \quad (j = 1,\dots,n-1) \tag{4.4}$$

where $g_j(x) = g(x+j-1)$, $x \in (0,1)$. We look for solutions of the form $u(x,t) = v(x)e^{-\lambda t}$ or, equivalently, of the form

$$\begin{pmatrix} u_1(x,t) \\ \vdots \\ u_n(x,t) \end{pmatrix} = \begin{pmatrix} v_1(x) \\ \vdots \\ v_n(x) \end{pmatrix} e^{-\lambda t}.$$

Equation (4.1) then reads

$$\frac{\sigma^2}{2}\begin{pmatrix} v_1'' \\ \vdots \\ v_n'' \end{pmatrix} + \mu \begin{pmatrix} v_1' \\ \vdots \\ v_n' \end{pmatrix} + T_n(c)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = -\lambda \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \tag{4.5}$$

and the boundary conditions (4.3), (4.4) are

$$v_1(0) = v_n(1) = 0, \tag{4.6}$$

$$v_j(1) = v_{j+1}(0), \quad v_j'(1) = v_{j+1}'(0) \quad (j = 1, \ldots, n-1). \tag{4.7}$$

Suppose the eigenvalues $\gamma_1, \ldots, \gamma_n$ of $T_n(c)$ are all simple. Then there is an invertible matrix $E = (E_{jk})_{j,k=1}^n$ such that

$$T_n(c) = E\Lambda E^{-1} \quad \text{with} \quad \Lambda = \operatorname{diag}(\gamma_1, \ldots, \gamma_n).$$

Put

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = E^{-1}\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}. \tag{4.8}$$

On multiplying (4.5) from the left by E^{-1} we arrive at the equivalent equation

$$\frac{\sigma^2}{2}\begin{pmatrix} y_1'' \\ \vdots \\ y_n'' \end{pmatrix} + \mu \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} + \Lambda\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = -\lambda \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

which can be written as

$$\frac{\sigma^2}{2}\, y_k'' + \mu y_k' + (\lambda + \gamma_k) y_k = 0 \quad (k = 1, \ldots, n). \tag{4.9}$$

Suppose the equation

$$z^2 + \frac{2\mu}{\sigma^2}\, z + \frac{2(\lambda+\gamma_k)}{\sigma^2} = 0 \tag{4.10}$$

has two distinct zeros

$$\alpha_k = -\frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} - \frac{2(\lambda+\gamma_k)}{\sigma^2}}, \quad \beta_k = -\frac{\mu}{\sigma^2} - \sqrt{\frac{\mu^2}{\sigma^4} - \frac{2(\lambda+\gamma_k)}{\sigma^2}}. \tag{4.11}$$

Then (4.9) is satisfied by

$$y_k(x) = a_k e^{\alpha_k x} + b_k e^{\beta_k x},$$

where a_k and b_k are arbitrary constants. These $2n$ constants can be determined from the $2n$ conditions (4.6), (4.7) and from (4.8). By virtue of (4.8),

$$\begin{aligned}
v_j(0) &= \sum_{k=1}^{n} E_{jk} y_k(0) = \sum_{k=1}^{n} E_{jk}(a_k + b_k), \\
v_j(1) &= \sum_{k=1}^{n} E_{jk} y_k(1) = \sum_{k=1}^{n} E_{jk}(e^{\alpha_k} a_k + e^{\beta_k} b_k), \\
v_j'(0) &= \sum_{k=1}^{n} E_{jk} y_k'(0) = \sum_{k=1}^{n} E_{jk}(\alpha_k a_k + \beta_k b_k), \\
v_j'(1) &= \sum_{k=1}^{n} E_{jk} y_k'(1) = \sum_{k=1}^{n} E_{jk}(\alpha_k e^{\alpha_k} a_k + \beta_k e^{\beta_k} b_k),
\end{aligned}$$

and hence (4.6), (4.7) is the $2n \times 2n$ system

$$\begin{aligned}
&\sum_{k=1}^{n} (E_{1k} a_k + E_{1k} b_k) = 0, \\
&\sum_{k=1}^{n} \left(E_{nk} e^{\alpha_k} a_k + E_{nk} e^{\beta_k} b_k\right) = 0, \\
&\sum_{k=1}^{n} \left((E_{jk} e^{\alpha_k} - E_{j+1,k})\, a_k + \left(E_{jk} e^{\beta_k} - E_{j+1,k}\right) b_k\right) = 0 \quad (j = 1, \dots, n-1) \\
&\sum_{k=1}^{n} \left(\alpha_k \,(E_{jk} e^{\alpha_k} - E_{j+1,k})\, a_k + \beta_k \left(E_{jk} e^{\beta_k} - E_{j+1,k}\right) b_k\right) = 0 \quad (j = 1, \dots, n-1).
\end{aligned}$$

Note that the E_{jk}'s depend on λ. Thus, the system is of the form

$$B_{2n}(\lambda) \begin{pmatrix} a_1 \\ b_1 \\ \vdots \\ a_n \\ b_n \end{pmatrix} = 0. \tag{4.12}$$

We first have to find the $\lambda's$ such that $\det B_{2n}(\lambda) = 0$ and then to find an eigenvector to $B_{2n}(\lambda)$ to the eigenvalue 0.

Suppose finally that the two zeros of (4.10) coincide and denote them by α_k. Then the general solution of (4.9) is

$$y_k(x) = a_k e^{\alpha_k x} + b_k x e^{\alpha_k x},$$

we have

$$\begin{aligned}
&y_k(0) = a_k, \quad y_k(1) = e^{\alpha_k} a_k + e^{\alpha_k} b_k, \\
&y_k'(0) = \alpha_k a_k + b_k, \quad y_k'(1) = \alpha_k e^{\alpha_k} a_k + (1 + \alpha_k) e^{\alpha_k} b_k,
\end{aligned}$$

and hence, for the indices k in question, we must make the following changes in the matrix $B_{2n}(\lambda)$:

$$E_{1k} a_k \longrightarrow E_{1k} a_k, \quad E_{1k} b_k \longrightarrow 0,$$

$$
\begin{aligned}
&E_{nk}e^{\alpha_k}a_k \longrightarrow E_{nk}e^{\alpha_k}a_k, \quad E_{nk}e^{\beta_k}b_k \longrightarrow E_{nk}e^{\alpha_k}b_k\\
&(E_{jk}e^{\alpha_k} - E_{j+1,k})\, a_k \longrightarrow (E_{jk}e^{\alpha_k} - E_{j+1,k})\, a_k,\\
&\big(E_{jk}e^{\beta_k} - E_{j+1,k}\big)\, b_k \longrightarrow E_{jk}e^{\alpha_k}b_k,\\
&\alpha_k\, (E_{jk}e^{\alpha_k} - E_{j+1,k})\, a_k \longrightarrow \alpha_k\, (E_{jk}e^{\alpha_k} - E_{j+1,k})\, a_k,\\
&\beta_k\, \big(E_{jk}e^{\beta_k} - E_{j+1,k}\big)\, b_k \longrightarrow \Big((1+\alpha_k)E_{jk}e^{\alpha_k} - E_{j+1,k}\Big) b_k.
\end{aligned}
$$

Let $\lambda_1, \dots, \lambda_L$ be solutions of $\det B_{2n}(\lambda) = 0$ and suppose we have the corresponding exponents $\alpha_{\ell,k}$ $\beta_{\ell,k}$ given by (4.11) and the corresponding solutions $a_{\ell,k}$, $b_{\ell,k}$ of (4.12). For $\ell = 1, \dots, L$, we define

$$
\begin{pmatrix} v_{\ell,1}(x) \\ \vdots \\ v_{\ell,n}(x) \end{pmatrix} = E \begin{pmatrix} a_{\ell,1}e^{\alpha_{\ell,1}x} + b_{\ell,1}e^{\beta_{\ell,1}x} \\ \vdots \\ a_{\ell,n}e^{\alpha_{\ell,n}x} + b_{\ell,n}e^{\beta_{\ell,n}x} \end{pmatrix}, \quad x \in (0,1)
$$

(with the obvious modification if $\alpha_{\ell,k} = \beta_{\ell,k}$). We denote by V_ℓ the function on $(0, n)$ given by

$$
V_\ell(x + k - 1) = v_{\ell,k}(x), \quad x \in (0,1).
$$

The function V_ℓ is in $C^2(0,n) \cap C[0,n]$ and satisfies $V_\ell(0) = V_\ell(n) = 0$. For arbitrary constants $C_1, \dots, C_L$, the function

$$
u_L(x,t) := \sum_{\ell=1}^{L} C_\ell V_\ell(x) e^{-\lambda_\ell t} \tag{4.13}
$$

satisfies (3.7) and (4.1). The constants $C_1, \dots, C_L$ have to be chosen so that

$$
u_L(x,0) = \sum_{\ell=1}^{L} C_\ell V_\ell(x) \approx g(x), \quad x \in (0,n), \tag{4.14}
$$

i.e., so that (3.6) is approximately satisfied. The function $u_L(x,t)$ obtained in this way is the desired approximation to the exact option price $u(x,t)$.

Let

$$
\Phi_0(x) = g(x) - u_L(x,0) = g(x) - \sum_{\ell=1}^{L} C_\ell V_\ell(x)
$$

be the error made in (4.14) and let

$$
\Phi_t(x) = u(x,t) - u_L(x,t) = u(x,t) - \sum_{\ell=1}^{L} C_\ell V_\ell(x) e^{-\lambda_\ell t}
$$

be the difference at time t between the exact solution of (3.5), (3.6), (3.7) and the approximate solution given by the right-hand side of (4.13) with the coefficients from (4.14). Theorem 3 implies that $\|\Phi_t\|_2 \le e^{-rt}\|\Phi_0\|_2$, where $\|\cdot\|_2$ is the L^2 norm on $(0, n)$.

To summarize, the algorithm is as follows. Compute the eigenvalues and eigenvectors of the Toeplitz matrix $T_n(c)$, that is, the numbers $\gamma_1, \dots, \gamma_n$ and the matrix

E. For λ on a grid on the real line, compute the numbers (4.11), construct the matrix $B_{2n}(\lambda)$ (taking into account the modifications if the two numbers (4.11) coincide), and check whether $B_{2n}(\lambda)$ is almost singular (e.g., by computing the determinant or the minimum of the absolute values of the eigenvalues). Refine the grid in neighborhoods of the λ's where $B_{2n}(\lambda)$ is close to singular to find λ's where $B_{2n}(\lambda)$ is actually (or almost actually) singular. Suppose we have L such λ's, $\lambda_1, \ldots, \lambda_L$. For these λ's, solve (4.12), which amounts to finding an eigenvector for the eigenvalue 0. Construct the functions $V_1, \ldots, V_L$ and finally determine $C_1, \ldots, C_L$ from (4.14).

We discuss the effectiveness of this algorithm. As $L \to \infty$, the series (4.14) converge in the L^2 sense. In more detail: a Gibbs's phenomenon appears; that is, the series (4.14) converges uniformly on the segment $[0, L-\varepsilon]$ for each positive ε, and the partial sums are uniformly bounded in neighborhood of the endpoint L. Therefore it is necessary use many coefficients C_ℓ to obtain a good approximation of the function $u_L(x, 0)$. But for $t > 0$ we note the exponential factors $\exp(-\lambda_\ell t)$ in the series (4.13), for which it is known that the λ_ℓ are asymptotically linear in ℓ. Therefore the series (4.13) converges very rapidly if t is not very small. The main drawback of our method is the solution of the nonlinear equation (4.12), which is time-consuming when n is large. However, for a fixed model (i.e., the values σ, μ, δ, p_j and L) we can calculate the λ_ℓ once and then reuse them as many times as desired for different times t and even for different payoffs.

5. Numerical example. Our calculations were performed with Mathematica (Wolfram) but could be reproduced easily in many standard programming languages or (with sufficient resourcefulness) on a spreadsheet. Assume that the upper barrier is at a 25% increase over the current stock price. We will arbitrarily set the jump probabilities at $q_{-1} = 0.6$, $q_0 = 0.1$, $q_1 = 0.3$ and define $\sigma = 0.45$, $\mu_0 = 0.12$, $r = 0.1$. Thus applying $n = 2$ and $b^- = 1$, $b^+ = 1.25$ in (1.3), we are looking at

$$S_\tau = S_0 e^{X_\tau^*} \tag{5.1}$$

where

$$X_\tau^* := \frac{\log(b^+/b^-)}{n} X_\tau = 0.11157 X_\tau.$$

The region within the barrier is $0 = \log b^- < x^* < \log b^+ = 0.22314$.

The EMM parameters (2.5) corresponding to $\varepsilon = 0$ are found to be $\mu = -0.00125$, $\delta = 0$, $p_{-1} = 0.804796$, $p_0 = 0.073704$, $p_1 = 0.121500$. (In these calculations $\theta_0 = -0.598765$ and $S = 1.35677$.) The resulting values of $\det B_{2n}(\lambda)$ are graphed in Figure 5.1. The zeroes $\lambda_1(0)$, $\lambda_2(0)$, ... are easily isolated numerically, using a coarse grid of points separated by, say, $\Delta\lambda = 0.5$; one may then refine the grid in those intervals containing roots, or even more easily, apply standard programs, to approximate these roots to any desired accuracy. For $\varepsilon > 0$ the values of θ_ε of Proposition 1 are likewise determined by a root-finding program, and shown in Figure 5.2. Applying these values one calculates the corresponding EMM parameters and then finds $\lambda_1(\varepsilon)$, $\lambda_2(\varepsilon)$, ... for any fixed ε. To provide some insight into the dependence of the eigenvalues on ε, we graph $\lambda_\ell(\varepsilon)$ in Figure 5.3.

In this example we use a European call with strike price equal to the initial stock value S_0; i.e., we take $h(s) = \max(s - S_0, 0)$. With the normalization $S_0 = 1$, this means $g(x) = (b^+)^{x/n} - 1$ for $0 \le x \le n$, since $s/S_0 - 1 \ge 0$ for these values of x. For each $\ell = 1, \ldots, L$, using the values $\alpha_{k,\ell}, \beta_{k,\ell}$ of (4.11) a null vector $(a_{\ell,1}, b_{\ell,1}, \ldots, a_{\ell,n}, b_{\ell,n})$ of (4.12) is obtained by linear algebra routines. With this we have the

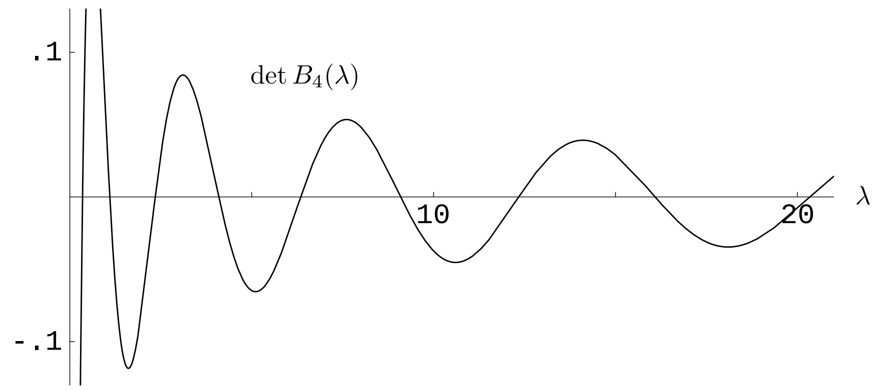

FIG. 5.1. *Location of the zeroes* $\lambda_j(0)$ *of* $\det B_{2n}(\lambda)$ *for* $n = 2$, $\varepsilon = 0$

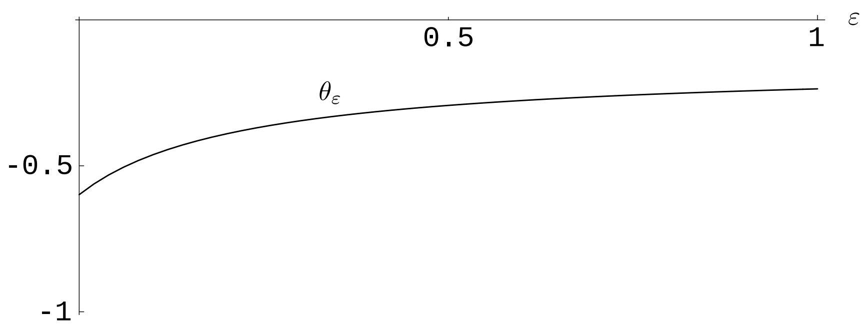

FIG. 5.2. θ_ε *as given by Proposition 1*

functions $V_1, \ldots, V_L$, each being defined by a separate formula of the type $ae^{\alpha x} + be^{\beta x}$ in the successive intervals $(0, 1), (1, 2), \ldots$. In the present calculations we have used $L = 34$, obtained by setting an upper limit of 300.0 for λ_ℓ. The coefficients $C_1, \ldots, C_L$ are obtained by a least-squares fit of (4.14) based on $g(x)$ for $8L$ equally spaced values of $x \in (0, n)$.

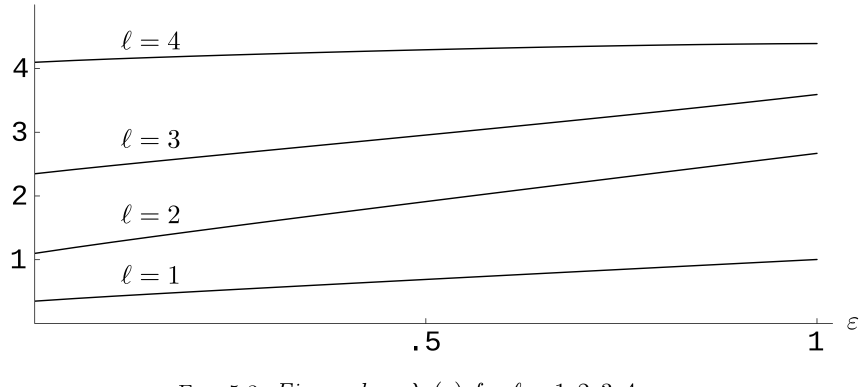

FIG. 5.3. *Eigenvalues* $\lambda_\ell(\varepsilon)$ *for* $\ell = 1, 2, 3, 4$

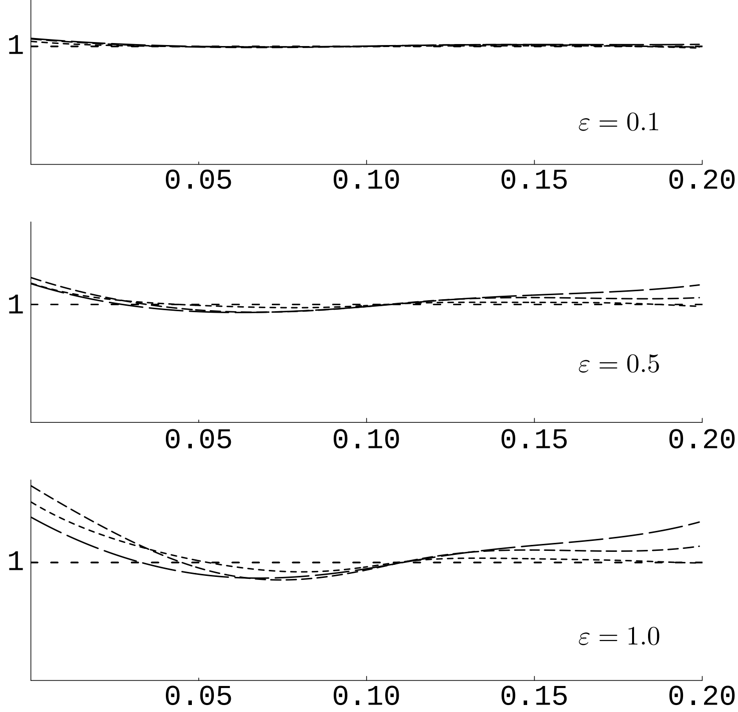

FIG. 5.4. *Variations in option price induced by assigning increasing weight* $\varepsilon = 0.1, 0.5, 1.0$ *to jumps. Ratios* $u_\varepsilon(x,t)/u_0(x,t)$ *are shown for time slices* $t = 0.1, 0.5, 1.0$ *(indicated by increasing lengths of dashes). The horizontal axis is scaled to the auxiliary variable* $x^* = x\log(b^+/b^-)/n$.

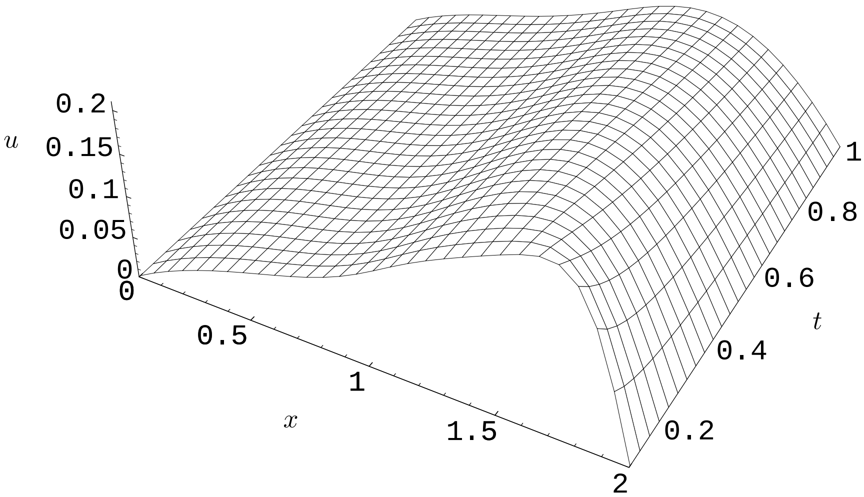

FIG. 5.5. *Option price u for* $\varepsilon = 1.0$ *as function of time t and logarithmic stock price x*

With C_ℓ in hand we have our approximation of $u_\varepsilon(x,t)$ via (4.13). To exhibit the dependence on ε, we plot in Figure 5.4 the ratios $u_\varepsilon(x,t)/u_0(x,t)$ for various values of ε and times t. The reference values $u_0(x,t)$ refer to a market in which the jump phenomenon is insignificant. Finally, the option price surface $u(x,t)$ for $\varepsilon = 1.0$ is drawn in Figure 5.5.

The approximate calculation times in seconds for the essential steps of this algorithm (all calculations done with standard precision) are listed in the following table. For fixed n some minor variation was observed with differences in the probabilities $\{p_i\}$; average values are reported. The time for calculating the coefficients C_ℓ (as well as the functions V_ℓ) does not depend significantly on the value of δ.

n	L	$\{\lambda_\ell\}$	$\{C_\ell\}$
3	34	2.0 sec	4.1 sec
5	30	5.7 sec	41.0 sec
7	28	44.0 sec	50.0 sec

We stress that for a given payoff, the data λ_ℓ, C_ℓ, and V_ℓ would be calculated only once. With these data, the calculation of $u(x,t)$ is extremely rapid.

6. Pure jumps. Let us consider the case where the Brownian component and the drift are absent. Thus,

$$X_\tau = \varepsilon \sum_{k=1}^{N_\tau} J_k, \quad E\left[e^{i\xi X_\tau}\right] = e^{-t\psi(\xi)}, \quad \psi(\xi) := \varepsilon\left(1 - \sum_{j=-\infty}^{\infty} q_j e^{ij\xi}\right),$$

$$(Af)(x) := (r+\delta)f(x) - \delta \sum_{j=-\infty}^{\infty} p_j f(x+j)\Big|_{(0,n)},$$

$$\delta := \varepsilon S, \quad p_j := \frac{q_j e^{j\theta_\varepsilon}}{S}, \quad S := \sum_{j=-\infty}^{\infty} q_j e^{j\theta_\varepsilon}, \tag{6.1}$$

In that case A is bounded on $L^2(0,n)$ and hence $-A$ generates a uniformly bounded semigroup. The system (4.1),(4.2) becomes

$$\begin{pmatrix} u_{1,t} \\ \vdots \\ u_{n,t} \end{pmatrix} = T_n(c) \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \tag{6.2}$$

$$u_j(x,0) = g_j(x) \quad \text{for } x \in (0,1), \tag{6.3}$$

and hence

$$\begin{pmatrix} u_1(x,t) \\ \vdots \\ u_n(x,t) \end{pmatrix} = e^{tT_n(c)} \begin{pmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{pmatrix} = E e^{t\Lambda} E^{-1} \begin{pmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{pmatrix},$$

which can be written in the form

$$u(j-1+x,t) = \sum_{\ell=1}^{n} g(\ell-1+x) \sum_{k=1}^{n} e^{t\gamma_k} E_{jk} (E^{-1})_{k\ell} \tag{6.4}$$

for $j = 1, \ldots, n$ and $x \in (0,1)$.

The case of a tridiagonal Toeplitz matrix is especially simple. Let $q_j = 0$ for $|j| \geq 2$ and define δ and p_{-1}, p_0, p_1 by (6.1). Suppose p_{-1} and p_1 are nonzero. Then

$$T_n(c) = \begin{pmatrix} -r-\delta(1-p_0) & \delta p_1 & 0 & \dots \\ \delta p_{-1} & -r-\delta(1-p_0) & \delta p_0 & \dots \\ 0 & \delta p_{-1} & -r-\delta(1-p_0) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

The eigenvalues of $T_n(c)$ are

$$\gamma_k = -r - \delta(1-p_0) + \delta\sqrt{p_1 p_{-1}}\,\cos\frac{\pi k}{n+1} \quad (k = 1,\dots,n) \tag{6.5}$$

and an eigenvector for γ_k is

$$e_k := \left(\frac{1}{\varrho}\,\sin\frac{\pi k}{n+1}, \frac{1}{\varrho^2}\,\sin\frac{2\pi k}{n+1}, \dots, \frac{1}{\varrho^n}\,\sin\frac{n\pi k}{n+1}\right)^\top$$

with $\varrho := \sqrt{p_1/p_{-1}}$. Thus, $T_n(c) = E\Lambda E^{-1}$ where E is the matrix whose kth column is e_k. The matrix E^{-1} is $2/(n+1)$ times the matrix whose kth row is

$$d_k := \left(\varrho\,\sin\frac{\pi k}{n+1}, \varrho^2\,\sin\frac{2\pi k}{n+1}, \dots, \varrho^n\,\sin\frac{n\pi k}{n+1}\right).$$

Inserting this in (6.4) we arrive at the following result.

Theorem 4. *If $T_n(c)$ is tridiagonal and $p_1 p_{-1} \neq 0$, then*

$$u(j-1+x,t) = \frac{2}{n+1}\,\frac{1}{\varrho^j}\sum_{\ell=1}^{n} g(\ell-1+x)\varrho^\ell \sum_{k=1}^{n} e^{t\gamma_k}\,\sin\frac{k\pi j}{n+1}\,\sin\frac{k\pi\ell}{n+1}$$

where γ_k is given by (6.5) and $\varrho = \sqrt{p_1/p_{-1}}$.

Letting $n \to \infty$, we get the solution in the single-barrier case: for $j = 1, 2, \dots$ and $x \in (0,1)$,

$$u(j-1+x,t) = \frac{1}{\varrho^j}\sum_{\ell=1}^{\infty} g(\ell-1+x)\varrho^\ell e^{tc_0}\int_0^1 e^{t\delta\sqrt{p_1 p_{-1}}\cos(\pi\xi)}\,\sin(\pi j\xi)\,\sin(\pi\ell\xi)\,d\xi$$

with $c_0 = -r - \delta(1-p_0)$. We finally mention that in the case of no barriers the solution is

$$u(j-1+x,t) = \sum_{\ell=-\infty}^{\infty} g(\ell-1+x)\,\frac{1}{2\pi}\int_0^{2\pi} e^{i\xi(\ell-j)}e^{tr+t\psi^Q(-\xi)}\,d\xi$$

where $\psi^Q(\xi) := \delta(1 - p_{-1}e^{-i\xi} - p_0 - p_1 e^{i\xi})$.

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