Visualization of Complex Functions
– Plea for the Phase Plot –

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This work has been inspired by the recent article "Möbius Transformations Revealed" by Douglas N. Arnold and Jonathan Rogness [2] in the Notices. The authors write:

"*Among the most insightful tools that mathematics has developed is the representation of a function of a real variable by its graph. . . . The situation is quite different for a function of a complex variable. The graph is then a surface in four dimensional space, and not so easily drawn. Many texts in complex analysis are without a single depiction of a function. Not it is unusual for average students to complete a course in the subject with little idea of what even simple functions, say trigonometric functions, 'look like'.*"

There are praiseworthy exceptions from this rule, like the textbooks by Tristan Needham [10] with its beautiful illustrations, and by Steven Krantz [7], with a chapter on computer packages for studying complex variables". And certainly some of us have invented their own techniques to visualize complex functions in teaching and research.

The objective of this paper is to promote one such method which is equally simple and powerful. It does not only allow to depict complex functions, but may also serve as a tool for their visual exploration. Sometimes it also provides a new view on known results and opens up new perspectives, as is demonstrated in this paper for a universality property of Riemann's Zeta function.

The Analytic Landscape

The visualization of functions is based on and limited by our intuitive understanding of geometric objects. The graph of a complex function $f : D \subset \mathbb{C} \to \mathbb{C}$ lives in four real dimensions, and since our imagination is trained in three dimensional space, most of us have difficulties to "see" such an object.

Of course one can separate complex-valued functions into their real and imaginary parts. This, however, destroys their mutual relation, and exactly this interplay, controlled by the Cauchy-Riemann equations, is essential for *analytic* functions.

Some old books on function theory used to have nice illustrations of complex functions. These figures show the *analytic landscape* of a function f, which is the graph of its modulus $|f|$.

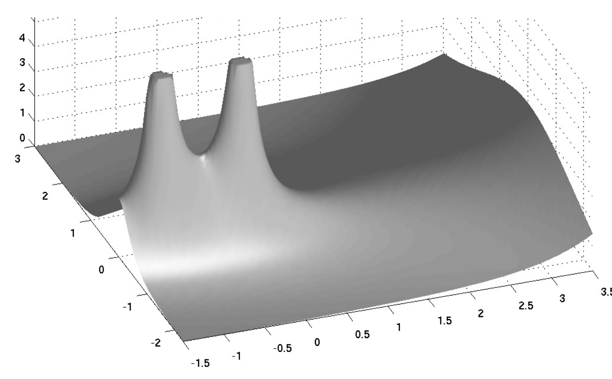

Figure 1: Analytic landscape of the complex Gamma function

Taking into account how much easier it is today to produce such illustrations, it seems to be strange that we do not find them more often in contemporary textbooks.

But in fact this may happen for good reasons. Analytic landscapes must be generated with care, otherwise discretization effects may result in pictures which rather hide what they should demonstrate.

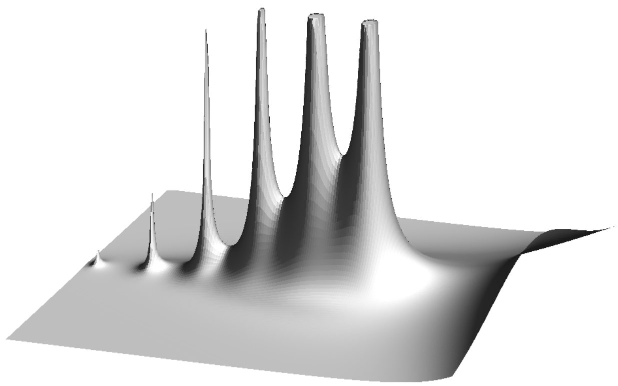

Figure 2: Some "poles" of the complex Gamma function

In order to understand what a pole is we need not really see the volcanoes of an analytic landscape, and students looking at Figure 2 may even be in doubt about the validity of the maximum principle.

Also, the perspective view of the surface embedded in three dimensional space does not allow to see things very precisely. For instance, it is almost impossible to read off location and degree of a zero from the analytic landscape. So what else can we do?

Colored Analytic Landscape

The analytic landscape involves only one part of the function f, namely its *modulus* $|f|$. The second part, its *argument* $\arg f$, is lost. How can we incorporate this missing information appropriately?

Recall that the argument of a complex number is only defined up to an additive multiple of 2π. In order to get a well-defined function, we frequently restrict it artificially to the interval $(-\pi, \pi]$, or, even worse, to $[0, 2\pi)$. This drawback vanishes if we replace φ with the *phase* $e^{i\varphi}$, which lives on the complex unit circle $\mathbb{T}$. And points on a circle can naturally be encoded by *colors*. So we let color serve as the lacking fourth dimension in representing graphs of complex-valued functions.

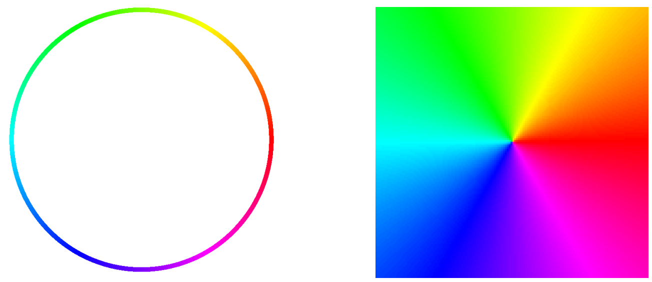

Figure 3: The color circle and the color encoded phase of points close to the origin

Though one usually does not distinguish between the notions of "argument" and "phase", this difference is essential for our purposes. Since the phase of $f(z)$ is just the quotient $f(z)/|f(z)|$, one need not worry about the annoying multi–valuedness of the argument.

The *colored analytic landscape* is the graph of $|f|$ colored according to the phase of f.

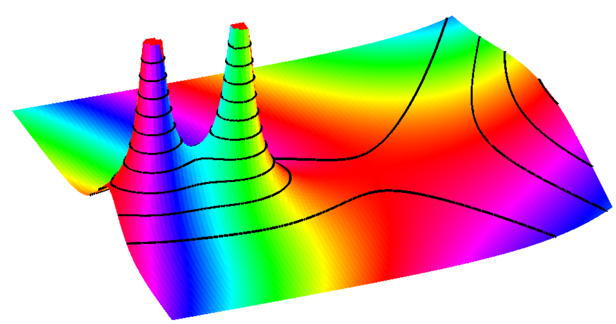

Figure 4: Colored analytic landscape of the complex Gamma function

Typically analytic functions grow very fast at infinity, at singularities or at the boundary of their (natural) domains. In such cases it is convenient to use the graph of $\ln|f|$ instead of $|f|$ for constructing the surface of the analytic landscape. Since $\ln|f|$ and $\arg f$ are harmonic conjugate functions the logarithmic representation is also more natural.

The Phase Plot

One drawback of the colored analytic landscapes is their three–dimensionality. The perspective view makes it difficult to locate points on the surface, and essential details may be invisible. Also, there is no standard view, which would allow to recognize functions visually by their canonical appearance, as we do with the real sine function, for instance.
But indeed there *is* such a distinguished perspective, namely the view “straight from the top”. What we see then is a flat color image which just displays the phase. As will be shown in a moment, this *phase plot* alone gives (almost) all relevant information about the depicted analytic function.

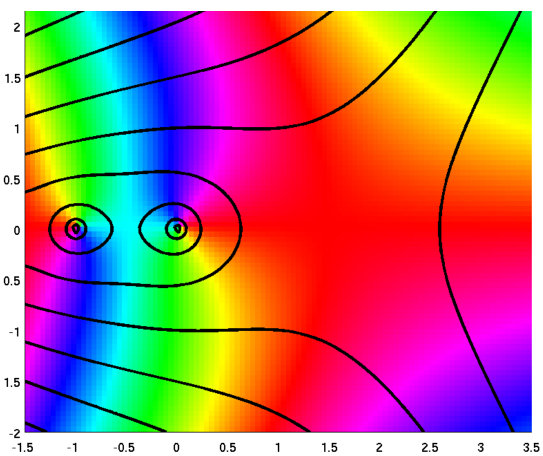

Figure 5: Phase plot of the Gamma function with logarithmically spaced level lines

And if one would not like to miss the information about $|f|$, it can easily be incorporated into the phase plot as (preferably logarithmically spaced) level lines.

How to Read It

Though the phase of a function f with domain D lives on the set

$$D_0 := \{z \in D : \ f(z) \in \mathbb{C} \setminus \{0\}\}$$

we shall nevertheless speak of phase plots on D, considering those points where the phase is undefined as singularities.

If f is analytic or, more generally, meromorphic in D, then its phase (plot) encodes the full information about the function up to a positive scaling factor.

Theorem 1. *If two (non–zero) meromorphic functions f and g on a connected domain D have the same phase plot, then f is a positive scalar multiple of g.*

Proof. Removing from D all zeros and poles of f and g we get a connected domain D_0. Since, by assumption, $f(z)/|f(z)| = g(z)/|g(z)|$ for all $z \in D_0$, the function f/g is holomorphic and real-valued in D_0, and so it must be a (positive) constant.

It is clear that the result extends to the case where the phases of f and g coincide on an open subset of D.
In order to check if two functions f and g with the same phase are equal, it suffices to compare their values at a single point which is neither a zero nor a pole.
There is also an intrinsic test which avoids values and works with phases alone: If f and g are not constant (in which case the phase plot is isochromatic), it follows from the open mapping principle that $f = g$ if the phases of $f + c_i$ and $g + c_i$ coincide for two distinct constants c_1 and c_2.

Zeros and Poles

Since the phases of zero and infinity are undefined, zeros and poles of f are singularities of its phase plot. How does the plot look like in a neighborhood of such points?
If a meromorphic function f has a zero of degree n at z_0 it can be represented as

$$f(z) = (z - z_0)^n \, g(z),$$

where g is meromorphic and $g(z_0) \neq 0$. It follows that the phase plot of f close to z_0 looks like the phase plot of z^n at 0, rotated by the angle $\arg g(z_0)$. The same reasoning, with a negative integer n, applies to poles.

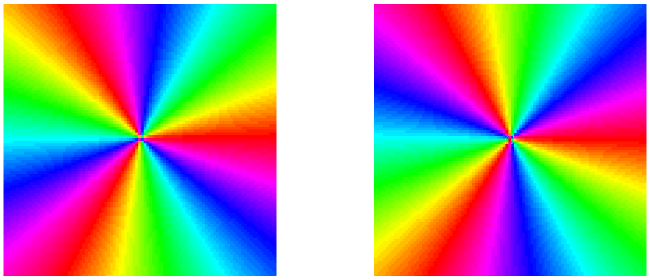

Figure 6: Phase plots in a neighborhood of a zero (left) and a pole (right) of third order

Note that the colors are arranged in different orders for zeros and poles. It is now clear that the phase plot does not only show the location of zeros and poles but also reveals their *degrees*.
A useful tool for locating zeros is the *argument principle*. In order to formulate it in the context of phase plots we translate the definition of winding numbers into the language of colors: Let $\gamma : \mathbb{T} \to D_0$ be a closed oriented path in the domain D_0 of a phase plot $P : D_0 \to \mathbb{D}$. Then the usual winding number (or index) of the mapping $P \circ \gamma : \mathbb{T} \to \mathbb{T}$ is called the *color index* of γ with respect to the phase plot P and is denoted by $\operatorname{cind} \gamma$.
Less formally, the color index counts how many times the color of the point $\gamma(t)$ moves around the color circle when $\gamma(t)$ traverses γ once in positive direction.
Now the argument principle can be rephrased as follows: *Let D be a Jordan domain with (positively oriented) boundary ∂D and assume that f is meromorphic in a neighborhood of D. If f has n zeros and p poles in D (counted with multiplicity), and none of them lies on ∂D, then*

$$n - p = \operatorname{cind} \partial D.$$

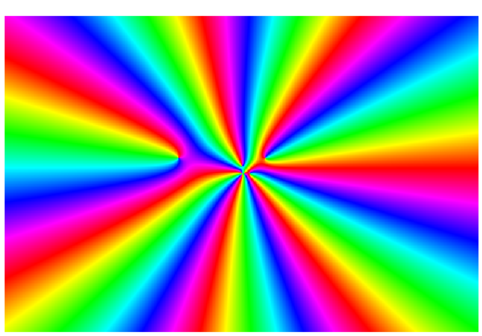

Figure 7: This function has no poles. How many zeros are in the displayed rectangle?

Looking at this picture in search for zeros immediately brings forth new questions, for example: Where do the isochromatic lines end up? Can these lines connect two zeros? If so, do they have a special meaning? What about "basins of attraction"? Is there always a natural (cyclic) ordering of zeros? What can be said about the global structure of phase plots?

Essential Singularities

Have you ever seen an essential singularity? Here is the picture which usually illustrates this situation.

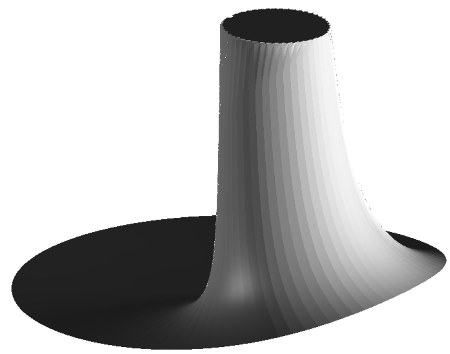

Figure 8: Analytic landscape of $f(z) = \mathrm{e}^{1/z}$

Despite of the massive tower this is not very impressive, and with regard to the Casorati-Weierstrass theorem, or even Picard's great theorem one would expect something much wilder. Now look at the colored logarithmic analytic landscape and the phase plot:

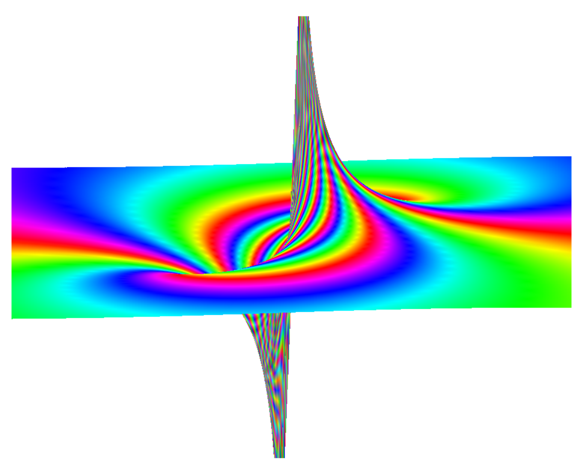

Figure 9: Colored logarithmic analytic landscape of $f(z) = \mathrm{e}^{1/z}$

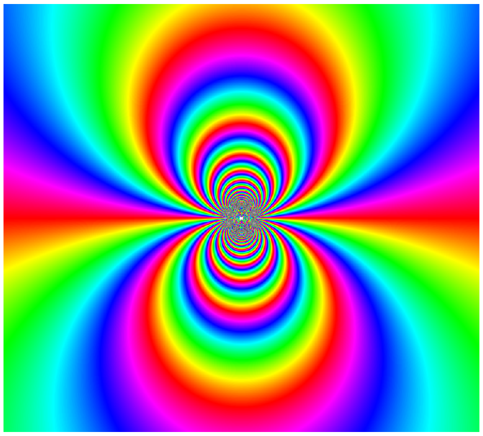

Figure 10: Phase plot of the essential singularity of $f(z) = \mathrm{e}^{1/z}$

The Logarithmic Derivative

Along the *isochromatic lines* of a phase plot the argument of f is constant. The Cauchy-Riemann equations for (any continuous branch of) the logarithm $\log f = \ln|f| + \mathrm{i} \arg f$ imply that these lines are orthogonal to the level lines of $|f|$, i.e. the isochromatic lines are parallel to the gradient of $|f|$. According to the chosen color scheme, we have red on the right and green on the left when walking on a yellow line in ascending direction.

To go a little beyond this qualitative result, we denote by s the unit vector parallel to the gradient of $|f|$ and set $n := \mathrm{i}s$. With $\varphi := \arg f$ and $\psi := \ln|f|$ the Cauchy-Riemann equations for $\log f$ yield that the directional derivatives of φ and ψ satisfy

$$\partial_s\psi = \partial_n\varphi > 0, \quad \partial_n\psi = -\partial_s\varphi = 0,$$

at all regular points of the phase plot. Since the absolute value of $\partial_n\varphi$ measures the density of the isochromatic lines we can visually estimate the growth of $\ln|f|$ along these lines from their density. Because the phase plot delivers no information on the absolute value, this does not say much about the growth of $|f|$. But taking into account the second Cauchy-Riemann equation and

$$|(\log f)'|^2 = (\partial_n\varphi)^2 + (\partial_s\varphi)^2,$$

we obtain the correct interpretation of the density $\partial_n\varphi$: it is the modulus of the *logarithmic derivative*,

$$\partial_n\varphi = |f'/f|\,. \tag{1}$$

So, finally, we need not worry about branches of the logarithm. It is worth mentioning that $\partial_n\varphi(z)$ behaves asymptotically like $k/|z - z_0|$ if z approaches a zero or pole of order k at z_0.

But this is not yet the end of the story. What about zeros of f'? Equation (1) indicates that something should be visible in the phase plot. Indeed points z_0 where $f'(z_0) = 0$ and $f(z_0) \neq 0$ are "*color saddles*", i.e. intersections of isochromatic lines. If f' has a zero of order k at z_0 then $2k + 2$ such lines of a specific color emanate from z_0, which is equivalent to saying that $k+1$ smooth isochromatic lines intersect at z_0.

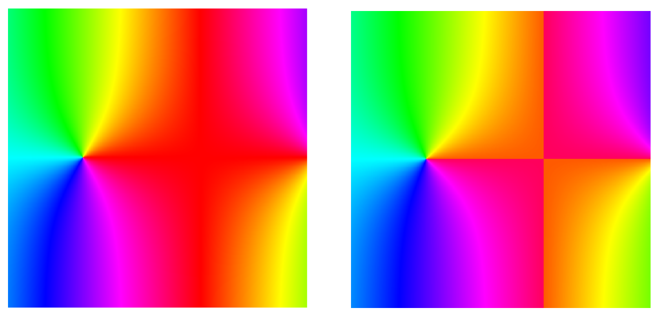

Figure 11: Zeros of f' are color saddles

Color saddles appear as diffuse spots and it needs some training to detect them. Here it helps to use a color scheme which has a jump at some point t of the unit circle. If we choose $t := f(z_0)/|f(z_0)|$, then the phase plot depicts a sharp saddle at the zero z_0 of f'. When t rotates through the full (color) circle, every saddle shows up clearly at some moment.

Periodic Functions

Obviously, the phase of a periodic function is periodic, but what about the converse? If, for example, a phase plot is doubly periodic, can we then be sure that it represents an elliptic function?

Though there are only *two* classes (simply and doubly periodic) of nonconstant periodic meromorphic functions on $\mathbb{C}$, we can observe *three* different types of periodic phase plots.

Figure 12: Phase plot of $f(z) = \mathrm{e}^z$

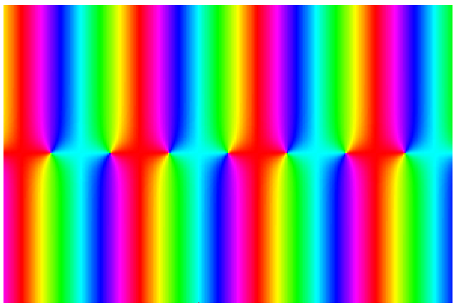

Figure 13: Phase plot of $f(z) = \sin z$

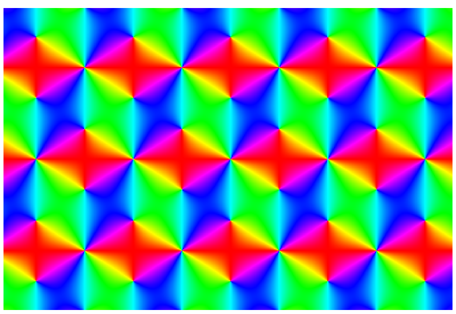

Figure 14: Phase plot of a Weierstrass $\wp$-function

Motivated by these pictures we give the following definition: A nonconstant map Φ is said to be

(i) *striped* if there exists $p_0 \neq 0$ such that for all $p = \alpha p_0$ with $\alpha \in \mathbb{R}$

$$\Phi(z+p) = \Phi(z) \text{ for all } z, \tag{2}$$

(ii) *simply periodic* if there exists $p_1 \neq 0$ such that (2) holds if and only if $p = k\,p_1$ for all $k \in \mathbb{Z}$,

(iii) *doubly periodic* if there exist $p_1, p_2 \neq 0$ with $p_1/p_2 \notin \mathbb{R}$ such that (2) holds if and only if $p = k_1 p_1 + k_2 p_2$ for all $k_1, k_2 \in \mathbb{Z}$.

As usual the numbers p_1, p_2 in (ii) and (iii) are referred to as *fundamental periods* of Φ.

Theorem 2. *Let f be a nonconstant meromorphic function on $\mathbb{C}$ with phase $\Phi := f/|f|$.*

(i) *The phase Φ is striped if and only if there exist $a, b \in \mathbb{C}$ with $a \neq 0$ such that*

$$f(z) = \exp(az + b).$$

(ii) *The phase Φ is simply periodic with fundamental period p if and only if there exist a simply periodic function $g : \mathbb{C} \to \mathbb{C}$ with period p and a real number α such that*

$$f(z) = \exp(\alpha z/p) \cdot g(z).$$

(iii) *The phase Φ is doubly periodic if and only if f is doubly periodic.*

Proof. The "if-direction" of all statements is easy to verify.
(i) After rotating the z-plane we may assume that p_0 is real. Then $\Phi(z)$ only depends on the imaginary part y of z. Writing Φ (locally) as $\Phi = \mathrm{e}^{\mathrm{i}\varphi}$ we get a harmonic function φ which depends only on y. Hence $\varphi(z) = \alpha y + \beta$. The function $-\ln|f|$ is (locally) a harmonic conjugate of φ, thus $\ln|f(z)| = \alpha x + \gamma$. Finally $f(z) = \mathrm{e}^{\alpha z + b}$ with $b = \gamma + \mathrm{i}\beta$. The case $\alpha = 0$ corresponds to a constant function with isochromatic phase plot, which was excluded.
(ii) If (2) holds for the phase of f we have

$$h(z) := \frac{f(z+p)}{f(z)} = \frac{|f(z+p)|}{|f(z)|} \in \mathbb{R}_+.$$

Since h is meromorphic on $\mathbb{C}$ it must be a positive constant e^{α}. It is easy to check that $g(z) := f(z) \cdot \mathrm{e}^{-\alpha z/p}$ defines a simply periodic function g with fundamental period p.

(iii) If p_1 and p_2 are fundamental periods of Φ, then it follows from the proof of (ii) that there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $f(z+p_j) = \mathrm{e}^{\alpha_j} f(z)$. The meromorphic function g defined by $g(z) := f'(z)/f(z)$ has only simple poles and zeros. Integration of $g = (\log f)'$ along a (straight) line from z_0 to $z_0 + p_j$ which contains no pole of g yields that

$$\alpha_j = \int_{z_0}^{z_0+p_j} g(z)\, dz.$$

Evaluating now the area integral $\iint_\Omega g\,dx\,dy$ over the parallelogram Ω with vertices at $0, p_1, p_2, p_1 + p_2$ by two different iterated integrals, we get $\alpha_2\,p_1 = \alpha_1\,p_2$. Since $\alpha_1, \alpha_2 \in \mathbb{R}$ and $p_1/p_2 \notin \mathbb{R}$ this implies that $\alpha_j = 0$.

Partial Sums of Power Series

Here is a strange image which, in similar form, occurred in an experiment. Since it looks so special, one could attribute it to a programming error.

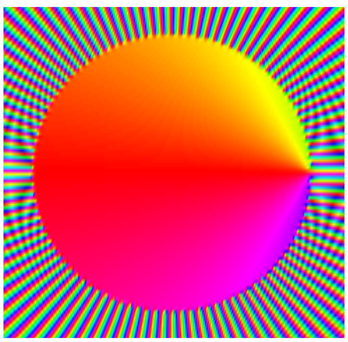

Figure 15: A Taylor polynomial of $f(z) = 1/(1-z)$

A moment's thought shows what is going on here, at least at an intuitive level. This example demonstrates that looking at phase plots can immediately bring forth new questions which would perhaps not have been posed otherwise. But in the case at hand the question has not only been posed, it has also been answered much earlier (see [12], Section 7.8) – and probably without looking at pictures.

Theorem 3 (Robert Jentzsch, 1914)**.** *If a power series* $a_0 + a_1 z + a_2 z^2 + \ldots$ *has a positive finite convergence radius* R*, then the zeros of its partial sums cluster at every point* z *with* $|z| = R$*.*

The reader interested in life and personality of Robert Jentzsch is referred to the recent paper [5] by P. Duren, A.-K. Herbig, and D. Khavinson.

Riemann Surfaces

Any landscape invites for exploration. An appropriate vehicle for a journey through analytic landscapes and phase plots is Weierstrass' disc chain method ('Kreiskettenverfahren') for analytic continuation of a function along a path γ. Phase plots are appropriate to visualize this method not only in its abstract setting with empty discs, but using concrete functions, like the square root or the logarithm. Such experiments are not only helpful to develop an intuitive understanding of Riemann surfaces, but can even guide students to the *discovery* of such objects.

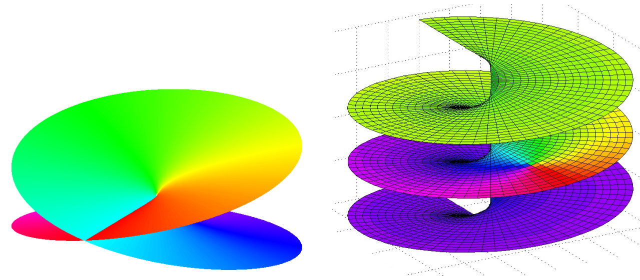

Figure 16: Square root and logarithm on their Riemann surfaces.

It is an advantage of the phase plot that it allows to visualize analytic function directly on their Riemann surface.

Boundary Value Problems

Experimenting with phase plots generates a number of new questions. One such problem is to find a criterion for deciding which color images are *analytic phase plots*, i.e. phase plots of analytic functions.
Since phase plots are painted with the restricted palette of *saturated* colors from the color circle, Leonardo's Mona Lisa will certainly never appear. But for analytic phase plots there are much stronger restrictions: By the uniqueness theorem for harmonic functions an arbitrarily small open piece determines the plot entirely.
So let us pose the question a little differently: What are appropriate data which can be prescribed to construct an analytic phase plot, say

in a Jordan domain D? Can we start, for instance, with given colors on the boundary ∂D? If so, can the boundary colors be prescribed arbitrarily or must they be subject to certain conditions?

In order to state these questions more precise we introduce the concept of a *colored set* K_C, which is a subset K of the complex plane together with a mapping $C : K \to \mathbb{T}$. Any such mapping is referred to as a *coloring* of K.

For the sake of simplicity we here consider only boundary value problems for phase plots with continuous colorings.

Let D be a Jordan domain and let B be a continuous coloring of its boundary ∂D. Find all continuous colorings C of $\overline{D}$ such that the restriction of C to ∂D coincides with B and the restriction of C to D is the phase (plot) of an analytic function f in D.

If such a coloring C exists, we say that B *admits a continuous extension to an analytic phase plot in $\overline{D}$.*

The restriction to continuous colorings automatically excludes zeros of f in D. It does, however, *not* imply that f must extend continuously onto $\overline{D}$ – and in fact it is essential not to require this continuity in order to get a nice result.

Theorem 4. *Let D be a Jordan domain with a continuous coloring B of its boundary ∂D. Then B admits a (unique) continuous extension to an analytic phase plot in $\overline{D}$ if and only if the color index of B is zero.*

Proof. If $C : \overline{D} \to \mathbb{T}$ is a continuous coloring, then a simple homotopy argument (contract ∂D inside D to a point) shows that the color index of its restriction to ∂D must vanish.

Conversely, any continuous coloring B of ∂D with color index zero can be represented as $B = \mathrm{e}^{\mathrm{i}\varphi}$ with a continuous function $\varphi : \partial D \to \mathbb{R}$. This function admits a (unique) continuous harmonic extension Φ to $\overline{D}$. If Ψ denotes a harmonic conjugate of Φ, then $f = \mathrm{e}^{\mathrm{i}\Phi - \Psi}$ is analytic in D. Its phase $C := \mathrm{e}^{\mathrm{i}\Phi}$ is continuous on $\overline{D}$ and coincides with B on ∂D.

Though the function f need not be continuous, it belongs to the Smirnov spaces $E^p(D)$ for all $p < \infty$.

The figure below shows the phase plots of $f_1(z) = -1$, $f_2(z) = (z+1)^2/(z-1)^2$, and $f_3(z) = (z^2 + \frac{5}{2}z + 1)/(z^2 - \frac{5}{2}z + 1)$ which all have constant phase -1 almost everywhere on $\mathbb{T}$. Though f_1 and f_2 are analytic and have continuous phase in $\mathbb{D}$, only f_1 is a solution of the boundary value problem in the sense of Theorem 4.

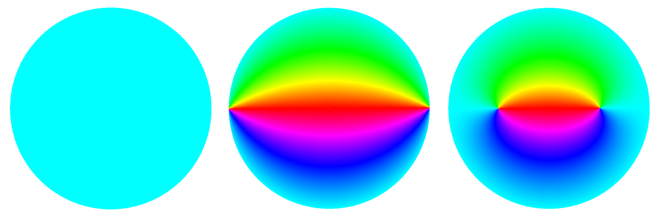

Figure 17: Three functions with phase -1 almost everywhere on $\mathbb{T}$

The theorem can be rephrased in yet another intuitive form, which avoids the color index, as follows.

Let $C : \mathbb{C} \to \mathbb{T}$ be any continuous coloring of the plane. Draw an arbitrary Jordan curve J and erase the color in the interior of J. Then the empty space can be filled in a unique way with the phase plot of an analytic function such that the resulting coloring of the plane is continuous again.

Theorem 4 *parametrizes* analytic phase plots which extend continuously on $\overline{D}$ by their boundary colorings. This result can be extended to phase plots which are continuous on $\overline{D}$ with the exception of finitely many singularities of zero or pole type in D. If we then admit boundary colorings B with arbitrary color index we get the following result:

For any finite collection of given zeros with orders $n_1, \ldots, n_j$ and poles of orders $p_1, \ldots, p_k$ the boundary value problem for analytic phase plots with prescribed singularities has a (unique) solution if and only if the (continuous) boundary coloring B satisfies

$$\operatorname{cind} B = n_1 + \ldots + n_j - p_1 - \ldots - p_k.$$

The Riemann Zeta-Function

After these preparations we are ready to pay a visit to "Zeta", the mother of all analytic functions. Here is a phase plot in the square $-40 \le \operatorname{Re} z \le 10$, $-2 \le \operatorname{Im} z \le 48$.

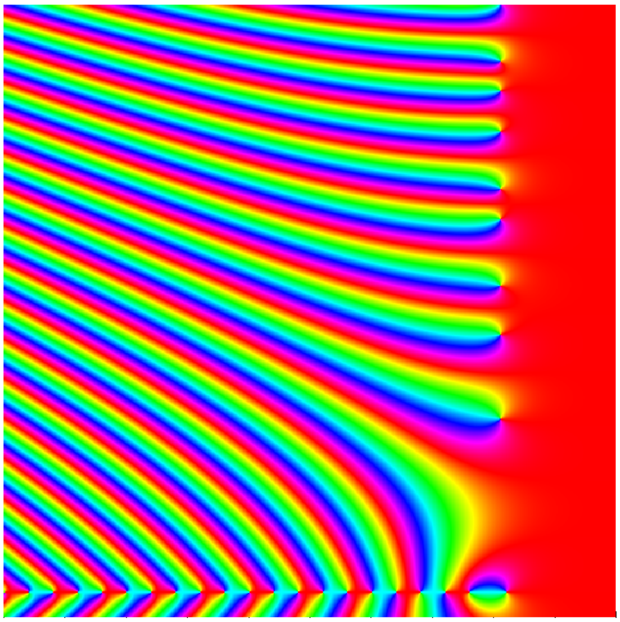

Figure 18: Riemann's Zeta function

We see the pole at $z = 1$, the trivial zeros at the points $-2, -4, -6, \ldots$ and several zeros on the critical line $\operatorname{Re} z = 1/2$. Also we observe that the isochromatic lines are quite regularly distributed in the left half plane.

The regular behavior of the phase of Zeta on the imaginary axis provides a basis for counting its zeros in the rectangle $0 < \operatorname{Re} z < 1$, $0 < \operatorname{Im} z < T$ by the argument principle. Further, the striking visual similarity between the phase plots of ζ and Γ in the left half plane, together with the location of poles of Gamma and zeros of Zeta, suggests to look at the product $\zeta(z)\,\Gamma(z/2)$. Observing then that the phase of this function is changing very slowly along the imaginary axis, we get a clue of the famous Riemann-Siegel formula (see [6]).

The saying that Zeta is the mother of all functions alludes to its *universality*. The objective of this section is to present a version of this result which can be communicated to (almost) everybody.[1].

Our starting point is the following variant of Voronin's universality theorem (Karatsuba and Voronin [8], see also Steuding [11]):

Let D be a Jordan domain such that $\overline{D}$ is contained in the strip

$$R := \{z \in \mathbb{C} : 1/2 < \operatorname{Re} z < 1\},$$

and let f be any function which is analytic in D, continuous on $\overline{D}$, and has no zeros in $\overline{D}$. Then f can be uniformly approximated on $\overline{D}$ by vertical shifts of Zeta, $\zeta_t(z) := \zeta(z + \mathrm{i}t)$ with $t \in \mathbb{R}$.

Recall that a continuously colored Jordan curve J_C is a continuous mapping $C : J \to \mathbb{T}$ from a simple closed curve J into the color circle $\mathbb{T}$. A *string* S is the equivalence class of all such colored curves with respect to rigid motions of the plane.

Strings can be classified by their *chromatic number*, which is the winding number of the color map of any representative. The figure depicts a string with chromatic number one.

We say that a string S *lives in a domain* D if it can be represented by a colored Jordan curve J_C with $J \subset D$. A string can *hide itself* in a phase plot $P : D \to \mathbb{T}$, if, for every $\varepsilon > 0$, it has a representative J_C such that $J \subset D$ and

$$\max_{z \in J} |C(z) - P(z)| < \varepsilon.$$

In less technical terms, a string can hide itself if it can move to a place where it is invisible since it looks almost like the background.

In conjunction with Theorem 4 the following universality result for the phase plot of the Riemann Zeta function can easily be derived from Voronin's theorem.

Theorem 5. *Let S be a string which lives in the strip R. Then S can hide itself in the phase plot of the Riemann Zeta function on R if it has chromatic number zero.*

[1] "Man hat eine Sache erst dann verstanden, wenn man sie seiner Großmutter erklären kann." (Albert Einstein)

In view of the extreme richness of Jordan curves and colorings this result is a real miracle. The three pictures below show phase plots of Zeta in the critical strip. The regions with saturated colors belong to R. The rightmost figure depicts the domain considered on p. 342 of Conrey's paper [4].

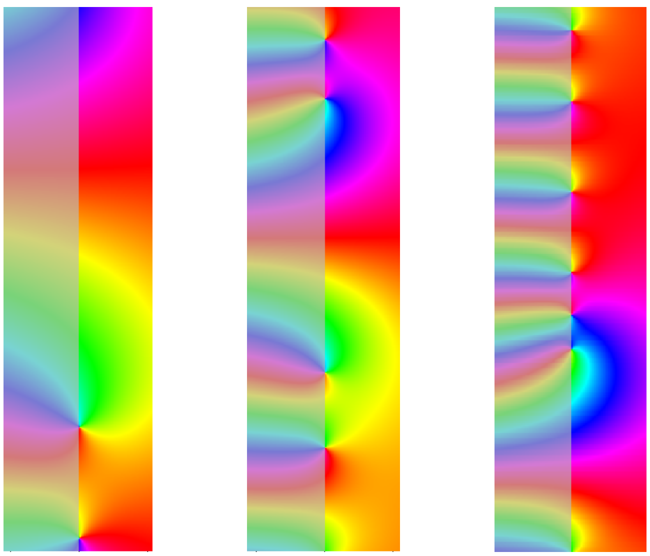

Figure 19: Riemann's Zeta function at $\operatorname{Im} z = 171,\ 8230$ and 121415

What about the converse of Theorem 5 ? If there existed strings with *nonzero chromatic number* which can hide themselves in the strip R, their potential hiding-places must be Jordan curves with non-vanishing color index in the phase plot. By the argument principle this would imply that Zeta has zeros in R. If we assume this, for a moment, then such strings indeed exist: They are perfectly hidden and wind themselves once around such a zero. So the converse of Theorem 5 holds if and only if R contains no zeros of Zeta, which is known to be equivalent to the *Riemann hypothesis* (see [4], [6]).

The Phase Plot as a Tool

Phase plots may be a useful tool for everybody who is working with complex–valued functions. Here are a few examples which also demonstrate some additional tricks.

If it is not clear which branch of a function is used in a certain software implementation of a special function a glimpse of the phase plot may help. In particular, if several functions are composed, implementations with different branch cuts can lead to completely different results. Figure 20 illustrates the difference between the MATHEMATICA functions `Log`(`Gamma`) and `LogGamma`.

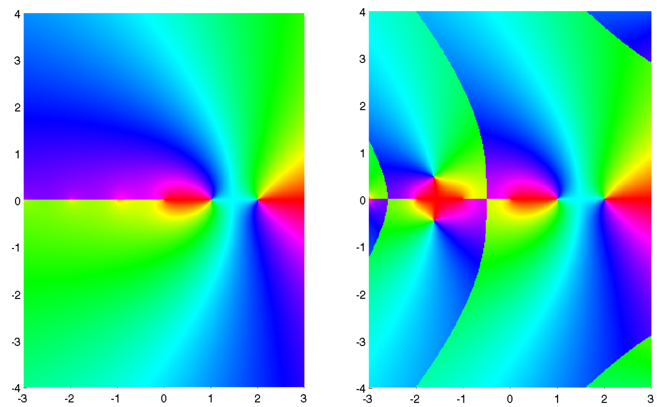

Figure 20: The MATHEMATICA implementations of `LogGamma` and `Log`(`Gamma`)

Another promising field of application is the visual analysis of *transfer functions* in systems theory and filter design.

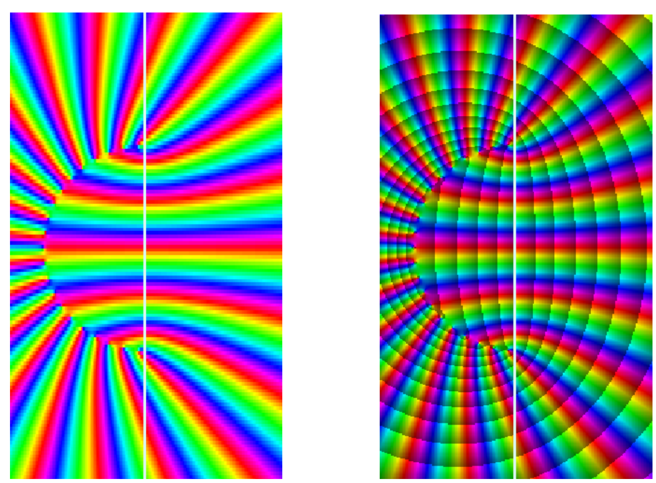

Figure 22: The transfer function of a Butterworth filter

The left part of the figure depicts the phase plot of a Butterworth filter. The frequency response is on the white line (imaginary axis). In the right figure the color scheme is a modified by a black component, which is a sawtooth function of $\ln |f|$. This allows to read off the gain of the filter directly from the plot. Numerically this is much more efficient than computing contour lines of the modulus.

Phase plots also allow to guess the *asymptotic behavior* of functions and to find *functional relations.* For example you may wish to compare the phase plots of $\exp z$ and $\sin z$ or, more challenging, reinvent Riemann's ξ-function and discover its functional relation from the phase plots of Zeta and Gamma.

Further potential applications are in the field of *Laplace* and complex *Fourier transforms*, in particular to the method of steepest descent (or stationary phase).

Of course, the utility of phase plots is not restricted to analytic functions. Figure 23 shows the phase plot of a harmonic polynomial (Wilmshurst's example [13], see also the paper on gravitational lenses by Khavinson and Neumann [9]).

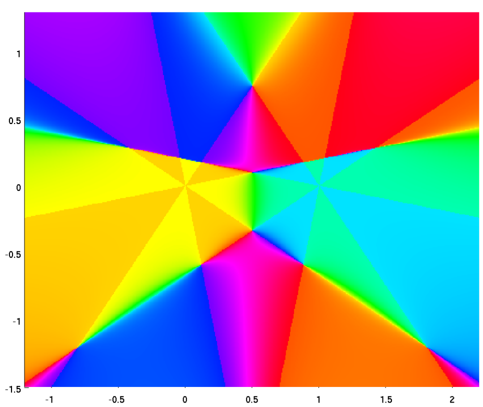

Figure 23: A modified phase plot of Wilmshurst's example for $n = 4$

To understand the construction of the depicted function it is important to keep track of the zeros of its real and imaginary parts. In the figure these (straight) lines are visualized using a modified color scheme which has jumps at the points $1, \mathrm{i}, -1$ and $-\mathrm{i}$ on the unit circle.

Finally, it should be mentioned that phase plots can easily be animated. For example, a spinning color wheel with a discontinuity is helpful to identify zeros of f' as color saddles.

In the phase plot of $f - c$, with a complex constant c, the c-values of f appear as singularities. Now let c move and observe what happens. Good examples are Blaschke products[2] or functions with an essential singularity.

Figure 24: A finite Blaschke product with randomly distributed zeros

Can you predict how the phase plot in Figure 24 will morph when c travels along the real axis from 0 to 1 ?

Even more interesting are animated phase plots of functions involving parameters. For example, varying the module of elliptic functions allows to visualize their metamorphosis from simply periodic to doubly periodic functions.

If you meet an interesting complex function in your own work, I recommend that you look at its phase plot. There is a good chance to discover new aspects of the subject. Here is a self-explaining MATLAB code[3]:

```
xmin=-0.5; xmax=0.5; ymin=-0.5; ymax=0.5;
xres = 400; yres = 400;
x = linspace(xmin,xmax,xres);
y = linspace(ymin,ymax,yres);
z = ones(yres,1)*x + i*y'*ones(1,xres);
f = exp(1./z);
p = surf(real(z),imag(z),0*f,angle(-f));
set(p,'EdgeColor','none');
caxis([-pi,pi]), colormap hsv(600)
view(0,90), axis equal
```

[2]For readers interested in other color representations of Blaschke products the paper [3] is a must.

[3]This is a tribute to N. Trefethen's proposal to communicate ideas by exchanging ten-lines computer code

Conclusion

Color encoding of function values has been customary for many decades (think about altitudes on maps or temperature and air pressure distribution in weather forecast). Also the idea to represent the argument of a complex function by a circular color scheme (its phase) is quite natural.

Since color printing is not yet standard in mathematical publications, analytic functions are usually either depicted by their analytic landscape or by images of grids (see Section 12 of [7]). The latter approach is perfect for conformal mappings, but the visualization of non-univalent function is problematic.

To the best of my knowledge the colored analytic landscape appears for the first time in printed form in the outstanding mathematics textbook [1] for engineering students.

In the internet, the page on special functions at Wolfram Research is a well-structured source for all kinds of information, including tools for graphical visualization of complex functions. There are a few colored analytic landscapes, but at the time of this writing phase plots play only a marginal rôle. A good source for aesthetic "phase plot like" pictures (see the comments below) is the Wikipedia page of Jan Homann.

Though I have been working with phase plots in education and research for several years, it took me some time to realize that they are much more than just one of several options to visualize analytic functions, and that forgetting about the modulus may even be an *advantage*.

The concept of phase plots starts with splitting the information about the function into two parts, such that one (phase $f/|f|$) can be easily represented, while the second one ($\ln|f|$) can be reconstructed from the same picture.

So why not separate f into its real and imaginary part? One reason is that often zeros are of special interest, which can easily be detected and characterized using the phase, but there is no way to find them from the real or imaginary part alone.

And what is the advantage of using $f/|f|$ instead of $\ln|f|$? Of course, zeros and poles can be seen in the analytic landscape, but they are much better represented in the phase plot. In fact there is a subtle asymmetry between modulus and argument (respectively phase). For example, Theorem 4 has no counterpart for the modulus of a function.

Another advantage of phase is its small range, the unit circle. So one standardized color scheme is appropriate for all functions.

Reducing the range by replacing the argument $\arg f$ with the phase $f/|f|$ has yet another interpretation: we periodize the values of a function. The same idea was applied to $\ln|f|$ in Figure 22 (Butterworth filter), where the gray scale was chosen according to the fractional part of $\ln|f|$. If a sawtooth function with high frequency is applied to both, the phase and the module, the resulting plots show the preimages of a polar grid, which yields a *conformal representation* of the function.

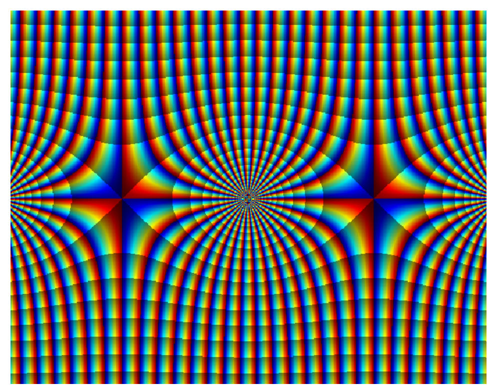

Figure 25: A conformal representation of the sine function

Finally, why do we not use a gray scale in general to represent the modulus and show the full information about f in a "value plot"?

The interested reader can find such pictures (where large values of $|f|$ are associated with dark colors and small values are bright) on the Wikipedia page of Jan Homann.

There is no doubt that value plots outperform phase plots in some applications. However, for *analytic functions* their main drawback is the *lack of a simple characterization* among all possible colorings of the domain. In particular,

there is no result about the boundary coloring corresponding to Theorem 4.
As a consequence the universality of Riemann's Zeta function for value plots needs the concept of "*analytic color patterns*". For a non-mathematician this does not say much, in particular it is not clear if this class is large or small. *For some applications value plots have too many colors.*

Technical Remark. All images of this article have been created using MATHEMATICA and MATLAB.

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