An inverse problem of digital signal processing

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An important problem of digital signal processing is the so-called frequency analysis problem: Let f be an anharmonic Fourier sum. Determine the different frequencies, the coefficients, and the number of frequencies from finitely many equispaced sampled data of f. This is a nonlinear inverse problem. In this paper, we present new results on an approximate Prony method which is based on [1, 2]. In contrast to [1, 2], we apply matrix perturbation theory such that we can describe the properties and the numerical behavior of the approximate Prony method in detail. Numerical experiments show the performance of our method.

Key words and phrases: frequency analysis problem, nonequispaced fast Fourier transform, digital signal processing, anharmonic Fourier sum, approximate Prony method, matrix perturbation theory, perturbed Hankel matrix, Vandermonde–type matrix.
AMS *Subject Classifications*: 42C15, 65T40, 65T50, 65F15, 65F20, 94A12.

1 Introduction

We consider an *anharmonic Fourier sum* of the form

$$f(x) := \frac{\alpha_0}{2} + \sum_{j=1}^{M} \big(\alpha_j\, \cos(\omega_j x) + \beta_j\, \sin(\omega_j x)\big) \quad (x \in \mathbb{R})$$

with real coefficients α_j, β_j $(\alpha_j^2 + \beta_j^2 > 0)$ and frequencies ω_j with

$$0 < \omega_1 < \ldots < \omega_M < \pi\,.$$

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For $\alpha_0 \neq 0$ let $\omega_0 = 0$ and $\beta_0 = 0$. For simplicity we consider only the case $\alpha_0 \neq 0$. The frequencies ω_j ($j = -M, \ldots, M$) are extended with the period 2π such that for example $\omega_{-M-1} := -2\pi + \omega_M$ and $\omega_{M+1} := 2\pi - \omega_M$. We denote the *separation distance* q of the frequency set $\{\omega_j : j = 0, \ldots, M+1\}$ by

$$q := \min_{\substack{j,k=0,\ldots,M+1 \\ j \neq k}} |\omega_k - \omega_j| \,.$$

Then we have $qM < \pi$. Let $N \in \mathbb{N}$ an integer with $N \geq 2M+1$. Assume that the sampled data $h_k := f(k)$ ($k = 0, \ldots, 2N$) are given. Since $\omega_M < \pi$, we infer that the Nyquist condition is fulfilled (see [3, p. 183]). From the $2N+1$ sampled data h_k ($k = 0, \ldots, 2N$) we have to determine the positive integer M, the real coefficients α_j, β_j and the frequencies $\omega_j \in [0, \pi)$ ($j = 0, \ldots, M$). This is the *real frequency analysis problem.*

Using

$$\cos(\omega_j x) = \frac{1}{2}\,(\mathrm{e}^{\mathrm{i}\omega_j x} + \mathrm{e}^{-\mathrm{i}\omega_j x}), \quad \sin(\omega_j x) = \frac{1}{2\mathrm{i}}\,(\mathrm{e}^{\mathrm{i}\omega_j x} - \mathrm{e}^{-\mathrm{i}\omega_j x}),$$

we obtain the complex representation of f in the form

$$f(x) = \rho_0 + \sum_{j=1}^{M} \left(\rho_j\, \mathrm{e}^{\mathrm{i}\omega_j x} + \bar{\rho}_j\, \mathrm{e}^{-\mathrm{i}\omega_j x}\right)$$

with the coefficients

$$\rho_j := \frac{1}{2}(\alpha_j - \mathrm{i}\,\beta_j) \quad (j = 0, \ldots, M).$$

We define

$$\omega_j := -\omega_{-j}, \quad \rho_j := \bar{\rho}_{-j} \quad (j = -M, \ldots, -1) \tag{1.1}$$

and write the anharmonic Fourier sum f in the complex form

$$f(x) = \sum_{j=-M}^{M} \rho_j\, \mathrm{e}^{\mathrm{i}\omega_j x} \,. \tag{1.2}$$

Then the *complex frequency analysis problem* reads as follows: Determine the positive integer M, the frequencies $\omega_j \in [0, \pi)$ and the coefficients $\rho_j \in \mathbb{C} \setminus \{0\}$ ($j = 0, \ldots, M$) with (1.1) such that

$$\sum_{j=-M}^{M} \rho_j\, \mathrm{e}^{\mathrm{i}\omega_j k} = h_k \quad (k = 0, \ldots, 2N) \,.$$

This is a nonlinear inverse problem which can be simplified by original ideas of G. R. de Prony. But the classical Prony method is numerically unstable such that numerous modifications were attempted to improve its numerical behavior. For the more general case with $w_j \in \mathbb{C}$ see e.g. [13] and the references therein. Our results are based on the papers [1, 2] of G. Beylkin and L. Monzón. The nonlinear problem of finding the frequencies and coefficients can be split into two problems. To obtain the frequencies,

we solve an eigenvalue problem of the Hankel matrix $\mathbf{H} = \big(f(k+l)\big)_{k,l=0}^{N}$ and find the frequencies via roots of an eigenpolynomial. To obtain the coefficients, we use the frequencies to solve an overdetermined linear Vandermonde–type system. In contrast to [1, 2], we present an approximate Prony method by means of matrix perturbation theory such that we can describe the properties and the numerical behavior of the approximate Prony method in detail.
In applications, perturbed values $\tilde{h}_k \in \mathbb{R}$ of the exact sampled data $h_k = f(k)$ are only known with the property

$$\tilde{h}_k = h_k + e_k\,, \quad |e_k| \le \varepsilon_1 \quad (k = 0,\, \ldots,\, 2N),$$

where the error terms e_k are bounded by certain accuracy $\varepsilon_1 > 0$. Also if the sampled values h_k are accurately determined, then we still have a small roundoff error due to the use of floating point arithmetic. Furthermore we assume that $|\rho_j| \ge \varepsilon_1$ $(j = 0, \ldots, M)$.

This paper is organized as follows. In Section 2, we discuss the classical Prony method. Then in Section 3, we shortly describe an improved Prony method founded by G. Beylkin and L. Monzón [2]. The core of this paper are the Sections 4 and 5 with our new results on an approximate Prony method. Using matrix perturbation theory, we discuss the properties of small eigenvalues and related eigenvectors of a real Hankel matrix formed by given noisy data. By means of the separation distance of the frequency set, we can describe the numerical behavior of the approximate Prony method also for clustered frequencies, if we use oversampling. Further we discuss the sensitivity of the approximate Prony method to perturbation. Finally, various numerical examples are presented in Section 6.

2 Classical Prony method

The classical Prony method works with exact sampled data. Following an idea of G. R. de Prony from 1795 (see e.g. [11, pp. 303–310]), we regard the sampled data $h_k = f(k)$ $(k = 0,\, \ldots,\, 2N)$ as solution of a *homogeneous linear difference equation with constant coefficients*. If

$$h_k = f(k) = \sum_{j=-M}^{M} \rho_j \left(\mathrm{e}^{\mathrm{i}\omega_j}\right)^k$$

with (1.1) is a solution of certain homogeneous linear difference equation with constant coefficients, then $\mathrm{e}^{\mathrm{i}\omega_j}$ $(j = -M,\, \ldots,\, M)$ must be zeros of the corresponding characteristic polynomial. Thus

$$\begin{aligned} P_0(z) \;&:=\; \prod_{j=-M}^{M} (z - \mathrm{e}^{\mathrm{i}\omega_j}) = (z-1) \prod_{j=1}^{M} (z^2 - 2z\,\cos\omega_j + 1) \\ &=\; p_{2M+1}\, z^{2M+1} + p_{2M}\, z^{2M} + \ldots + p_1\, z + p_0 \quad (z \in \mathbb{C}) \qquad (2.1) \end{aligned}$$

with $p_{2M+1} = -p_0 = 1$ is the monic characteristic polynomial of minimal degree. With the real coefficients p_k $(k = 0, \ldots, 2M+1)$ we compose the homogeneous linear difference equation

$$\sum_{l=0}^{2M+1} x_{l+m}\, p_l = 0 \quad (m = 0,\, 1,\, \ldots), \tag{2.2}$$

which obviously has P_0 as characteristic polynomial. Consequently, (2.2) has the real general solution

$$x_m = \sum_{j=-M}^{M} \rho_j\, \mathrm{e}^{\mathrm{i}\omega_j m} \quad (m = 0,\, 1,\, \ldots)$$

with arbitrary coefficients $\rho_0 \in \mathbb{R}$ and $\rho_j \in \mathbb{C}$ $(j = 1, \ldots, M)$ with (1.1). Then we determine ρ_j $(j = 0, \ldots, M)$ in such a way that $x_k \approx h_k$ $(k = 0, \ldots, 2N)$. To this end, we compute the least squares solution of the overdetermined linear Vandermonde–type system

$$\sum_{j=-M}^{M} \rho_j\, \mathrm{e}^{\mathrm{i}\omega_j k} = h_k \quad (k = 0, \ldots, 2N).$$

The Prony method is based on following assertions:

Lemma 2.1 *Let $M,\, N \in \mathbb{N}$ with $N \geq 2M + 1$ and*

$$h_k = f(k) = \sum_{j=-M}^{M} \rho_j\, \mathrm{e}^{\mathrm{i}\omega_j k} \quad (k = 0, \ldots, 2N) \tag{2.3}$$

with $\rho_0 \in \mathbb{R} \setminus \{0\}$, $\rho_j \in \mathbb{C} \setminus \{0\}$ $(j = 1, \ldots, M)$ and let $\omega_0 = 0 < \omega_1 < \ldots < \omega_M < \pi$ be given with (1.1)*. Furthermore let*

$$P(z) = \sum_{k=0}^{N} u_k\, z^k \quad (z \in \mathbb{C})$$

be a polynomial with real coefficients u_k $(k = 0, \ldots, N)$ and with $2M + 1$ different zeros $\mathrm{e}^{\mathrm{i}\omega_j}$ $(j = -M, \ldots, M)$ on the unit circle.
Then the equations

$$\sum_{l=0}^{N} h_{l+m}\, u_l = 0 \quad (m = 0,\, 1,\, \ldots) \tag{2.4}$$

are fulfilled and 0 *is an eigenvalue of the real Hankel matrix $\mathbf{H} = \big(h_{l+m}\big)_{m,l=0}^{N}$ with the eigenvector $\mathbf{u} := (u_l)_{l=0}^{N}$.*

Proof. We compute the sum (2.4) by using (2.3) and obtain for $m = 0,\, 1,\, \ldots$

$$\begin{aligned}\sum_{l=0}^{N} h_{l+m}\, u_l &= \sum_{l=0}^{N} u_l \Big(\sum_{j=-M}^{M} \rho_j\, \mathrm{e}^{\mathrm{i}\omega_j (l+m)} \Big) \\ &= \sum_{j=-M}^{M} \rho_j\, \mathrm{e}^{\mathrm{i}\omega_j m}\, P(\mathrm{e}^{\mathrm{i}\omega_j}) = 0.\end{aligned}$$

Therefore we get $\mathbf{Hu} = \mathbf{o}$, where $\mathbf{o} \in \mathbb{R}^{N+1}$ denotes the zero vector. ■

Lemma 2.2 *Let M, $N \in \mathbb{N}$ with $N \geq 2M+1$ be given. Furthermore let h_k be given by (2.3) with $\rho_0 \in \mathbb{R} \setminus \{0\}$, $\rho_j \in \mathbb{C} \setminus \{0\}$ $(j = 1, \ldots, M)$, $\omega_0 = 0 < \omega_1 < \ldots < \omega_M < \pi$ and (1.1). Assume that the Hankel matrix $\mathbf{H} = (h_{l+m})_{l,m=0}^N$ has the eigenvalue 0 with an eigenvector $\mathbf{u} = (u_n)_{n=0}^N \in \mathbb{R}^{N+1}$.*
Then the corresponding eigenpolynom

$$P(z) = \sum_{k=0}^{N} u_k\, z^k \quad (z \in \mathbb{C}),$$

has the values $\mathrm{e}^{\mathrm{i}\omega_j}$ $(j = -M, \ldots, M)$ as zeros, i.e., P_0 defined by (2.1) is a divisor of P.

Proof. By assumption we have

$$\sum_{l=0}^{N} h_{l+m}\, u_l = 0 \quad (m = 0, \ldots, N)$$

such that we obtain for arbitrary $z \in \mathbb{C}$

$$\sum_{m=0}^{N} \Big(\sum_{l=0}^{N} h_{l+m}\, u_l \Big)\, z^m = 0.$$

We use (2.3), change the order of summation and obtain

$$\sum_{j=-M}^{M} \rho_j\, P(\mathrm{e}^{\mathrm{i}\omega_j})\, Q(\mathrm{e}^{\mathrm{i}\omega_j} z) = 0 \tag{2.5}$$

with

$$Q(z) := \sum_{m=0}^{N} z^m.$$

The $2M+1$ polynomials $Q(\mathrm{e}^{\mathrm{i}\omega_j} z)$ $(j = -M, \ldots, M)$ are linearly independent, since by comparison of coefficients from

$$\sum_{j=-M}^{M} \alpha_j\, Q(\mathrm{e}^{\mathrm{i}\omega_j} z) = 0 \quad (\alpha_j \in \mathbb{C})$$

we obtain the following linear system

$$\sum_{j=-M}^{M} \alpha_j\, \mathrm{e}^{\mathrm{i}k\omega_j} = 0 \quad (k = 0, \ldots, 2M)$$

with the regular Vandermonde matrix $\big(\mathrm{e}^{\mathrm{i}k\omega_j}\big)_{k=0,j=-M}^{2M,M}$ such that $\alpha_j = 0$ $(j = -M, \ldots, M)$. Hence we infer from (2.5) and $\rho_j \neq 0$

$$P(\mathrm{e}^{\mathrm{i}\omega_j}) = 0 \quad (j = -M, \ldots, M),$$

but this means that P_0 divides P. ■

The idea of G. R. de Prony is based on the separation of the unknown frequencies ω_j from the unknown coefficients ρ_j by means of a homogeneous linear difference equation (2.2). With the $2N + 1$ sampled data $h_k \in \mathbb{R}$ we form the Hankel matrix $\mathbf{H} = (h_{l+m})_{l,m=0}^{N} \in \mathbb{R}^{(N+1)\times(N+1)}$. Using the coefficients p_k $(k = 0, \ldots, 2M + 1)$ of (2.1), we construct the vector $\mathbf{p} := (p_k)_{k=0}^{N}$, where $p_{2M+2} = \ldots = p_N := 0$. By $\mathbf{S} := \big(\delta_{k-l-1}\big)_{k,l=0}^{N}$ we denote the forward shift matrix, where δ_k is the Kronecker symbol.

Lemma 2.3 *Let M, $N \in \mathbb{N}$ with $N \geq 2M+1$ be given. Furthermore let $h_k \in \mathbb{R}$ be given by* (2.3) *with $\rho_0 \in \mathbb{R} \setminus \{0\}$, $\rho_j \in \mathbb{C} \setminus \{0\}$ $(j = 1, \ldots, M)$, $\omega_0 = 0 < \omega_1 < \ldots < \omega_M < \pi$ and* (1.1).
Then the Hankel matrix $\mathbf{H} = (h_{l+m})_{l,m=0}^{N}$ has the eigenvalue 0 with multiplicity $N - 2M$ and

$$\dim(\ker \mathbf{H}) = N - 2M, \quad \operatorname{rank} \mathbf{H} = 2M + 1$$

with the kernel

$$\ker \mathbf{H} = \operatorname{span}\{\mathbf{p}, \mathbf{S}\mathbf{p}, \ldots, \mathbf{S}^{N-2M-1}\mathbf{p}\}.$$

Proof. 1. From

$$\sum_{l=0}^{2M+1} h_{l+m}\, p_l = 0 \quad (m = 0, \ldots, N - 2M - 1)$$

it follows that

$$\mathbf{H}\big(\mathbf{S}^j\mathbf{p}\big) = \mathbf{o} \quad (j = 0, \ldots, N - 2M - 1).$$

By $p_0 = -1$ we see immediately that the vectors $\mathbf{S}^j\mathbf{p}$ $(j = 0, \ldots, N-2M-1)$ are linearly independent and located in the kernel $\ker \mathbf{H}$.
2. We prove that $\ker \mathbf{H}$ is contained in the span of the vectors $\mathbf{S}^j\mathbf{p}$ $(j = 0, \ldots, N-2M-1)$. Let $\mathbf{u} \in \mathbb{R}^{N+1}$ be an arbitrary eigenvector of $\mathbf{H}$ of the eigenvalue 0 and let P be the related eigenpolynomial. Using Lemma 2.2, we infer that $P(z) = P_0(z)\, P_1(z)$ with a polynomial

$$P_1(z) = \beta_0 + \beta_1 z + \ldots + \beta_{N-2M-1} z^{N-2M-1},$$

where the coefficients β_k are real. But this means that

$$\mathbf{u} = \beta_0\, \mathbf{p} + \beta_1\, \mathbf{S}\mathbf{p} + \ldots + \beta_{N-2M-1}\, \mathbf{S}^{N-2M-1}\mathbf{p}.$$

Hence it follows that the vectors $\mathbf{S}^j\mathbf{p}$ $(j = 0, \ldots, N - 2M - 1)$ compose a basis of $\ker \mathbf{H}$ and we obtain $\dim(\ker \mathbf{H}) = N - 2M$.

3. Since $\mathbf{H}$ is real and symmetric, we can represent $\mathbb{R}^{N+1}$ as the orthogonal sum of the kernel $\ker \mathbf{H}$ and the image $\operatorname{im} \mathbf{H}$ such that

$$\operatorname{rank} \mathbf{H} = \dim (\operatorname{im} \mathbf{H}) = (N+1) - (N-2M) = 2M+1\,.$$

This completes the proof. ■

We summaries:

Algorithm 2.4 *(Classical Prony Method)*
Input: $N \in \mathbb{N}$ $(N \gg 1)$, $h_k = f(k)$ $(k = 0, \ldots, 2N)$, $0 < \varepsilon \ll 1$.
1. Compute an eigenvector $\mathbf{u} = (u_l)_{l=0}^N$ *corresponding to the eigenvalue* 0 *of the exact Hankel matrix* $\mathbf{H} = (h_{l+m})_{l,m=0}^N$.
2. Form the corresponding eigenpolynomial

$$P(z) = \sum_{k=0}^{N} u_k\, z^k$$

and evaluate all zeros $\mathrm{e}^{\mathrm{i}\omega_j}$ $(j = 1, \ldots, \tilde{M})$ *with* $\omega_j \in (0,\, \pi)$ *and* (1.1) *lying on the unit circle. Note that* $N \geq 2\tilde{M} + 1$.
3. Compute $\rho_0 \in \mathbb{R}$ *and* $\rho_j \in \mathbb{C}$ $(j = 1, \ldots, \tilde{M})$ *with* (1.1) *as least squares solution of the overdetermined linear Vandermonde–type system*

$$\sum_{j=-\tilde{M}}^{\tilde{M}} \rho_j\, \mathrm{e}^{\mathrm{i}\omega_j k} = h_k \quad (k = 0, \ldots, 2N). \tag{2.6}$$

4. Cancel all that pairs $(\omega_l,\, \rho_l)$ $(l \in \{1, \ldots, \tilde{M}\})$ *with* $|\rho_l| \leq \varepsilon$ *and denote the remaining set by* $\{(\omega_j,\, \rho_j) :\, j = 1, \ldots, M\}$ *with* $M \leq \tilde{M}$.
Output: $M \in \mathbb{N}$, $\rho_0 \in \mathbb{R}$, $\rho_j \in \mathbb{C}$, $\omega_j \in (0,\, \pi)$ $(j = 1, \ldots, M)$.

Remark 2.5 *Let* $N > 4M + 2$. *If one knows* M *or a good approximation of* M, *then one can use the following least squares Prony method, see e.g. [5]. Since the leading coefficient* p_{2M+1} *of the characteristic polynomial* P_0 *is equal to 1, from (2.2) it follows the overdetermined linear system*

$$\sum_{l=0}^{2M} h_{l+m}\, p_l = -h_{2M+1+m}\, p_{2M+1} = -h_{2M+1+m} \quad (m = 0, \ldots, N - 2M - 1)\,,$$

which can be solved by a least squares method. See also the relation to the classic Yule-Walker system [4] □

3 Improved Prony method

The classical Prony method is known to perform poorly when noisy data are given. Therefore numerous modifications were attempted to improve the numerical behavior of the classical Prony method. Recently, a very interesting approach is described by G. Beylkin and L. Monzón [2], where the more general problem of approximation by exponential sums is considered. Since our research is mainly motivated by these results, we sketch this *improved Prony method* which is based on the following

Theorem 3.1 (*see* [2]) *Let* $\sigma \in [-\varepsilon, \varepsilon]$ $(0 < \varepsilon \ll 1)$ *be a small eigenvalue of the Hankel matrix* $\mathbf{H} = (h_{k+l})_{k,l=0}^{N} \in \mathbb{R}^{(N+1)\times(N+1)}$ *with a corresponding eigenvector* $\mathbf{u} = (u_l)_{l=0}^{N} \in \mathbb{R}^{N+1}$*. Assume that the eigenpolynomial* P *related to* $\mathbf{u}$ *has* N *pairwise distinct zeros* $\gamma_n \in \mathbb{C}$ $(n = 1, \ldots, N)$*. Further let* $L > 2N$*.*
Then there exists a unique vector $(\nu_n)_{n=1}^{N} \in \mathbb{C}^N$ *such that*

$$h_k = \sum_{n=1}^{N} \nu_n \, \gamma_n^k + \sigma \, d_k \quad (k = 0, \ldots, 2N)\,,$$

where the normed vector $\big(d_k\big)_{k=0}^{L-1} \in \mathbb{C}^L$ *is defined by*

$$d_k := \frac{1}{L} \sum_{l=0}^{L-1} \hat{d}_l \, \mathrm{e}^{2\pi \mathrm{i} kl/L} \quad (k = 0, \ldots, L-1) \tag{3.1}$$

with

$$\begin{aligned} \hat{d}_l &:= \begin{cases} \hat{u}_l / \bar{\hat{u}}_l & \text{if } \hat{u}_l \neq 0, \\ 1 & \text{if } \hat{u}_l = 0, \end{cases} \\ \hat{u}_l &:= \sum_{k=0}^{N} u_k \, \mathrm{e}^{-2\pi \mathrm{i} kl/L} \quad (l = 0, \ldots, L-1)\,. \end{aligned}$$

Furthermore

$$|h_k - \sum_{n=1}^{N} \nu_n \, \gamma_n^k| \le |\sigma| \le \varepsilon \quad (k = 0, \ldots, 2N)\,.$$

Remark 3.2 *The Theorem 3.1 yields a different representation for each* $L > 2N$ *even though* γ_n *and* σ *remain the same. If* L *is chosen as power of 2, then the entries* $\hat{u}_l$ *and* d_k *can be computed by fast Fourier transforms. Note that the least squares solution* $\big(\rho_n\big)_{n=1}^{N}$ *of the overdetermined linear system*

$$\sum_{n=1}^{N} \rho_n \, \gamma_n^k = h_k \quad (k = 0, \ldots, 2N) \tag{3.2}$$

has an error with Euclidean norm less than ε, since

$$\begin{aligned}\sum_{k=0}^{2N} |h_k - \sum_{n=1}^{N} \rho_n \, \gamma_n^k|^2 \quad &\le \quad \sum_{k=0}^{2N} |h_k - \sum_{n=1}^{N} \nu_n \, \gamma_n^k|^2 \\ &= \quad \sum_{k=0}^{2N} |\sigma \, d_k|^2 \le |\sigma|^2 \sum_{k=0}^{L-1} |d_k|^2 \le \varepsilon^2.\end{aligned}$$

□

The proof of Theorem 3.1 implies that one can obtain the vector $(\nu_n)_{n=1}^N$ as the unique solution of the linear Vandermonde system

$$\sum_{n=1}^{N} \nu_n \, \gamma_n^k = h_k - \sigma \, d_k \quad (k = 0, \ldots, N-1) \, .$$

Since this equation is also valid for $k = N, \ldots, 2N$, it follows that the least squares solution $\big(\rho_n\big)_{n=1}^N$ of the overdetermined linear Vandermonde–type system

$$\sum_{n=1}^{N} \rho_n \, \gamma_n^k = h_k \quad (k = 0, \ldots, 2N)$$

has an error with Euclidean norm less than ε. Thus an algorithm of the improved Prony method [2] reads as follows:

Algorithm 3.3 *(Improved Prony method)*
Input: $N \in \mathbb{N}$ ($N \gg 1$), L power of 2 with $L > 2N$, $h_k = f(k)$ ($k = 0, \ldots, 2N$) , accuracies ε, ε_1.

1. Compute a small eigenvalue $\sigma \in [-\varepsilon, \, \varepsilon]$ and a corresponding eigenvector $\mathbf{u} = \big(u_l\big)_{l=0}^N \in \mathbb{R}^{N+1}$ of the Hankel matrix $\mathbf{H} = \big(h_{l+m}\big)_{l,m=0}^N$.
2. Determine all zeros $\gamma_n \in \mathbb{C}$ of the corresponding eigenpolynomial

$$P(z) = \sum_{l=0}^{N} u_l \, z^l \, .$$

Assume that all N zeros of P are simple.
3. Determine the least squares solution $\big(\rho_n\big)_{n=1}^N \in \mathbb{C}^N$ of the overdetermined linear Vandermonde–type system

$$\sum_{n=1}^{N} \rho_n \, \gamma_n^k = h_k \quad (k = 0, \ldots, 2N) \, .$$

4. Denote by γ_j ($j = -M, \ldots, M$) with $\gamma_j = \bar{\gamma}_{-j}$ ($j = 0, \ldots, M$) all that zeros of P which are close to the unit circle and for which $|\rho_j| \ge \varepsilon_1$. Set $\omega_j = \arg \gamma_j \in [0, \, \pi)$

$(j = 0, \ldots, M)$.

Output: $M \in \mathbb{N}$, $\rho_j \in \mathbb{C}$ $(\rho_j = \bar{\rho}_{-j})$, $\omega_j \in [0,\, \pi)$ $(j = 0, \ldots, M)$.

In [2], G. Beylkin and L. Monzón are not interested in exact representations of the sampled values

$$h_k = \sum_{n=1}^{N} \rho_n\, \gamma_n^k \quad (k = 0, \ldots, 2N)$$

but rather in *approximate representations*

$$\Big| h_k - \sum_{j=-M}^{M} \rho_j\, \gamma_j^k \Big| \le \varepsilon \quad (k = 0, \ldots, 2N)$$

for very small accuracy ε and minimal number $2M + 1$ of nontrivial terms.

4 Approximate Prony method

In contrast to [1, 2], we present a new approximate Prony method by means of matrix perturbation theory. In praxis, only perturbed values $\tilde{h}_k := h_k + e_k$ $(k = 0, \ldots, 2N)$ of the exact sampled data h_k are known. Here we assume that $|e_k| \le \varepsilon_1$ with certain accuracy $\varepsilon_1 > 0$ such that the error Hankel matrix

$$\mathbf{E} := \big(e_{k+l}\big)_{k,l=0}^{N}$$

has a small spectral norm by

$$\|\mathbf{E}\|_2 \le \sqrt{\|\mathbf{E}\|_1\, \|\mathbf{E}\|_\infty} = \max_{l=0,\ldots,N} \sum_{k=0}^{N} |e_{k+l}| \le (N+1)\, \varepsilon_1. \tag{4.1}$$

Then the perturbed Hankel matrix can be represented by

$$\tilde{\mathbf{H}} := \big(\tilde{h}_{k+l}\big)_{k,l=0}^{N} = \mathbf{H} + \mathbf{E}\,. \tag{4.2}$$

Using the Theorem of H. Weyl (see [7, p. 181]), we receive two–sided bounds for small eigenvalues of $\tilde{\mathbf{H}}$. More precisely, $N - 2M$ eigenvalues of $\tilde{\mathbf{H}}$ are contained in $[-\|\mathbf{E}\|_2,\, \|\mathbf{E}\|_2]$, if the modulus of each nonzero eigenvalue of $\mathbf{H}$ is greater than $2\,\|\mathbf{E}\|_2$. In the following we use this property and evaluate a small real eigenvalue σ $(|\sigma| \le \varepsilon_2)$ and a corresponding eigenvector of the perturbed Hankel matrix $\tilde{\mathbf{H}}$.

Lemma 4.1 *Let* $M,\, N \in \mathbb{N}$ *with* $N \ge 2M + 1$ *given. Furthermore let* $\omega_0 = 0 < \omega_1 < \ldots < \omega_M < \pi$ *with* (1.1) *and a separation distance*

$$q > \frac{1}{N+1} \sqrt{\frac{\pi^3}{3}}$$

be given. Let $\mathbf{D} := \operatorname{diag}\big(1 - |k|/(N+1)\big)_{k=-N}^{N}$ *be a diagonal matrix and let*

$$\mathbf{V} := \big(\mathrm{e}^{\mathrm{i}k\omega_j}\big)_{k=0,j=-M}^{2N,M} \in \mathbb{C}^{(2N+1)\times(2M+1)} \tag{4.3}$$

be a Vandermonde–type matrix.
Then for arbitrary $\mathbf{r} \in \mathbb{C}^{2M+1}$*, the following inequality*

$$\big(N + 1 - \frac{\pi^3}{3q^2(N+1)}\big)\|\mathbf{r}\|^2 \le \|\mathbf{D}^{1/2}\mathbf{V}\mathbf{r}\|_2^2 \le \big(N + 1 + \frac{\pi^3}{3q^2(N+1)}\big)\|\mathbf{r}\|^2$$

is fulfilled. Further the squared spectral norm of the left inverse $\mathbf{L} := (\mathbf{V}^{\mathrm{H}}\mathbf{D}\mathbf{V})^{-1}\mathbf{V}^{\mathrm{H}}\mathbf{D}$ *of* $\mathbf{V}$ *can be estimated by*

$$\|\mathbf{L}\|_2^2 \le \frac{3q^2(N+1)}{3q^2(N+1)^2 - \pi^3}\,. \tag{4.4}$$

Proof. 1. The rectangular Vandermonde–type matrix $\mathbf{V} \in \mathbb{C}^{(2N+1)\times(2M+1)}$ has full rank $2M+1$, since the submatrix

$$\big(\mathrm{e}^{\mathrm{i}k\omega_j}\big)_{k=0,j=-M}^{2M,M}$$

is a regular Vandermonde matrix. Hence we infer that $\mathbf{V}^{\mathrm{H}}\mathbf{D}\mathbf{V}$ is Hermitian and positive definite such that all eigenvalues of $\mathbf{V}^{\mathrm{H}}\mathbf{D}\mathbf{V}$ are positive.
2. We introduce the 2π–periodic Fejér kernel F_N by

$$F_N(x) := \sum_{k=-N}^{N} \Big(1 - \frac{|k|}{N+1}\Big)\mathrm{e}^{\mathrm{i}kx}\,.$$

Then we obtain

$$\big(\mathbf{V}^{\mathrm{H}}\mathbf{D}\mathbf{V}\big)_{j,l} = \mathrm{e}^{-\mathrm{i}N\omega_j}\, F_N(\omega_j - \omega_l)\, \mathrm{e}^{\mathrm{i}N\omega_l} \quad (j,l = -M,\ldots,M)$$

for the (j,l)-th entry of the matrix $\mathbf{V}^{\mathrm{H}}\mathbf{D}\mathbf{V}$. We use Gershgorin's Disk Theorem (see [7, p. 344]) such that for an arbitrary eigenvalue λ of the matrix $\mathbf{V}^{\mathrm{H}}\mathbf{D}\mathbf{V}$ we preserve the estimate (see also [10, Theorem 4.1])

$$|\lambda - F_N(0)| = |\lambda - N - 1| \le \max\Big\{\sum_{\substack{j=-M\\ j\ne l}}^{M} |F_N(\omega_j - \omega_l)|;\ l = -M,\ldots,M\Big\}. \tag{4.5}$$

3. Now we estimate the right–hand side of (4.5). As known, the Fejér kernel F_N can be written in the form

$$0 \le F_N(x) = \frac{1}{N+1}\left(\frac{\sin((N+1)x/2)}{\sin(x/2)}\right)^2.$$

Thus we obtain the estimate

$$\sum_{\substack{j=-M\\ j\ne l}}^{M} |F_N(\omega_j - \omega_l)| \le \frac{1}{N+1}\sum_{\substack{j=-M\\ j\ne l}}^{M} |\sin((\omega_j - \omega_l)/2)|^{-2}$$

for $l \in \{-M, \ldots, M\}$. In the case $l = 0$, we use (1.1), $qM < \pi$, $\omega_j \geq jq$ $(j = 1, \ldots, M)$ and

$$\sin x \geq \frac{2}{\pi}\, x \quad (x \in [0,\, \pi/2])$$

and then we estimate the above sum by

$$\begin{aligned} \sum_{\substack{j=-M \\ j\neq 0}}^{M} |\sin(\omega_j/2)|^{-2} &= 2 \sum_{j=1}^{M} \big(\sin(\omega_j/2)\big)^{-2} \leq 2\pi \sum_{j=1}^{M} \omega_j^{-2} \\ &\leq \frac{2\pi}{q^2} \sum_{j=1}^{M} j^{-2} < \frac{\pi^3}{3q^2}. \end{aligned}$$

By similar arguments, we obtain

$$\sum_{\substack{j=-M \\ j\neq l}}^{M} |\sin((\omega_j - \omega_l)/2)|^{-2} < \frac{\pi^3}{3q^2}$$

in case $l \in \{\pm 1, \ldots, \pm M\}$. Note that we use the 2π–periodization of the frequencies ω_j $(j = -M, \ldots, M)$ such that

$$|\sin((\omega_j - \omega_l)/2)| = |\sin(\pm\pi + (\omega_j - \omega_l)/2)|\,.$$

Hence it follows in each case that

$$|\lambda - N - 1| < \frac{\pi^3}{3q^2(N+1)}\,. \tag{4.6}$$

4. Let $\lambda_{\min}$ and $\lambda_{\max}$ be the smallest and greatest eigenvalue of $\mathbf{V}^{\mathrm{H}}\mathbf{D}\mathbf{V}$, respectively. Using (4.6), we receive

$$N + 1 - \frac{\pi^3}{3q^2(N+1)} \leq \lambda_{\min} \leq N+1 \leq \lambda_{\max} \leq N + 1 + \frac{\pi^3}{3q^2(N+1)}\,,$$

where we have by assumption

$$N + 1 - \frac{\pi^3}{3q^2(N+1)} > 0\,.$$

Using the variational characterization of the Rayleigh–Ritz ratio for the Hermitian matrix $\mathbf{V}^{\mathrm{H}}\mathbf{D}\mathbf{V}$ (see [7, p. 176]), we obtain for arbitrary $\mathbf{r} \in \mathbb{C}^{2M+1}$ that

$$\lambda_{\min}\, \|\mathbf{r}\|^2 \leq \|\mathbf{D}^{1/2}\mathbf{V}\mathbf{r}\|^2 \leq \lambda_{\max}\, \|\mathbf{r}\|^2\,.$$

From

$$\big(\mathbf{V}^{\mathrm{H}}\mathbf{D}\mathbf{V}\big)^{-1}\, \mathbf{V}^{\mathrm{H}}\mathbf{D}^{1/2}\mathbf{D}^{1/2}\mathbf{V} = \mathbf{I}$$

it follows that the singular values of $\mathbf{D}^{1/2}\mathbf{V}$ lie in $[\sqrt{\lambda_{\min}}, \sqrt{\lambda_{\max}}]$. Hence we obtain for the squared spectral norm of the left inverse $\mathbf{L}$ (see also [15, Theorem 4.2])

$$\begin{aligned}\|\mathbf{L}\|_2^2 &\leq \|\left(\mathbf{V}^{\mathrm{H}}\mathbf{D}\mathbf{V}\right)^{-1}\mathbf{V}^{\mathrm{H}}\mathbf{D}^{1/2}\|_2^2\,\|\mathbf{D}^{1/2}\|_2^2 \leq \lambda_{\min}^{-1} \\ &\leq \frac{3q^2(N+1)}{3q^2(N+1)^2-\pi^3}\,.\end{aligned}$$

This completes the proof. ■

Corollary 4.2 *Let $M \in \mathbb{N}$ be given. Furthermore let $\omega_0 = 0 < \omega_1 < \ldots < \omega_M < \pi$ and (1.1) with separation distance q.*
Then for each $N \in \mathbb{N}$ with

$$N > \frac{2\pi}{q} + 1$$

the squared spectral norm of the left inverse $\mathbf{L}$ is bounded, i.e.,

$$\|\mathbf{L}\|_2^2 \leq \frac{3}{2N+2}\,.$$

Proof. By $N > \frac{2\pi}{q} + 1$ and $q\,M < \pi$, we see immediately that

$$2M+1 < \frac{2\pi}{q} + 1 < N\,, \quad q^2 > \pi^3\,(N+1)^{-2}\,.$$

Using the upper estimate (4.4) of $\|\mathbf{L}\|_2^2$, we obtain the result. ■

Corollary 4.3 *Let $M \in \mathbb{N}$ be given. Furthermore let $\omega_0 = 0 < \omega_1 < \ldots < \omega_M < \pi$ and (1.1) with separation distance q. Then the squared spectral norm of the left inverse $\mathbf{L}$ is bounded, i.e.,*

$$\|\mathbf{V}\|_2^2 \leq 2N + 2 + \frac{2\pi}{q}\,(1 + \ln\frac{\pi}{q}).$$

Proof. We follow the lines of the proof of Lemma 4.1 with the trivial diagonal matrix $\mathbf{D} := \mathrm{diag}\,(1)_{k=-N}^{N}$. Instead of the Fejér kernel F_N we use the modified Dirichlet kernel

$$D_N(x) := \sum_{k=0}^{2N} \mathrm{e}^{\mathrm{i}kx} = \mathrm{e}^{\mathrm{i}Nx}\,\frac{\sin((2N+1)x/2)}{\sin(x/2)}$$

and we obtain for the (j,l)-th entry of $\mathbf{V}^{\mathrm{H}}\mathbf{V}$

$$\left(\mathbf{V}^{\mathrm{H}}\mathbf{V}\right)_{j,l} = D_N(\omega_j - \omega_l) \quad (j,l = -M,\ldots,M)\,.$$

We proceed with

$$|\lambda - D_N(0)| = |\lambda - 2(N+1)| \leq \max\Big\{\sum_{\substack{j=-M\\ j\neq l}}^{M} |D_N(\omega_j-\omega_l)|;\, l = -M,\ldots,M\Big\}\,,$$

use the estimate

$$\sum_{\substack{j=-M\\ j\neq l}}^{M} |D_N(\omega_j - \omega_l)| \le \sum_{\substack{j=-M\\ j\neq l}}^{M} |\sin((\omega_j - \omega_l)/2)|^{-1}$$

and infer

$$\begin{aligned}\sum_{\substack{j=-M\\ j\neq 0}}^{M} |\sin(\omega_j/2)|^{-1} &= 2\sum_{j=1}^{M}\big(\sin(\omega_j/2)\big)^{-1} \le 2\pi \sum_{j=1}^{M} \omega_j^{-1} \\ &\le \frac{2\pi}{q}\sum_{j=1}^{M} j^{-1} < \frac{2\pi}{q}\,(1+\ln M) < \frac{2\pi}{q}\,\big(1+\ln\frac{\pi}{q}\big).\end{aligned}$$

Finally we obtain

$$\lambda_{\max}(\mathbf{V}^{\mathrm{H}}\mathbf{V}) \le 2N + 2 + \frac{2\pi}{q}\,(1+\ln\frac{\pi}{q})$$

and hence the assertion. ■

Theorem 4.4 *Let $M,\, N \in \mathbb{N}$ with $N \ge 2M+1$ be given. Furthermore let $h_k \in \mathbb{C}$ be given in* (2.3) *with $\rho_0 \in \mathbb{R}\backslash\{0\}$, $\rho_j \in \mathbb{C}\backslash\{0\}$ $(j = 1, \ldots, M)$, $\omega_0 = 0 < \omega_1 < \ldots < \omega_M < \pi$ and* (1.1)*. Assume that the perturbed Hankel matrix $\tilde{\mathbf{H}} = (\tilde{h}_{l+m})_{l,m=0}^{N}$ has $\sigma \in [-\varepsilon_2,\, \varepsilon_2]$ as eigenvalue with the corresponding normed eigenvector $\tilde{\mathbf{u}} = (\tilde{u}_n)_{n=0}^{N} \in \mathbb{R}^{N+1}$ and the related eigenpolynom*

$$\tilde{P}(z) = \sum_{k=0}^{N} \tilde{u}_k\, z^k \quad (z \in \mathbb{C}).$$

Then the rectangular Vandermode–type matrix (4.3) *has a left inverse $\mathbf{L} = (\mathbf{V}^{\mathrm{H}}\mathbf{D}\mathbf{V})^{-1}\mathbf{V}^{\mathrm{H}}\mathbf{D}$. Further the values $\tilde{P}(\mathrm{e}^{\mathrm{i}\omega_j})$ $(j = -M, \ldots, M)$ fulfill the estimate*

$$\rho_0^2\, \tilde{P}(1)^2 + 2\sum_{j=1}^{M} |\rho_j|^2 |\tilde{P}(\mathrm{e}^{\mathrm{i}\omega_j})|^2 \le (\varepsilon_2 + \|\mathbf{E}\|_2)^2\, \|\mathbf{L}\|_2^2\,.$$

Proof. 1. By assumption we have $\tilde{\mathbf{H}}\,\tilde{\mathbf{u}} = \sigma\,\tilde{\mathbf{u}}$, i.e.,

$$\sum_{l=0}^{N} \tilde{h}_{l+m}\, \tilde{u}_l = \sigma\, \tilde{u}_m \quad (m = 0,\, \ldots,\, N)$$

such that for arbitrary $z \in \mathbb{C}$ it follows that

$$\sum_{m=0}^{N} \Big(\sum_{l=0}^{N} \tilde{h}_{l+m}\, \tilde{u}_l\Big)\, z^m = \sigma\, \tilde{P}(z).$$

Using (2.3) and $\tilde{h}_k = h_k + e_k$, we obtain

$$\sum_{j=-M}^{M} \rho_j \, \tilde{P}(\mathrm{e}^{\mathrm{i}\omega_j}) \, Q(\mathrm{e}^{\mathrm{i}\omega_j} z) = \sigma \, \tilde{P}(z) - \sum_{m=0}^{N} \big(\sum_{l=0}^{N} e_{l+m} \, \tilde{u}_l\big) \, z^m$$

with

$$Q(z) := \sum_{m=0}^{N} z^m .$$

By comparison of coefficients, we receive the $N+1$ equations

$$\sum_{j=-M}^{M} \rho_j \, \tilde{P}(\mathrm{e}^{\mathrm{i}\omega_j}) \, \mathrm{e}^{\mathrm{i}k\omega_j} = \sigma \, \tilde{u}_k - \sum_{l=0}^{N} e_{l+k} \, \tilde{u}_l \quad (k = 0, \ldots, N). \tag{4.7}$$

2. Using the matrix–vector notation of (4.7) with the rectangular Vandermonde–type matrix $\mathbf{V}$ given by (4.3), we obtain

$$\mathbf{V} \, \big(\rho_j \, \tilde{P}(\mathrm{e}^{\mathrm{i}\omega_j})\big)_{j=-M}^{M} = \sigma \, \tilde{\mathbf{u}} - \mathbf{E} \, \tilde{\mathbf{u}}$$

with $\tilde{\mathbf{u}} := (\tilde{u}_k)_{k=0}^{N}$. By $N > 2M$ the matrix $\mathbf{V}$ has full rank $2M+1$ (see step 1 of the proof of Lemma 4.1). Hence $\mathbf{V}^{\mathrm{H}}\mathbf{D}\mathbf{V}$ is Hermitian and positive definite. The matrix $\mathbf{V}$ has a left inverse $\mathbf{L} = (\mathbf{V}^{\mathrm{H}}\mathbf{D}\mathbf{V})^{-1}\mathbf{V}^{\mathrm{H}}\mathbf{D}$ such that

$$\big(\rho_j \, \tilde{P}(\mathrm{e}^{\mathrm{i}\omega_j})\big)_{j=-M}^{M} = \sigma \, \mathbf{L} \, \tilde{\mathbf{u}} - \mathbf{L} \, \mathbf{E} \, \tilde{\mathbf{u}}$$

and hence

$$\rho_0^2 \, \tilde{P}(1)^2 + 2 \sum_{j=1}^{M} |\rho_j|^2 \, |\tilde{P}(\mathrm{e}^{\mathrm{i}\omega_j})|^2 \le (|\sigma| + \|\mathbf{E}\|_2)^2 \, \|\mathbf{L}\|_2^2 \, \|\tilde{\mathbf{u}}\|^2 \, .$$

This completes the proof. ■

Lemma 4.5 *If the assumptions of Theorem* 4.4 *are fulfilled with sufficiently small accuracies* ε_1, $\varepsilon_2 > 0$ *and if* $\delta > 0$ *is the smallest singular value* $\neq 0$ *of* $\mathbf{H}$, *then*

$$\|\tilde{\mathbf{u}} - \mathbf{P} \, \tilde{\mathbf{u}}\| \le \frac{\varepsilon_2 + (N+1) \, \varepsilon_1}{\delta} \, , \tag{4.8}$$

where $\mathbf{P}$ *is the orthogonal projector of* $\mathbb{R}^{N+1}$ *onto* $\ker \mathbf{H}$. *Furthermore the eigenpolynomial* $\tilde{P}$ *related to* $\tilde{\mathbf{u}}$ *has zeros close to* $\mathrm{e}^{\mathrm{i}\omega_j}$ $(j = -M, \ldots, M)$, *where*

$$\tilde{P}(1)^2 + 2 \sum_{j=1}^{M} |\tilde{P}(\mathrm{e}^{\mathrm{i}\omega_j})|^2 \le \Big(2N + 2 + \frac{2\pi}{q} \, (1 + \ln \frac{\pi}{q})\Big) \, \Big(\frac{\varepsilon_2 + (N+1)\varepsilon_1}{\delta}\Big)^2 \, .$$

Proof. 1. Let $\tilde{\mathbf{u}}$ be a normed eigenvector of $\tilde{\mathbf{H}}$ with respect to the eigenvalue $\sigma \in [-\varepsilon_2,\, \varepsilon_2]$. Using the Rayleigh–Ritz Theorem (see [7, pp. 176–178]), we receive

$$\delta = \min_{\substack{\mathbf{u} \neq \mathbf{0} \\ \mathbf{u} \perp \ker \mathbf{H}}} \frac{\|\mathbf{H}\mathbf{u}\|}{\|\mathbf{u}\|} = \min_{\substack{\tilde{\mathbf{u}} - \mathbf{u} \neq \mathbf{0} \\ \tilde{\mathbf{u}} - \mathbf{u} \perp \ker \mathbf{H}}} \frac{\|\mathbf{H}(\tilde{\mathbf{u}} - \mathbf{u})\|}{\|\tilde{\mathbf{u}} - \mathbf{u}\|}\,,$$

i.e., the following estimate

$$\delta\, \|\tilde{\mathbf{u}} - \mathbf{u}\| \le \|\mathbf{H}(\tilde{\mathbf{u}} - \mathbf{u})\|$$

is valid for all $\mathbf{u} \in \mathbb{R}^{N+1}$ with $\tilde{\mathbf{u}} - \mathbf{u} \perp \ker \mathbf{H}$. Especially for $\mathbf{u} = \mathbf{P}\,\tilde{\mathbf{u}}$, we see that $\tilde{\mathbf{u}} - \mathbf{P}\,\tilde{\mathbf{u}} \perp \ker \mathbf{H}$ and hence by (4.1)

$$\delta\, \|\tilde{\mathbf{u}} - \mathbf{P}\,\tilde{\mathbf{u}}\| \le \|\mathbf{H}\,\tilde{\mathbf{u}}\| = \|(\tilde{\mathbf{H}} - \mathbf{E})\,\tilde{\mathbf{u}}\| = \|\sigma\,\tilde{\mathbf{u}} - \mathbf{E}\,\tilde{\mathbf{u}}\| \le |\sigma| + (N+1)\,\varepsilon_1$$

such that (4.8) follows.
2. Thereby $\mathbf{u} = \mathbf{P}\,\tilde{\mathbf{u}}$ is an eigenvector of $\mathbf{H}$ with respect to the eigenvalue 0. Thus the corresponding eigenpolynomial P has the values $\mathrm{e}^{\mathrm{i}\omega_j}$ $(j = -M, \ldots, M)$ as zeros by Lemma 2.2. By (4.8), the coefficients of P differ only a little from the coefficients of $\tilde{P}$ with respect to $\tilde{\mathbf{u}}$. Consequently, the zeros of $\tilde{P}$ lie nearby the zeros of P, i.e., $\tilde{P}$ has zeros close to $\mathrm{e}^{\mathrm{i}\omega_j}$ $(j = -M, \ldots, M)$ (see [7, pp. 539–540]).
By $\|\mathbf{V}\|_2 = \|\mathbf{V}^{\mathrm{H}}\|_2$, (4.8), and Corollary 4.3, we obtain the estimate

$$\begin{aligned}\sum_{j=-M}^{M} |\tilde{P}(\mathrm{e}^{\mathrm{i}\omega_j})|^2 &= \sum_{j=-M}^{M} |P(\mathrm{e}^{\mathrm{i}\omega_j}) - \tilde{P}(\mathrm{e}^{\mathrm{i}\omega_j})|^2 = \|\mathbf{V}^{\mathrm{H}}(\mathbf{u} - \tilde{\mathbf{u}})\|^2 \\ &\le \|\mathbf{V}^{\mathrm{H}}\|_2^2\, \|\mathbf{u} - \tilde{\mathbf{u}}\|^2 \le \Big(2N + 2 + \frac{2\pi}{q}\,(1 + \ln \frac{\pi}{q})\Big)\Big(\frac{\varepsilon_2 + (N+1)\varepsilon_1}{\delta}\Big)^2 .\end{aligned}$$

This completes the proof. ■

Corollary 4.6 *Let $M \in \mathbb{N}$. Furthermore let $\omega_0 = 0 < \omega_1 < \ldots < \omega_M < \pi$ with* (1.1) *and the separation distance q be given. Further let $N \in \mathbb{N}$ with*

$$N > \frac{2\pi}{q} + 1$$

be given. Assume that the perturbed Hankel matrix $\tilde{\mathbf{H}} = (\tilde{h}_{l+m})_{l,m=0}^{N}$ has an eigenvalue $\sigma \in [-\varepsilon_2,\, \varepsilon_2]$ with a corresponding normed eigenvector $\tilde{\mathbf{u}} = (\tilde{u}_n)_{n=0}^{N} \in \mathbb{R}^{N+1}$.
Then the values $|\tilde{P}(\mathrm{e}^{\mathrm{i}\omega_j})|$ $(j = -M, \ldots, M)$ can be estimated by

$$\rho_0^2\, \tilde{P}(1)^2 + 2 \sum_{j=1}^{M} |\rho_j|^2\, |\tilde{P}(\mathrm{e}^{\mathrm{i}\omega_j})|^2 \le \frac{3}{2}\Big(\frac{\varepsilon_2}{\sqrt{N+1}} + \sqrt{N+1}\,\varepsilon_1\Big)^2$$

i.e., the values $|\tilde{P}(\mathrm{e}^{\mathrm{i}\omega_j})|$ $(j = -M, \ldots, M)$ are small. Further the eigenpolynomial $\tilde{P}$ has zeros close to $\mathrm{e}^{\mathrm{i}\omega_j}$ $(j = -M, \ldots, M)$.

Proof. The assertion follows immediately by the assumption on N, the estimate (4.1), Theorem 4.4, and Lemma 4.5. ■

Now we can formulate the approximate Prony method.

Algorithm 4.7 *(Approximate Prony method)*
Input: $N \in \mathbb{N}$ $(N \gg 1)$, $\tilde{h}_k = f(k) + e_k$ $(k = 0, \ldots, 2N)$ *with* $|e_k| \le \varepsilon_1$, *accuracies* ε_1, ε_2, ε_3, ε_4.

1. Compute an eigenvector $\tilde{\mathbf{u}}^{(1)} = (\tilde{u}_l^{(1)})_{l=0}^N$ *corresponding to an eigenvalue* $\sigma^{(1)} \in [-\varepsilon_2,\, \varepsilon_2]$ *of the perturbed Hankel matrix* $\tilde{\mathbf{H}} = (\tilde{h}_{l+m})_{l,m=0}^N$.
2. Form the corresponding eigenpolynomial

$$\tilde{P}^{(1)}(z) = \sum_{k=0}^{N} \tilde{u}_k^{(1)}\, z^k$$

and evaluate all zeros $r_j^{(1)}\, \mathrm{e}^{\mathrm{i}\omega_j^{(1)}}$ $(j = 1, \ldots, M^{(1)})$ *with* $\omega_j^{(1)} \in (0,\, \pi)$, (1.1) *and* $|r_j^{(1)} - 1| \le \varepsilon_4$. *Note that* $N \ge 2M^{(1)} + 1$.
3. Compute an eigenvector $\tilde{\mathbf{u}}^{(2)} = (\tilde{u}_l^{(2)})_{l=0}^N$ *corresponding to an eigenvalue* $\sigma^{(2)} \in [-\varepsilon_2,\, \varepsilon_2]$ $(\sigma^{(1)} \ne \sigma^{(2)})$ *of the perturbed Hankel matrix* $\tilde{\mathbf{H}} = (\tilde{h}_{l+m})_{l,m=0}^N$.
4. Form the corresponding eigenpolynomial

$$\tilde{P}^{(2)}(z) = \sum_{k=0}^{N} \tilde{u}_k^{(2)}\, z^k$$

and evaluate all zeros $r_k^{(2)}\, \mathrm{e}^{\mathrm{i}\omega_k^{(2)}}$ $(k = 1, \ldots, M^{(2)})$ *with* $\omega_k^{(2)} \in (0,\, \pi)$, (1.1) *and* $|r_k^{(2)} - 1| \le \varepsilon_4$. *Note that* $N \ge 2M^{(2)} + 1$.
5. Determine all frequencies

$$\tilde{\omega}_l := \frac{1}{2}\left(\omega_{j(l)}^{(1)} + \omega_{k(l)}^{(2)}\right) \quad (l = 1 \ldots, \tilde{M}),$$

if there exist indices $j(l) \in \{1, \ldots, M^{(1)}\}$ *and* $k(l) \in \{1, \ldots, M^{(2)}\}$ *such that* $|\omega_{j(l)}^{(1)} - \omega_{k(l)}^{(2)}| \le \varepsilon_3$. *Replace* $r_{j(l)}^{(1)}$ *and* $r_{k(l)}^{(2)}$ *by* 1. *Note that* $N \ge 2\tilde{M} + 1$.
6. Compute $\tilde{\rho}_0 \in \mathbb{R}$ *and* $\tilde{\rho}_j \in \mathbb{C}$ $(j = 1, \ldots, \tilde{M})$ *with* (1.1) *as least squares solution of the overdetermined linear Vandermonde–type system*

$$\sum_{j=-\tilde{M}}^{\tilde{M}} \tilde{\rho}_j\, \mathrm{e}^{\mathrm{i}\tilde{\omega}_j k} = \tilde{h}_k \quad (k = 0, \ldots, 2N) \tag{4.9}$$

with the diagonal preconditioner $\mathbf{D} = \mathrm{diag}\big(1 - |k|/(N+1)\big)_{k=-N}^N$. *For very large* M *and* N *use the CGNR method, where the multiplication of the Vandermonde–type matrix*

$$\tilde{\mathbf{V}} := \left(\mathrm{e}^{\mathrm{i}k\tilde{\omega}_j}\right)_{k=0, j=-\tilde{M}}^{2N,\, \tilde{M}}$$

is realized in each iteration step by the nonequispaced fast Fourier transform (see [14, 9]).
7. Cancel all that pairs $(\tilde{\omega}_l, \tilde{\rho}_j)$ $(l \in \{1, \ldots, \tilde{M}\})$ with $|\tilde{\rho}_l| \le \varepsilon_1$ and denote the remaining frequency set by $\{\omega_j : j = 1, \ldots, M\}$ with $M \le \tilde{M}$.
8. Repeat step 6 and solve the overdetermined linear Vandermonde–type system

$$\sum_{j=-M}^{M} \rho_j \, \mathrm{e}^{\mathrm{i}\omega_j k} = \tilde{h}_k \quad (k = 0, \ldots, 2N)$$

with respect to the new frequency set $\{\omega_j : j = 1, \ldots, M\}$ again.

Output: $M \in \mathbb{N}$, $\rho_0 \in \mathbb{R}$, $\rho_j \in \mathbb{C}$, $(\rho_j = \bar{\rho}_{-j})$, $\omega_j \in (0, \pi)$ $(j = 1, \ldots, M)$.

5 Sensitivity analysis of the approximate Prony method

In this section, we discuss the sensitivity of step 6 of our Algorithm 4.7. Assume that $N \ge 2M + 1$ and $\tilde{M} = M$. Then solve the overdetermined linear Vandermonde–type system (4.9) with $M = \tilde{M}$ as weighted least squares problem

$$\big\|\mathbf{D}^{1/2}\big(\tilde{\mathbf{V}}\,\tilde{\boldsymbol{\rho}} - \tilde{\mathbf{h}}\big)\big\| = \min\,. \tag{5.1}$$

Here $\tilde{\mathbf{h}} = \big(\tilde{h}_k\big)_{k=0}^{2N}$ is the perturbed data vector and

$$\tilde{\mathbf{V}} = \Big(\mathrm{e}^{\mathrm{i}k\tilde{\omega}_j}\Big)_{k=0, j=-M}^{2N, M}$$

is the Vandermonde–type matrix with the computed frequencies $\tilde{\omega}_j$ $(j = -M, \ldots, M)$, where $0 = \tilde{\omega}_0 < \omega_1 < \ldots < \tilde{\omega}_M < \pi$, $\tilde{\omega}_j = -\tilde{\omega}_{-j}$ $(j = -M, \ldots, -1)$ and $\tilde{q}$ is the separation distance of the computed frequency set $\{\tilde{\omega}_j : j = 0, \ldots, M+1\}$ with $\tilde{\omega}_{M+1} = 2\pi - \tilde{\omega}_M$. Note that $\tilde{\mathbf{V}}$ has full rank and that the unique solution of (5.1) is given by $\tilde{\boldsymbol{\rho}} = \tilde{\mathbf{L}}\,\tilde{\mathbf{h}}$ with the left inverse $\tilde{\mathbf{L}} = \big(\tilde{\mathbf{V}}^{\mathrm{H}}\mathbf{D}\,\tilde{\mathbf{V}}\big)^{-1}\tilde{\mathbf{V}}^{\mathrm{H}}\mathbf{D}$ of $\tilde{\mathbf{V}}$.
We begin with a normwise perturbation result, if all frequencies are exactly determined, i.e., $\omega_j = \tilde{\omega}_j$ $(j = -M, \ldots, M)$, and if a perturbed data vector $\tilde{\mathbf{h}}$ is given.

Lemma 5.1 *Assume that $\omega_j = \tilde{\omega}_j$ $(j = -M, \ldots, M)$ and that $|\tilde{h}_k - h_k| \le \varepsilon_1$. Let $\mathbf{V}$ be given by (4.3). Further let $\mathbf{V}\,\boldsymbol{\rho} = \mathbf{h}$ and $\tilde{\boldsymbol{\rho}} = \mathbf{L}\,\tilde{\mathbf{h}}$, where $\mathbf{L} = \big(\mathbf{V}^{\mathrm{H}}\mathbf{D}\,\mathbf{V}\big)^{-1}\mathbf{V}^{\mathrm{H}}\mathbf{D}$ is a left inverse of $\mathbf{V}$. If the assumptions of Corollary 4.2 are fulfilled, then for each $N \in \mathbb{N}$ with $N > 2\pi\, q^{-1} + 1$ the condition number $\kappa(\mathbf{V}) := \|\mathbf{L}\|_2\, \|\mathbf{V}\|_2$ is uniformly (with respect to N) bounded by*

$$\kappa(\mathbf{V}) \le \sqrt{3 + \frac{3\pi q}{2\pi + 2q}\left(1 + \ln\frac{\pi}{q}\right)}\,. \tag{5.2}$$

Furthermore, the following stability inequalities are fulfilled

$$\|\boldsymbol{\rho} - \tilde{\boldsymbol{\rho}}\| \;\le\; \sqrt{\frac{3}{2N+2}}\,\|\mathbf{h} - \tilde{\mathbf{h}}\| \le \sqrt{3}\,\varepsilon_1\,, \tag{5.3}$$

$$\frac{\|\boldsymbol{\rho} - \tilde{\boldsymbol{\rho}}\|}{\|\boldsymbol{\rho}\|} \;\le\; \kappa\,(\mathbf{V})\;\frac{\|\mathbf{h} - \tilde{\mathbf{h}}\|}{\|\mathbf{h}\|}\,. \tag{5.4}$$

Proof. 1. The condition number $\kappa(\mathbf{V})$ of the rectangular Vandermonde–type matrix $\mathbf{V}$ is defined as the number $\|\mathbf{L}\|_2\,\|\mathbf{V}\|_2$. Note that $\kappa(\mathbf{V})$ does not coincide with the condition number of $\mathbf{V}$ related to the spectral norm, since the left inverse $\mathbf{L}$ is not the Moore–Penrose pseudoinverse of $\mathbf{V}$ by $\mathbf{D} \neq \mathbf{I}$. Applying the Corollaries 4.2 and 4.3, we receive that

$$\kappa(\mathbf{V}) \leq \sqrt{3 + \frac{6\pi}{2N+2}\left(1 + \ln\frac{\pi}{q}\right)}\,.$$

This provides (5.2), since by assumption $(N+1)^{-1} < q\,(2\pi + 2q)^{-1}$.
2. The inequality (5.3) follows immediately from $\|\boldsymbol{\rho} - \tilde{\boldsymbol{\rho}}\| \leq \|\mathbf{L}\|_2\,\|\mathbf{h} - \tilde{\mathbf{h}}\|$ and Corollary 4.2. The estimate (5.4) arises from (5.3) by multiplication with $\|\mathbf{V}\boldsymbol{\rho}\|\,\|\mathbf{h}\|^{-1} = 1$. ■

Now we consider the general case, where the weighted least squares problem (5.1) is solved for perturbed data $\tilde{h}_k$ $(k = 0, \ldots, 2N)$ and computed frequencies $\tilde{\omega}_j$ $(j = -M, \ldots, M)$.

Theorem 5.2 *Assume that* $|\tilde{h}_k - h_k| \leq \varepsilon_1$ $(k = 0, \ldots, 2N)$ *and* $|\tilde{\omega}_j - \omega_j| \leq \delta$ $(j = 0, \ldots, M)$*. Let* $\mathbf{V}$ *be given by* (4.3) *and let*

$$\tilde{\mathbf{V}} := \left(\mathrm{e}^{\mathrm{i}k\tilde{\omega}_j}\right)_{k=0,\, j=-M}^{2N,\, M}.$$

Further let $\mathbf{V}\boldsymbol{\rho} = \mathbf{h}$ *and* $\tilde{\boldsymbol{\rho}} = \tilde{\mathbf{L}}\,\tilde{\mathbf{h}}$*, where* $\tilde{\mathbf{L}} = \left(\tilde{\mathbf{V}}^{\mathrm{H}}\mathbf{D}\,\tilde{\mathbf{V}}\right)^{-1}\tilde{\mathbf{V}}^{\mathrm{H}}\mathbf{D}$ *is a left inverse of* $\tilde{\mathbf{V}}$*. If the assumptions of Corollary* 4.2 *are fulfilled, then for each* $N \in \mathbb{N}$ *with*

$$N > 2\pi\,\max\{q^{-1},\, \tilde{q}^{-1}\} + 1 \tag{5.5}$$

the following estimate

$$\|\boldsymbol{\rho} - \tilde{\boldsymbol{\rho}}\| \leq \sqrt{(6N+6)\,(2M+1)}\,\|\mathbf{h}\|\,\delta + \sqrt{3}\,\varepsilon_1$$

is fulfilled.

Proof. 1. Using the matrix–vector notation, we can write (2.6) in the form $\mathbf{V}\,\boldsymbol{\rho} = \mathbf{h}$. The overdetermined linear system (4.9) with $\tilde{M} = M$ reads $\tilde{\mathbf{V}}\,\tilde{\boldsymbol{\rho}} = \tilde{\mathbf{h}}$. Using the left inverse $\tilde{\mathbf{V}}$, the solution $\tilde{\boldsymbol{\rho}}$ of the weighted least squares problem

$$\|\mathbf{D}^{1/2}\left(\tilde{\mathbf{V}}\,\tilde{\boldsymbol{\rho}} - \tilde{\mathbf{h}}\right)\| = \min$$

is $\tilde{\boldsymbol{\rho}} = \tilde{\mathbf{L}}\,\tilde{\mathbf{h}}$. Thus it follows that

$$\begin{aligned}
\|\boldsymbol{\rho} - \tilde{\boldsymbol{\rho}}\| &= \|\boldsymbol{\rho} - \tilde{\mathbf{L}}\tilde{\mathbf{h}}\| \\
&= \|\tilde{\mathbf{L}}\,\tilde{\mathbf{V}}\,\boldsymbol{\rho} - \tilde{\mathbf{L}}\,\mathbf{V}\,\boldsymbol{\rho} - \tilde{\mathbf{L}}\left(\tilde{\mathbf{h}} - \mathbf{h}\right)\| \\
&\leq \|\tilde{\mathbf{L}}\|_2\,\|\tilde{\mathbf{V}} - \mathbf{V}\|_2\,\|\boldsymbol{\rho}\| + \|\tilde{\mathbf{L}}\|_2\,\|\tilde{\mathbf{h}} - \mathbf{h}\|\,.
\end{aligned}$$

2. Now we estimate the squared spectral norm of $\tilde{\mathbf{V}} - \mathbf{V}$ by means of the Frobenius norm

$$\begin{aligned}
\|\mathbf{V} - \tilde{\mathbf{V}}\|_2^2 &\le \|\mathbf{V} - \tilde{\mathbf{V}}\|_{\mathrm{F}}^2 = \sum_{k=0}^{2N} \sum_{j=-M}^{M} \left|\mathrm{e}^{\mathrm{i}k\omega_j} - \mathrm{e}^{\mathrm{i}k\tilde{\omega}_j}\right|^2 \\
&= \sum_{j=-M}^{M} \Big(\sum_{k=0}^{2N} \big[2 - 2\cos((\omega_j - \tilde{\omega}_j)k)\big]\Big) \\
&= \sum_{j=-M}^{M} \Big(4N + 1 - \frac{\sin\left((2N+1/2)(\omega_j - \tilde{\omega}_j)\right)}{\sin((\omega_j - \tilde{\omega}_j)/2))}\Big) .
\end{aligned}$$

Using the special property of the Dirichlet kernel

$$\frac{\sin(4N+1)x/2}{\sin x/2} \ge 4\,N + 1 + \Big(-\frac{1}{3}\,\big(2N + \frac{1}{2}\big)^3 + \frac{N}{6} + \frac{1}{24}\Big)\,x^2 ,$$

we infer

$$\begin{aligned}
\|\mathbf{V} - \tilde{\mathbf{V}}\|_2^2 &\le \sum_{j=-M}^{M} \Big(\frac{1}{3}\,\big(2N + \frac{1}{2}\big)^3 - \frac{N}{6} - \frac{1}{24}\Big)\,\delta^2 \\
&\le \frac{(2N+1/2)^3(2M+1)}{3}\,\delta^2 . \qquad (5.6)
\end{aligned}$$

Thus we receive

$$\|\boldsymbol{\rho} - \tilde{\boldsymbol{\rho}}\| \le \frac{1}{\sqrt{3}}\,\big(2N + \frac{1}{2}\big)^{3/2}\,\sqrt{2M+1}\,\|\tilde{\mathbf{L}}\|\,\|\boldsymbol{\rho}\|\,\delta + \|\tilde{\mathbf{L}}\|_2\,\|\tilde{\mathbf{h}} - \mathbf{h}\| .$$

From Corollary 4.2 it follows that for each $N \in \mathbb{N}$ with (5.5)

$$\|\tilde{\mathbf{L}}\|_2 \le \sqrt{\frac{3}{2N+2}} .$$

Finally we use

$$\begin{aligned}
\|\tilde{\mathbf{h}} - \mathbf{h}\| &\le \sqrt{2N+1}\,\|\tilde{\mathbf{h}} - \mathbf{h}\|_\infty \le \sqrt{2N+1}\,\varepsilon_1 , \\
\|\boldsymbol{\rho}\| &= \|\mathbf{L}\,\mathbf{h}\| \le \|\mathbf{L}\|_2\,\|\mathbf{h}\| \le \sqrt{\frac{3}{2N+2}}\,\|\mathbf{h}\|
\end{aligned}$$

and obtain the result. ■

By Theorem 5.2, we see that we have to compute the frequencies very carefully. This is the reason that we repeat the computation of the frequencies in the steps 3–4 of Algorithm 4.7.

6 Numerical examples

Finally, we apply Algorithm 4.7 to various examples. We have implemented our algorithm of the approximate Prony method in Matlab with IEEE double precision arithmetic. In order to evaluate the zeros of the corresponding eigenpolynomial, we use the Matlab command `roots`, which is based on computing the eigenvalues of the companion matrix related to the eigenpolynomial.

Example 6.1 *We sample the anharmonic Fourier sum*

$$\begin{aligned} f_1(x) \;:=\;& 14 - 8\cos(0.453\,x) + 9\sin(0.453\,x) + 4\cos(0.979\,x) \\ & + 8\sin(0.979\,x) - 2\cos(0.981\,x) + 2\cos(1.847\,x) \\ & - 3\sin(1.847\,x) + 0.1\cos(2.154\,x) - 0.3\sin(2.154\,x) \end{aligned} \qquad (6.1)$$

at the equidistant nodes $x = k$ $(k = 0, \ldots, 2N)$. Then we apply Algorithm 4.7 with exact sampled data $h_k = f_1(k)$, i.e., $e_k = 0$ $(k = 0, \ldots, 2N)$. Thus the accuracy ε_1 can be chosen as the unit roundoff $\varepsilon_1 = 2^{-53} \approx 1.11 \times 10^{-16}$ (see [6, p. 45]). Note that the frequencies $\omega_2 = 0.979$ and $\omega_3 = 0.981$ are very closely such that $q = 0.002$. Using Corollary 4.6, we have to choose $N > \frac{2\pi}{q} + 1$, i.e., $N \geq 3143$. However for $N = 22$ and for the accuracies $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 10^{-7}$ we obtain all nonzero frequencies ω_j $(j = 1, \ldots, 10)$ with 11 correct decimal places and all coefficients ρ_j $(j = 0, \ldots, 10)$ with 8 correct decimal places. Furthermore we determine the absolute error $e_{\mathrm{abs}}(f_1) := \max|f_1(x) - \tilde{f}_1(x)|$ computed on 10000 equidistant points on the interval $[0, 2N]$ and obtain $e_{\mathrm{abs}}(f_1) \leq 6.8 \cdot 10^{-13}$. Here $\tilde{f}_1$ denotes the reconstructed function. $\square$

Example 6.2 *We use again the function (6.1). Now we consider noisy sampled data $\tilde{h}_k = f_1(k) + e_k$ $(k = 0, \ldots, 2N)$, where e_k is a random error with $|e_k| \leq \varepsilon_1 = 10^{-3}$, more precisely we add pseudorandom values drawn from the standard uniform distribution on $(0, 10^{-3})$. Now we choose $\varepsilon_4 < 10^{-3}$. In the case $N = 22$, we obtain the correct number $M = 10$, all nonzero frequencies ω_j $(j = 1, \ldots, 10)$ and all coefficients ρ_j $(j = 0, \ldots, 10)$ with 2 correct decimal places. The absolute error $e_{\mathrm{abs}}(f_1)$ is bounded by $1.8 \cdot 10^{-3}$.*
For $N = 100$, we obtain the correct number $M = 10$, all nonzero frequencies ω_j $(j = 1, \ldots, 10)$ and all coefficients ρ_j $(j = 0, \ldots, 10)$ with an 3 correct decimal places. The absolute error is now $e_{\mathrm{abs}}(f_1) \leq 7.1 \cdot 10^{-4}$. We see that for noisy sampled data oversampling, i.e., increasing N improves the results. $\square$

Example 6.3 *As in [8], we sample the 24–periodic function*

$$f_2(x) := 2\,\cos(\pi x/6) + 200\,\cos(\pi x/4) + 2\,\cos(\pi x/2) + 2\,\cos(5\pi x/6)$$

at the equidistant nodes $x = k$ $(k = 0, \ldots, 2N)$. In [8], only the dominant frequency $\omega_2 = \pi/4$ is iteratively computed via zeros of Szegö polynomials. For $N = 50$, ω_2 is determined to 2 correct decimal places. For $N = 200$ and $N = 500$, the frequency ω_2 is computed to 4 correct decimal places in [8].

Now we apply Algorithm 4.7 with the same parameters as in Example 6.1*, but we use only the* 37 *sampled data* $f_2(k)$ $(k = 0, \ldots, 36)$*. In this case we are able to compute all frequencies and all coefficients with* 12 *correct decimal places.* □

Example 6.4 *A different method for finding the frequencies based on detection of singularities is suggested in* [12]*. Similarly as in* [12]*, we consider the* 8*-periodic test function*

$$f_3(x) := 34 + 300\,\cos(\pi x/4) + \cos(\pi x/2)$$

and sample f_3 *at the nodes* $x = k$ $(k = 0, \ldots, 32)$*. Note that in* [12]*, uniform noise in the range* $[0, 3]$ *is added and this experiment is repeated* 500*-times. Then the frequencies are recovered very accurately. In this case, the noise exceeds the lowest coefficient of* f_3*. Now we compare this with Algorithm 4.7, where we add only noise in the range* $[0, 1]$ *to the sampled data* $f_3(k)$ $(k = 0, \ldots, 64)$*. We choose the accuracies as follows* $\varepsilon_1 = 0.9$*,* $\varepsilon_2 = \varepsilon_3 = 10^{-7}$ *and* $\varepsilon_4 = 0.1$*. In this case, we are able to compute all frequencies and all coefficients with 3 correct decimal places and the absolute error is* $e_{\text{abs}}(f_3) \le 0.6$*.* □

Example 6.5 *Finally, we consider the function*

$$f_4(x) := \sum_{j=1}^{80} \cos(\omega_j x)\,,$$

where we choose random frequencies ω_j *drawn from the standard uniform distribution on* $(0, \pi)$*. We sample the function* f_4 *at the equidistant nodes* $x = k$ $(k = 0, \ldots, 2N)$ *with* $N = 175$*. Using the Algorithm 4.7, we receive all frequencies* ω_j *and all coefficients* $\rho_j = 1/2$ *with* 2 *correct decimal places. For the absolute error* $e_{\text{abs}}(f_4) := \max |f_4(x) - \tilde{f}_4(x)|$ *computed on* 10000 *equidistant points, we obtain* $e_{\text{abs}}(f_4) \le 1.3\cdot 10^{-4}$*. Note that without the diagonal preconditioner* $\mathbf{D}$ *the overdetermined linear Vandermonde–type system is numerically not solvable due to the ill–conditioning of the Vandermonde–type system. Of course, we can improve the accuracy of the results by oversampling, i.e., increasing* N*.* □

In summary, we obtain very accurate results already for relatively few sampled data. We can analyze both periodic and nonperiodic functions without preprocessing (as filtering or windowing). The approximate Prony method works correctly for noisy sampled data and for clustered frequencies assumed that the separation distance of the frequencies is not too small and the number $2N+1$ of sampled data is sufficiently large. Oversampling improves the results. The numerical examples show that one can choose the number $2N + 1$ of sampled data less than expected by the theoretical results. However we have essentially improved the stability of our approximate Prony method algorithm by using a weighted least squares method. Our numerical examples confirm that the proposed approximate Prony method is robust with respect to noise.

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