Inequalities of the Markov type for partial derivatives of polynomials in several variables

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First order asymptotic formulas for the best constants in inequalities of the Markov type with L^2 norms for partial derivatives of polynomials in several variables are derived. The principal coefficient in the leading term of the formulas is identified as the operator norm of a Volterra integral operator and is studied in detail.

Keywords: Markov inequality, orthogonal polynomials, Volterra integral operators, partial differential operator, polynomials of several variables

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1 Introduction and main results

Markov-type inequalities give upper bounds for the derivatives of an algebraic polynomial by the polynomial itself. To be more precise, they provide a constant C such that $\|D^\nu f\| \le C\|f\|$ for all polynomials of degree at most n, where D is the operator of differentiation. The constant C depends on n, on the order ν of the derivative, and on the norm $\|\cdot\|$. We here consider the case where $\|\cdot\|$ is one of the classical L^2 norms and study the problem of extending such inequalities to the situation when f is a polynomial of several variables and D^ν is replaced by a partial differential operator.

Let $\mathcal{P}_n$ be the linear space of all polynomials $f(t) = \sum_{j=0}^n f_j t^j$ of degree at most n with complex coefficients f_j. We equip $\mathcal{P}_n$ with one of the classical Hermite, Laguerre, or Gegenbauer norms. These are defined by

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 e^{-t^2}\, dt, \tag{1}$$

$$\|f\|^2 = \int_0^{\infty} |f(t)|^2 t^\alpha e^{-t}\, dt, \tag{2}$$

$$\|f\|^2 = \int_{-1}^{1} |f(t)|^2 (1-t^2)^\alpha\, dt, \tag{3}$$

where $\alpha > -1$ is a parameter. Given a polynomial

$$p(\xi) = \xi^m + p_{m-1}\xi^{m-1} + \ldots + p_0,$$

we can consider the differential operator $p(D)$ on $\mathcal{P}_n$. The best constant C such that $\|p(D)f\| \le C\|f\|$ for all $f \in \mathcal{P}_n$ is clearly nothing but the norm of the operator $p(D) : \mathcal{P}_n \to \mathcal{P}_n$. This constant will be denoted by $\eta_n(p(D))$, $\lambda_n(p(D))$, $\gamma_n(p(D))$ in dependence on whether the norm $\|\cdot\|$ is (1), (2), (3). In [4], we showed that

$$\begin{aligned}
\lim_{n\to\infty} \frac{\eta_n(p(D))}{n^{m/2}} &= 2^{m/2},\\
\lim_{n\to\infty} \frac{\lambda_n(p(D))}{n^{m}} &= \|L_{m,\alpha}\|_\infty,\\
\lim_{n\to\infty} \frac{\gamma_n(p(D))}{n^{2m}} &= \|G_{m,\alpha}\|_\infty,
\end{aligned}$$

where $L_{m,\alpha}$ and $G_{m,\alpha}$ are the Volterra integral operators on $L^2(0,1)$ that are given by

$$(L_{m,\alpha}f)(x) = \frac{1}{\Gamma(m)} \int_x^1 x^{\alpha/2} y^{-\alpha/2} (y-x)^{m-1} f(y)\, dy, \tag{4}$$

$$(G_{m,\alpha}f)(x) = \frac{1}{2^{m-1}\Gamma(m)} \int_x^1 x^{1/2+\alpha} y^{1/2-\alpha} (y^2-x^2)^{m-1} f(y)\, dy \tag{5}$$

and $\|\cdot\|_\infty$ denotes the operator norm. Note that these operators are just the iterates (= powers) of their $m = 1$ versions, that is, $L_{m,\alpha} = L_{1,\alpha}^m$ and $G_{m,\alpha} = G_{1,\alpha}^m$.

Let $[0,1]^N$ be the N-dimensional cube. Given a closed subset E of $[0,1]^N$, we define $\mathcal{P}_n(E)$ as the linear space of all polynomials f of the form

$$f(t_1, \ldots, t_N) = \sum_{(i_1/n, \ldots, i_N/n) \in E} f_{i_1,\ldots,i_N} t_1^{i_1} \ldots t_N^{i_N}$$

with complex coefficients. We will always assume that E contains a point in the interior of $[0,1]^N$ and that E contains together with each of its points $(x_1, \ldots, x_N)$ also the cube $[0, x_1] \times \ldots \times [0, x_N]$. The most canonical choice of E is

$$\Omega_N := \{(x_1, \ldots, x_N) \in [0,1]^N : x_1 + \ldots + x_N \le 1\}.$$

For $\delta > 0$, we define $E^\delta = \{(x_1^\delta, \ldots, x_N^\delta) : (x_1, \ldots, x_N) \in E\}$. For example,

$$\begin{aligned}
\Omega_N^{1/2} &= \{(x_1, \ldots, x_N) \in [0,1]^N : x_1^2 + \ldots + x_N^2 \le 1\},\\
\Omega_N^{2} &= \{(x_1, \ldots, x_N) \in [0,1]^N : \sqrt{x_1} + \ldots + \sqrt{x_N} \le 1\}.
\end{aligned}$$

We endow $\mathcal{P}_n(E)$ with the N-dimensional versions of the norms (1), (2), (3):

$$\|f\|^2 = \int_{\mathbf{R}^N} |f(t_1, \ldots, t_N)|^2 e^{-t_1^2} \ldots e^{-t_N^2}\, dt_1 \ldots dt_N, \tag{6}$$

$$\|f\|^2 = \int_{(0,\infty)^N} |f(t_1, \ldots, t_N)|^2 t_1^{\alpha_1} \ldots t_N^{\alpha_N} e^{-t_1} \ldots e^{-t_N}\, dt_1 \ldots dt_N, \tag{7}$$

$$\|f\|^2 = \int_{(-1,1)^N} |f(t_1, \ldots, t_N)|^2 (1-t_1^2)^{\alpha_1} \ldots (1-t_N^2)^{\alpha_N}\, dt_1 \ldots dt_N, \tag{8}$$

where $\alpha_j > -1$ for all j.

Take a polynomial

$$p(\xi_1, \ldots, \xi_N) = \sum_{\nu_1+\ldots+\nu_N \le M} p_{\nu_1,\ldots,\nu_N} \xi_1^{\nu_1} \ldots \xi_N^{\nu_N}. \tag{9}$$

Here $\nu_1+\ldots+\nu_n \le M$ means that the sum is taken over all N-tuples $(\nu_1, \ldots, \nu_N)$ of nonnegative integers ν_j whose sum does not exceed M. The differential operator on $\mathcal{P}_n(E)$ given by

$$p(\partial_1, \ldots, \partial_N) = \sum_{\nu_1+\ldots+\nu_N \le M} p_{\nu_1,\ldots,\nu_N} \partial_1^{\nu_1} \ldots \partial_N^{\nu_N}$$

may be written in the form

$$p(\partial_1, \ldots, \partial_N) = \sum_{\nu_1+\ldots+\nu_N \le M} p_{\nu_1,\ldots,\nu_N} D^{\nu_1} \otimes \ldots \otimes D^{\nu_N} \,|\, \mathcal{P}_n(E),$$

where $|$ denotes restriction to a subspace. In dependence of the choice of the norm $\|\cdot\|$ from (6), (7), (8), we let

$$\eta(p(\partial_1, \ldots, \partial_N) \,|\, \mathcal{P}_n(E)),\ \lambda(p(\partial_1, \ldots, \partial_N) \,|\, \mathcal{P}_n(E)),\ \gamma(p(\partial_1, \ldots, \partial_N) \,|\, \mathcal{P}_n(E))$$

denote the best constant C for which

$$\|p(\partial_1, \ldots, \partial_N) f\| \le C \|f\| \quad \text{for all} \quad f \in \mathcal{P}_n(E).$$

Of course, this constant is just the norm of $p(\partial_1, \ldots, \partial_N)$ as an operator on $\mathcal{P}_n(E)$. Our standing assumption that E contains the cube $[0, x_1] \times \ldots \times [0, x_N]$ together with each of its points $(x_1, \ldots, x_N)$ guarantees that $p(\partial_1, \ldots, \partial_N)$ maps $\mathcal{P}_n(E)$ into itself.

If $A_1, \ldots, A_k$ are operators on $L^2(0,1)$, their tensor product $A := A_1 \otimes \ldots \otimes A_k$ on $L^2((0,1)^k)$ is defined in the usual way. If $L^2(E)$ is an invariant subspace for A, we denote by $A \,|\, L^2(E)$ the restriction of A to $L^2(E)$. The notation $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$.

We need one more definition. Let $\nu := (\nu_1, \ldots, \nu_N)$ be an N-tuple of nonnegative integers. We put $|\nu| = \nu_1 + \ldots + \nu_N$ and always suppose that $|\nu| \geq 1$. Let $\nu_{j_1}, \ldots, \nu_{j_k}$ $(j_1 < \ldots < j_k)$ denote the nonzero integers among $\nu_1, \ldots, \nu_N$. Given a point $(x_1, \ldots, x_k) \in [0,1]^k$, we define $(x_1, \ldots, x_k)_\nu$ as the point in $[0,1]^N$ whose ℓth coordinate is 0 if $\nu_\ell = 0$ and is x_m if $\ell = j_m$. For example, if $\nu = (0, \nu_2, 0, 0, \nu_5)$ with nonzero ν_2 and ν_5, then $k = 2, j_1 = 2, j_2 = 5$ and $(x_1, x_2)_\nu = (0, x_1, 0, 0, x_2)$. Finally, given $E \subset [0,1]^N$, we put

$$E_\nu = \{(x_1, \ldots, x_k) \in [0,1]^k : (x_1, \ldots, x_k)_\nu \in E\}.$$

Note that if E is $[0,1]^N$, Ω_N, Ω_N^δ, then E_ν is simply $[0,1]^k$, Ω_k, Ω_k^δ, respectively.

Theorem 1.1 *The best constants for $\partial_1^{\nu_1} \ldots \partial_N^{\nu_N}$ have the asymptotic behavior*

$$\eta(\partial_1^{\nu_1} \ldots \partial_N^{\nu_N} \,|\, \mathcal{P}_n(E)) \sim n^{|\nu|/2} \max_{(x_1, \ldots, x_k) \in E_\nu} (2x_1)^{\nu_{j_1}/2} \ldots (2x_k)^{\nu_{j_k}/2}, \tag{10}$$

$$\lambda(\partial_1^{\nu_1} \ldots \partial_N^{\nu_N} \,|\, \mathcal{P}_n(E)) \sim n^{|\nu|} \, \|L_{\nu_{j_1}, \alpha_{j_1}} \otimes \ldots \otimes L_{\nu_{j_k}, \alpha_{j_k}} \,|\, L^2(E_\nu)\|_\infty, \tag{11}$$

$$\gamma(\partial_1^{\nu_1} \ldots \partial_N^{\nu_N} \,|\, \mathcal{P}_n(E)) \sim n^{2|\nu|} \, \|G_{\nu_{j_1}, \alpha_{j_1}} \otimes \ldots \otimes G_{\nu_{j_k}, \alpha_{j_k}} \,|\, L^2(E_\nu)\|_\infty. \tag{12}$$

If E is $[0,1]^N$, then the maximum in (10) is $2^{|\nu|/2}$, and if $E = \Omega_N^\delta$, then this maximum equals

$$\left(\frac{2^{1/\delta} \nu_{j_1}}{|\nu|}\right)^{\delta \nu_{j_1}/2} \ldots \left(\frac{2^{1/\delta} \nu_{j_k}}{|\nu|}\right)^{\delta \nu_{j_k}/2}.$$

Thus, in the Hermite case the coefficient in the asymptotic formula is explicitly available. Theorem 1.2 will show that the Gegenbauer case can be reduced to the Laguerre case.

When dealing with the coefficients in the asymptotics, we have k dimensions instead of N. To avoid double subscripts, we assume in this context that we are given a k-tuple $\nu = (\nu_1, \ldots, \nu_k)$ of positive integers and a k-tuple $\alpha = (\alpha_1, \ldots, \alpha_k)$ of real numbers such that $\alpha_j > -1$ for all j. The set E is now a closed subset of $[0,1]^k$ which contains a point of $(0,1)^k$ and which contains $[0, x_1] \times \ldots \times [0, x_k]$ together with each of its points $(x_1, \ldots, x_k)$. Recall that $E^2 = \{(x_1^2, \ldots, x_k^2) : (x_1, \ldots, x_k) \in E\}$. Henceforth we also make use of the abbreviations

$$L_{\nu,\alpha} = L_{\nu_1,\alpha_1} \otimes \ldots \otimes L_{\nu_k,\alpha_k}, \quad G_{\nu,\alpha} = G_{\nu_1,\alpha_1} \otimes \ldots \otimes G_{\nu_k,\alpha_k}.$$

Theorem 1.2 *The Gegenbauer and Laguerre cases are related by the equality*

$$\|G_{\nu,\alpha} \,|\, L^2(E)\|_\infty = 2^{-|\nu|} \, \|L_{\nu,\alpha} \,|\, L^2(E^2)\|_\infty.$$

Bounds for $\|L_{\nu,\alpha} \,|\, L^2(E)\|_\infty$ are delivered by the trivial inequalities

$$\frac{\|L_{\nu,\alpha}^* L_{\nu,\alpha} \,|\, L^2(E)\|_2}{\|L_{\nu,\alpha} \,|\, L^2(E)\|_2} \leq \|L_{\nu,\alpha} \,|\, L^2(E)\|_\infty \leq \|L_{\nu,\alpha} \,|\, L^2(E)\|_2, \tag{13}$$

where $\|\cdot\|_2$ stands for the Hilbert-Schmidt norm. Here $L_{\nu,\alpha}^*$ is the adjoint of $L_{\nu,\alpha} \,|\, L^2(E)$, that is, by $L_{\nu,\alpha}^*$ we actually mean $(L_{\nu,\alpha} \,|\, L^2(E))^*$.

Theorem 1.3 *We have*

$$\|L_{\nu,\alpha} \,|\, L^2(\Omega_k^\delta)\|_2^2 = \frac{\delta^k}{\Gamma(2\delta|\nu|+1)} \prod_{j=1}^{k} \frac{\Gamma(\alpha_j+1)\Gamma(2\nu_j-1)\Gamma(2\delta\nu_j)}{\Gamma(\alpha_j+2\nu_j)\Gamma(\nu_j)^2}$$

and

$$\|L_{\nu,\alpha}^* L_{\nu,\alpha} \,|\, L^2(\Omega_k^\delta)\|_2^2 \ge \frac{(2\delta)^k}{\Gamma(4\delta|\nu|+1)} \prod_{j=1}^{k} \frac{\Gamma(\alpha_j+1)^2\Gamma(2\nu_j-1)^2\Gamma(4\delta\nu_j)}{\Gamma(\alpha_j+2\nu_j)^2(\alpha_j+2\nu_j+1)\Gamma(\nu_j)^4}.$$

Inserting the expressions from Theorem 1.3 in (13) we get bounds

$$b_1(\nu,\alpha,\Omega_k^\delta) \le \|L_{\nu,\alpha} \,|\, L^2(\Omega_k^\delta)\|_\infty \le b_2(\nu,\alpha,\Omega_k^\delta).$$

If $k, \delta, \alpha_1, \ldots, \alpha_k$ remain fixed and $\nu_j \to \infty$ for all j, then

$$\frac{b_2(\nu,\alpha,\Omega_k^\delta)}{b_1(\nu,\alpha,\Omega_k^\delta)} \sim (2\pi\delta)^{(k-1)/4} \frac{\nu_1^{1/4} \ldots \nu_k^{1/4}}{(\nu_1+\ldots+\nu_k)^{1/4}}. \tag{14}$$

For $k=1$, the right-hand side of (14) is 1, but for $k \ge 2$ it increases (though moderately). The following result reveals that the upper bound $b_2(\nu,\alpha,\Omega_k^\delta)$ is asymptotically sharp as ν goes to infinity along a straight line.

Theorem 1.4 *Let $\nu = (\varrho_1\tau, \ldots, \varrho_k\tau)$ where $\varrho_1, \ldots, \varrho_k$ are positive real numbers. Then as $\tau \to \infty$,*

$$\|L_{\nu,\alpha} \,|\, L^2(\Omega_k^\delta)\|_\infty \sim \|L_{\nu,\alpha} \,|\, L^2(\Omega_k^\delta)\|_2.$$

If, for example, $k=2$, $\delta=1$, $\alpha_1=\alpha_2=0$, $\nu_1=\nu_2=:\nu$, then Theorems 1.1, 1.3, 1.4 yield

$$\lambda(\partial_1^\nu\partial_2^\nu \,|\, \mathcal{P}_n(\Omega_2)) \sim \frac{(2\pi\nu)^{1/4}}{4} \frac{1}{2^{2\nu}\nu^2} \frac{1}{\Gamma(\nu)^2} n^{2\nu}. \tag{15}$$

In the case $k=1$ we infer from the same theorems that

$$\lambda(\partial_1^\nu \,|\, P_n(\Omega_2)) = \lambda(\partial_2^\nu \,|\, P_n(\Omega_2)) \sim \frac{1}{2\nu\Gamma(\nu)} n^\nu. \tag{16}$$

It is easily seen that always

$$\lambda(\partial_1^\nu\partial_2^\nu \,|\, \mathcal{P}_n(\Omega_2)) \le \lambda(\partial_1^\nu \,|\, P_n(\Omega_2))\lambda(\partial_2^\nu \,|\, P_n(\Omega_2)). \tag{17}$$

However, from (15) and (16) we obtain that

$$\frac{\lambda(\partial_1^\nu\partial_2^\nu \,|\, \mathcal{P}_n(\Omega_2))}{\lambda(\partial_1^\nu \,|\, P_n(\Omega_2))\lambda(\partial_2^\nu \,|\, P_n(\Omega_2))} \sim \frac{(2\pi\nu)^{1/4}}{2^{2\nu}},$$

which is much smaller than 1 if ν is large and hence much sharper than (17).

We finally turn to linear combinations of partial derivatives. Let

$$p_0(\xi_1, \ldots, \xi_N) = \sum_{\nu_1+\ldots+\nu_N=M} p_{\nu_1,\ldots,\nu_N} \xi_1^{\nu_1} \ldots \xi_N^{\nu_N}$$

be the principal part of polynomial (9). We write $a_n \simeq b_n$ if there exist constants $0 < c_1 < c_2 < \infty$ such that $c_1 b_n \le a_n \le c_2 b_n$ for all sufficiently large n. The following theorem reveals that the asymptotic behavior of the best constants for linear partial differential operators with constant coefficients is completely determined by their principal parts.

Theorem 1.5 *Let C stand for η, λ, or γ and put $\sigma = 1/2$ in the Hermite case, $\sigma = 1$ in the Laguerre case, and $\sigma = 2$ in the Gegenbauer case. Then*

$$C(p_0(\partial_1, \ldots, \partial_N) \,|\, \mathcal{P}_n(E)) \simeq n^{\sigma M} \tag{18}$$

and

$$C(p(\partial_1, \ldots, \partial_N) \,|\, \mathcal{P}_n(E)) \sim C(p_0(\partial_1, \ldots, \partial_N) \,|\, \mathcal{P}_n(E)). \tag{19}$$

The paper is organized as follows. Section 2 is devoted to comments on previous work and the results and methods of this paper. Sections 3 to 7 contain the proofs of Theorems 1.1 to 1.5 and in Section 8 we list some problems we have to leave open.

2 Remarks on the history

The problem of finding upper bounds for the derivatives of functions in terms of the functions themselves has a long and rich history. Nowadays one speaks of Bernstein-type inequalities if the functions are trigonometric polynomials and of Markov-type inequalities in the case of algebraic polynomials. The Markov brothers [24], [25] found the best constant C in the inequality $\|D^m f\| \le C\|f\|$ when $\|\cdot\|$ is the L^∞ norm on some finite interval. We refer to [2], [15], [26], [27], [29], [30] for more on the subject and here confine ourselves to the asymptotics of the best constants in Markov-type inequalities with L^2 norms.

An L^2 version of a Markov-type inequality was first established by Erhard Schmidt [31] and subsequently for L^p norms by Hille, Szegö, and Tamarkin [17]. In 1944, Schmidt [32] proved that

$$\eta_n(D) = \sqrt{2n}, \quad \lambda_n(D) \sim \frac{2}{\pi}\, n, \quad \gamma_n(D) \sim \frac{1}{\pi}\, n^2,$$

assuming $\alpha = 0$, and even provided the next two terms in the asymptotics of $\lambda_n(D)$ and $\gamma_n(D)$. Schmidt also observed that for the Hermite weight the problem is more or less trivial. Shampine [33] then showed that, again for $\alpha = 0$,

$\lambda_n(D^2) \sim n^2/\mu_0^2$ and $\gamma_n(D^2) \sim n^4/(4\mu_0^2)$ where μ_0 is the smallest positive root of the equation $1+\cos\mu\,\cosh\mu = 0$. For the exact values of $\lambda_n(D)$ and $\gamma_n(D)$ in the case $\alpha = 0$ see [17], [21], [36]. The idea that the best constants in question are the largest singular value (= operator norm) of certain matrices was developed in [7], [8], [9] and used to derive bounds for

$$\liminf_{n\to\infty} \frac{\lambda_n(D^m)}{n^m}, \quad \limsup_{n\to\infty} \frac{\lambda_n(D^m)}{n^m}$$

for general m and α. In [3], [4] we proved that $\lambda_n(D^m)/n^m$ and $\gamma_n(D^m)/n^{2m}$ converge to a limit as $n \to \infty$ and identified these limits as the operator norms of certain Volterra integral operators.

The appearance of Volterra operators in this context connects us with another field of research. Paul Halmos [16] was probably the first to state explicitly that the operator norm of the operator

$$(L_{1,0}f)(x) = \int_x^1 f(y)\,dy$$

on $L^2(0,1)$ is $2/\pi$. Combining this result with our formula $\lambda_n(D) \sim \|L_{1,0}\|_\infty\, n$ reproduces Schmidt's formula $\lambda_n(D) \sim (2/\pi)\,n$. Halmos also raised the question of determining the operator norms of the iterates $L_{1,0}^m = L_{m,0}$. This problem was subsequently studied by many authors, including [1], [12], [18], [23], [22], [35]. The much earlier paper [14] was detected by Thorpe [35] to be also of relevancy in connection with the matter. The reader may consult [3] and [4] for details. In the course of these investigations sharp bounds for $\|L_{m,0}\|_\infty$ and the asymptotic formula $\|L_{m,0}\|_\infty \sim 1/(2\,m!)$ were established. In [4] we solved the corresponding problems for the norms $\|L_{m,\alpha}\|_\infty$ and $\|G_{m,\alpha}\|_\infty$. In particular, the $N = 1$ versions of Theorems 1.2, 1.3, 1.4 are already in [4]. We also want to note that Halmos' $\|L_{1,0}\|_\infty = 2/\pi$ was in [10] and [4] extended to the equality $\|L_{1,\alpha}\|_\infty = 1/j_0(\alpha)$ where $j_0(\alpha)$ is the smallest positive zero of the Bessel function $J_{(\alpha-1)/2}$.

The literature on multivariate Markov-type inequalities is immense, a main topic of research being inequalities for the L^∞ norm on multidimensional regions and manifolds. See, for example, [19], [20], [28]. However, we are not aware of publications dealing with best constants in multivariate Markov-type inequalities with the L^2 norm and for partial derivatives of arbitrary order. Note that even the one-dimensional versions of the results of this paper were established only in [3], [4]. Clearly, for $E = [0,1]^N$ our Theorems 1.1 to 1.4 simply follow from their one-dimensional counterparts by taking tensor products. However, passage to the simplex $E = \Omega_N$ makes things nontrivial. Moreover, Theorem 1.2 even motivates consideration of the entire scale $E = \Omega_N^\delta$.

The approach employed in [3], [4] and also in Section 4 of this paper is based on an idea by Harold Widom [37], [38], [39], which was independently also introduced by Lawrence Shampine [33], [34]. In order to find the asymptotic behavior of spectral quantities of a sequence of $n \times n$ matrices A_n, they associated an integral operator W_{A_n} on $L^2(0,1)$ with each matrix A_n and then studied whether, after appropriate scaling, the operators W_{A_n} converge uniformly to some limiting operator. In this way Widom and Shampine were able to express asymptotic properties of A_n in terms of the limiting operator. In particular, Shampine [33] considered W_{A_n} where A_n is the matrix representation of the operator $(D^2)^* D^2$. For $(D^m)^* D^m$ with $m \geq 2$, the matrices become more complicated and hence we have the limitation to $m = 2$ in [33]. What is new in our approach is that we exploit the fact that the replacement $A_n \mapsto W_{A_n}$ is an algebraic homomorphism that preserves also tensor products. Thus, we simply consider the operator W_{A_n} for A_n being the matrix representation of D and show that, after scaling, W_{A_n} converges to some limiting operator. Once this has been done, we can easily pass to adjoints, sums, products, and tensor products.

The reasoning in Section 5 is similar to the one of [3], [4]. The argument used in Section 6 is different from [3], [4] and based on a strategy that was in another context pursued in [6].

3 Hermite weights

In this section we prove the assertion of Theorem 1.1 that concerns the Hermite case. This case is particularly simple.

An orthonormal basis in $\mathcal{P}_n$ with the norm (1) is $\{h_0, h_1, \ldots, h_n\}$ where h_k is the kth normalized Hermite polynomial. We have

$$D^\nu h_i = 2^{\nu/2} \sqrt{\frac{\Gamma(i+1)}{\Gamma(i-\nu+1)}}\, h_{i-\nu} \tag{20}$$

for $\nu \leq i$. As usual, $h_{i_1} \otimes \ldots \otimes h_{i_N}$ is defined by

$$(h_{i_1} \otimes \ldots \otimes h_{i_N})(t_1, \ldots, t_N) = h_{i_1}(t_1) \ldots h_{i_N}(t_N).$$

Then $\{h_{i_1} \otimes \ldots \otimes h_{i_N} : (i_1/n, \ldots, i_N/n) \in E\}$ is an orthonormal basis in $\mathcal{P}_n(E)$. For

$$f = \sum f_{i_1,\ldots,i_N} h_{i_1} \otimes \ldots \otimes h_{i_N} \in \mathcal{P}_n(E)$$

we then get

$$\begin{aligned}
&\|(D^{\nu_1} \otimes \ldots \otimes D^{\nu_N}) f\|^2 = ((D^{\nu_1} \otimes \ldots \otimes D^{\nu_N}) f, (D^{\nu_1} \otimes \ldots \otimes D^{\nu_N}) f) \\
&= \Big(\sum f_{i_1,\ldots,i_N}\, D^{\nu_1} h_{i_1} \otimes \ldots \otimes D^{\nu_N} h_{i_N}, \sum f_{k_1,\ldots,k_N}\, D^{\nu_1} h_{k_1} \otimes \ldots \otimes D^{\nu_N} h_{k_N} \Big) \\
&= \sum \sum f_{i_1,\ldots,i_N} \overline{f_{k_1,\ldots,k_N}} (D^{\nu_1} h_{i_1}, D^{\nu_1} h_{k_1}) \ldots (D^{\nu_N} h_{i_N}, D^{\nu_N} h_{k_N}).
\end{aligned} \tag{21}$$

Using (20) and the orthonormality of $h_0, h_1, \ldots, h_n$ we see that (21) equals

$$\sum |f_{i_1,\ldots,i_N}|^2\, 2^{\nu_1} \frac{\Gamma(i_1+1)}{\Gamma(i_1-\nu_1+1)} \ldots 2^{\nu_N} \frac{\Gamma(i_N+1)}{\Gamma(i_N-\nu_N+1)},$$

the sum over $i_1 \geq \nu_1, \ldots, i_N \geq \nu_N, (i_1/n, \ldots, i_N/n) \in E$. It follows that the operator norm of $D^{\nu_1} \otimes \ldots \otimes D^{\nu_N}$ on $\mathcal{P}_n(E)$ is given by

$$\|D^{\nu_1} \otimes \ldots \otimes D^{\nu_N} \,|\, \mathcal{P}_n(E)\|_\infty^2 = \max 2^{|\nu|} \frac{\Gamma(i_1+1)}{\Gamma(i_1-\nu_1+1)} \cdots \frac{\Gamma(i_N+1)}{\Gamma(i_N-\nu_N+1)}$$
$$= 2^{|\nu|} \max i_1(i_1-1)\ldots(i_1-\nu_1+1)\ldots i_N(i_N-1)\ldots(i_N-\nu_N+1),$$

the maximum over $i_1 \geq \nu_1, \ldots, i_N \geq \nu_N, (i_1/n, \ldots, i_N/n) \in E$. Consequently,

$$\eta^2(\partial_1^{\nu_1} \ldots \partial_N^{\nu_N} \,|\, \mathcal{P}_n(E))/(2n)^{|\nu|}$$
$$= \max \frac{i_1}{n}\left(\frac{i_1}{n} - \frac{1}{n}\right) \cdots \left(\frac{i_1}{n} - \frac{\nu_1 - 1}{n}\right) \cdots \frac{i_N}{n}\left(\frac{i_N}{n} - \frac{1}{n}\right) \cdots \left(\frac{i_N}{n} - \frac{\nu_N - 1}{n}\right)$$
$$= \max \left(\frac{i_1}{n}\right)^{\nu_1} \cdots \left(\frac{i_N}{n}\right)^{\nu_N} \left(1 + O\left(\frac{1}{n}\right)\right). \tag{22}$$

The limit of (22) is

$$\max_{(x_1,\ldots,x_N)\in E} x_1^{\nu_1} \ldots x_N^{\nu_N} = \max_{(x_1,\ldots,x_k)\in E_\nu} x_1^{\nu_{j_1}} \ldots x_k^{\nu_{j_k}},$$

which proves (10).

4 Laguerre and Gegenbauer weights

This section is devoted to the proof of Theorem 1.1 in the Laguerre and Gegenbauer cases and to the proof of Theorem 1.2.

Suppose first that $\nu_j \geq 1$ for all j. In that case $E_\nu = E$. We denote by $p_{\alpha,i}$ the ith normalized Laguerre or Gegenbauer polynomial in the norm (2) or (3), respectively. Then $\{p_{\alpha,0}, p_{\alpha,1}, \ldots, p_{\alpha,n}\}$ is an orthonormal basis in $\mathcal{P}_n$. Let $D_{\alpha,n}$ be the matrix representation of the operator $D : \mathcal{P}_n \to \mathcal{P}_n$ in this basis. Given a matrix $A_n = (a_{ik})_{i,k=0}^n$, we denote by W_{A_n} the integral operator on $L^2(0,1)$ with the piecewise constant kernel $(n+1)a_{[(n+1)x],[(n+1)y]}$, where $[\xi]$ is the integral part of ξ. Recall that $L_{m,\alpha}$ and $G_{m,\alpha}$ are given by (4) and (5). Put $\sigma = 1$ and $T_{m,\alpha} = L_{m,\alpha}$ in the Laguerre case and $\sigma = 2$ and $T_{m,\alpha} = G_{m,\alpha}$ in the Gegenbauer case. In [4] we showed that

$$\|(n+1)^{-\sigma m} W_{D_{\alpha,n}^m} - T_{m,\alpha}\|_\infty \to 0 \quad \text{as} \quad n \to \infty. \tag{23}$$

An orthonormal basis in $\mathcal{P}_n([0,1]^N)$ is

$$\mathcal{F} := \{p_{\alpha_1,i_1} \otimes \ldots \otimes p_{\alpha_N,i_N} : (i_1, \ldots, i_N) \in S_n^N\}$$

where $S_n := \{0, 1, \ldots, n\}$. If A is a linear operator on $\mathcal{P}_n([0,1]^N)$, we denote by $A_n = (a_{i,k})_{i,k\in S_n^N}$ its matrix representation in this basis. Thus, if $i = (i_1, \ldots, i_N)$ and $k = (k_1, \ldots, k_N)$, then

$$a_{i,k} = (A(p_{\alpha_1,k_1} \otimes \ldots \otimes p_{\alpha_N,k_N}), p_{\alpha_1,i_1} \otimes \ldots \otimes p_{\alpha_N,i_N}).$$

In the case where $A = \partial_1^{\nu_1} \dots \partial_N^{\nu_N} = D^{\nu_1} \otimes \dots \otimes D^{\nu_N}$, the matrix representation A_n is just the Kronecker product $D^{\nu_1}_{\alpha_1,n} \otimes \dots \otimes D^{\nu_N}_{\alpha_N,n}$. We associate with A_n the integral operator W_{A_n} on $L^2((0,1)^N)$ given by

$$\begin{aligned}&(W_{A_n} f)(x_1, \dots, x_N)\\ &= (n+1)^N \int_{(0,1)^N} a_{[(n+1)x_1],\dots,[(n+1)x_N],[(n+1)y_1],\dots,[(n+1)y_N]} f(y_1, \dots, y_N)\, dy.\end{aligned}$$

Throughout what follows $\|A\|_\infty$ denotes the operator norm if A is an operator and the spectral norm in case A is a matrix.

Lemma 4.1 (Widom and Shampine) *If $C_1, \dots, C_N$ are linear operators on $\mathcal{P}_n$, A, B are linear operators on $\mathcal{P}_n([0,1]^N)$, and $\alpha, \beta \in \mathbf{C}$, then*

$$\begin{aligned}&W_{\alpha A_n + \beta B_n} = \alpha W_{A_n} + \beta W_{B_n}, \quad W_{A_n B_n} = W_{A_n} W_{B_n},\\ &W_{(C_1)_n \otimes \dots \otimes (C_N)_n} = W_{(C_1)_n} \otimes \dots \otimes W_{(C_N)_n}, \quad \|W_{A_n}\|_\infty = \|A_n\|_\infty.\end{aligned}$$

Proof. Let I_k be the interval $(k/(n+1), (k+1)/(n+1))$, denote by χ_k the characteristic function of I_k, and consider the operators

$$\begin{aligned}&R : \mathbf{C}^{n+1} \to L^2(0,1), \quad \{x_k\}_{k=0}^n \mapsto \sqrt{n+1} \sum_{k=0}^n x_k \chi_k,\\ &S : L^2(0,1) \to \mathbf{C}^{n+1}, \quad f \mapsto \left\{ \sqrt{n+1} \int_{I_k} f(x)dx \right\}_{k=0}^n.\end{aligned}$$

It can be verified straightforwardly that $\|R\|_\infty = \|S\|_\infty = 1$, that $SR = I$, and that $R(C_i)_n S = W_{(C_i)_n}$ and $(R \otimes \dots \otimes R) X_n (S \otimes \dots \otimes S) = W_{X_n}$ for every linear operator X on $\mathcal{P}_n([0,1]^N)$. It follows that $W_{\alpha A_n + \beta B_n} = \alpha W_{A_n} + \beta W_{B_n}$, $W_{A_n B_n} = W_{A_n} W_{B_n}$, and $W_{(C_1)_n \otimes \dots \otimes (C_N)_n} = W_{(C_1)_n} \otimes \dots \otimes W_{(C_N)_n}$. Since

$$\begin{aligned}\|W_{A_n}\|_\infty &= \|(R \otimes \dots \otimes R) A_n (S \otimes \dots S)\|_\infty \le \|A_n\|_\infty\\ &= \|(S \otimes \dots \otimes S) W_{A_n} (R \otimes \dots \otimes R)\|_\infty \le \|W_{A_n}\|_\infty,\end{aligned}$$

we arrive at the equality $\|W_{A_n}\|_\infty = \|A_n\|_\infty$. $\square$

From (23) we infer that

$$\|(n+1)^{-\sigma|\nu|}\, W_{D^{\nu_1}_{\alpha_1,n}} \otimes \dots \otimes W_{D^{\nu_N}_{\alpha_N,n}} - T_{\nu_1,\alpha_1} \otimes \dots \otimes T_{\nu_N,\alpha_N}\|_\infty \to 0,$$

and from Lemma 4.1 we therefore deduce that

$$\|(n+1)^{-\sigma|\nu|}\, W_{D^{\nu_1}_{\alpha_1,n} \otimes \dots \otimes D^{\nu_N}_{\alpha_N,n}} - T_{\nu_1,\alpha_1} \otimes \dots \otimes T_{\nu_N,\alpha_N}\|_\infty \to 0. \tag{24}$$

Lemma 4.2 *Let X be a Banach space and suppose X is the direct sum of two closed subspaces U and V, $X = U \oplus V$. Let K be a bounded linear operator on X which has U as an invariant subspace. Then, with P_U denoting the projection of X onto U parallel to V,*

$$\|K \,|\, U\|_\infty = \|K P_U\|_\infty.$$

Proof. The decomposition $X = U \oplus V$ allows us to represent K by a 2×2 operator matrix. Since U is an invariant subspace, the $2,1$ entry of this matrix is zero. Thus,

$$K = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

Clearly, $K\,|\,U = A$ and

$$KP_U = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

This shows that $\|K\,|\,U\|_\infty = \|KP_U\|_\infty$. $\square$

We may think of $\mathcal{P}_n(E)$ as a subspace of $\mathcal{P}_n([0,1]^N)$. Moreover, in the orthonormal basis $\mathcal{F}$ we may identify $\mathcal{P}_n([0,1]^N)$ with $\ell^2(S_n^N)$ and $\mathcal{P}_n(E)$ with $\ell^2(\Pi_n)$ where $S_n = \{0, 1, \ldots, n\}$ and

$$\Pi_n = \{(i_1, \ldots, i_N) \in S_n^N : (i_1/n, \ldots, i_N/n) \in E\}.$$

We are interested in the norm $\|D^{\nu_1}_{\alpha_1,n} \otimes \ldots \otimes D^{\nu_N}_{\alpha_N,n}\,|\,\ell^2(\Pi_n)\|_\infty$. Let P_{Π_n} be the orthogonal projection of $\ell^2(S_n^N)$ onto $\ell^2(\Pi_n)$. By Lemma 4.2,

$$\|D^{\nu_1}_{\alpha_1,n} \otimes \ldots \otimes D^{\nu_N}_{\alpha_N,n}\,|\,\ell^2(\Pi_n)\|_\infty = \|(D^{\nu_1}_{\alpha_1,n} \otimes \ldots \otimes D^{\nu_N}_{\alpha_N,n})P_{\Pi_n}\|_\infty$$

where the norm on the right is taken over $\ell^2(S_n^N)$. We denote the matrix representation of P_{Π_n} in the orthonormal basis $\mathcal{F}$ also by P_{Π_n}, although strict use of notation would require to denote it by $(P_{\Pi_n})_n$. A little thought reveals that

$$(P_{\Pi_n})_{i,k} = \begin{cases} 1 & \text{if } i = k \in \Pi_n, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

Lemma 4.3 *The operators $W_{P_{\Pi_n}}$ converge strongly (= pointwise) on $L^2((0,1)^N)$ to the operator*

$$P_E : L^2((0,1)^N) \to L^2((0,1)^N), \quad (P_E f)(x) = \begin{cases} f(x) & \textit{if } x \in E, \\ 0 & \textit{if } x \notin E. \end{cases}$$

Proof. By Lemma 4.1, $\|W_{P_{\Pi_n}}\|_\infty = \|P_{\Pi_n}\|_\infty = 1$. Hence it suffices to prove that $W_{P_{\Pi_n}} f \to P_E f$ for all f in some dense subset of $L^2((0,1)^N)$, say for $f \in C([0,1]^N)$. Thus, fix $f \in C([0,1]^N)$ and $\varepsilon > 0$. For $i = (i_1, \ldots, i_N) \in \{0, 1, \ldots, n\}^N$, put

$$Q_i = \left[\frac{i_1}{n+1}, \frac{i_1+1}{n+1}\right) \times \ldots \times \left[\frac{i_N}{n+1}, \frac{i_N+1}{n+1}\right).$$

We have

$$\|W_{P_{\Pi_n}} f - P_E f\|^2 = \sum_{i_1,\ldots,i_N=0}^{n} \int_{Q_i} |W_{P_{\Pi_n}} f(x) - P_E f(x)|^2 dx. \tag{26}$$

If $x = (x_1, \ldots, x_N) \in Q_i$, then $[(n+1)x_1] = i_1, \ldots, [(n+1)x_N] = i_N$ and hence

$$\begin{aligned}
&(W_{P_{\Pi_n}} f)(x) \\
&\quad = (n+1)^N \int_{(0,1)^N} (P_{\Pi_n})_{[(n+1)x_1],\ldots,[(n+1)x_N],[(n+1)y_1],\ldots,[(n+1)y_N]} f(y)\, dy \\
&\quad = (n+1)^N \int_{(0,1)^N} (P_{\Pi_n})_{i_1,\ldots,i_N,[(n+1)y_1],\ldots,[(n+1)y_N]} f(y)\, dy.
\end{aligned}$$

By virtue of (25),

$$(W_{P_{\Pi_n}} f)(x) = (n+1)^N \int_{Q_i} (P_{\Pi_n})_{i_1,\ldots,i_N,i_1,\ldots,i_N} f(y)\, dy.$$

If $(i_1/(n+1), \ldots, i_N/(n+1)) \notin E$ then $(i_1/n, \ldots, i_N/n) \notin E$. Thus, in this case we have $i = (i_1, \ldots, i_N) \notin \Pi_n$ and $Q_i \subset [0,1]^N \setminus E$. Consequently, $(P_{\Pi_n})_{i,i} = 0$ and $(P_E f)(x) = 0$ for $x \in Q_i$, which implies that

$$\int_{Q_i} |W_{P_{\Pi_n}} f(x) - P_E f(x)|^2 dx = 0.$$

Suppose $((i_1+1)/n, \ldots, (i_N+1)/n) \in E$. Then $i = (i_1, \ldots, i_N) \in \Pi_n$ and since $((i_1+1)/(n+1), \ldots, (i_N+1)/(n+1)) \in E$, it follows that $Q_i \subset E$. Thus, $(P_{\Pi_n})_{i,i} = 1$ and $(P_E f)(x) = f(x)$ for $x \in Q_i$. This gives

$$\begin{aligned}
&\int_{Q_i} |(W_{P_{\Pi_n}} f)(x) - (P_E f)(x)|^2 dx \\
&\quad \le \frac{1}{(n+1)^N} \sup_{x \in Q_i} |(W_{P_{\Pi_n}} f)(x) - (P_E f)(x)|^2 \\
&\quad = \frac{1}{(n+1)^N} \sup_{x \in Q_i} \left| (n+1)^N \int_{Q_i} (f(y) - f(x))\, dy \right|^2 \\
&\quad \le \frac{1}{(n+1)^N} \sup_{x,y \in Q_i} |f(y) - f(x)|^2 \le \frac{\varepsilon}{2\, n^N}
\end{aligned}$$

if only $n \ge n_1 = n_1(\varepsilon)$. It follows that the sum of the terms

$$\int_{Q_i} |W_{P_{\Pi_n}} f(x) - P_E f(x)|^2 dx$$

over $((i_1+1)/n, \ldots, (i_N+1)/n) \in E$ is at most $n^N(\varepsilon/(2\, n^N)) = \varepsilon/2$ for all $n \ge n_1$. We are left with the case where

$$\left(\frac{i_1}{n+1}, \ldots, \frac{i_N}{n+1} \right) \in E, \quad \left(\frac{i_1+1}{n}, \ldots, \frac{i_N+1}{n} \right) \notin E.$$

These points $(i_1,\ldots,i_N)$ are all in a small shell around the boundary of nE and hence their number is $O(n^{N-1})$. Summing up over these points we get

$$\begin{aligned}
&\sum_i \int_{Q_i} |(W_{P_{\pi_n}} f)(x) - (P_E f)(x)|^2 dx \\
&\le \sum_i \left(2 \max_{x\in[0,1]^N} |f(x)|\right)^2 \frac{1}{(n+1)^N} = O(n^{N-1}) \left(2 \max_{x\in[0,1]^N} |f(x)|\right)^2 \frac{1}{(n+1)^N},
\end{aligned}$$

which is smaller than $\varepsilon/2$ if $n \ge n_2 = n_2(\varepsilon)$. In summary, (26) is smaller that ε if $n \ge \max(n_1, n_2)$. $\square$

We are now in a position to prove (11) and (12) of Theorem 1.1. It is well known that if K is a compact operator, $\|K_n - K\|_\infty \to 0$, and $C_n^* \to C^*$ strongly (the asterisk denoting the adjoint), then $\|K_n C_n - KC\|_\infty \to 0$ (see, e.g., [5, Lemma 2.8]). Put

$$\begin{aligned}
&K_n = (n+1)^{-\sigma|\nu|} W_{D^{\nu_1}_{\alpha_1,n}\otimes\ldots\otimes D^{\nu_N}_{\alpha_N,n}}, \quad K = T_{\nu_1,\alpha_1} \otimes \ldots \otimes T_{\nu_N,\alpha_N} \\
&C_n = C_n^* = W_{P_{\Pi_n}}, \quad C = C^* = P_E.
\end{aligned}$$

It is easily seen that all T_{ν_j,α_j} are Hilbert-Schmidt operators. (Here we are using our assumption that $\nu_j \ge 1$ for all j.) This implies that $T_{\nu_1,\alpha_1}\otimes\ldots\otimes T_{\nu_N,\alpha_N}$ is also Hilbert-Schmidt and thus compact. From (24) we know that $\|K_n - K\|_\infty \to 0$, and Lemma 4.3 states that $C_n^* \to C^*$ strongly. Consequently,

$$\|(n+1)^{-\sigma|\nu|} W_{D^{\nu_1}_{\alpha_1,n}\otimes\ldots\otimes D^{\nu_N}_{\alpha_N,n}} W_{P_{\Pi_n}} - (T_{\nu_1,\alpha_1} \otimes \ldots \otimes T_{\nu_N,\alpha_N}) P_E\|_\infty \to 0.$$

This yields that

$$(n+1)^{-\sigma|\nu|} \|W_{D^{\nu_1}_{\alpha_1,n}\otimes\ldots\otimes D^{\nu_N}_{\alpha_N,n}} W_{P_{\Pi_n}}\|_\infty \to \|(T_{\nu_1,\alpha_1} \otimes \ldots \otimes T_{\nu_N,\alpha_N}) P_E\|_\infty.$$

From Lemma 4.1 we infer that

$$\|W_{D^{\nu_1}_{\alpha_1,n}\otimes\ldots\otimes D^{\nu_N}_{\alpha_N,n}} W_{P_{\Pi_n}}\|_\infty = \|(D^{\nu_1}_{\alpha_1,n} \otimes \ldots \otimes D^{\nu_N}_{\alpha_N,n}) P_{\Pi_n}\|_\infty,$$

and since $n/(n+1) \to 1$, Lemma 4.2 gives the desired result.

We are left with the case where some of the numbers $\nu_1,\ldots,\nu_N$ are zero. We assume without loss of generality that $\nu_1,\ldots,\nu_k \ge 1$ and $\nu_{k+1} = \ldots = \nu_N = 0$. Note that then

$$E_\nu = \{(x_1,\ldots,x_k) \in [0,1]^k : (x_1,\ldots,x_k,0,\ldots,0) \in E\}.$$

For the sake of definiteness, we consider the Laguerre case. Let

$$C_n = \lambda(\partial_1^{\nu_1} \ldots \partial_k^{\nu_k} \,|\, \mathcal{P}_n(E_\nu))$$

be the norm of $\partial_1^{\nu_1}\dots\partial_k^{\nu_k}$ on $\mathcal{P}_n(E_\nu)$. We have already proved that

$$C_n \sim n^{|\nu|}\,\|L_{\nu_1,\alpha_1}\otimes\dots\otimes L_{\nu_k,\alpha_k}\,|\,L^2(E_\nu)\|_\infty. \tag{27}$$

There exists a polynomial $g\in\mathcal{P}_n(E_\nu)$ such that $\|\partial_1^{\nu_1}\dots\partial_k^{\nu_k}g\| = C_n\|g\|$. Define $f\in\mathcal{P}_n(E)$ by $f(t_1,\dots,t_N)=g(t_1,\dots,t_k)$. Then $\partial_1^{\nu_1}\dots\partial_N^{\nu_N}f=\partial_1^{\nu_1}\dots\partial_k^{\nu_k}g$ and hence

$$\begin{aligned}&\int_{(0,\infty)^k}|(\partial_1^{\nu_1}\dots\partial_N^{\nu_N}f)(t_1,\dots,t_N)|^2 t_1^{\alpha_1}\dots t_k^{\alpha_k}e^{-t_1}\dots e^{-t_k}dt_1\dots dt_k\\ &= C_n^2\int_{(0,\infty)^k}|f(t_1,\dots,t_N)|^2 t_1^{\alpha_1}\dots t_k^{\alpha_k}e^{-t_1}\dots e^{-t_k}dt_1\dots dt_k\end{aligned}$$

for each point $(t_{k+1},\dots,t_N)\in(0,\infty)^{N-k}$. Multiplying this equality by

$$t_{k+1}^{\alpha_{k+1}}\dots t_N^{\alpha_N}e^{-t_{k+1}}\dots e^{-t_N} \tag{28}$$

and integrating the result over $(0,\infty)^{N-k}$ we get $\|\partial_1^{\nu_1}\dots\partial_N^{\nu_N}f\|^2=C_n^2\|f\|^2$. Thus, $\lambda(\partial_1^{\nu_1}\dots\partial_N^{\nu_N}\,|\,\mathcal{P}_n(E))\ge C_n$.

On the other hand, every polynomial $f\in\mathcal{P}_n(E)$ may be written as

$$f(t_1,\dots,t_N)=\sum p_{i_1,\dots,i_k}(t_{k+1},\dots,t_N)t_1^{i_1}\dots t_k^{i_k} \tag{29}$$

where

$$p_{i_1,\dots,i_k}(t_{k+1},\dots,t_N)=\sum p_{i_1,\dots,i_k,\ell_{k+1},\dots,\ell_N}t_{k+1}^{\ell_{k+1}}\dots t_N^{\ell_N}$$

and $(i_1/n,\dots,i_k/n,\ell_{k+1}/n,\dots,\ell_N/n)\in E$. This implies that $(i_1/n,\dots,i_k/n)\in E_\nu$ and that hence the polynomial (29) belongs to $\mathcal{P}_n(E_\nu)$ for each fixed point $(t_{k+1},\dots,t_N)\in(0,\infty)^{N-k}$. We obtain that, for fixed $(t_{k+1},\dots,t_N)\in(0,\infty)^{N-k}$,

$$\begin{aligned}&\int_{(0,\infty)^k}|(\partial_1^{\nu_1}\dots\partial_N^{\nu_N}f)(t_1,\dots,t_N)|^2 t_1^{\alpha_1}\dots t_k^{\alpha_k}e^{-t_1}\dots e^{-t_k}dt_1\dots dt_k\\ &\le C_n^2\int_{(0,\infty)^k}|f(t_1,\dots,t_N)|^2 t_1^{\alpha_1}\dots t_k^{\alpha_k}e^{-t_1}\dots e^{-t_k}dt_1\dots dt_k,\end{aligned}$$

which after multiplication by (28) and integration over $(0,\infty)^{N-k}$ becomes the inequality $\|\partial_1^{\nu_1}\dots\partial_N^{\nu_N}f\|^2\le C_n^2\|f\|^2$. This proves that $\lambda(\partial_1^{\nu_1}\dots\partial_N^{\nu_N}\,|\,\mathcal{P}_n(E))\le C_n$.

In summary, we have $\lambda(\partial_1^{\nu_1}\dots\partial_N^{\nu_N}\,|\,\mathcal{P}_n(E))=C_n$, which in conjunction with (27) completes the proof of Theorem 1.1.

Here is the proof of Theorem 1.2. The operator

$$U: L^2(E^2)\to L^2(E),\quad (Uf)(t_1,\dots,t_k)=2^{k/2}t_1^{1/2}\dots t_k^{1/2}f(t_1^2,\dots,t_k^2)$$

is an isometry and the inverse operator acts by the rule

$$(U^{-1}g)(x_1,\dots,x_k)=2^{-k/2}x_1^{-1/4}\dots x_k^{-1/4}g(x_1^{1/2},\dots,x_k^{1/2}).$$

The kernel of the integral operator $G_{\nu,\alpha}\,|\,L^2(E)$ is

$$\kappa(x_1,\dots,x_k,y_1,\dots,y_k)=\prod_{j=1}^{k}\frac{x_j^{1/2+\alpha_j}y_j^{1/2-\alpha_j}}{2^{\nu_j-1}\Gamma(\nu_j)}\,(y_j^2-x_j^2)^{\nu_j-1}\chi(y_j-x_j)$$

where $\chi(\xi)=1$ for $\xi>0$ and $\chi(\xi)=0$ for $\xi<0$. Thus, for $x=(x_1,\dots,x_k)\in E$,

$$\begin{aligned}&(U^{-1}(G_{\nu,\alpha}\,|\,L^2(E))Uf)(x)\\&=2^{-k/2}\prod_{j=1}^{k}x_j^{-1/4}\int_E\kappa(x_1^{1/2},\dots,x_k^{1/2},t_1,\dots,t_k)(Uf)(t_1,\dots,t_k)\,dt\end{aligned}$$

and the integral equals

$$\begin{aligned}&\int_{E^2}\kappa(x_1^{1/2},\dots,x_k^{1/2},y_1^{1/2},\dots,y_k^{1/2})(Uf)(y_1^{1/2},\dots,y_k^{1/2})\,\frac{1}{2^k}\prod_{j=1}^{k}y_j^{-1/2}\,dy\\&=\int_{E^2}\kappa(x_1^{1/2},\dots,x_k^{1/2},y_1^{1/2},\dots,y_k^{1/2})\,2^{k/2}\prod_{j=1}^{k}y_j^{1/4}f(y_1,\dots,y_k)\,\frac{1}{2^k}\prod_{j=1}^{k}y_j^{-1/2}\,dy.\end{aligned}$$

Consequently, $U^{-1}G_{\nu,\alpha}U$ is the integral operator on $L^2(E^2)$ with the kernel

$$\begin{aligned}&\frac{1}{2^k}\left(\prod_{j=1}^{k}x_j^{-1/4}y_j^{-1/4}\right)\kappa(x_1^{1/2},\dots,x_k^{1/2},y_1^{1/2},\dots,y_k^{1/2})\\&=\frac{1}{2^k}\prod_{j=1}^{k}\frac{1}{2^{\nu_j-1}\Gamma(\nu_j)}\,x_j^{-1/4}y_j^{-1/4}x_j^{1/4+\alpha_j/2}y_j^{1/4-\alpha_j/2}(y_j-x_j)^{\nu_j-1}\chi(y_j^{1/2}-x_j^{1/2})\\&=\frac{1}{2^{|\nu|}}\prod_{j=1}^{k}\frac{1}{\Gamma(\nu_j)}\,x_j^{\alpha_j/2}y_j^{-\alpha_j/2}(y_j-x_j)^{\nu_j-1}\chi(y_j-x_j).\end{aligned}$$

As this is just $1/2^{|\nu|}$ times the kernel of $L_{\nu,\alpha}$, the proof of Theorem 1.2 is complete.

5 Bounds

In this section we prove Theorem 1.3.

Thus, let $\nu=(\nu_1,\dots,\nu_k)$ with natural numbers $\nu_j\ge 1$ and $\alpha=(\alpha_1,\dots,\alpha_k)$ with real numbers $\alpha_j>-1$. The operator $L_{\nu,\alpha}$ has the kernel

$$\prod_{j=1}^{k}\frac{x_j^{\alpha_j/2}y_j^{-\alpha_j/2}}{\Gamma(\nu_j)}(y_j-x_j)^{\nu_j-1}\chi(y_j-x_j).$$

The adjoint of the operator $L_{\nu,\alpha} \,|\, L^2(E)$ is $L^*_{\nu,\alpha} \,|\, L^2(E)$ where $L^*_{\nu,\alpha}$ is the integral operator with the kernel

$$\prod_{j=1}^{k} \frac{x_j^{-\alpha_j/2} y_j^{\alpha_j/2}}{\Gamma(\nu_j)} (x_j - y_j)^{\nu_j - 1} \chi(x_j - y_j).$$

Recall that $\Omega_k^\delta = \{x \in [0,1]^k : x_1^{1/\delta} + \ldots + x_k^{1/\delta} \le 1\}$. In what follows we will make frequent use of Euler's formula

$$\int_0^1 t^{p-1}(1-t)^{q-1}\, dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

and of Dirichlet's formula

$$\int_{\Omega_k^\delta} x_1^{p_1-1} \ldots x_k^{p_k-1}\, dx = \frac{\delta^k\, \Gamma(\delta p_1) \ldots \Gamma(\delta p_k)}{\Gamma(\delta p_1 + \ldots + \delta p_k + 1)}. \tag{30}$$

The Hilbert-Schmidt norm $\|K \,|\, L^2(E)\|_2$ of an integral operator is the L^2 norm of its kernel over $E \times E$. We therefore have

$$\left(\prod_{j=1}^{k} \Gamma(\nu_j)^2\right) \|L_{\nu,\alpha} \,|\, L^2(\Omega_k^\delta)\|_2^2 = \left(\prod_{j=1}^{k} \Gamma(\nu_j)^2\right) \|L^*_{\nu,\alpha} \,|\, L^2(\Omega_k^\delta)\|_2^2$$

$$= \int_{\Omega_k^\delta} \int_{\Omega_k^\delta} \prod_{j=1}^{k} x_j^{-\alpha_j} y_j^{\alpha_j} (x_j - y_j)^{2\nu_j - 2} \chi(x_j - y_j)\, dy dx. \tag{31}$$

If $(x_1, \ldots, x_k) \in \Omega_k^\delta$ and $y_j < x_j$ for all j, then $(y_1, \ldots, y_j)$ is automatically in Ω_k^δ. Thus, (31) is

$$\int_{\Omega_k^\delta} \left(\prod_{j=1}^{k} x_j^{-\alpha_j} \int_0^{x_j} y_j^{\alpha_j} (x_j - y_j)^{2\nu_j - 2} dy_j\right) dx$$

$$= \int_{\Omega_k^\delta} \left(\prod_{j=1}^{k} x_j^{-\alpha_j} \int_0^1 x_j^{\alpha_j} t^{\alpha_j} x_j^{2\nu_j - 2} (1-t)^{2\nu_j - 2} x_j\, dt\right) dx$$

$$= \left(\prod_{j=1}^{k} \frac{\Gamma(\alpha_j + 1)\Gamma(2\nu_j - 1)}{\Gamma(\alpha_j + 2\nu_j)}\right) \int_{\Omega_k^\delta} \prod_{j=1}^{k} x_j^{2\nu_j - 1}\, dx$$

and formula (30) now implies the equality asserted in Theorem 1.3.

To prove the inequality stated in Theorem 1.3 note first that the kernel of the integral operator $L^*_{\nu,\alpha} L_{\nu,\alpha} \,|\, L^2(\Omega_k^\delta)$ is

$$\int_{\Omega_k^\delta} g(t,x) g(t,y)\, dt$$

where

$$g(t,z)=\prod_{j=1}^{k}\frac{t_j^{\alpha_j/2}z_j^{-\alpha_j/2}}{\Gamma(\nu_j)}\,(z_j-t_j)^{\nu_j-1}\chi(z_j-t_j).$$

Consequently,

$$\|L^*_{\nu,\alpha}L_{\nu,\alpha}\,|\,L^2(\Omega_k^\delta)\|_2^2=\int_{\Omega_k^\delta}\int_{\Omega_k^\delta}\left(\int_{\Omega_k^\delta}g(t,x)g(t,y)\,dt\right)^2dxdy. \tag{32}$$

The inner integral in (32) is

$$\begin{aligned}I_1:=&\left(\prod_{j=1}^{k}\frac{x_j^{-\alpha_j/2}y_j^{-\alpha_j/2}}{\Gamma(\nu_j)^2}\right)\times\\&\times\int_{\Omega_k^\delta}\left(\prod_{j=1}^{k}t_j^{\alpha_j}(y_j-t_j)^{\nu_j-1}(x_j-t_j)^{\nu_j-1}\chi(y_j-t_j)\chi(x_j-t_j)\right)dt.\end{aligned} \tag{33}$$

Put $u_j=\min(x_j,y_j)$. If x and y are in Ω_k^δ, then $(u_1,\ldots,u_k)$ is also in Ω_k^δ. Therefore the integral in (33) is

$$I_2:=\prod_{j=1}^{k}\int_0^{u_j}t_j^{\alpha_j}(y_j-t_j)^{\nu_j-1}(x_j-t_j)^{\nu_j-1}dt_j.$$

If $y_j\le x_j$ then $u_j=y_j$ and

$$\begin{aligned}&\int_0^{u_j}t_j^{\alpha_j}(y_j-t_j)^{\nu_j-1}(x_j-t_j)^{\nu_j-1}dt_j\\&=\int_0^{y_j}t_j^{\alpha_j}(y_j-t_j)^{\nu_j-1}(x_j-t_j)^{\nu_j-1}dt_j\\&=\int_0^{1}y_j^{\alpha_j}s^{\alpha_j}y_j^{\nu_j-1}(1-s)^{\nu_j-1}(x_j-y_js)^{\nu_j-1}y_j\,ds.\end{aligned} \tag{34}$$

Since $x_j-y_js\ge x_j-x_js$ for $0\le y_j\le x_j$ and $0\le s\le 1$, integral (34) is at least

$$\begin{aligned}&\int_0^1 y_j^{\alpha_j+\nu_j}x_j^{\nu_j-1}s^{\alpha_j}(1-s)^{2\nu_j-2}ds\\&=\frac{\Gamma(\alpha_j+1)\Gamma(2\nu_j-1)}{\Gamma(\alpha_j+2\nu_j)}\,y_j^{\alpha_j+\nu_j}x_j^{\nu_j-1}\\&=\frac{\Gamma(\alpha_j+1)\Gamma(2\nu_j-1)}{\Gamma(\alpha_j+2\nu_j)}\,u_j^{\alpha_j+\nu_j}w_j^{\nu_j-1},\end{aligned} \tag{35}$$

where $w_j=\max(x_j,y_j)$. The integral I_2 is symmetric in x and y and hence (35) will also result in the case $y_j>x_j$. Thus,

$$I_2\ge\Gamma_{\nu,\alpha}\prod_{j=1}^{k}u_j^{\alpha_j+\nu_j}w_j^{\nu_j-1},\quad \Gamma_{\nu,\alpha}:=\prod_{j=1}^{k}\frac{\Gamma(\alpha_j+1)\Gamma(2\nu_j-1)}{\Gamma(\alpha_j+2\nu_j)}.$$

Letting $C_\nu = \prod_{j=1}^{k}(1/\Gamma(\nu_j))$, we obtain that

$$\begin{aligned} I_1^2 &\geq C_\nu^4 \Gamma_{\nu,\alpha}^2 \prod_{j=1}^{k} x_j^{-\alpha_j} y_j^{-\alpha_j} u_j^{2\alpha_j+2\nu_j} w_j^{2\nu_j-2} \\ &= C_\nu^4 \Gamma_{\nu,\alpha}^2 \prod_{j=1}^{k} u_j^{\alpha_j+2\nu_j} w_j^{2\nu_j-\alpha_j-2}. \end{aligned}$$

Consequently, a lower bound for (32) is

$$C_\nu^4 \Gamma_{\nu,\alpha}^2 \int_{\Omega_k^\delta} \int_{\Omega_k^\delta} \left(\prod_{j=1}^{k} u_j^{\alpha_j+2\nu_j} w_j^{2\nu_j-\alpha_j-2} \right) dx dy. \tag{36}$$

For each j, we have, up to sets of measure zero, the two possibilities $y_j < x_j$ or $y_j > x_j$. This gives 2^k possibilities for all j. Accordingly, we may partition $\Omega_k^\delta \times \Omega_k^\delta$ into 2^k domains B_i ($i = 1, \ldots, 2^k$). For a fixed j, we have either $u_j = x_j$ and $w_j = y_j$ or $u_j = y_j$ and $w_j = x_j$ throughout each domain B_i. Thus,

$$dx_1 \ldots dx_k \, dy_1 \ldots dy_k = du_1 \ldots du_k \, dw_1 \ldots dw_k$$

in each B_i. This implies that each B_i makes the same contribution to (36) and that hence (36) equals

$$2^k C_\nu^4 \Gamma_{\nu,\alpha}^2 \int_{B_1} \left(\prod_{j=1}^{k} u_j^{\alpha_j+2\nu_j} w_j^{2\nu_j-\alpha_j-2} \right) du dw \tag{37}$$

where $B_1 = \{(u,w) \in \Omega_k^\delta \times \Omega_k^\delta : u_j < w_j \text{ for all } j\}$. If $w \in \Omega_k^\delta$ and $u_j < w_j$ for all j, then u is automatically in Ω_k^δ. This shows that (37) is equal to

$$\begin{aligned} & 2^k C_\nu^4 \Gamma_{\nu,\alpha}^2 \int_{\Omega_k^\delta} \left(\prod_{j=1}^{k} \int_0^{w_j} u_j^{\alpha_j+2\nu_j} w_j^{2\nu_j-\alpha_j-2} du_j \right) dw \\ &= 2^k C_\nu^4 \Gamma_{\nu,\alpha}^2 \int_{\Omega_k^\delta} \left(\prod_{j=1}^{k} \frac{w_j^{4\nu_j-1}}{\alpha_j + 2\nu_j + 1} \right) dw. \end{aligned}$$

From (30) we infer that

$$\int_{\Omega_k^\delta} \prod_{j=1}^{k} w_j^{4\nu_j-1} dw = \frac{\delta^k}{\Gamma(4\delta|\nu|+1)} \prod_{j=1}^{k} \Gamma(4\delta\nu_j).$$

Putting things together we arrive at the inequality in Theorem 1.3.

Here is, for the sake of completeness, a proof of (14). The squares of the bounds $b_1 := b_1(\nu, \alpha, \Omega_k^\delta)$ and $b_2 := b_2(\nu, \alpha, \Omega_k^\delta)$ are

$$b_2^2 = \frac{\delta^k}{\Gamma(2\delta|\nu|+1)} \prod_{j=1}^{k} \frac{\Gamma(\alpha_j+1)\Gamma(2\nu_j-1)\Gamma(2\delta\nu_j)}{\Gamma(\alpha_j+2\nu_j)\Gamma(\nu_j)^2},$$

$$b_1^2 = \frac{2^k\Gamma(2\delta|\nu|+1)}{\Gamma(4\delta|\nu|+1)} \prod_{j=1}^{k} \frac{\Gamma(\alpha_j+1)\Gamma(2\nu_j-1)\Gamma(4\delta\nu_j)}{\Gamma(\alpha_j+2\nu_j)(\alpha_j+2\nu_j+1)\Gamma(\nu_j)^2\Gamma(2\delta\nu_j)},$$

and the quotient of these two bounds is

$$\frac{b_2^2}{b_1^2} = \left(\frac{\delta}{2}\right)^k \frac{\Gamma(4\delta|\nu|+1)}{\Gamma(2\delta|\nu|+1)^2} \prod_{j=1}^{k} (\alpha_j+2\nu_j+1) \frac{\Gamma(2\delta\nu_j)^2}{\Gamma(4\delta\nu_j)}. \tag{38}$$

Suppose $k, \delta, \alpha_1, \ldots, \alpha_k$ remain fixed and $\nu_j \to \infty$ for all j. Taking into account that

$$\frac{\Gamma(2\mu)}{\Gamma(\mu)^2} \sim 2^{2\mu}\sqrt{\frac{\mu}{4\pi}} \quad \text{as} \quad \mu \to \infty,$$

we see that (38) is

$$\begin{aligned}
&\frac{\delta^k}{2^k} \frac{4\delta|\nu|\,\Gamma(4\delta|\nu|)}{4\delta^2|\nu|^2\,\Gamma(2\delta|\nu|)^2} \prod_{j=1}^{k} (\alpha_j+2\nu_j+1) \frac{\Gamma(2\delta\nu_j)^2}{\Gamma(4\delta\nu_j)} \\
&\sim \frac{\delta^k}{2^k} \frac{1}{\delta|\nu|}\, 2^{4\delta|\nu|} \left(\frac{2\delta|\nu|}{4\pi}\right)^{1/2} 2^k \prod_{j=1}^{k} \nu_j \frac{1}{2^{4\delta\nu_j}} \left(\frac{4\pi}{2\delta\nu_j}\right)^{1/2} \\
&= (2\pi\delta)^{(k-1)/2} \frac{1}{|\nu|^{1/2}} \prod_{j=1}^{k} \nu_j^{1/2},
\end{aligned}$$

which is the same as (14).

6 Asymptotics

In this section we present a proof of Theorem 1.4.

Let $\varrho_1, \ldots, \varrho_k$ be positive real numbers. For a real number $\tau > 0$, we consider the integral operator L_τ with the kernel

$$g_\tau(x,y) = \prod_{j=1}^{k} x_j^{-\alpha_j/2} y_j^{\alpha_j/2} (x_j - y_j)^{\varrho_j\tau - 1} \chi(x_j - y_j)$$

on $L^2(\Omega_k^\delta)$. Clearly, L_τ is nothing but

$$\left(\prod_{j=1}^{k} \Gamma(\varrho_j\tau)\right) (L^*_{\varrho_1\tau,\alpha_1} \otimes \ldots \otimes L^*_{\varrho_k\tau,\alpha_k}) \,|\, L^2(\Omega_k^\delta).$$

The function $\prod_{j=1}^k x_j^{\varrho_j}$ attains its maximum on Ω_k^δ at the point $(p_1, \ldots, p_k)$ and nowhere else. This point lies on the boundary of Ω_k^δ and is given by

$$p_j = \left(\frac{\varrho_j}{|\varrho|}\right)^\delta, \quad |\varrho| := \varrho_1 + \ldots + \varrho_k.$$

We denote by H_τ the integral operator on $L^2(\Omega_k^\delta)$ whose kernel is

$$h_\tau(x,y) = \prod_{j=1}^k x_j^{-\alpha_j/2} y_j^{\alpha_j/2} x_j^{\varrho_j\tau-1}\left(1-\frac{y_j}{p_j}\right)^{\varrho_j\tau-1}\chi(p_j-y_j).$$

As $h_\tau(x,y)$ is of the form $a(x)b(y)$, the operator norm of H_τ is the product of the L^2 norms of a and b. Thus, using the Dirichlet and Euler formulas,

$$\begin{aligned}\|H_\tau\|_\infty^2 &= \int_{\Omega_k^\delta}\prod_{j=1}^k x_j^{2\varrho_j\tau-\alpha_j-2}dx \prod_{j=1}^k\int_0^{p_j} y_j^{\alpha_j}\left(1-\frac{y_j}{p_j}\right)^{2\varrho_j\tau-2}dy_j \\ &= \frac{\delta^k\prod_{j=1}^k\Gamma(2\delta\varrho_j\tau-(\alpha_j+1)\delta)}{\Gamma(2\delta|\varrho|\tau-\sum(\alpha_j+1)\delta+1)}\prod_{j=1}^k p_j^{\alpha_j+1}\frac{\Gamma(\alpha_j+1)\Gamma(2\varrho_j\tau-1)}{\Gamma(\alpha_j+2\varrho_j\tau)}, \qquad (39)\end{aligned}$$

the sum over $j = 1, \ldots, k$. From Theorem 1.3 we infer that

$$\|L_\tau\|_2^2 = \frac{\delta^k}{\Gamma(2\delta|\varrho|\tau+1)}\prod_{j=1}^k\frac{\Gamma(\alpha_j+1)\Gamma(2\varrho_j\tau-1)\Gamma(2\delta\varrho_j\tau)}{\Gamma(\alpha_j+2\varrho_j\tau)}.$$

Consequently, by Stirling's formula,

$$\begin{aligned}\frac{\|H_\tau\|_\infty^2}{\|L_\tau\|_2^2} &= \frac{\Gamma(2\delta|\varrho|\tau+1)}{\Gamma(2\delta|\varrho|\tau-\sum(\alpha_j+1)\delta+1)}\prod_{j=1}^k p_j^{\alpha_j+1}\frac{\Gamma(2\delta\varrho_j\tau-(\alpha_j+1)\delta)}{\Gamma(2\delta\varrho_j\tau)} \\ &\sim (2\delta|\varrho|\tau)^{\sum(\alpha_j+1)\delta}\prod_{j=1}^k p_j^{\alpha_j+1}(2\delta\varrho_j\tau)^{-(\alpha_j+1)\delta} \\ &= |\varrho|^{\sum(\alpha_j+1)\delta}\prod_{j=1}^k p_j^{\alpha_j+1}\varrho_j^{-(\alpha_j+1)\delta} \\ &= |\varrho|^{\sum(\alpha_j+1)\delta}\prod_{j=1}^k\left(\frac{\varrho_j}{|\varrho|}\right)^{\delta(\alpha_j+1)}\varrho_j^{-(\alpha_j+1)\delta} = 1,\end{aligned}$$

that is, $\|H_\tau\|_\infty = \|L_\tau\|_2(1+o(1))$. Now suppose we had shown that

$$\|L_\tau - H_\tau\|_2^2 = o(\|H_\tau\|_\infty^2). \qquad (40)$$

It would follow that $\|L_\tau - H_\tau\|_\infty = o(\|H_\tau\|_\infty)$ and hence

$$\begin{aligned} \|L_\tau\|_\infty &= \|H_\tau\|_\infty + o(\|H_\tau\|_\infty) \\ &= \|H_\tau\|_\infty(1+o(1)) = \|L_\tau\|_2(1+o(1)), \end{aligned}$$

as desired. Thus, we are left with (40).

We have

$$\begin{aligned} \|L_\tau - H_\tau\|_2^2 &= \int_{\Omega_k^\delta}\int_{\Omega_k^\delta} (g_\tau^2 - 2\, g_\tau h_\tau + h_\tau^2)\, dydx \\ &= \|L_\tau\|_2^2 - 2\int_{\Omega_k^\delta}\int_{\Omega_k^\delta} g_\tau h_\tau\, dydx + \|H_\tau\|_2^2. \end{aligned}$$

We already showed that $\|L_\tau\|_2^2/\|H_\tau\|_\infty^2 \to 1$, and since the kernel of H_τ is of the form $a(x)b(y)$ we may conclude that $\|H_\tau\|_2^2/\|H_\tau\|_\infty^2 = 1$. Consequently, estimate (40) will follow as soon as we have proved that

$$\int_{\Omega_k^\delta}\int_{\Omega_k^\delta} g_\tau h_\tau\, dydx/\|H_\tau\|_\infty^2 \to 1. \tag{41}$$

The integral in (41) equals

$$\int_{\Omega_k^\delta} \prod_{j=1}^k x_j^{\varrho_j\tau-\alpha_j-1} \left(\prod_{j=1}^k \int_0^{p_j} y_j^{\alpha_j} \left(1-\frac{y_j}{p_j}\right)^{\varrho_j\tau-1} \chi(x_j-y_j)(x_j-y_j)^{\varrho_j\tau-1}\, dy_j \right) dx.$$

To tackle the inner integrals, we use that if $\alpha > -1$ and $0 \le \beta \le 1$, then

$$\int_0^1 t^\alpha (1-\beta t)^\lambda (1-t)^\lambda dt = \frac{\Gamma(\alpha+1)}{(1+\beta)^{\alpha+1}}\, \frac{1}{\lambda^{\alpha+1}} (1+o(1))$$

as $\lambda \to \infty$, the $o(1)$ being unform in $\beta \in [0,1]$; see, for example, [11, Section 2.4]. If $x_j < p_j$, the jth inner integral is

$$\begin{aligned} &\int_0^{x_j} y_j^{\alpha_j} \left(1-\frac{y_j}{p_j}\right)^{\varrho_j\tau-1} (x_j-y_j)^{\varrho_j\tau-1} dy_j \\ &= x_j^{\alpha_j+\varrho_j\tau} \int_0^1 t^{\alpha_j} \left(1-\frac{x_j t}{p_j}\right)^{\varrho_j\tau-1} (1-t)^{\varrho_j\tau-1} dt \\ &= x_j^{\alpha_j+\varrho_j\tau}\, \frac{\Gamma(\alpha_j+1)}{(1+x_j/p_j)^{\alpha_j+1}}\, \frac{1}{(\varrho_j\tau)^{\alpha_j+1}}\, (1+o(1)), \end{aligned}$$

while for $x_j > p_j$ it is

$$\begin{aligned}
&\int_0^{p_j} y_j^{\alpha_j}\left(1-\frac{y_j}{p_j}\right)^{\varrho_j\tau-1}(x_j-y_j)^{\varrho_j\tau-1}dy_j\\
&= x_j^{\varrho_j\tau-1}p_j^{\alpha_j+1}\int_0^1 t^{\alpha_j}(1-t)^{\varrho_j\tau-1}\left(1-\frac{p_jt}{x_j}\right)^{\varrho_j\tau-1}dt\\
&= x_j^{\varrho_j\tau-1}p_j^{\alpha_j+1}\,\frac{\Gamma(\alpha_j+1)}{(1+p_j/x_j)^{\alpha_j+1}}\,\frac{1}{(\varrho_j\tau)^{\alpha_j+1}}\,(1+o(1))\\
&= x_j^{\alpha_j+\varrho_j\tau}\,\frac{\Gamma(\alpha_j+1)}{(1+x_j/p_j)^{\alpha_j+1}}\,\frac{1}{(\varrho_j\tau)^{\alpha_j+1}}\,(1+o(1)).
\end{aligned}$$

Thus, the integral in (41) is asymptotically equal to

$$\prod_{j=1}^{k}\frac{\Gamma(\alpha_j+1)}{(\varrho_j\tau)^{\alpha_j+1}}\int_{\Omega_k^\delta}\prod_{j=1}^{k}\frac{x_j^{2\varrho_j\tau-1}}{(1+x_j/p_j)^{\alpha_j+1}}\,dx. \tag{42}$$

Taking into account that $\prod x_j^{\varrho_j}$ has its maximum at $(p_1,\dots,p_k)$ one can employ standard methods, for example, such as in [13, Section II.4], to show that the integral in (42) is asymptotically equal to

$$\int_{\Omega_k^\delta}\prod_{j=1}^{k}\frac{x_j^{2\varrho_j\tau-1}}{(1+p_j/p_j)^{\alpha_j+1}}\,dx=\prod_{j=1}^{k}\frac{1}{2^{\alpha_j+1}}\int_{\Omega_k^\delta}\prod_{j=1}^{k}x_j^{2\varrho_j\tau-1}dx. \tag{43}$$

By Dirichlet's formula (30), the integral on the right of (43) is

$$\frac{\delta^k}{\Gamma(2\delta|\varrho|\tau+1)}\prod_{j=1}^{k}\Gamma(2\delta\varrho_j\tau).$$

In summary, the integral in (41) equals

$$\frac{\delta^k}{\Gamma(2\delta|\varrho|\tau+1)}\prod_{j=1}^{k}\frac{\Gamma(\alpha_j+1)\Gamma(2\delta\varrho_j\tau)}{2^{\alpha_j+1}(\varrho_j\tau)^{\alpha_j+1}}\,(1+o(1)).$$

This in conjunction with (39) gives that the left-hand side of (41) is asymptotically equal to

$$\begin{aligned}
&\frac{\Gamma(2\delta|\varrho|\tau-\sum(\alpha_j+1)\delta+1)}{\Gamma(2\delta|\varrho|\tau+1)}\\
&\quad\times\prod_{j=1}^{k}\frac{\Gamma(2\delta\varrho_j\tau)\Gamma(\alpha_j+2\varrho_j\tau)}{(2p_j\varrho_j\tau)^{\alpha_j+1}\Gamma(2\delta\varrho_j\tau-(\alpha_j+1)\delta)\Gamma(2\varrho_j\tau-1)}\\
&\sim\frac{1}{(2\delta|\varrho|\tau)^{\sum(\alpha_j+1)\delta}}\prod_{j=1}^{k}\frac{(2\delta\varrho_j\tau)^{(\alpha_j+1)\delta}(2\varrho_j\tau)^{\alpha_j+1}}{(2p_j\varrho_j\tau)^{\alpha_j+1}}\\
&=\prod_{j=1}^{k}\frac{1}{p_j^{\alpha_j+1}}\left(\frac{\varrho_j}{|\varrho|}\right)^{(\alpha_j+1)\delta}=\prod_{j=1}^{k}\frac{1}{p_j^{\alpha_j+1}}\,p_j^{\alpha_j+1}=1,
\end{aligned}$$

which completes the proof.

7 Linear combinations

Here is the proof of Theorem 1.5.

Let $m \ge 1$ be an integer and $\alpha > -1$ be a real number. We put $T_{m,\alpha} = L_{m,\alpha}$ in the Laguerre case and $T_{m,\alpha} = G_{m,\alpha}$ in the Gegenbauer case. Let $M_{(2x)^{m/2}}$ be the operator of multiplication by $(2x)^{m/2}$ on $L^2(0,1)$ and put $T_{m,\alpha} = M_{(2x)^{m/2}}$ in the Hermite case; clearly, in that case $T_{m,\alpha}$ does actually not depend on α. In either case, we let $T_{0,\alpha}$ be the identity operator. We finally use the abbreviation

$$C_n(p) = C(p(\partial_1, \ldots, \partial_N) \,|\, \mathcal{P}_n(E)). \tag{44}$$

In Section 4 we introduced the matrix representations $D_{\alpha,n}$ of the operator of differentiation on $\mathcal{P}_n$ in the Laguerre and Gegenbauer cases. We use the notation $D_{\alpha,n}$ also for the matrix representation of the operator $D : \mathcal{P}_n \to \mathcal{P}_n$ in the Hermite basis (where in fact there is no dependence on α).

Lemma 7.1 *We have*

$$(n+1)^{-\sigma|\nu|} W_{D^{\nu_1}_{\alpha_1,n}} \otimes \ldots \otimes W_{D^{\nu_N}_{\alpha_N,n}} \to T_{\nu_1,\alpha_1} \otimes \ldots \otimes T_{\nu_N,\alpha_N}$$

strongly on $L^2((0,1)^N)$ as $n \to \infty$.

Proof. Let I_n be the $(n+1) \times (n+1)$ identity matrix. It is easily seen that $W_{I_n} \to I$ strongly. In the Laguerre and Gegenbauer cases, we know from (23) that $(n+1)^{-\sigma m} W_{D^m_{\alpha,n}} \to T_{m,\alpha}$ in the norm provided $m \ge 1$. This implies the lemma in these two cases. Let us consider the Hermite case. For $x \in Q_j := [j/(n+1), (j+1)/(n+1))$, we obtain from (20) that

$$\begin{aligned}
(W_{D^m_{\alpha,n}} f)(x) &= (n+1) \sum_{i=0}^{n} \int_{Q_i} (D^m_{\alpha,n})_{j,i} f(y)\, dy \\
&= (n+1) \int_{Q_{j+m}} (D^m_{\alpha,n})_{j,j+m} f(y)\, dy \\
&= (n+1) \int_{Q_{j+m}} 2^{m/2} \sqrt{\frac{\Gamma(j+m+1)}{\Gamma(j+1)}}\, f(y)\, dy \\
&= (n+1) \int_{Q_{j+m}} 2^{m/2} \sqrt{(j+m)\ldots(j+1)}\, f(y)\, dy,
\end{aligned}$$

provided $j + m \le n$. Thus,

$$\begin{aligned}
&(n+1)^{-m/2} (W_{D^m_{\alpha,n}} f)(x) \\
&= (n+1) \cdot 2^{m/2} \int_{Q_{j+m}} \sqrt{\left(\frac{j}{n+1} + \frac{m}{n+1}\right) \ldots \left(\frac{j}{n+1} + \frac{1}{n+1}\right)}\, f(y)\, dy.
\end{aligned}$$

If f is in $C[0,1]$ and n is large, then the right-hand side of this equality is approximately equal to $2^{m/2}x^{m/2}f(x)$. It is not difficult to make this precise and to show that

$$\|(n+1)^{-m/2}W_{D^m_{\alpha,n}}f - M_{(2x)^{m/2}}f\| \to 0$$

for every $f \in C[0,1]$. Since, by Lemma 4.1 and the result known for the one-dimensional case,

$$\|(n+1)^{-m/2}W_{D^m_{\alpha,n}}\|_\infty = (n+1)^{-m/2}\|D^m_{\alpha,n}\|_\infty = (n+1)^{-m/2}\eta_n(D^m) \sim 2^{m/2},$$

it follows that $(n+1)^{-m/2}W_{D^m_{\alpha,n}} \to M_{(2x)^{m/2}}$ strongly on $L^2(0,1)$. Tensoring we get the assertion in the Hermite case. □

Since $(n+1)/n \to 1$, Lemmas 4.1, 4.3, and 7.1 give that

$$\sum_{\nu_1+\ldots+\nu_N\le M} n^{-\sigma|\nu|}p_{\nu_1,\ldots,\nu_N}W_{(D^{\nu_1}_{\alpha_1,n}\otimes\ldots\otimes D^{\nu_N}_{\alpha_N,n})P_{\Pi_n}} \to \sum_{\nu_1+\ldots+\nu_N\le M} p_{\nu_1,\ldots,\nu_N}(T_{\nu_1,\alpha_1}\otimes\ldots\otimes T_{\nu_N,\alpha_N})P_E$$

strongly, which in turn implies that

$$n^{-\sigma M}\sum_{\nu_1+\ldots+\nu_N=M} p_{\nu_1,\ldots,\nu_N}W_{(D^{\nu_1}_{\alpha_1,n}\otimes\ldots\otimes D^{\nu_N}_{\alpha_N,n})P_{\Pi_n}} \to \sum_{\nu_1+\ldots+\nu_N=M} p_{\nu_1,\ldots,\nu_N}(T_{\nu_1,\alpha_1}\otimes\ldots\otimes T_{\nu_N,\alpha_N})P_E \tag{45}$$

strongly. From the Banach-Steinhaus theorem we therefore obtain that

$$\liminf_{n\to\infty} n^{-\sigma M}\left\|\sum_{\nu_1+\ldots+\nu_N=M} p_{\nu_1,\ldots,\nu_N}W_{(D^{\nu_1}_{\alpha_1,n}\otimes\ldots\otimes D^{\nu_N}_{\alpha_N,n})P_{\Pi_n}}\right\|_\infty \ge \left\|\sum_{\nu_1+\ldots+\nu_N=M} p_{\nu_1,\ldots,\nu_N}(T_{\nu_1,\alpha_1}\otimes\ldots\otimes T_{\nu_N,\alpha_N})P_E\right\|_\infty$$

The right-hand side of this inequality is strictly positive and the left-hand side is just $\liminf n^{-\sigma M}C_n(p_0)$ due to Lemma 4.1. Hence

$$\liminf n^{-\sigma M}C_n(p_0) > 0.$$

Lemma 7.2 *If p and q are any two polynomials, then $C_n(p+q) \le C_n(p)+C_n(q)$.*

Proof. Obvious. □

Lemma 7.2 in conjunction with Theorem 1.1 shows that

$$\limsup n^{-\sigma M}C_n(p_0) < \infty.$$

Thus, at this point we have proved (18). From Lemma 7.2 we also get

$$C_n(p) \le C_n(p_0) + \sum_{\nu_1+\ldots+\nu_N \le M-1} |p_{\nu_1,\ldots,\nu_N}| C(\partial_1^{\nu_1}\ldots\partial_N^{\nu_n}\,|\,\mathcal{P}_n(E)),$$
$$C_n(p_0) \le C_n(p) + \sum_{\nu_1+\ldots+\nu_N \le M-1} |p_{\nu_1,\ldots,\nu_N}| C(\partial_1^{\nu_1}\ldots\partial_N^{\nu_n}\,|\,\mathcal{P}_n(E)),$$

and Theorem 1.1 therefore yields that $C_n(p) = C_n(p_0) + O(n^{\sigma(M-1)})$, which together with (18) gives (19) and thus completes the proof of Theorem 1.5.

Remark 7.3 The arguments used in this section reveal the difference between the Hermite case on the one hand and the Laguerre and Gegenbauer cases on the other. In contrast to the Laguerre and Gegenbauer cases, the limiting operators in the Hermite case are no longer (compact) integral operators, but multiplication operators, and secondly, in the Hermite case the convergence is no longer uniform, but only strong. In this light it comes as a fortune that the Hermite case can be disposed of by the simple reasoning presented in Section 3.

8 Open problems

Problem 8.1 Let $N=2$ and consider the operator

$$p(\partial_1,\partial_2) = p_{30}\partial_1^3 + p_{21}\partial_1^2\partial_2 + p_{12}\partial_1\partial_2^2 + p_{03}\partial_2^3 + \sum_{\nu_1+\nu_2\le 2} p_{\nu_1\nu_2}\partial_1^{\nu_1}\partial_2^{\nu_2}.$$

The principal part is $p_0(\partial_1,\partial_2) = p_{30}\partial_1^3 + p_{21}\partial_1^2\partial_2 + p_{12}\partial_1\partial_2^2 + p_{03}\partial_2^3$. Theorem 1.5 tells us that, with abbreviation (44),

$$C_n(p) \sim C_n(p_0) \simeq n^{3\,\sigma},$$

and as long as one of the terms $p_{30}\partial_1^3$ and $p_{03}\partial_2^3$ is present, we cannot say more. However, if $p_{30} = p_{03} = 0$ and if we are in the Laguerre or Gegenbauer cases, then the convergence in (45) is uniform because all occurring T_{ν_i,α_i} are compact integral operators. Consequently, in these cases there is no need in having recourse to the Banach-Steinhaus theorem, since we can rather conclude straightforwardly that

$$C_n(p_0) \sim n^{3\,\sigma}\, \|p_{21}T_{2,\alpha_1}\otimes T_{1,\alpha_2} + p_{12}T_{1,\alpha_1}\otimes T_{2,\alpha_2}\,|\,L^2(E)\|_\infty.$$

However, in the general case we must the replacement of "$\simeq$" in (18) by "$\sim$ times a constant" leave as an open problem.

Problem 8.2 The most embarrassing message is that our approach fails for the Laplace operator. This failure is of course connected with Problem 8.1. Let, for example, $N=2$ and $\Delta = \partial_1^2 + \partial_2^2$. From Theorem 1.5 we deduce that

$$C(\Delta\,|\,\mathcal{P}_n(E)) \simeq n^{2\,\sigma},$$

but we cannot even prove that $C(\Delta \,|\, \mathcal{P}_n(E))/n^{2\,\sigma}$ converges to a limit.

On the credit side we have a few modest estimates. From (45) we get

$$\liminf_{n\to\infty} \frac{C(\Delta \,|\, \mathcal{P}_n(E))}{n^{2\,\sigma}} \ge \|T_{2,\alpha_1} \otimes I + I \otimes T_{2,\alpha_2} \,|\, L^2(E)\|_\infty, \tag{46}$$

while combination of Theorem 1.1 and Lemma 7.2 yields that

$$\limsup_{n\to\infty} \frac{C(\Delta \,|\, \mathcal{P}_n(E))}{n^{2\,\sigma}} \le \|T_{2,\alpha_1} \,|\, L^2(E_{2,0})\|_\infty + \|T_{2,\alpha_2} \,|\, L^2(E_{0,2})\|_\infty. \tag{47}$$

Suppose $\alpha_1 = \alpha_2 = 0$ and $E = \Omega_2$. In that case $E_{2,0} = E_{0,2} = [0,1]$.

For the Hermite weight, the right-hand sides of (46) and (47) become

$$\|M_{2x_1+2x_2} \,|\, L^2(\Omega_2)\|_\infty = 2 \quad \text{and} \quad 2\,\|M_{2x} \,|\, L^2(0,1)\|_\infty = 4,$$

respectively, which results in the estimates

$$2 \le \liminf_{n\to\infty} \frac{\eta(\Delta \,|\, \mathcal{P}_n(\Omega_2))}{n} \le \limsup_{n\to\infty} \frac{\eta(\Delta \,|\, \mathcal{P}_n(\Omega_2))}{n} \le 4.$$

Let us turn to the Laguerre weight. In that case the right-hand side of (47) is

$$2\,\|L_{2,0} \,|\, L^2(0,1)\|_\infty = 2 \times 0.284\ldots < 0.569$$

(see [33] or [3]). On the right of (46) we now have the operator norm of the operator $L_{2,0} \otimes I + I \otimes L_{2,0}$ on $L^2(\Omega_2)$. Let f be identically 1 on Ω_2. Then $\|f\|^2 = 1/2$ and for $(x_1, x_2) \in \Omega_2$,

$$\begin{aligned}
&((L_{2,0}^* \otimes I + I \otimes L_{2,0}^*) f)(x_1, x_2) \\
&= \int_0^{x_1} (x_1 - y_1) f(y_1, x_2)\, dy_1 + \int_0^{x_2} (x_2 - y_2) f(x_1, y_2)\, dy_2 \\
&= \int_0^{x_1} (x_1 - y_1)\, dy_1 + \int_0^{x_2} (x_2 - y_2)\, dy_2 = \frac{x_1^2 + x_2^2}{2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|L_{2,0} \otimes I + I \otimes L_{2,0} \,|\, L^2(\Omega_2)\|_\infty^2 = \|L_{2,0}^* \otimes I + I \otimes L_{2,0}^* \,|\, L^2(\Omega_2)\|_\infty^2 \\
&\ge 2 \int_{\Omega_2} \left(\frac{x_1^2 + x_2^2}{2} \right)^2 dx_1 dx_2 = \frac{7}{180} > 0.197^2.
\end{aligned}$$

It follows that

$$0.197 \le \liminf_{n\to\infty} \frac{\lambda(\Delta \,|\, \mathcal{P}_n(\Omega_2))}{n^2} \le \limsup_{n\to\infty} \frac{\lambda(\Delta \,|\, \mathcal{P}_n(\Omega_2))}{n^2} < 0.569.$$

In the Legendre case (= Gegenbauer case for $\alpha_1 = \alpha_2 = 0$) we obtain analogously that

$$0.091 \le \liminf_{n\to\infty} \frac{\gamma(\Delta \,|\, \mathcal{P}_n(\Omega_2))}{n^4} \le \limsup_{n\to\infty} \frac{\gamma(\Delta \,|\, \mathcal{P}_n(\Omega_2))}{n^4} < 0.143.$$

We remark that for the N-dimensional Laplace operator $\Delta = \partial_1^2 + \ldots + \partial_N^2$ the estimates obtained in this way become worse and worse: in the Hermite case the right-hand sides of (46) and (47) are 2 and $2N$, respectively, whereas in the Laguerre and Gegenbauer cases the upper bounds go to infinity and the lower bounds approach zero as $N \to \infty$. Thus, in general we cannot answer even the question whether

$$\liminf_{n\to\infty} \frac{C(\Delta \,|\, \mathcal{P}_n(\Omega_N))}{n^{2\,\sigma}}, \quad \limsup_{n\to\infty} \frac{C(\Delta \,|\, \mathcal{P}_n(\Omega_N))}{n^{2\,\sigma}}$$

converge to zero, increase to infinity, or remain bounded and bounded away from zero as $N \to \infty$.

Problem 8.3 Consider the wave operator $\Box = \partial_1^2 - \partial_2^2$ on $\mathcal{P}_n(\Omega_2)$. The linear operator

$$S : \mathcal{P}_n(\Omega_2) \to \mathcal{P}_n(\Omega_2), \quad (Sf)(t_1, t_2) = f\left(\frac{t_1 + t_2}{\sqrt{2}}, \frac{t_1 - t_2}{\sqrt{2}}\right)$$

is an isometry when taking the Hermite norm. Moreover, the factorization

$$\partial_1^2 - \partial_2^2 = (\partial_1 + \partial_2)(\partial_1 - \partial_2)$$

may be written as the identity $2\,\partial_1\partial_2\, S = S(\partial_1^2 - \partial_2^2)$. It follows that

$$\begin{aligned}\eta(\Box \,|\, \mathcal{P}_n(\Omega_2)) &= \max_{\|f\|=1} \|(\partial_1^2 - \partial_2^2)f\| = \max_{\|Sf\|=1} \|S(\partial_1^2 - \partial_2^2)f\| \\ &= \max_{\|Sf\|=1} \|2\,\partial_1\partial_2\, Sf\| = \max_{\|g\|=1} \|2\,\partial_1\partial_2\, g\| = 2\,\eta(\partial_1\partial_2 \,|\, \mathcal{P}_n(\Omega_2))\end{aligned}$$

and from Theorem 1.1 we obtain that

$$2\,\eta(\partial_1\partial_2 \,|\, \mathcal{P}_n(\Omega_2)) \sim 2n \max_{(x_1,x_2)\in\Omega_2} \sqrt{4x_1x_2} = 2n.$$

The following problems remain open. What can be said about $C(\Box \,|\, \mathcal{P}_n(\Omega_2))$ if $C = \lambda$ or $C = \gamma$? What happens if $\Box$ is the more general operator $\partial_1^2 - c^2\partial_2^2$? Can one tackle the Laplace operator by using the factorization $\Delta = (\partial_1 + i\partial_2)(\partial_1 - i\partial_2)$?

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