Interplay of source conditions and variational inequalities for nonlinear ill-posed problems

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March 20, 2009

Abstract

In the last years convergence rates results for Tikhonov regularization of nonlinear ill-posed problems in Banach spaces have been published, where the classical concept of source conditions was replaced with variational inequalities holding on some level sets. Also this advanced essentially the analysis of non-smooth situations with respect to forward operators and solutions. In fact, such variational inequalities combine both structural conditions on the nonlinearity of the operator and smoothness properties of the solution. Varying exponents in the variational inequalities correspond to different levels of convergence rates. In this paper, we discuss the range of occurring exponents in the Banach space setting. To lighten the cross-connections between generalized source conditions, degree of nonlinearity of the forward operator and associated variational inequalities we study the Hilbert space situation and even prove some converse result for linear operators.

MSC2000 subject classification: 47J06, 65J20, 47J20, 47A52

Keywords: Nonlinear ill-posed problems, regularization, variational inequalities, source conditions, degree of nonlinearity, convergence rates.

1 Introduction

After turn of the millennium there seems to be a substantial progress in regularization theory for the stable approximate solution of ill-posed inverse problems. On the one hand, partially motivated by specific applications in imaging and by a growing interest in sparsity of solutions as well as in new types of stabilizing terms in variational regularization, the Banach space treatment of linear and nonlinear operator equations and occurring difficulties in this context came into the focus of recent papers and books. On the other hand, Bregman distances for measuring the regularization error and variational

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inequalities for replacing the standard form of source conditions offer now good prospects for proving convergence rates results also for non-smooth situations with respect to solution and forward operator. For recent results we refer to the monograph [17] and in an exemplary manner to the papers [1, 2, 6, 7, 8, 9, 12, 14, 15, 16, 18, 19, 20].

This paper is devoted to the utility of variational inequalities combining both structural conditions on the nonlinearity of the operator and smoothness properties of the solution. Varying exponents in the variational inequalities correspond to different levels of convergence rates. We are going to discuss the range of occurring exponents in the Banach space setting and the interplay of general source conditions and variational inequalities in Banach and Hilbert spaces.

The paper is organized as follows: In Section 2 we describe the Tikhonov type regularization for the stable approximate solution of nonlinear ill-posed operator equations in a Banach space setting under basic assumptions which follow the corresponding assumptions of the papers [8, 9]. As in the previous papers the focus is again on level sets for the Tikhonov sum functional, and the majority of conditions under consideration have to hold on such sets. Section 3 summarizes propositions on convergence and convergence rates under variational inequalities. Moreover, we recall the concept of a degree of nonlinearity for characterizing the local structural nonlinearity conditions in the solution point. The range of occurring exponents in the variational inequalities is discussed in Section 4. Here the forward operator and the stabilizing functional are assumed to be Gâteaux differentiable. We distinguish three typical cases of exponents and make assertions for all of them. Some open questions cannot be answered currently for the general Banach space setting. Therefore we restrict our considerations in the concluding Section 5 to the classical nonlinear Tikhonov regularization in Hilbert spaces. Under that restriction we are able to formulate assertions on the interplay of variational inequalities and Hölder source conditions with fractional exponents including some converse result for the subcase of linear operators.

2 Problem, notation, and basic assumptions

We are going to study ill-posed operator equations

$$F(u) = v \tag{2.1}$$

expressing inverse problems with an in general nonlinear forward operator $F : \mathcal{D}(F) \subseteq U \to V$ possessing the domain $\mathcal{D}(F)$ and mapping between normed real linear spaces U and V with norms $\|\cdot\|_U$ and $\|\cdot\|_V$, respectively. Based on noisy data v^δ of the exact right-hand side $v = v^0 \in F(\mathcal{D}(F))$ with

$$\|v^\delta - v\|_V \le \delta \tag{2.2}$$

and noise level $\delta \ge 0$ we consider stable approximate solutions u_α^δ as minimizers over U of the Tikhonov type functional

$$T_\alpha^{v^\delta}(u) := \|F(u) - v^\delta\|_V^p + \alpha\,\Omega(u) \tag{2.3}$$

with a prescribed norm exponent

$$1 < p < \infty$$

and a regularization parameter $\alpha > 0$. In this context, let $\Omega : U \to [0, +\infty]$ be a stabilizing functional with

$$\mathcal{D}(\Omega) := \{u \in U : \Omega(u) \neq +\infty\} \neq \emptyset$$

and set $T_\alpha^{v^\delta}(u) = \infty$ if $u \notin \mathcal{D}(F)$.

Throughout this paper we make the following assumptions:

Assumption 2.1

1. *U and V are reflexive Banach spaces with duals U^* and V^*, respectively. In U and V we consider in addition to the norm convergence the associated weak convergence. That means in U*

$$u_k \rightharpoonup u \quad \Longleftrightarrow \quad \langle f, u_k \rangle_{U^*,U} \to \langle f, u \rangle_{U^*,U} \;\; \forall f \in U^*$$

   for the dual pairing $\langle \cdot, \cdot \rangle_{U^*,U}$ with respect to U^* and U. The weak convergence in V is defined in an analog manner.

2. *$F : \mathcal{D}(F) \subseteq U \to V$ is weakly continuous and $\mathcal{D}(F)$ is weakly sequentially closed, i.e.,*

$$u_k \rightharpoonup u \quad \text{in} \quad U \;\; \text{with} \;\; u_k \in \mathcal{D}(F) \quad \Longrightarrow \quad u \in \mathcal{D}(F) \quad \text{and} \quad F(u_k) \rightharpoonup F(u) \quad \text{in} \quad V.$$

3. *The functional Ω is convex and weakly sequentially lower semi-continuous.*

4. *The domain $\mathcal{D} := \mathcal{D}(F) \cap \mathcal{D}(\Omega)$ is non-empty.*

5. *For every $\alpha > 0$, $c \geq 0$, and for the exact right-hand side $v = v^0$ of (2.1) the sets*

$$\mathcal{M}_\alpha^v(c) := \{u \in \mathcal{D} : T_\alpha^v(u) \leq c\} , \tag{2.4}$$

   whenever they are non-empty, are weakly sequentially pre-compact in the following sense: every sequence $\{u_k\}_{k=1}^\infty$ in $\mathcal{M}_\alpha^v(c)$ has a subsequence, which is weakly convergent in U to some element from U.

Under the stated assumptions existence and stability of regularized solutions u_α^δ can be shown (cf. [9, §3] and [17, Theorems 3.22 and 3.23]).

In the Banach space theory of Tikhonov type regularization methods, regularization errors are frequently measured, for the convex functional Ω with subdifferential $\partial\Omega$, by means of Bregman distances

$$D_\xi(\tilde{u}, u) := \Omega(\tilde{u}) - \Omega(u) - \langle \xi, \tilde{u} - u \rangle_{U^*,U} , \quad \tilde{u} \in \mathcal{D}(\Omega) \subseteq U ,$$

at $u \in \mathcal{D}(\Omega) \subseteq U$ and $\xi \in \partial\Omega(u) \subseteq U^*$. The set

$$\mathcal{D}_B(\Omega) := \{u \in \mathcal{D}(\Omega) : \partial\Omega(u) \neq \emptyset\}$$

is called Bregman domain. For more details see, e.g., [17, Lemmas 3.16 and 3.17].

An element $u^\dagger \in \mathcal{D}$ is called an *Ω-minimizing solution* to (2.1) if

$$\Omega(u^\dagger) = \min \{\Omega(u) : F(u) = v, \; u \in \mathcal{D}\} < \infty .$$

Such Ω-minimizing solutions exist under Assumption 2.1 if (2.1) has a solution $u \in \mathcal{D}$ (see [17, Theorem 3.25]), and by [17, Theorem 3.26]

3 Convergence, convergence rates, the degree of nonlinearity, and variational inequalities

As the following proposition shows, all regularized solutions associated with data possessing a sufficiently small noise level δ belong to a common pre-compact level set of type $\mathcal{M}_\alpha^v(c)$ whenever the regularization parameters $\alpha = \alpha(\delta)$ are chosen such that weak convergence to Ω-minimizing solutions $u^\dagger$ is enforced.

Proposition 3.1 *Consider an a priori choice* $\alpha = \alpha(\delta) > 0,\ 0 < \delta < \infty$, *for the regularization parameter in (2.3) depending on the noise level* δ *such that*

$$\alpha(\delta) \to 0 \quad and \quad \frac{\delta^p}{\alpha(\delta)} \to 0, \quad as \quad \delta \to 0. \tag{3.1}$$

Then under Assumption 2.1 every sequence $\{u_n\}_{n=1}^\infty := \{u_{\alpha(\delta_n)}^{\delta_n}\}_{n=1}^\infty$ *of regularized solutions corresponding to a sequence* $\{v^{\delta_n}\}_{n=1}^\infty$ *of data with* $\lim\limits_{n\to\infty} \delta_n = 0$ *has a subsequence* $\{u_{n_k}\}_{k=1}^\infty$*, which is weakly convergent in* U*, i.e.* $u_{n_k} \rightharpoonup u^\dagger$*, and its limit* $u^\dagger$ *is an* Ω*-minimizing solution of (2.1) with* $\Omega(u^\dagger) = \lim\limits_{k\to\infty} \Omega(u_{n_k})$.

For given $\alpha_{max} > 0$ *let* $u^\dagger$ *denote an* Ω*-minimizing solution of (2.1). If we set*

$$\rho := 2^{p-1}\alpha_{max}(1 + \Omega(u^\dagger))\,, \tag{3.2}$$

then we have $u^\dagger \in \mathcal{M}_{\alpha_{max}}^v(\rho)$ *and there exists some* $\delta_{max} > 0$ *such that*

$$u_{\alpha(\delta)}^\delta \in \mathcal{M}_{\alpha_{max}}^v(\rho) \qquad for\ all \qquad 0 \le \delta \le \delta_{max}\,. \tag{3.3}$$

Proof: The first part of the proposition concerning convergence replicates only the result of [17, Theorem 3.26] and we refer to the proof ibidem. The second part can be proven as follows: Owing to (3.1) there exists some $\delta_{max} > 0$ such that $\alpha(\delta) \le \alpha_{max}$ and $\frac{\delta^p}{\alpha(\delta)} \le \frac{1}{2}$ for all $0 < \delta \le \delta_{max}$. Then for such δ, by writing for simplicity α instead of $\alpha(\delta)$, we have with $(a+b)^p \le 2^{p-1}(a^p + b^p)$ $(a, b \ge 0,\ p > 1)$ and $T_\alpha^v(u_\alpha^\delta) \le T_\alpha^v(u^\dagger)$ $(\alpha > 0)$ the estimate

$$T_{\alpha_{max}}^v(u_\alpha^\delta) \le 2^{p-1}\left[\left\|F(u_\alpha^\delta) - v^\delta\right\|_V^p + \delta^p + \alpha_{max}\Omega(u_\alpha^\delta)\right]$$

$$= 2^{p-1}\left[\left\|F(u_\alpha^\delta) - v^\delta\right\|_V^p + \alpha\Omega(u_\alpha^\delta) + (\alpha_{max} - \alpha)\Omega(u_\alpha^\delta) + \delta^p\right]$$

$$\le 2^{p-1}\left[T_\alpha^{v^\delta}(u^\dagger) + (\alpha_{max} - \alpha)\Omega(u_\alpha^\delta) + \delta^p\right] \le 2^{p-1}\left[\delta^p + \alpha\Omega(u^\dagger) + (\alpha_{max} - \alpha)\Omega(u_\alpha^\delta) + \delta^p\right].$$

Because of $\Omega(u_\alpha^\delta) \le \frac{\delta^p}{\alpha} + \Omega(u^\dagger)$ and $\frac{\delta^p}{\alpha} \le \frac{1}{2}$ this yields

$$T_{\alpha_{max}}^v(u_\alpha^\delta) \le 2^{p-1}\left[\delta^p + \alpha_{max}\frac{\delta^p}{\alpha} + \alpha_{max}\Omega(u^\dagger)\right] \le 2^{p-1}\left[2\alpha_{max}\frac{\delta^p}{\alpha} + \alpha_{max}\Omega(u^\dagger)\right] \le \rho$$

and hence proves (3.3). Evidently, it holds $T_{\alpha_{max}}^v(u^\dagger) = \alpha_{max}\Omega(u^\dagger) \le 2^{p-1}\alpha_{max}\Omega(u^\dagger)$ for all $p > 1$, which implies $u^\dagger \in \mathcal{M}_{\alpha_{max}}^v(\rho)$ and completes the proof. $\square$

For the analysis of nonlinear problems both the smoothness of Ω-minimizing solutions $u^\dagger$ and the smoothness of the forward operator F in a neighbourhood of $u^\dagger$ are essential ingredients. In this context, the term 'smoothness' has to be considered in a very general sense. With respect to the operator we recall the concept of a degree of nonlinearity from [8, Definition 2.5] which represents a Banach space update of Definition 1 from [10].

Definition 3.2 *Let $c_1, c_2 \geq 0$ and $c_1 + c_2 > 0$. We define F to be nonlinear of degree (c_1, c_2) for the Bregman distance $D_\xi(\cdot, u^\dagger)$ of Ω at a solution $u^\dagger \in \mathcal{D}_B(\Omega) \subseteq U$ of (2.1) with $\xi \in \partial\Omega(u^\dagger) \subseteq U^*$ if there is a constant $K > 0$ such that*

$$\left\|F(u) - F(u^\dagger) - F'(u^\dagger)(u - u^\dagger)\right\|_V \leq K \left\|F(u) - F(u^\dagger)\right\|_V^{c_1} D_\xi(u, u^\dagger)^{c_2} \tag{3.4}$$

for all $u \in \mathcal{M}^v_{\alpha_{max}}(\rho)$.

In recent publications the distinguished role of variational inequalities

$$\left\langle \xi, u^\dagger - u \right\rangle_{U^*,U} \leq \beta_1 D_\xi(u, u^\dagger) + \beta_2 \left\|F(u) - F(u^\dagger)\right\|_V^\kappa, \quad \text{for all} \quad u \in \mathcal{M}^v_{\alpha_{max}}(\rho) \tag{3.5}$$

with some $\xi \in \partial\Omega(u^\dagger)$, two multipliers $0 \leq \beta_1 < 1$, $\beta_2 \geq 0$ and an exponent $\kappa > 0$ for obtaining convergence rates was elaborated. The subsequent proposition outlines the chances of such variational inequalities for ensuring convergence rates in Tikhonov type regularization. Here we summarize convergence rates results from [8], [9], and [17, Section 3.2].

Proposition 3.3 *Assume that $F, \Omega, \mathcal{D}, U$ and V satisfy the Assumption 2.1 and that there is an Ω-minimizing solution from the Bregman domain $u^\dagger \in \mathcal{D}_B(\Omega)$. If there exist an element $\xi \in \partial\Omega(u^\dagger)$ and constants $0 \leq \beta_1 < 1$, $\beta_2 \geq 0$, and $0 < \kappa \leq 1$ such that the variational inequality (3.5) holds with ρ from (3.2), then we have the convergence rate*

$$D_\xi(u^\delta_{\alpha(\delta)}, u^\dagger) = \mathcal{O}(\delta^\kappa) \quad \text{as} \quad \delta \to 0 \tag{3.6}$$

for an a priori parameter choice $\alpha(\delta) \asymp \delta^{p-\kappa}$.

Proof: We write again for simplicity α instead of $\alpha(\delta)$ and note that the parameter choice rule $\alpha \asymp \delta^{p-\kappa}$ satisfies the condition (3.1) with the consequence that Proposition 3.1 is applicable. Then by using $T^{v^\delta}_\alpha(u^\delta_\alpha) \leq T^{v^\delta}_\alpha(u^\dagger)$, (2.2), and the definition of the Bregman distance we can estimate as follows:

$$\left\|F(u^\delta_\alpha) - v^\delta\right\|_V^p + \alpha D_\xi(u^\delta_\alpha, u^\dagger) \leq \delta^p + \alpha\left(\Omega(u^\dagger) - \Omega(u^\delta_\alpha) + D_\xi(u^\delta_\alpha, u^\dagger)\right) . \tag{3.7}$$

Moreover, by exploiting the inequality $(a+b)^\kappa \leq a^\kappa + b^\kappa$ $(a, b > 0,\ 0 < \kappa \leq 1)$ because of (3.3) we obtain from the variational inequality (3.5) that

$$\begin{aligned} &\Omega(u^\dagger) - \Omega(u^\delta_\alpha) + D_\xi(u^\delta_\alpha, u^\dagger) = -\left\langle \xi, u^\delta_\alpha - u^\dagger \right\rangle_{U^*,U} \\ &\leq \beta_1 D_\xi(u^\delta_\alpha, u^\dagger) + \beta_2 \left\|F(u^\delta_\alpha) - F(u^\dagger)\right\|_V^\kappa \\ &\leq \beta_1 D_\xi(u^\delta_\alpha, u^\dagger) + \beta_2 \left(\left\|F(u^\delta_\alpha) - v^\delta\right\|_V^\kappa + \delta^\kappa\right) . \end{aligned}$$

Therefore from (3.7) it follows that

$$\left\|F(u_\alpha^\delta) - v^\delta\right\|_V^p + \alpha D_\xi(u_\alpha^\delta, u^\dagger) \le \delta^p + \alpha\left(\beta_1 D_\xi(u_\alpha^\delta, u^\dagger) \;+\; \beta_2\left(\left\|F(u_\alpha^\delta) - v^\delta\right\|_V^\kappa + \delta^\kappa\right)\right). \quad (3.8)$$

Using the variant

$$a\,b \le \varepsilon\, a^{p_1} + \frac{b^{p_2}}{(\varepsilon\, p_1)^{p_2/p_1} p_2} \qquad (a, b \ge 0,\;\; \varepsilon > 0\;\; p_1, p_2 > 1 \;\text{ with }\; \frac{1}{p_1} + \frac{1}{p_1} = 1) \qquad (3.9)$$

of Young's inequality twice with $\varepsilon := 1$, $p_1 := p/\kappa$, $p_2 := p/(p-\kappa)$ and $b := \alpha\beta_2$, on the one hand with $a := \|F(u_\alpha^\delta) - u^\dagger\|_V^\kappa$ and on the other hand with $a := \delta^\kappa$, the inequality

$$\alpha D_\xi(u_\alpha^\delta, u^\dagger) \le 2\delta^p + \alpha\beta_1 D_\xi(u_\alpha^\delta, u^\dagger) + \frac{2(p-\kappa)}{(p/\kappa)^{\kappa/(p-\kappa)}\,p}(\alpha\beta_2)^{p/(p-\kappa)}$$

follows from (3.8). Because of $0 \le \beta_1 < 1$ this provides us with the estimate

$$D_\xi(u_\alpha^\delta, u^\dagger) \le \frac{2\delta^p + \frac{2(p-\kappa)}{(p/\kappa)^{\kappa/(p-\kappa)}\,p}\,(\alpha\,\beta_2)^{p/(p-\kappa)}}{\alpha\,(1-\beta_1)} \qquad (3.10)$$

for sufficiently small $\delta > 0$, which yields (3.6) for the a priori parameter choice $\alpha \asymp \delta^{p-\kappa}$ and proves the proposition. As a by-product from formula (3.10) we obtain for the case $\delta = 0$ of noiseless data the corresponding estimate

$$D_\xi(u_\alpha^0, u^\dagger) \le \hat{C}\,\alpha^{\frac{\kappa}{p-\kappa}} \qquad (3.11)$$

with some constant $\hat{C} > 0$. □

The Proposition 3.3 shows the formidable capability of variational inequalities (3.5) for obtaining convergence rates without any additional requirements on the solution smoothness and on the nonlinearity structure of the forward operator. In this sense, the validity of such variational inequality (3.5) on the associated level set embodies an advantageous combination of properties on $u^\dagger$ and F in a neighbourhood of $u^\dagger$. Necessary and sufficient conditions for (3.5) are given in the literature only in a fragmented manner, mostly expressing the interplay with classical source conditions. In the next two sections we discuss the limited variability of exponents $\kappa > 0$ in (3.5) and we analyze in a Hilbert space situation the interplay of general source conditions, the degree of nonlinearity and required variational inequalities.

4 A case distinction for the exponent in the variational inequality

We specify the general Assumption 2.1 to Assumption 4.1 by additional requirements for local use in this section.

Assumption 4.1

1. *$F, \Omega, \mathcal{D}, U$ and V satisfy the Assumption 2.1.*

2. *Let $u^\dagger \in \mathcal{D}$ be an Ω-minimizing solution of (2.1).*

3. *The operator F is Gâteaux differentiable in $u^\dagger$ with Gâteaux derivative $F'(u^\dagger)$.*

4. *The functional Ω is Gâteaux differentiable in $u^\dagger$ with Gâteaux derivative $\xi = \Omega'(u^\dagger)$, i.e., the subdifferential $\partial\Omega(u^\dagger) = \{\xi\}$ is a singleton.*

Remark 4.2 The Gâteaux differentiability of F and Ω in $u^\dagger$ implies that there is a ball $B_r(u^\dagger)$ with center $u^\dagger$ and radius $r > 0$ such that $B_r(u^\dagger) \subseteq \mathcal{D}$, i.e., $u^\dagger$ is an inner point of $\mathcal{D}(F)$ and $\mathcal{D}(\Omega)$.

Case $\boldsymbol{\kappa} > 1$:

The following proposition shows that exponents $\kappa > 1$ in the variational inequality (3.5) under Assumption 4.1 in principle cannot occur.

Proposition 4.3 *Under the Assumption 4.1 the variational inequality (3.5) cannot hold with $\xi = \Omega'(u^\dagger) \neq 0$ and multipliers $\beta_1, \beta_2 \geq 0$ whenever $\kappa > 1$.*

Proof: To prove the proposition we assume that the variational inequality (3.5) holds for $\xi = \Omega'(u^\dagger) \neq 0$ and some $\kappa > 1$ with multipliers $\beta_1, \beta_2 \geq 0$ and for all $u \in \mathcal{M}^v_{\alpha_{max}}(\rho)$. Then there is an element $u_\xi \in U$ with $\langle \xi, u_\xi \rangle_{U^*,U} > 0$ such that $u^\dagger - tu_\xi \in \mathcal{M}^v_{\alpha_{max}}(\rho)$ for all $0 \leq t \leq 1$. Hence we have for all $0 < t \leq 1$

$$0 < \langle \xi, tu_\xi \rangle_{U^*,U} \leq \beta_1 D_\xi(u^\dagger - tu_\xi, u^\dagger) \; + \; \beta_2 \left\| F(u^\dagger - tu_\xi) - F(u^\dagger) \right\|_V^\kappa$$

and dividing by $t > 0$

$$\langle \xi, u_\xi \rangle_{U^*,U} \leq \beta_1 \left[\frac{\Omega(u^\dagger - tu_\xi) - \Omega(u^\dagger)}{t} + \langle \xi, u_\xi \rangle_{U^*,U} \right] + \beta_2 \left\| \frac{F(u^\dagger - tu_\xi) - F(u^\dagger)}{t} \right\|_V \left\| F(u^\dagger - tu_\xi) - F(u^\dagger) \right\|_V^{\kappa-1}. \tag{4.1}$$

The left-hand side of inequality (4.1) is a positive constant. The right-hand side, however, tends to zero as $t \to 0$. Precisely, it holds $\lim\limits_{t\to 0} \frac{\Omega(u^\dagger - tu_\xi) - \Omega(u^\dagger)}{t} = -\langle \xi, u_\xi \rangle_{U^*,U}$, $\lim\limits_{t\to 0} \left\| \frac{F(u^\dagger - tu_\xi) - F(u^\dagger)}{t} \right\|_V = \|F'(u^\dagger) u_\xi\|_V < \infty$ and $\lim\limits_{t\to 0} \left\| F(u^\dagger - tu_\xi) - F(u^\dagger) \right\|_V^{\kappa-1} = 0$ because of the Gâteaux-differentiability of F and Ω in $u^\dagger$ taking into account that Gâteaux-differentiability of F at some point implies strong continuity of F in that point. This contradicts the assumption and proves the proposition. □

Case $\boldsymbol{\kappa} = 1$:

As the next proposition shows the variational inequality (3.5) is closely connected with the source condition $\xi \in \mathcal{R}(F'(u^\dagger)^*)$, where $\mathcal{R}(A)$ denotes the range of a linear operator A. The assertion a) of Proposition 4.4 repeats the Proposition 3.38 from [17], but reflects in contrast to the original the fact that the proof ibidem does not need the condition $\beta_1 < 1$. Note that the proof given there is similar to the proof of Proposition 4.3 presented above. On the other hand, for the assertion b) of Proposition 4.4 and its proof we refer to Proposition 3.35 in [17].

Proposition 4.4 *Under the Assumption 4.1 the following two assertions hold:*
a) The validity of a variational inequality

$$\langle \xi, u^\dagger - u \rangle_{U^*,U} \le \beta_1 D_\xi(u,u^\dagger) + \beta_2 \|F(u) - F(u^\dagger)\|_V \quad \text{for all} \quad u \in \mathcal{M}^v_{\alpha_{max}}(\rho) \qquad (4.2)$$

for $\xi = \Omega'(u^\dagger)$ and two multipliers $\beta_1, \beta_2 \ge 0$ implies the source condition

$$\xi \;=\; F'(u^\dagger)^*\, w, \qquad w \in V^*. \qquad (4.3)$$

b) Let F be nonlinear of degree $(0,1)$ for the Bregman distance $D_\xi(\cdot, u^\dagger)$ of Ω at $u^\dagger$, i.e., we have

$$\|F(u) - F(u^\dagger) - F'(u^\dagger)(u - u^\dagger)\|_V \;\le\; K\; D_\xi(u,u^\dagger) \qquad (4.4)$$

for a constant $K > 0$ and all $u \in \mathcal{M}^v_{\alpha_{max}}(\rho)$. Then the source condition (4.3) together with the smallness condition

$$K\, \|w\|_{V^*} \;<\; 1 \qquad (4.5)$$

imply the validity of a variational inequality (4.2) with $\xi = \Omega'(u^\dagger)$ and multipliers $0 \le \beta_1 = K\|w\|_{V^*} < 1,\; \beta_2 = \|w\|_{V^*} \ge 0$.

Case $0 < \kappa \le 1$:

The following proposition extends the result b) of Proposition 4.4 to a wider class of degrees of nonlinearity. The particular case $\kappa = 1$ discussed above occurs here only for the complementary situation $c_1 > 0$.

Proposition 4.5 *Under the Assumption 4.1 let F be nonlinear of degree (c_1, c_2) with $0 < c_1 \le 1$, $0 \le c_2 < 1$, $c_1 + c_2 \le 1$ for the Bregman distance $D_\xi(\cdot, u^\dagger)$ of Ω at $u^\dagger$, i.e., we have*

$$\|F(u) - F(u^\dagger) - F'(u^\dagger)(u - u^\dagger)\|_V \;\le\; K\, \|F(u) - F(u^\dagger)\|_V^{c_1}\, D_\xi(u,u^\dagger)^{c_2} \qquad (4.6)$$

for a constant $K > 0$ and all $u \in \mathcal{M}^v_{\alpha_{max}}(\rho)$. Then the source condition (4.3) without any additional condition implies the validity of a variational inequality (3.5) with

$$\kappa \;=\; \frac{c_1}{1 - c_2}\,, \qquad (4.7)$$

$\xi = \Omega'(u^\dagger)$ and multipliers $0 \le \beta_1 < 1,\; \beta_2 \ge 0$.

Proof: We can estimate for $u \in \mathcal{M}^v_{\alpha_{max}}(\rho)$

$$\langle \xi, u^\dagger - u \rangle_{U^*,U} = \langle F'(u^\dagger)^*\, w, u^\dagger - u \rangle_{U^*,U} = \langle w, F'(u^\dagger)(u^\dagger - u) \rangle_{V^*,V} \le \|w\|_{V^*} \|F'(u^\dagger)(u^\dagger - u)\|_V$$

$$\le \|w\|_{V^*} \|F(u) - F(u^\dagger) - F'(u^\dagger)(u - u^\dagger)\|_V + \|w\|_{V^*} \|F(u) - F(u^\dagger)\|_V$$

$$\le K \|w\|_{V^*} \|F(u) - F(u^\dagger)\|_V^{c_1}\; D_\xi(u,u^\dagger)^{c_2} + \|w\|_{V^*} \|F(u) - F(u^\dagger)\|_V.$$

Taking into account that $\|F(u) - F(u^\dagger)\|_V \le \rho^{1/p}$ for $u \in \mathcal{M}^v_{\alpha_{max}}(\rho)$ this implies for the case $c_2 = 0$ and $0 < c_1 \le 1$ the variational inequality (3.5) with $\beta_1 = 0, \beta_2 =$

$\|w\|_{V^*}(K+\rho^{\frac{1-c_1}{p}})$ and $\kappa = c_1$. On the other hand, for $0 < c_2 < 1$ and $0 < c_1 \le 1$ the variant (3.9) of Young's inequality with $p_1 := \frac{1}{c_2}$, $p_2 := \frac{1}{1-c_2}$, $\varepsilon := c_2$, $a := D_\xi(u,u^\dagger)^{c_2}$ and $b := K\|w\|_{V^*}\|F(u)-F(u^\dagger)\|_V^{c_1}$ yields here

$$K\|w\|_{V^*}\|F(u)-F(u^\dagger)\|_V^{c_1}\, D_\xi(u,u^\dagger)^{c_2} \le c_2\, D_\xi(u,u^\dagger)+(1-c_2)(K\|w\|_{V^*})^{\frac{1}{1-c_2}}\|F(u)-F(u^\dagger)\|_V^{\frac{c_1}{1-c_2}}.$$

and hence the validity of a variational inequality (3.5) with $\kappa = \frac{c_1}{1-c_2}$ and multipliers

$$0 \le \beta_1 = c_2 < 1, \quad \beta_2 = \rho^{\frac{1-\kappa}{p}}\|w\|_{V^*} + (1-c_2)K^{\frac{1}{1-c_2}}\|w\|_{V^*}^{\frac{1}{1-c_2}}.$$

This proves the proposition. □

Note that essential ingredients for this proposition and its proof have already been presented in [8, Lemma 3.1]. The proposition shows that the variational inequality (3.5) holds with the maximum exponent $\kappa = 1$ if either c_1 itself is maximal, i.e., $c_1 = 1$, or its defect in the case $0 < c_1 < 1$ can be compensated by $c_2 > 0$ whenever we have $c_1 + c_2 = 1$.

We close this section with three questions, which cannot be answered in a moment for the used general Banach space setting under consideration in this paper:
I. Are there alternative sufficient conditions for obtaining a variational inequality (3.5) with exponents $0 < \kappa < 1$ when ξ fails to satisfy a source condition (4.3)?
II. What combinations of c_1 and c_2 in the degree of nonlinearity do really occur?
In particular:
III. Are the degrees of nonlinearity (c_1, c_2) with $c_1 + c_2 > 1$ of interest?
These questions, however, will be partially answered in the subsequent Section 5 for the standard Tikhonov regularization in a Hilbert space setting.

5 Extended results for a Hilbert space situation

In Assumption 5.1 we specify now the requirements expressing the setting of this section.

Assumption 5.1

1. *Set $p := 2$ and let U, V be Hilbert spaces. Moreover, set $\Omega(u) := \|u-u^*\|_U^2$ with fixed reference element $u^* \in U$ and $\mathcal{D}(\Omega) = U$.*

2. *The operator F, $\mathcal{D}(F)$, $u^\dagger$ and ξ are chosen such that they satisfy togehther with U, V and Ω the Assumption 4.1.*

Remark 5.2 Under Assumption 5.1 the Ω-minimizing solutions and the classical u^*-minimum solutions (cf. [4, 5]) coincide. Moreover, we have $\mathcal{D} = \mathcal{D}(F)$ and for ξ and $D_\xi(\tilde{u}, u)$ the simple structure

$$\xi = 2(u^\dagger - u^*), \qquad D_\xi(\tilde{u}, u) = \|\tilde{u} - u\|_U^2 \tag{5.1}$$

with Bregman domain $\mathcal{D}_B(\Omega) = U$. Regularized solutions u_α^δ are minimizers over U of the classical Tikhonov functional of Hilbert space type

$$T_\alpha^{v^\delta}(u) := \|F(u) - v^\delta\|_V^2 + \alpha\,\|u - u^*\|_U^2$$

comprehensively studied in [4, Chapter 10].

Consequently, we have to specify the Definition 3.2 as follows:

Definition 5.3 *Let $c_1, c_2 \geq 0$ and $c_1 + c_2 \geq 0$. We define F to be nonlinear of degree (c_1, c_2) at a solution $u^\dagger \in \mathcal{D}(F)$ of (2.1) if there is a constant $K > 0$ such that*

$$\left\|F(u) - F(u^\dagger) - F'(u^\dagger)(u - u^\dagger)\right\|_V \;\leq\; K\,\left\|F(u) - F(u^\dagger)\right\|_V^{c_1}\,\|u - u^\dagger\|_U^{2c_2} \tag{5.2}$$

for all $u \in \mathcal{M}_{\alpha_{max}}^v(\rho)$.

Remark 5.4 In this Hilbert space setting we can formulate conditions for admissible pairs (c_1, c_2) in formula (5.2) of Definition 5.3 and study the smoothness background of such degrees of nonlinearity.

A sufficient condition for the classical case $c_1 = 0$, $c_2 = 1$, assumed for example in [17, Section 3.2]), is the Lipschitz continuity

$$\|F'(u) - F'(u^\dagger)\|_{\mathcal{L}(U,V)} \leq L\|u - u^\dagger\|_U$$

of F' for all u in a neighbourhood of $u^\dagger$. On the other hand, the case $c_1 = 1$, $c_2 = 0$ characterized by a tangential cone condition is frequently discussed in the theory of iterative regularization (cf. [4, Chapter 11] and [11]). In [8] the focus is on the case $c_1 > 0$, $0 < c_1 + c_2 \leq 1$, but it is well-known that numerous applications of ill-posed nonlinear inverse problems occur, where $c_1 = 1$, $c_2 = 1/2$ can be shown, i.e. $1 < c_1 + c_2 \leq 2$. We conjecture that the conditions

$$0 \leq c_1, c_2 \leq 1, \quad 0 < c_1 + 2c_2 \leq 2 \tag{5.3}$$

characterize all really occurring situations apart from singular cases.

As already mentioned in [10] the pairs (c_1, c_2) of the degree of nonlinearity are not necessarily uniquely determined. Namely, under a local Lipschitz condition

$$\|F(u) - F(u^\dagger)\|_V \;\leq\; C\,\|u - u^\dagger\|_U \tag{5.4}$$

for all u in a neighbourhood of $u^\dagger$ a degree (c_1, c_2) evidently implies the degree $(0, c_1/2 + c_2)$. Then $c_1 + 2c_2 > 2$ would lead to some $\varepsilon > 0$ such that

$$\left\|F(u) - F(u^\dagger) - F'(u^\dagger)(u - u^\dagger)\right\|_V \;\leq\; K\,\|u - u^\dagger\|_U^{2+\varepsilon}$$

for all u from appropriate level sets. If the operator F is continuously twice differentiable in a neighbourhood of $u^\dagger$ with bilinear operators $F''(u) : U \times U \to V$ using the integral representation of the second order Taylor remainder this would imply $\|F''(u^\dagger)(h, h)\|_V = 0$ for all $h \in U$ indicating a singular case.

Using similar arguments as in the proof of Proposition 4.3 we can easily see that for $F'(u^\dagger) \neq 0$ an inequality

$$\left\|F'(u^\dagger)(u-u^\dagger)\right\|_V \le K \left\|F(u)-F(u^\dagger)\right\|_V^{c_1}$$

cannot hold for all $u \in \mathcal{M}_{\alpha_{max}}(\rho)$ whenever $c_1 > 1$. Then because of $c_1 - 1 > 0$ an ansatz $u := u^\dagger + th$ with $h \neq 0$ and $\|h\|_U$ sufficiently small would after division by $t > 0$ lead to

$$\left\|F'(u^\dagger)h\right\|_V \le K \left\|F'(u^\dagger)h\right\|_V \lim_{t\to 0} \left\|F(u)-F(u^\dagger)\right\|_V^{c_1-1} = 0$$

in the limit case for $t \to 0$. However, we have no stringent proof for the limitation $c_1 \le 1$ which shows that in regular cases of nonlinear operators F

$$\left\|F(u)-F(u^\dagger)-F'(u^\dagger)(u-u^\dagger)\right\|_V \le K \left\|F(u)-F(u^\dagger)\right\|_V^{c_1}$$

will not hold for all $u \in \mathcal{M}_{\alpha_{max}}(\rho)$ whenever $c_1 > 1$.

Proposition 5.5 *Under the Assumption 5.1 let the operator F mapping between the Hilbert spaces U and V be nonlinear of degree (c_1, c_2) at $u^\dagger$ with $c_1 > 0$ and let $\xi = 2(u^\dagger - u^*)$ satisfy the general source condition*

$$\xi = (F'(u^\dagger)^* F'(u^\dagger))^{\eta/2} w, \qquad 0 < \eta < 1, \quad w \in U. \tag{5.5}$$

Then we have the variational inequality (3.5) with exponent

$$\kappa = \min\left\{ \frac{2\eta c_1}{1+\eta(1-2c_2)}, \frac{2\eta}{1+\eta} \right\} \tag{5.6}$$

for all $u \in \mathcal{M}_{\alpha_{max}}(\rho)$ and multipliers $0 \le \beta_1 < 1$, $\beta_2 \ge 0$.

Proof: Under the general source condition (5.5) we can estimate for all $u \in \mathcal{M}_{\alpha_{max}}(\rho)$ with the interpolation inequality [4, formula (2.49), p. 47]

$$\left\langle \xi, u^\dagger - u\right\rangle_U \le \left\langle w, (F'(u^\dagger)^* F'(u^\dagger))^{\eta/2}(u^\dagger - u)\right\rangle_U$$

$$\le \|w\|_U \|(F'(u^\dagger)^* F'(u^\dagger))^{\eta/2}(u^\dagger-u)\|_U^{\eta} \|u^\dagger - u\|_U^{1-\eta} = \|w\|_U \|F'(u^\dagger)(u^\dagger-u)\|_V^{\eta} \|u^\dagger-u\|_U^{1-\eta},$$

where $\langle \cdot, \cdot \rangle_U$ denotes the inner product in the Hilbert space U. Now we use the degree of nonlinearity in order to estimate the term $\|F'(u^\dagger)(u^\dagger - u)\|_V^{\eta}$ from above for $u \in \mathcal{M}_{\alpha_{max}}(\rho)$. Owing to $0 < \eta < 1$ we have

$$\|F'(u^\dagger)(u^\dagger-u)\|_V^{\eta} \le \|F(u)-F(u^\dagger)-F'(u^\dagger)(u-u^\dagger)\|_V^{\eta} + \|F(u^\dagger)-F(u)\|_V^{\eta}$$

and hence with some constants $K_1, K_2 > 0$

$$\left\langle \xi, u^\dagger - u\right\rangle_U \le \|w\|_U \left(\|F(u)-F(u^\dagger)-F'(u^\dagger)(u-u^\dagger)\|_V^{\eta} + \|F(u^\dagger)-F(u)\|_V^{\eta}\right) \|u^\dagger - u\|_U^{1-\eta}$$

$$\le K_1 \|F(u^\dagger)-F(u)\|_V^{c_1\eta} \|u^\dagger-u\|_U^{1-\eta+2c_2\eta} + K_2 \|F(u^\dagger)-F(u)\|_V^{\eta} \|u^\dagger-u\|_U^{1-\eta}.$$

Applying again Young's inequality (3.9) twice with $\varepsilon := 1/4$ such that terms $\frac{1}{4}\|u^\dagger - u\|_U^2$ occur in a sum with powers of $\|F(u^\dagger) - F(u)\|_V$ we obtain with some constants $C_1, C_2 > 0$

$$\langle \xi, u^\dagger - u \rangle_U \le \frac{1}{2}\|u^\dagger - u\|_U^2 + C_1 \|F(u^\dagger) - F(u)\|_V^{\frac{2c_1\eta}{1+\eta(1-2c_2)}} + C_2 \|F(u^\dagger) - F(u)\|_V^{\frac{2\eta}{1+\eta}}.$$

Taking into account that there is a constant $\bar{K} > 0$ such that $\|F(u^\dagger) - F(u)\|_V \le \bar{K}$ for all $u \in \mathcal{M}_{\alpha_{max}}(\rho)$ we have the variational inequality

$$\langle \xi, u^\dagger - u \rangle_U \le \frac{1}{2}\|u^\dagger - u\|_U^2 + \beta_2 \|F(u^\dagger) - F(u)\|_V^\kappa$$

with κ from (5.6) for all such u and some constant $\beta_2 > 0$. This completes the proof. □

Remark 5.6 An exponent $\kappa = \frac{2\eta}{1+\eta}$ in Proposition 5.5 indicates order optimal convergence rates with respect to the general source condition (5.5). This is the case if the condition

$$1 + \eta(1 - 2c_2 - c_1) \le c_1 \tag{5.7}$$

already occurring in [10] is satisfied. Note that the condition (5.7) holds for $0 < \eta < 1$ only if either $c_1 = 1$ or for $0 < c_1 < 1$ if $c_1 + c_2 > 1$ and η is large enough.

The author has no general answer to the question whether one can formulate converse assertions concluding in the Hilbert space setting from a variational inequality (3.5) with exponents $0 < \kappa < 1$ and nonlinear forward operator F to Hölder source conditions of type (5.5). However, for the subcase of a continuous linear operator

$$F := A \in \mathcal{L}(U, V) \tag{5.8}$$

we can prove a converse result in the following proposition (cf. also [3, Section 3] with respect to approximate source conditions). Since the conditions $\xi \in \mathcal{R}(F'(u^\dagger)^*)$ and $\xi \in \mathcal{R}((F'(u^\dagger)^* F'(u^\dagger))^{1/2})$ are equivalent this result complements for the subcase the assertion a) of our Proposition 4.4 and of Proposition 3.38 in [17] which just handle the case $\kappa = 1$. We should mention here that for linear operators (5.8) no structural condition (degree of nonlinearity) is required and Proposition 5.5 always yields the implication from

$$\xi = (A^*A)^{\eta/2} w, \qquad 0 < \eta < 1, \quad w \in U, \tag{5.9}$$

to a variational inequality

$$\langle \xi, u^\dagger - u \rangle_U \le \beta_1 \|u^\dagger - u\|_U^2 \; + \; \beta_2 \big\| A(u^\dagger - u) \big\|_V^\kappa \tag{5.10}$$

with exponent

$$\kappa \; = \; \frac{2\eta}{1+\eta} \in (0, 1) \tag{5.11}$$

for all $u \in \mathcal{M}_{\alpha_{max}}(\rho)$ and multipliers $0 \le \beta_1 < 1$, $\beta_2 \ge 0$.

Proposition 5.7 *If for linear forward operators (5.8) a variational inequality (5.10) holds for all* $u \in \mathcal{M}_{\alpha_{max}}(\rho)$ *and multipliers* $0 \le \beta_1 < 1$, $\beta_2 \ge 0$, *then under Assumption 5.1 a general source condition (5.9) is valid for all exponents* $\eta > 0$ *satisfying the inequality* $\eta < \frac{\kappa}{2-\kappa}$.

Proof: Under Assumption 5.1 we obtain from (5.10) and (3.11) for noiseless data the estimate

$$\|u_\alpha^0 - u^\dagger\|_U \le \hat{C}\, \alpha^{\frac{\kappa}{2(2-\kappa)}}.$$

This allows us to apply the converse result of [13] for linear Tikhonov regularization which provides a Hölder source condition (5.9) for all exponents $\eta > 0$ satisfying the inequality $\eta < \frac{\kappa}{2-\kappa}$. This completes the proof. □

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