On the structure of the eigenvectors of large Hermitian Toeplitz band matrices

A. Böttcher, S. M. Grudsky[1], and E. A. Maksimenko

The paper is devoted to the asymptotic behavior of the eigenvectors of banded Hermitian Toeplitz matrices as the dimension of the matrices increases to infinity. The main result, which is based on certain assumptions, describes the structure of the eigenvectors in terms of the Laurent polynomial that generates the matrices up to an error term that decays exponentially fast. This result is applicable to both extreme and inner eigenvectors.

Keywords: Toeplitz matrix, eigenvector, asymptotic expansions

Mathematics Subject Classification (2000): Primary 47B35; Secondary 15A18, 41A25, 65F15

1 Introduction and main results

Given a function a in L^1 on the complex unit circle $\mathbf{T}$, we denote by a_ℓ the ℓth Fourier coefficient,

$$a_\ell = \frac{1}{2\pi}\int_0^{2\pi} a(e^{ix})e^{-i\ell x}dx \quad (\ell \in \mathbf{Z}),$$

and by $T_n(a)$ the $n \times n$ Toeplitz matrix $(a_{j-k})_{j,k=1}^n$. We assume that a is real-valued, in which case the matrices $T_n(a)$ are all Hermitian. Let

$$\lambda_1^{(n)} \le \lambda_2^{(n)} \le \ldots \le \lambda_n^{(n)}$$

be the eigenvalues of $T_n(a)$ and let

$$\{v_1^{(n)}, v_2^{(n)}, \ldots, v_n^{(n)}\}$$

[1]This work was partially supported by CONACYT project 80503, Mexico.

be an orthonormal basis of eigenvectors such that $T_n(a)v_j^{(n)} = \lambda_j^{(n)} v_j^{(n)}$. The present paper is dedicated to the asymptotic behavior of the eigenvectors $v_j^{(n)}$ as $n \to \infty$.

To get an idea of the kind of results we will establish, consider the function $a(e^{ix}) = 2 - 2\cos x$. The range $a(\mathbf{T})$ is the segment $[0, 4]$. It is well known that the eigenvalues and eigenvectors of $T_n(a)$ are given by

$$\lambda_j^{(n)} = 2 - 2\cos\frac{\pi j}{n+1}, \quad x_j^{(n)} = \sqrt{\frac{2}{n+1}} \left(\sin\frac{m\pi j}{n+1}\right)_{m=1}^{n}. \tag{1}$$

(We denote the eigenvectors in this reference case by $x_j^{(n)}$ and reserve the notation $v_j^{(n)}$ for the general case.) Let φ be the function

$$\varphi : [0, 4] \to [0, \pi], \quad \varphi(\lambda) = \arccos\frac{2-\lambda}{2}.$$

We have $\varphi(\lambda_j^{(n)}) = \pi j/(n+1)$ and hence, apart from the normalization factor $\sqrt{2/(n+1)}$, $x_{j,m}^{(n)}$ is the value of $\sin(m\varphi(\lambda))$ at $\lambda = \lambda_j^{(n)}$. In other words, an eigenvector for λ is given by $(\sin(m\varphi(\lambda)))_{m=1}^{n}$. A speculative question is whether in the general case we can also find functions Ω_m such that, at least asymptotically, $(\Omega_m(\lambda))_{m=1}^{n}$ is an eigenvector for λ. It turns out that this is in general impossible but that after a slight modification the answer to the question is in the affirmative. Namely, we will prove that, under certain assumptions, there are functions Ω_m, Φ_m and real-valued functions σ, η such that an eigenvector for $\lambda = \lambda_j^{(n)}$ is always of the form

$$\left(\Omega_m(\lambda) + \Phi_m(\lambda) + (-1)^{j+1} e^{-i(n+1)\sigma(\lambda)} e^{-i\eta(\lambda)} \overline{\Phi_{n+1-m}(\lambda)} + \text{error term}\right)_{m=1}^{n}. \tag{2}$$

The error term will be shown to decrease to zero exponentially fast and uniformly in j and m as $n \to \infty$. Moreover, we will show that $\Omega_m(\lambda)$ is an oscillating function of m for each fixed λ and that $\Phi_m(\lambda)$ decays exponentially fast to zero as $m \to \infty$ for each λ (which means that $\Phi_{n+1-m}(\lambda)$ is an exponentially increasing function of m for each λ). Finally, it will turn out that

$$\sum_{m=1}^{n} |\Phi_m(\lambda)|^2 \Big/ \sum_{m=1}^{n} |\Omega_m(\lambda)|^2 = O\left(\frac{1}{n}\right)$$

as $n \to \infty$, uniformly in λ. Thus, the dominant term in (2) is $\Omega_m(\lambda)$, while the terms containing $\Phi_m(\lambda)$ and $\Phi_{n+1-m}(\lambda)$ may be viewed as twin babies.

If a is also an even function, $a(e^{ix}) = a(e^{-ix})$ for all x, then all the matrices $T_n(a)$ are real and symmetric. In [4], we conjectured that then, again under additional but reasonable assumptions, the appropriately rotated extreme eigenvectors $v_j^{(n)}$ are all close to the vectors $x_j^{(n)}$. To be more precise, we conjectured that if $n \to \infty$ and j (or $n-j$) remains fixed, then there are complex numbers $\tau_j^{(n)}$ of modulus 1 such that

$$\left\| \tau_j^{(n)} v_j^{(n)} - x_j^{(n)} \right\|_2 = o(1), \tag{3}$$

where $\|\cdot\|_2$ is the ℓ^2 norm. Several results related to this conjecture were established in [3] and [4]. We here prove this conjecture under assumptions that will be specified in the following paragraph. We will even be able to show that the $o(1)$ in (3) is $O(j/n)$ if $j/n \to 0$ and $O(1-j/n)$ if $j/n \to 1$.

Throughout what follows we assume that a is a Laurent polynomial

$$a(t) = \sum_{k=-r}^{r} a_k t^k \quad (t = e^{ix} \in \mathbf{T})$$

with $r \geq 2$, $a_r \neq 0$, and $\overline{a_k} = a_{-k}$ for all k. The last condition means that a is real-valued on $\mathbf{T}$. We assume without loss of generality that $a(\mathbf{T}) = [0, M]$ with $M > 0$ and that $a(1) = 0$ and $a(e^{i\varphi_0}) = M$ for some $\varphi_0 \in (0, 2\pi)$. We require that the function $g(x) := a(e^{ix})$ is strictly increasing on $(0, \varphi_0)$ and strictly decreasing on $(\varphi_0, 2\pi)$ and that the second derivatives of g at $x = 0$ and $x = \varphi_0$ are nonzero. Finally, we denote by $[\alpha, \beta] \subset [0, M]$ a segment such that if $\lambda \in [\alpha, \beta]$, then the $2r-2$ zeros of the Laurent polynomial $a(z) - \lambda$ that lie in $\mathbf{C} \setminus \mathbf{T}$ are pairwise distinct.

Note that we exclude the case $r = 1$, because in this case the eigenvalues and eigenvectors of $T_n(a)$ are explicitly available. Also notice that if $r = 2$, which is the case of pentadiagonal matrices, then for every $\lambda \in [0, M]$ the polynomial $a(z) - \lambda$ has two zeros on $\mathbf{T}$, one zero outside $\mathbf{T}$, and one zero inside $\mathbf{T}$. Thus, in this situation the last requirement of the previous paragraph is automatically satisfied for $[\alpha, \beta] = [0, M]$.

The asymptotic behavior of the extreme eigenvalues and eigenvectors of $T_n(a)$, that is, of $\lambda_j^{(n)}$ and $v_j^{(n)}$ when j or $n-j$ remain fixed, has been studied by several authors. As for extreme eigenvalues, the pioneering works are [7], [9], [11], [12], [18], while recent papers on the subject include [3], [6], [8], [10], [13], [14], [15], [19], [20]. See also the books [1] and [5]. Much less is known about the asymptotics of the eigenvectors. Part of the results of [4] and [19] may be interpreted as results on the behavior of the eigenvectors "in the mean" on the one hand and as insights into what happens if eigenvectors are replaced by pseudomodes on the other. In [3], we investigated the asymptotics of the extreme eigenvectors of certain Hermitian (and not necessarily banded) Toeplitz matrices. Our paper [2] may be considered as a first step to the understanding of the asymptotic behavior of individual inner eigenvalues of Toeplitz matrices. In the same vein, this paper intends to understand the nature of individual eigenvectors as part of the whole, independently of whether they are extreme or inner ones.

To state our main results, we need some notation. Let $\lambda \in [0, M]$. Then there are uniquely defined $\varphi_1(\lambda) \in [0, \varphi_0]$ and $\varphi_2(\lambda) \in [\varphi_0 - 2\pi, 0]$ such that

$$g(\varphi_1(\lambda)) = g(\varphi_2(\lambda)) = \lambda;$$

recall that $g(x) := a(e^{ix})$. We put

$$\varphi(\lambda) = \frac{\varphi_1(\lambda) - \varphi_2(\lambda)}{2}, \quad \sigma(\lambda) = \frac{\varphi_1(\lambda) + \varphi_2(\lambda)}{2}.$$

We have

$$\begin{aligned} a(z) - \lambda &= z^{-r}\Big(a_r z^{2r} + \ldots + (a_0 - \lambda)z^r + \ldots + a_{-r}\Big) \\ &= a_r z^{-r} \prod_{k=1}^{2r} (z - z_k(\lambda)), \end{aligned}$$

and our assumptions imply that we can label the zeros $z_k(\lambda)$ so that the collection $\mathcal{Z}(\lambda)$ of the zeros may be written as

$$\begin{aligned} &\{z_1(\lambda), \ldots, z_{r-1}(\lambda), z_r(\lambda), z_{r+1}(\lambda), z_{r+2}(\lambda), \ldots, z_{2r}(\lambda)\} \\ &= \{u_1(\lambda), \ldots, u_{r-1}(\lambda), e^{i\varphi_1(\lambda)}, e^{i\varphi_2(\lambda)}, 1/\overline{u}_1(\lambda), \ldots, 1/\overline{u}_{r-1}(\lambda)\} \end{aligned} \tag{4}$$

where $|u_\nu(\lambda)| > 1$ for $1 \le \nu \le r - 1$ and each $u_\nu(\lambda)$ depends continuously on $\lambda \in [0, M]$. Here and in similar places below we write $\overline{u}_k(\lambda) := \overline{u_k(\lambda)}$. We define $\delta_0 > 0$ by

$$e^{\delta_0} = \min_{\lambda \in [0,M]} \min_{1 \le \nu \le r-1} |u_\nu(\lambda)|.$$

Throughout the following, δ stands for any number in $(0, \delta_0)$. Further, we denote by h_λ the function

$$h_\lambda(z) = \prod_{\nu=1}^{r-1} \left(1 - \frac{z}{u_\nu(\lambda)}\right).$$

The function $\Theta(\lambda) = h_\lambda(e^{i\varphi_1(\lambda)})/h_\lambda(e^{i\varphi_2(\lambda)})$ is continuous and nonzero on $[0, M]$ and we have $\Theta(0) = \Theta(M) = 1$. In [2], it was shown that the closed curve

$$[0, M] \to \mathbf{C} \setminus \{0\}, \quad \lambda \mapsto \Theta(\lambda)$$

has winding number zero. Let $\theta(\lambda)$ be the continuous argument of $\Theta(\lambda)$ for which $\theta(0) = \theta(M) = 0$.

In [2], we proved that if n is large enough, then the function

$$f_n : [0, M] \to [0, (n + 1)\pi], \quad f_n(\lambda) = (n + 1)\varphi(\lambda) + \theta(\lambda)$$

is bijective and increasing and that if $\lambda_{j,*}^{(n)}$ is the unique solution of the equation $f_n(\lambda_{j,*}^{(n)}) = \pi j$, then the eigenvalues $\lambda_j^{(n)}$ satisfy

$$|\lambda_j - \lambda_{j,*}^{(n)}| \le K\, e^{-\delta n}$$

for all $j \in \{1, \ldots, n\}$, where K is a finite constant depending only on a. Thus, we have

$$(n+1)\varphi(\lambda_j^{(n)}) + \theta(\lambda_j^{(n)}) = \pi j + O(e^{-\delta n}), \tag{5}$$

uniformly in $j \in \{1, \ldots, n\}$.

Now take λ from (α, β). For $j \in \{1, \ldots, n\}$ and $\nu \in \{1, \ldots, r-1\}$, we put

$$\begin{aligned}
A(\lambda) &= \frac{e^{i\sigma(\lambda)}}{2i\, h_\lambda(e^{i\varphi_1(\lambda)})}, \quad B(\lambda) = \frac{e^{i\sigma(\lambda)}}{2i\, h_\lambda(e^{i\varphi_2(\lambda)})}, \\
D_\nu(\lambda) &= \frac{e^{2i\sigma(\lambda)} \sin\varphi(\lambda)}{(u_\nu(\lambda) - e^{i\varphi_1(\lambda)})(u_\nu(\lambda) - e^{i\varphi_2(\lambda)}) h'_\lambda(u_\nu(\lambda))}, \\
F_\nu(\lambda) &= \frac{\sin\varphi(\lambda)}{(\overline{u}_\nu(\lambda) - e^{-i\varphi_1(\lambda)})(\overline{u}_\nu(\lambda) - e^{-i\varphi_2(\lambda)}) \overline{h'_\lambda(u_\nu(\lambda))}} \times \\
&\quad \times \frac{|h_\lambda(e^{i\varphi_1(\lambda)}) h_\lambda(e^{i\varphi_2(\lambda)})|}{h_\lambda(e^{i\varphi_1(\lambda)}) h_\lambda(e^{i\varphi_2(\lambda)})}
\end{aligned}$$

and define the vector $w_j^{(n)}(\lambda) = (w_{j,m}^{(n)}(\lambda))_{m=1}^n$ by

$$\begin{aligned}
w_{j,m}^{(n)}(\lambda) &= A(\lambda) e^{-im\varphi_1(\lambda)} - B(\lambda) e^{-im\varphi_2(\lambda)} \\
&\quad + \sum_{\nu=1}^{r-1} \left(D_\nu(\lambda) \frac{1}{u_\nu(\lambda)^m} + F_\nu(\lambda) \frac{(-1)^{j+1} e^{-i(n+1)\sigma(\lambda)}}{\overline{u}_\nu(\lambda)^{n+1-m}} \right).
\end{aligned}$$

The assumption that zeros $u_\nu(\lambda)$ are all simple guarantees that $h'(u_\nu) \neq 0$. We denote by $\|\cdot\|_2$ and $\|\cdot\|_\infty$ the ℓ^2 and ℓ^∞ norms on $\mathbf{C}^n$, respectively.

Here are our main results.

Theorem 1.1 *As $n \to \infty$ and if $\lambda_j^{(n)} \in (\alpha, \beta)$,*

$$\|w_j^{(n)}(\lambda_j^{(n)})\|_2^2 = \frac{n}{4} \left(\frac{1}{|h_\lambda(e^{i\varphi_1(\lambda)})|^2} + \frac{1}{|h_\lambda(e^{i\varphi_2(\lambda)})|^2} \right)\Bigg|_{\lambda = \lambda_j^{(n)}} + O(1),$$

uniformly in j.

Theorem 1.2 *Let $n \to \infty$ and suppose $\lambda_j^{(n)} \in (\alpha, \beta)$. Then the eigenvectors $v_j^{(n)}$ are of the form*

$$v_j^{(n)} = \tau_j^{(n)} \left(\frac{w_j^{(n)}(\lambda_j^{(n)})}{\|w_j^{(n)}(\lambda_j^{(n)})\|_2} + O_\infty(e^{-\delta n}) \right)$$

where $\tau_j^{(n)} \in \mathbf{T}$ and $O_\infty(e^{-\delta n})$ denotes vectors $\xi_j^{(n)} \in \mathbf{C}^n$ such that $\|\xi_j^{(n)}\|_\infty \leq Ke^{-\delta n}$ for all j and n with some finite constant K independent of j and n.

Note that the previous theorem gives (2) with

$$\Omega_m(\lambda) = A(\lambda)e^{-im\varphi_1(\lambda)} - B(\lambda)e^{-im\varphi_2(\lambda)}, \quad \Phi_m(\lambda) = \sum_{\nu=1}^{r-1} \frac{D_\nu(\lambda)}{u_\nu(\lambda)^m},$$

$$e^{-i\eta(\lambda)} = \frac{|h_\lambda(e^{i\varphi_1(\lambda)})h_\lambda(e^{i\varphi_2(\lambda)})|}{h_\lambda(e^{i\varphi_1(\lambda)})h_\lambda(e^{i\varphi_2(\lambda)})}.$$

Things can be a little simplified for symmetric matrices. Thus, suppose all a_k are real and $a_k = a_{-k}$ for all k. We will show that then $\{u_1(\lambda), \ldots, u_{r-1}(\lambda)\} = \{\overline{u}_1(\lambda), \ldots, \overline{u}_{r-1}(\lambda)\}$. Put

$$Q_\nu(\lambda) = \frac{|h_\lambda(e^{i\varphi(\lambda)})| \sin\varphi(\lambda)}{(u_\nu(\lambda) - e^{i\varphi(\lambda)})(u_\nu(\lambda) - e^{-i\varphi(\lambda)})h'_\lambda(u_\nu(\lambda))}$$

and let $y_j^{(n)}(\lambda) = (y_{j,m}^{(n)}(\lambda))_{m=1}^n$ be given by

$$y_{j,m}^{(n)}(\lambda) = \sin\left(m\varphi(\lambda) + \frac{\theta(\lambda)}{2}\right) - \sum_{\nu=1}^{r-1} Q_\nu(\lambda)\left(\frac{1}{u_\nu(\lambda)^m} + \frac{(-1)^{j+1}}{u_\nu(\lambda)^{n+1-m}}\right). \qquad (6)$$

Theorem 1.3 *Let $n \to \infty$ and suppose $\lambda_j^{(n)} \in (\alpha, \beta)$. If $a_k = a_{-k}$ for all k, then*

$$\|y_j^{(n)}(\lambda_j^{(n)})\|_2^2 = \frac{n}{2} + O(1)$$

uniformly in j, and the eigenvectors $v_j^{(n)}$ are of the form

$$v_j^{(n)} = \tau_j^{(n)}\left(\frac{y_j^{(n)}(\lambda_j^{(n)})}{\|y_j^{(n)}(\lambda_j^{(n)})\|_2} + O_\infty(e^{-\delta n})\right)$$

where $\tau_j^{(n)} \in \mathbf{T}$ and $O_\infty(e^{-\delta n})$ is as in the previous theorem.

Let J be the $n \times n$ matrix with ones on the counterdiagonal and zeros elsewhere. Thus, $(Jv)_m = v_{n+1-m}$. A vector v is called symmetric if $Jv = v$ and skew-symmetric if $Jv = -v$. Trench [17] showed that the eigenvectors $v_1^{(n)}, v_3^{(n)}, \ldots$ are all symmetric and that the eigenvectors $v_2^{(n)}, v_4^{(n)}, \ldots$ are all skew-symmetric. From (5) we infer that

$$\sin\left((n+1-m)\varphi(\lambda_j^{(n)}) + \frac{\theta(\lambda_j^{(n)})}{2}\right)$$
$$= (-1)^{j+1} \sin\left(m\varphi(\lambda_j^{(n)}) + \frac{\theta(\lambda_j^{(n)})}{2}\right) + O(e^{-\delta n})$$

and hence (6) implies that

$$(Jy_j^{(n)}(\lambda_j^{(n)}))_m = (-1)^{j+1} y_{j,m}^{(n)}(\lambda_j^{(n)}) + O(e^{-\delta n}).$$

Consequently, apart from the term $O(e^{-\delta n})$, the vectors $y_j^{(n)}(\lambda_j^{(n)})$ are symmetric for $j = 1, 3, \ldots$ and skew-symmetric for $j = 2, 4, \ldots$. This is in complete accordance with Trench's result.

Due to (5), we also have

$$\sin\left(m\varphi(\lambda_j^{(n)}) + \frac{\theta(\lambda_j^{(n)})}{2}\right) = \sin\left(\left(m - \frac{n+1}{2}\right)\varphi(\lambda_j^{(n)})\right) + O(e^{-\delta n}).$$

Thus, Theorem 1.3 remains valid with (6) replaced by

$$\begin{aligned} y_{j,m}^{(n)}(\lambda) &= \sin\left(\left(m - \frac{n+1}{2}\right)\varphi(\lambda) + \frac{\pi j}{2}\right) \\ &\quad - \sum_{\nu=1}^{r-1} Q_\nu(\lambda)\left(\frac{1}{u_\nu(\lambda)^m} + \frac{(-1)^{j+1}}{u_\nu(\lambda)^{n+1-m}}\right). \end{aligned} \tag{7}$$

In this expression, the function θ has disappeared.

Define $y_j^{(n)}$ again by (6). The following theorem in conjunction with Theorem 1.3 proves (3).

Theorem 1.4 *Let $n \to \infty$ and suppose $\lambda_j^{(n)} \in (\alpha, \beta)$. If $a_k = a_{-k}$ for all k, then*

$$\left\| \frac{y_j^{(n)}(\lambda_j^{(n)})}{\|y_j^{(n)}(\lambda_j^{(n)})\|_2} - x_j^{(n)} \right\|_2 = O\left(\frac{j}{n}\right).$$

The rest of the paper is as follows. We approach eigenvectors by using the elementary observation that if λ is an eigenvalue of $T_n(a)$, then every nonzero column of the adjugate matrix of $T_n(a) - \lambda I = T_n(a - \lambda)$ is an eigenvector for λ. In Section 2 we employ "exact" formulas by Trench and Widom for the inverse and the determinant of a banded Toeplitz matrix to get a representation of the first column of the adjugate matrix of $T_n(a-\lambda)$ that will be convenient for asymptotic analysis. This analysis is carried out in Section 3. On the basis of these results, Theorems 1.1 and 1.2 are proved in Section 4, while the proofs of Theorems 1.3 and 1.4 are given in Section 4. Section 6 contains numerical results.

2 The first column of the adjugate matrix

The adjugate matrix $\operatorname{adj} B$ of an $n \times n$ matrix $B = (b_{jk})_{j,k=1}^n$ is defined by

$$(\operatorname{adj} B)_{jk} = (-1)^{j+k} \det M_{kj}$$

where M_{kj} is the $(n-1)\times(n-1)$ matrix that results from B by deleting the kth row and the jth column. We have

$$(A-\lambda I)\,\mathrm{adj}\,(A-\lambda I) = (\det(A-\lambda I))I.$$

Thus, if λ is an eigenvalue of A, then each nonzero column of $\mathrm{adj}\,(A-\lambda I)$ is an eigenvector. For an invertible matrix B,

$$\mathrm{adj}\,B = (\det B)B^{-1}. \tag{8}$$

Formulas for $\det T_n(b)$ and $T_n^{-1}(b)$ were established by Widom [18] and Trench [16], respectively. The purpose of this section is to transform Trench's formula for the first column of $T_n^{-1}(b)$ into a form that will be convenient for further analysis.

Theorem 2.1 *Let*

$$b(t) = \sum_{k=-p}^{q} b_k t^k = b_p t^{-q} \prod_{j=1}^{p+q}(t-z_j) \quad (t\in\mathbf{T})$$

where $p\ge 1$, $q\ge 1$, $b_p\ne 0$, and $z_1,\ldots,z_{p+q}$ are pairwise distinct nonzero complex numbers. If $n>p+q$ and $1\le m\le n$, then the mth entry of the first column of of $\mathrm{adj}\,T_n(b)$ *is*

$$[\mathrm{adj}\,T_n(b)]_{m,1} = \sum_{J\subset\mathcal{Z},|J|=p} C_J W_J^n \sum_{z\in J} S_{m,J,z} \tag{9}$$

where $\mathcal{Z}=\{z_1,\ldots,z_{p+q}\}$, the sum is over all sets $J\subset\mathcal{Z}$ of cardinality p, and, with $\overline{J}:=\mathcal{Z}\setminus J$,

$$C_J = \prod_{z\in J} z^q \prod_{z\in J, w\in\overline{J}} \frac{1}{z-w}, \quad W_J = (-1)^p b_p \prod_{z\in J} z,$$

$$S_{m,J,z} = -\frac{1}{b_p}\frac{1}{z^m}\prod_{w\in J\setminus\{z\}}\frac{1}{z-w}.$$

Proof. It suffices to prove (9) under the assumption that $\det T_n(b)\ne 0$ because both sides of (9) are continuous functions of $z_1,\ldots,z_{p+q}$. Thus, let $\det T_n(b)\ne 0$. We will employ (8) with $B=T_n(b)$.

Trench [16] proved that $[T_n^{-1}(b)]_{m,1}$ equals

$$-\frac{1}{b_p}\,\frac{D_{\{1,\ldots,p+q\}}(0,\ldots,q-1,q+n,\ldots,q+n+p-2,q+n-m)}{D_{\{1,\ldots,p+q\}}(0,\ldots,q-1,q+n,\ldots,q+n+p-1)} \tag{10}$$

where $D_{\{j_1,\ldots,j_k\}}(s_1,\ldots,s_k)$ denotes the determinant

$$\det\begin{pmatrix} z_{j_1}^{s_1} & z_{j_1}^{s_2} & \ldots & z_{j_1}^{s_k} \\ z_{j_2}^{s_1} & z_{j_2}^{s_2} & \ldots & z_{j_2}^{s_k} \\ \vdots & \vdots & & \vdots \\ z_{j_k}^{s_1} & z_{j_k}^{s_2} & \ldots & z_{j_k}^{s_k} \end{pmatrix}.$$

Note that

$$D_J(s_1+\xi,\dots,s_k+\xi) = \left(\prod_{j\in J} z_j^{\xi}\right) D_J(s_1,\dots,s_k),$$
$$D_{\{1,2,\dots,k\}}(0,1,\dots,k-1) = \prod_{\substack{j,\ell\in J\\ \ell>j}} (z_\ell - z_j).$$

We first consider the denominator of (10). Put $Z=\{1,\dots,p+q\}$. Laplace expansion along the last p columns gives

$$\begin{aligned}
&D_Z(0,\dots,q-1,q+n,\dots,q+n+p-1)\\
&= \sum_{J\subset Z,|J|=p} (-1)^{\mathrm{inv}(\overline{J},J)} D_J(q+n,\dots,q+n+p-1) D_{\overline{J}}(0,\dots,q-1)\\
&= \sum_{J\subset Z,|J|=p} (-1)^{\mathrm{inv}(\overline{J},J)} \prod_{k\in J} z_k^{q+n} \prod_{\substack{k,\ell\in J\\ \ell>k}} (z_\ell - z_k) \prod_{\substack{k,\ell\in \overline{J}\\ \ell>k}} (z_\ell - z_k),
\end{aligned}$$

where $\mathrm{inv}(\overline{J},J)$ is the number of inversions in the permutation of length $p+q$ whose first q elements are the elements of the set $\overline{J}$ in increasing order and whose last p elements are the elements of the set J in increasing order. A little thought reveals that $\mathrm{inv}(\overline{J},J)$ is just the number of pairs (k,ℓ) with $k\in J$, $\ell\in\overline{J}$, $k<\ell$. We have

$$\begin{aligned}
\prod_{j\in J,s\in\overline{J}} (z_j - z_s) &= \prod_{\substack{\ell\in J,k\in\overline{J}\\ \ell>k}} (z_\ell - z_k) \prod_{\substack{k\in J,\ell\in\overline{J}\\ \ell>k}} (z_k - z_\ell)\\
&= (-1)^{\mathrm{inv}(\overline{J},J)} \prod_{\substack{\ell\in J,k\in\overline{J}\\ \ell>k}} (z_\ell - z_k) \prod_{\substack{\ell\in\overline{J},k\in J\\ \ell>k}} (z_\ell - z_k) \qquad (11)
\end{aligned}$$

and hence the denominator is equal to

$$R_n \sum_{J\subset\mathcal{Z},|J|=p} C_J W_J^n \quad \text{with} \quad R_n := \frac{(-1)^{pn}}{b_p^n} \prod_{\ell>k} (z_\ell - z_k).$$

A formula by Widom [18], which can also be found in [1], says that

$$\det T_n(b) = \sum_{J\subset\mathcal{Z},|J|=p} C_J W_J^n.$$

Consequently, the denominator of (10) is nothing but $R_n \det T_n(b)$.

Let us now turn to the numerator of (10). This time Laplace expansion along the last p columns yields

$$\begin{aligned} &D_Z(0,\dots,q-1,q+n,\dots,q+n+p-2,q+n-m) \\ &= \sum_{J\subset Z,|J|=p} (-1)^{\mathrm{inv}(\overline{J},J)} D_J(q+n,\dots,q+n+p-1,q+n-m) D_{\overline{J}}(0,\dots,q-1) \\ &= \sum_{J\subset Z,|J|=p} (-1)^{\mathrm{inv}(\overline{J},J)} D_{\overline{J}}(0,\dots,q-1) \left(\prod_{j\in J} z_j^{q+n}\right) D_J(0,\dots,p-2,-m). \end{aligned}$$

Expanding $D_J(0,\dots,p-2,-m)$ by its last column we get

$$D_J(0,\dots,p-2,-m) = \sum_{j\in J} (-1)^{\mathrm{inv}(J\setminus\{j\},j)} z_j^{-m} D_{J\setminus\{j\}}(0,\dots,p-2) \qquad (12)$$

with $\mathrm{inv}(J\setminus\{j\},j)$ being the number of $s\in J\setminus\{j\}$ such that $s>j$. Thus, (12) is

$$\begin{aligned} &\sum_{j\in J} (-1)^{\mathrm{inv}(J\setminus\{j\},j)} z_j^{-m} \prod_{\substack{k,\ell\in J\setminus\{j\} \\ \ell>k}} (z_\ell - z_k) \\ &= \sum_{j\in J} z_j^{-m} \prod_{\substack{k,\ell\in J \\ \ell>k}} (z_\ell - z_k) \prod_{s\in J\setminus\{j\}} \frac{1}{z_j - z_s}. \end{aligned}$$

This in conjunction with (11) shows that the numerator of (9) equals

$$-b_p R_n \sum_{J\subset\mathcal{Z},|J|=p} C_J W_J^n \sum_{z\in J} S_{m,J,z}.$$

In summary, from (10) we obtain that

$$[T_n^{-1}(b)]_{m,1} = \frac{1}{\det T_n(b)} \sum_{J\subset\mathcal{Z},|J|=p} C_J W_J^n \sum_{z\in J} S_{m,J,z},$$

which after multiplication by $\det T_n(b)$ becomes (9). $\square$

3 The main terms of the first column

We now apply Theorem 2.1 to

$$b(t) = a(t) - \lambda = a_r t^{-r} \prod_{k=1}^{2r} (t - z_k(\lambda)) \qquad (13)$$

where $\lambda \in (\alpha,\beta)$. The set $\mathcal{Z} = \mathcal{Z}(\lambda)$ is given by (4). Let

$$d_0(\lambda) = (-1)^r a_r e^{i\sigma(\lambda)} \prod_{k=1}^{r-1} u_k(\lambda). \tag{14}$$

In [2], we showed that $d_0(\lambda) > 0$ for all $\lambda \in (0, M)$. The dependence on λ will henceforth frequently be suppressed in notation. Let

$$J_1 = \{u_1, \ldots, u_{r-1}, e^{i\varphi_1}\}, \quad J_2 = \{u_1, \ldots, u_{r-1}, e^{i\varphi_2}\}$$

and for $\nu \in \{1, \ldots, r-1\}$, put

$$J_\nu^0 = \{u_1, \ldots, u_{r-1}, 1/\overline{u}_\nu\}.$$

Lemma 3.1 *If $J \subset \mathcal{Z}$, $|J| = r$, $J \notin \{J_1, J_2, J_1^0, \ldots, J_{r-1}^0\}$, then*

$$|C_J W_j^n S_{m,J,z}| \le K \,\frac{d_0^n}{\sin\varphi} e^{-\delta n}$$

for all $z \in J$, $n \ge 1$, $1 \le m \le n$, $\lambda \in (\alpha,\beta)$ with some finite constant K that does not depend on z, n, m, λ.

Proof. If both $e^{i\varphi_1}$ and $e^{i\varphi_2}$ belong to J, then

$$J = \{u_{\nu_1}, \ldots, u_{\nu_k}, e^{i\varphi_1}, e^{i\varphi_2}, 1/\overline{u}_{s_1}, \ldots, 1/\overline{u}_{s_\ell}\}$$

with $k + \ell = r - 2$. Since

$$\min_{\lambda\in[\alpha,\beta]} \min_{j_1\ne j_2} |u_{j_1}(\lambda) - u_{j_2}(\lambda)| > 0,$$

we conclude that $|C_J| \le K_1$. Here and in the following K_i denotes a finite constant that is independent of $\lambda \in [\alpha,\beta]$. We have $k \le r-2$ and thus

$$|W_J| = |a_r|\,\frac{|u_{\nu_1}\ldots u_{\nu_k}|}{|u_{s_1}\ldots u_{s_\ell}|} \le \frac{d_0 e^{-\delta}}{|u_{s_1}\ldots u_{s_\ell}|}. \tag{15}$$

If $z \in \{u_{\nu_1}, \ldots, u_{\nu_k}, e^{i\varphi_1}, e^{i\varphi_2}\}$, then obviously $|S_{m,J,z}| \le K_2/\sin\varphi$ and hence

$$|C_J W_J^n S_{m,J,z}| \le K_1 K_2 \,\frac{d_0^n e^{-\delta n}}{\sin\varphi}.$$

In case $z \in \{1/\overline{u}_{s_1}, \ldots, 1/\overline{u}_{s_\ell}\}$, say $z = 1/\overline{u}_{s_1}$, we have $|S_{m,J,z}| \le K_3 |u_{\nu_1}|^m$, which gives

$$|C_J W_J^n S_{m,J,z}| \le K_1 K_3 d_0^n e^{-\delta n}\frac{|u_{\nu_1}|^m}{|u_{\nu_1}|^n} \le K_1 K_3 d_0^n e^{-\delta n} \le K_1 K_3 \,\frac{d_0^n e^{-\delta n}}{\sin\varphi}.$$

The only other possibility for J is to be of the type

$$J = \{u_{\nu_1}, \ldots, u_{\nu_k}, e^{i\varphi_1}, 1/\overline{u}_{s_1}, \ldots, 1/\overline{u}_{s_\ell}\}$$

with $k + \ell \le r - 1$, $k \le r - 2$, $\ell \ge 1$. (The case where $e^{i\varphi_1}$ is replaced by $e^{i\varphi_2}$ is completely analogous.) This time, $|C_J| \le K_4/\sin\varphi$ and (15) holds again. For $z \in \{u_{\nu_1}, \ldots, u_{\nu_k}, e^{i\varphi_1}\}$ we have $|S_{m,J,z}| \le K_5$ and thus get the assertion. If $z = 1/\overline{u}_s$ for some $s \in \{s_1, \ldots, s_\ell\}$, say $s = s_1$, then $|S_{m,J,z}| \le K_6 |u_{s_1}|^m$, and the assertion follows as above, too. $\square$

Let

$$d_1(\lambda) = \frac{1}{|h_\lambda(e^{i\varphi_1(\lambda)}) h_\lambda(e^{i\varphi_2(\lambda)})|} \prod_{k,s=1}^{r-1} \left(1 - \frac{1}{u_k(\lambda)\overline{u}_s(\lambda)}\right)^{-1}.$$

It is easily seen that $d_1(\lambda) > 0$ for all $\lambda \in [0, M]$.

Lemma 3.2 *If $\lambda = \lambda_j^{(n)} \in (0, M)$, then*

$$C_{J_1} W_{J_1}^n S_{m,J_1,e^{i\varphi_1}} = \frac{d_1 d_0^{n-1}}{\sin\varphi} \left[(-1)^j A e^{-im\varphi_1} + O(e^{-\delta n})\right],$$
$$C_{J_2} W_{J_2}^n S_{m,J_2,e^{i\varphi_2}} = \frac{d_1 d_0^{n-1}}{\sin\varphi} \left[(-1)^{j+1} B e^{-im\varphi_2} + O(e^{-\delta n})\right]$$

uniformly in m and λ.

Proof. We abbreviate $\prod_{k=1}^{r-1}$ to $\prod_k$. Clearly,

$$W_{j_1} = (-1)^r a_r \left(\prod_k u_k\right) e^{i\varphi_1} = (-1)^r a_r \left(\prod_k u_k\right) e^{i\sigma} e^{i\varphi} = d_0 e^{i\varphi}.$$

We have

$$\begin{aligned} C_{J_1} &= \frac{\left(\prod_k u_k^r\right) e^{ir\varphi_1}}{(e^{i\varphi_1} - e^{i\varphi_2}) \prod_{k,s}\left(u_k - \frac{1}{\overline{u}_s}\right) \prod_k (u_k - e^{i\varphi_2}) \prod_k \left(e^{i\varphi_1} - \frac{1}{\overline{u}_k}\right)} \\ &= \frac{e^{ir\varphi_1}}{e^{i\sigma}(e^{i\varphi} - e^{-i\varphi}) \prod_{k,s}\left(1 - \frac{1}{u_k\overline{u}_s}\right) \prod_k \left(1 - \frac{e^{i\varphi_2}}{u_k}\right) e^{i(r-1)\varphi_1} \prod_k \left(1 - \frac{e^{-i\varphi_1}}{\overline{u}_k}\right)} \\ &= \frac{e^{i\varphi}}{2i \sin\varphi \prod_{k,s}\left(1 - \frac{1}{u_k\overline{u}_s}\right) h(e^{i\varphi_2}) \overline{h(e^{i\varphi_1})}} \end{aligned}$$

and because

$$h(e^{i\varphi_2}) \overline{h(e^{i\varphi_1})} = |h(e^{i\varphi_1}) h(e^{i\varphi_2})| e^{-i\theta}, \qquad (16)$$

it follows that

$$C_{J_1} = \frac{d_1 e^{i(\varphi+\theta)}}{2i \sin\varphi}.$$

Furthermore,

$$\begin{aligned}
S_{m,J_1,e^{i\varphi_1}} &= -\frac{1}{a_r}\,\frac{1}{e^{im\varphi_1}\prod_k(e^{i\varphi_1}-u_k)} \\
&= -\frac{1}{a_r}\,\frac{e^{-im\varphi_1}}{(-1)^{r-1}\left(\prod_k u_k\right)\prod_k(1-e^{i\varphi_1}/u_k)} \\
&= \frac{e^{-im(\sigma+\varphi)}}{(-1)^r a_r\left(\prod_k u_k\right)h(e^{i\varphi_1})} = \frac{e^{-im(\sigma+\varphi)}e^{i\sigma}}{d_0 h(e^{i\varphi_1})}
\end{aligned}$$

Putting things together we arrive at the formula

$$C_{J_1}W_{J_1}^n S_{m,J_1,e^{i\varphi_1}} = \frac{d_1 d_0^{n-1}}{\sin\varphi}\,A\,e^{-im(\sigma+\varphi)}e^{i((n+1)\varphi+\theta)}.$$

Obviously, $\sigma+\varphi=\varphi_1$. By virtue of (5),

$$e^{i((n+1)\varphi+\theta)} = e^{i\pi j}(1+O(e^{-\delta n})) = (-1)^j(1+O(e^{-\delta n})).$$

This proves the first of the asserted formulas. Analogously,

$$W_{J_2}=d_0e^{-i\varphi},\quad C_{J_2}=-\frac{d_1e^{-i(\varphi+\theta)}}{2i\,\sin\varphi},\quad S_{m,J_2,e^{i\varphi_2}}=\frac{e^{-im(\sigma-\varphi)}e^{i\sigma}}{d_0h(e^{i\varphi_2})},$$

which gives the second formula. □

Lemma 3.3 *If $1\le\nu\le r-1$ and $\lambda=\lambda_j^{(n)}\in(\alpha,\beta)$, then*

$$C_{J_1}W_{J_1}^n S_{m,J_1,u_\nu}+C_{J_2}W_{J_2}^n S_{m,J_2,u_\nu} = \frac{d_1d_0^{n-1}}{\sin\varphi}\left[(-1)^jD_\nu\,\frac{1}{u_\nu^m}+O(e^{-\delta n})\right]$$

uniformly in m and λ.

Proof. By definition,

$$\begin{aligned}
S_{m,J_1,u_\nu} &= -\frac{1}{a_r}\,\frac{1}{u_\nu^m(u_\nu-e^{i\varphi_1})\prod_{s\ne\nu}(u_\nu-u_s)} \\
&= \frac{u_\nu^{-m}}{(-1)^{r-1}\left(\prod_k u_k\right)a_r(u_\nu-e^{i\varphi_1})h'(u_\nu))}
\end{aligned}$$

Since $-h'(z)$ equals

$$\frac{1}{u_1}\left(1-\frac{z}{u_2}\right)\dots\left(1-\frac{z}{u_{r-1}}\right)+\dots+\frac{1}{u_{r-1}}\left(1-\frac{z}{u_1}\right)\dots\left(1-\frac{z}{u_{r-2}}\right),$$

we obtain that

$$h'(u_\nu) = -\frac{1}{u_\nu}\prod_{s\ne\nu}\left(1-\frac{u_\nu}{u_s}\right).$$

Thus,

$$S_{m,J_1,u_\nu} = \frac{u_\nu^{-m}}{(-1)^r a_r \left(\prod_k u_k\right)(u_\nu - e^{i\varphi_1})h'(u_\nu)} = \frac{u_\nu^{-m} e^{i\sigma}}{d_0(u_\nu - e^{i\varphi_1})h'(u_\nu)}.$$

Changing φ_1 to φ_2 we get

$$S_{m,J_2,u_\nu} = \frac{u_\nu^{-m} e^{i\sigma}}{d_0(u_\nu - e^{i\varphi_2})h'(u_\nu)}.$$

These two expressions along with the expressions for $C_{J_1}, W_{J_1}, C_{J_2}, W_{J_2}$ derived in the proof of Lemma 3.2 show that the sum under consideration is

$$\frac{d_1 d_0^{n-1}}{2i\,\sin\varphi}\,\frac{u_\nu^{-m} e^{i\sigma}}{h'(u_\nu)}\left[\frac{e^{i((n+1)\varphi+\theta)}}{u_\nu - e^{i\varphi_1}} - \frac{e^{-i((n+1)\varphi+\theta)}}{u_\nu - e^{i\varphi_2}}\right].$$

Because of (5), the term in brackets equals

$$\begin{aligned}&(-1)^j\left[\frac{1}{u_\nu - e^{i\varphi_1}} - \frac{1}{u_\nu - e^{i\varphi_2}} + O(e^{-\delta n})\right]\\ &= (-1)^j \frac{e^{i\sigma} 2i\sin\varphi}{(u_\nu - e^{i\varphi_1})(u_\nu - e^{i\varphi_2})} + O(e^{-\delta n}). \quad \square\end{aligned}$$

Lemma 3.4 *For $1 \le \nu \le r-1$ and $\lambda \in (\alpha, \beta)$,*

$$C_{J_\nu^0} W_{J_\nu^0}^n S_{m,J_\nu^0,1/\overline{u}_\nu} = -\frac{d_1 d_0^{n-1}}{\sin\varphi}\, F_\nu\, \frac{e^{-i(n+1)\sigma}}{\overline{u}_\nu^{n+1-m}}.$$

Proof. We have $C_{J_\nu^0} = \left(\prod_k u_k^r\right)/(\overline{u}_\nu^r P_1 P_2 P_3)$ with

$$\begin{aligned}
P_1 &= \left(\frac{1}{\overline{u}_\nu} - e^{i\varphi_1}\right)\left(\frac{1}{\overline{u}_\nu} - e^{i\varphi_2}\right) = \frac{(\overline{u}_\nu - e^{-i\varphi_1})(\overline{u}_\nu - e^{-i\varphi_2})}{\overline{u}_\nu^2 e^{-2i\sigma}},\\
P_2 &= \prod_k (u_k - e^{i\varphi_1}) \prod_k (u_k - e^{i\varphi_2}) = \left(\prod_k u_k^2\right) h(e^{i\varphi_1}) h(e^{i\varphi_2}),\\
P_3 &= \prod_{s\ne\nu}\left(\frac{1}{\overline{u}_\nu} - \frac{1}{\overline{u}_s}\right) \prod_k \prod_{s\ne\nu}\left(u_k - \frac{1}{\overline{u}_s}\right)\\
&= \frac{1}{\overline{u}_\nu^{r-2}} \prod_{s\ne\nu}\left(1 - \frac{\overline{u}_\nu}{\overline{u}_s}\right)\left(\prod_k u_k^{r-2}\right) \prod_k \prod_{s\ne\nu}\left(1 - \frac{1}{u_k \overline{u}_s}\right)\\
&= -\frac{1}{\overline{u}_\nu^{r-3}}\,\overline{h'(u_\nu)}\left(\prod_k u_k^{r-2}\right) \frac{1}{d_1 |h(e^{i\varphi_1}) h(e^{i\varphi_2})|}\,\frac{1}{\prod_k (1 - 1/(u_k \overline{u}_\nu))}.
\end{aligned}$$

Thus, $C_{J_\nu^0}$ equals

$$-\frac{d_1 e^{-2i\sigma} |h(e^{i\varphi_1}) h(e^{i\varphi_2})|}{\overline{u}_\nu (\overline{u}_\nu - e^{-i\varphi_1})(\overline{u}_\nu - e^{-i\varphi_2})\overline{h'(u_\nu)} h(e^{i\varphi_1}) h(e^{i\varphi_2})} \prod_k \left(1 - \frac{1}{u_k \overline{u}_\nu}\right).$$

Since $W_{J_\nu^0} = d_0 e^{-i\sigma}/\overline{u}_\nu$ and

$$\begin{aligned} S_{m,J_\nu^0,1/\overline{u}_\nu} &= -\frac{1}{a_r}\,\overline{u}_\nu^m\,\frac{1}{\prod_k (1/\overline{u}_\nu - u_k)} \\ &= \frac{\overline{u}_\nu^m}{(-1)^r a_r \left(\prod_k u_k\right) \prod_k \left(1 - \frac{1}{u_k \overline{u}_\nu}\right)} = \frac{\overline{u}_\nu^m e^{i\sigma}}{d_0 \prod_k \left(1 - \frac{1}{u_k \overline{u}_\nu}\right)}, \end{aligned}$$

we obtain that $C_{J_\nu^0} W_{J_\nu^0}^n S_{m,J_\nu^0,1/\overline{u}_\nu}$ is equal to

$$-\frac{d_1 d_0^{n-1}}{\overline{u}^{n+1-m}}\,\frac{e^{-i\sigma} e^{-in\sigma} |h(e^{i\varphi_1}) h(e^{i\varphi_2})|}{(\overline{u}_\nu - e^{-i\varphi_1})(\overline{u}_\nu - e^{-i\varphi_2})\overline{h'(u_\nu)} h(e^{i\varphi_1}) h(e^{i\varphi_2})}. \quad \square$$

Lemma 3.5 *If $1 \le k \le r-1$ and $\lambda \in (\alpha, \beta)$,*

$$C_{J_\nu^0} W_{J_\nu^0}^n S_{m,J_\nu^0,u_k} = \frac{d_1 d_0^{n-1}}{\sin\varphi}\, O(e^{-\delta n})$$

uniformly in m and λ.

Proof. This time

$$\begin{aligned} C_{J_\nu^0} W_{J_\nu^0}^n S_{m,J_\nu^0,u_k} &= -\frac{1}{a_r}\,\frac{1}{u_k^m}\,\frac{1}{(u_k - 1/\overline{u}_\nu)\prod_{s\ne k}(u_k - u_s)} \\ &= \frac{u_k^{-m}}{(-1)^{r-1} a_r u_k \left(1 - \frac{1}{u_k \overline{u}_\nu}\right) \left(\prod_{s\ne k} u_s\right) \prod_{s \ne k} \left(1 - \frac{u_k}{u_s}\right)} \\ &= -\frac{1}{d_0 u_k^m}\,\frac{1}{\left(1 - \frac{1}{u_k \overline{u}_\nu}\right) \prod_{s\ne k}\left(1 - \frac{u_k}{u_s}\right)} \end{aligned}$$

Expressions for $C_{J_\nu^0}$ and $W_{J_\nu^0}$ were given in the proof of Lemma 3.4. It follows that

$$C_{J_\nu^0} W_{J_\nu^0}^n S_{m,J_\nu^0,u_k} = G_{\nu,k}\,\frac{d_1 d_0^{n-1}}{\sin\varphi}\,\frac{1}{\overline{u}_\nu^{n+1} u_k^m}$$

where $G_{\nu,k}$ equals

$$\frac{e^{2i\sigma} e^{-in\sigma} |h(e^{i\varphi_1}) h(e^{i\varphi_2})| \sin\varphi}{(\overline{u}_\nu - e^{-i\varphi_1})(\overline{u}_\nu - e^{-i\varphi_2})\overline{h'(u_\nu)} h(e^{i\varphi_1}) h(e^{i\varphi_2})}\,\frac{\prod_{s\ne k}\left(1 - \frac{1}{u_s \overline{u}_\nu}\right)}{\prod_{s\ne k}\left(1 - \frac{u_k}{u_s}\right)}.$$

Since

$$\overline{h'(u_\nu)} = -\frac{1}{\overline{u}_\nu} \prod_{s \ne \nu} \left(1 - \frac{\overline{u}_\nu}{\overline{u}_s}\right),$$

we see that $G_{\nu,k}$ remains bounded on $[\alpha, \beta]$. Finally,

$$\frac{1}{|\overline{u}_\nu^{n+1} u_k^m|} \le \frac{1}{|u_\nu|^n} \le e^{-\delta n}. \quad \square$$

Corollary 3.6 *If* $\lambda = \lambda_j^{(n)} \in (\alpha, \beta)$, *then*

$$[\operatorname{adj} T_n(a-\lambda)]_{m,1} = (-1)^j \, \frac{d_1(\lambda) d_0^{n-1}(\lambda)}{\sin \varphi(\lambda)} [w_{j,m}(\lambda) + O(e^{-\delta n})]$$

uniformly in m *and* λ.

Proof. This follows from Theorem 2.1 and Lemmas 3.1 to 3.5 along with the fact that d_1 is bounded and bounded away from zero on $[\alpha, \beta]$. □

4 The asymptotics of the eigenvectors

We now prove Theorem 1.1. There is a finite constant K_1 such that $|D_\nu| \le K_1$ and $|F_\nu| \le K_1$ for all ν and all $\lambda \in (\alpha, \beta)$. Thus, summing up two finite geometric series, we get

$$\sum_{m=1}^{n} \left| D_\nu \, \frac{1}{u_\nu^m} + F_\nu \, \frac{(-1)^{j+1} e^{-i(n+1)\sigma}}{\overline{u}_\nu^{n+1-m}} \right|^2 \le 2K_1^2 \frac{1}{|u_\nu|^2} \, \frac{1 - 1/|u_\nu|^{2(n+1)}}{1 - 1/|u_\nu|^2} \le K_2$$

for all ν, n, λ. We further have

$$\begin{aligned} \sum_{m=1}^{n} \left| A e^{-im\varphi_1} - B e^{-im\varphi_2} \right|^2 &= \sum_{m=1}^{n} \left| \frac{e^{-im\varphi_1}}{2h(e^{i\varphi_1})} - \frac{e^{-im\varphi_2}}{2h(e^{i\varphi_2})} \right|^2 \\ = \sum_{m=1}^{n} &\left(\frac{1}{4|h(e^{i\varphi_1})|^2} + \frac{1}{4|h(e^{i\varphi_1})|^2} \right) \\ &- \sum_{m=1}^{n} \left(\frac{e^{-2im\varphi}}{4h(e^{i\varphi_1})\overline{h(e^{i\varphi_2})}} + \frac{e^{2im\varphi}}{4\overline{h(e^{i\varphi_1})} h(e^{i\varphi_2})} \right). \end{aligned}$$

The first sum is of the form $\sum_{m=1}^{n} (\gamma/4)$ and therefore equals $(n/4)\gamma$. Hence, because of (16) we are left to prove that

$$\left| \sum_{m=1}^{n} e^{i\theta(\lambda_j^{(n)})} e^{2im\varphi(\lambda_j^{(n)})} \right| \le K_3 \qquad (17)$$

for all n and j such that $\lambda_j^{(n)} \in (\alpha, \beta)$. The sum in (17) is

$$e^{i[(n+1)\varphi(\lambda_j^{(n)}) + \theta(\lambda_j^{(n)})]} \, \frac{\sin n\varphi(\lambda_j^{(n)})}{\sin \varphi(\lambda_j^{(n)})}.$$

Thus, (17) will follow as soon as we have shown that

$$\left| \frac{\sin n\varphi(\lambda_j^{(n)})}{\sin \varphi(\lambda_j^{(n)})} \right| \le K_3$$

for all n and j in question. From (5) we infer that

$$n\varphi(\lambda_j^{(n)}) = \pi j - \varphi(\lambda_j^{(n)}) - \theta(\lambda_j^{(n)}) + O(e^{-\delta n}),$$

which implies that

$$\sin n\varphi(\lambda_j^{(n)}) = (-1)^{j+1} \sin\left(\varphi(\lambda_j^{(n)}) + \theta(\lambda_j^{(n)})\right) + O(e^{-\delta n}).$$

Suppose first that $0 < \varphi(\lambda_j^{(n)}) \le \pi/2$. Then

$$\left|\frac{\sin\left(\varphi(\lambda_j^{(n)}) + \theta(\lambda_j^{(n)})\right)}{\sin\varphi(\lambda_j^{(n)})}\right| \le \frac{\pi}{2}\,\frac{|\varphi(\lambda_j^{(n)}) + \theta(\lambda_j^{(n)})|}{|\varphi(\lambda_j^{(n)})|} \le \frac{\pi}{2}\left(1 + \frac{|\theta(\lambda_j^{(n)})|}{|\varphi(\lambda_j^{(n)})|}\right). \tag{18}$$

In [2], we proved that $|\theta/\varphi|$ is bounded on $(0, M)$. Thus, the right-hand side of (18) is bounded by some K_3 for all n and j. If $\pi/2 < \varphi(\lambda_j^{(n)}) < \pi$, we may replace (18) by the upper bound

$$\frac{\pi}{2}\left(1 + \frac{|\theta(\lambda_j^{(n)})|}{|\pi - \varphi(\lambda_j^{(n)})|}\right).$$

We know again from [2] that $|\theta/(\pi - \varphi)|$ is bounded on $(0, M)$. This completes the proof of Theorem 1.1.

Here is the proof of Theorem 1.2. By virtue of Theorem 1.1, $\|w_j(\lambda_j^{(n)})\|_2 > 1$ whenever n is sufficiently large. Corollary 3.6 therefore implies that the first column of $\operatorname{adj} T_n(a - \lambda_j^{(n)})$ is nonzero and thus an eigenvector for $\lambda_j^{(n)}$ for all $n \ge n_0$ and all $1 \le j \le n$ such that $\lambda_j^{(n)} \in (\alpha, \beta)$. Again by Corollary 3.6, the mth entry of this column is

$$\frac{d_1(\lambda) d_0^{n-1}(\lambda)}{\sin\varphi(\lambda)}\,[w_{j,m}(\lambda) + \xi_{j,m}^{(n)}]\Big|_{\lambda=\lambda_j^{(n)}}$$

where $|\xi_{j,m}^{(n)}| \le K e^{-\delta n}$ for all n and j under consideration and K does not depend on m, n, j. It follows that

$$w_j(\lambda_j^{(n)}) + \left(\xi_{j,m}^{(n)}\right)_{m=1}^{n} = w_j(\lambda_j^{(n)}) + O_\infty(e^{-\delta n})$$

is also an eigenvector for $\lambda_j^{(n)}$. Consequently,

$$\frac{w_j(\lambda_j^{(n)}) + O_\infty(e^{-\delta n})}{\|w_j(\lambda_j^{(n)}) + O_\infty(e^{-\delta n})\|_2} = \frac{w_j(\lambda_j^{(n)})}{\|w_j(\lambda_j^{(n)})\|_2} + O_\infty(e^{-\delta n}) \tag{19}$$

is a normalized eigenvector for $\lambda_j^{(n)}$. From (5) we deduce that all eigenvalues of $T_n(a)$ are simple. Thus, $v_j^{(n)}$ is a scalar multiple of modulus 1 of (19). This completes the proof of Theorem 1.2.

5 Symmetric matrices

The matrices $T_n(a)$ are all symmetric if and only if all a_k are real and $a_k = a_{-k}$ for all k. Obviously, this is equivalent to the requirement that the real-valued function $g(x) := a(e^{ix})$ be even, that is, $g(x) = g(-x)$ for all x. Thus, suppose g is even. In that case

$$\varphi_0 = \pi, \quad \varphi_1(\lambda) = -\varphi_2(\lambda) = \varphi(\lambda), \quad \sigma(\lambda) = 0.$$

Moreover, for $t \in \mathbf{T}$ we have

$$\begin{aligned} a_r t^{-r} \prod_{k=1}^{2r}(t - z_k(\lambda)) &= a(t) - \lambda = a(1/t) - \lambda \\ &= a_r t^r \prod_{k=1}^{2r}(1/t - z_k(\lambda)) = a_r \left(\prod_{k=1}^{2r} z_k(\lambda)\right) t^{-r} \prod_{k=1}^{2r}(t - 1/z_k(\lambda)), \end{aligned}$$

which in conjunction with (4) implies that

$$\{u_1(\lambda), \ldots, u_{r-1}(\lambda)\} = \{\overline{u}_1(\lambda), \ldots, \overline{u}_{r-1}(\lambda)\}. \tag{20}$$

The coefficients of the polynomial $h_\lambda(t)$ are symmetric functions of

$$1/u_1(\lambda), \ldots, 1/u_{r-1}(\lambda).$$

From (20) we therefore see that these coefficients are real. It follows in particular that $h_\lambda(e^{-i\varphi(\lambda)}) = \overline{h_\lambda(e^{i\varphi(\lambda)})}$, which gives $\theta(\lambda) = 2 \arg h_\lambda(e^{i\varphi(\lambda)})$ and thus

$$h_\lambda(e^{i\varphi(\lambda)}) = |h_\lambda(e^{i\varphi(\lambda)})| e^{i\theta(\lambda)/2}, \quad h_\lambda(e^{-i\varphi(\lambda)}) = |h_\lambda(e^{i\varphi(\lambda)})| e^{-i\theta(\lambda)/2}.$$

We are now in a position to prove Theorem 1.3. To do so, we use Theorem 1.2. Consider the vector $w_j(\lambda_j^{(n)})$. We now have

$$\begin{aligned} A e^{-im\varphi_1} - B e^{-im\varphi_2} &= \frac{e^{-im\varphi}}{2ih(e^{i\varphi})} - \frac{e^{im\varphi}}{2ih(e^{-i\varphi})} \\ &= \frac{1}{2i|h(e^{i\varphi})|}\left(\frac{e^{-im\varphi}}{e^{i\theta/2}} - \frac{e^{im\varphi}}{e^{-i\theta/2}}\right) = -\frac{1}{|h(e^{i\varphi})|} \sin\left(m\varphi + \frac{\theta}{2}\right). \end{aligned}$$

Furthermore,

$$\begin{aligned} D_\nu &= \frac{\sin\varphi}{(u_\nu - e^{i\varphi})(u_\nu - e^{-i\varphi})h'(u_\nu)} = \frac{Q_\nu}{|h(e^{i\varphi})|}, \\ F_\nu &= \frac{\sin\varphi}{(\overline{u}_\nu - e^{i\varphi})(\overline{u}_\nu - e^{-i\varphi})\overline{h'(u_\nu)}} \frac{|h(e^{i\varphi})h(e^{-i\varphi})|}{h(e^{i\varphi})h(e^{-i\varphi})} \\ &= \frac{\sin\varphi}{(\overline{u}_\nu - e^{i\varphi})(\overline{u}_\nu - e^{-i\varphi})h'(\overline{u}_\nu)}. \end{aligned}$$

Consequently, from (20) we infer that

$$\sum_{\nu=1}^{r-1} \frac{F_\nu}{\overline{u}_\nu^{n+1-m}} = \sum_{\nu=1}^{r-1} \frac{D_\nu}{u_\nu^{n+1-m}} = \sum_{\nu=1}^{r-1} \frac{Q_\nu}{|h(e^{i\varphi})|\,\overline{u}_\nu^{n+1-m}}.$$

In summary, it follows that

$$w_j(\lambda_j^{(n)}) = -\frac{1}{|h(e^{i\varphi(\lambda_j^{(n)})})|}\, y_j(\lambda_j^{(n)}). \tag{21}$$

Thus, the representation

$$v_j^{(n)} = \tau_j^{(n)} \left[\frac{y_j(\lambda_j^{(n)})}{\|y_j(\lambda_j^{(n)})\|_2} + O_\infty(e^{-\delta n}) \right]$$

is immediate from Theorem 1.2. Finally, put $h_{j,n} = |h(e^{i\varphi(\lambda_j^{(n)})})| = |h(e^{-i\varphi(\lambda_j^{(n)})})|$. Theorem 1.1 shows that

$$\|w_j(\lambda_j^{(n)})\|_2^2 = \frac{n}{4}\left(\frac{1}{|h(e^{i\varphi})|^2} + \frac{1}{|h(e^{-i\varphi})|^2} \right)\Bigg|_{\lambda=\lambda_j^{(n)}} + O(1) = \frac{n}{2h_{j,n}^2} + O(1),$$

whence, by (21), $\|y_j(\lambda_j^{(n)})\|_2^2 = h_{j,n}^2 \|w_j(\lambda_j^{(n)})\|_2^2 = n/2 + O(1)$. The proof of Theorem 1.3 is complete.

Here is the proof of Theorem 1.4. We first estimate the "small terms" in $y_j^{(n)}$. Summing up finite geometric series and using the assumption that $|u_\nu(\lambda)|$ are separated from 1 we come to

$$\sum_{m=1}^{n} \left| \sum_{\nu=1}^{r-1} Q_\nu(\lambda) \left(\frac{1}{u_\nu(\lambda)^m} + \frac{(-1)^{j+1}}{u_\nu(\lambda)^{n+1-m}} \right) \right|^2$$
$$\leq \sum_{\nu=1}^{r-1} \frac{4(r-1)|Q_\nu(\lambda)|^2}{1-|u_\nu(\lambda)|^2} \leq K \sin^2 \varphi(\lambda)$$

where K is some positive number depending only on a. since $\varphi(\lambda_j^{(n)}) = O\,(j/n)$, it follows that

$$\left\| \left(\sum_{\nu=1}^{r-1} Q_\nu(\lambda) \left(\frac{1}{u_\nu(\lambda)^m} + \frac{(-1)^{j+1}}{u_\nu(\lambda)^{n+1-m}} \right) \right)_{m=1}^{n} \right\|_2 = O\left(\frac{j}{n}\right). \tag{22}$$

We next consider the difference between the "main term" of $y_j^{(n)}$ and $\sin \frac{mj\pi}{n+1}$. Using the elementary estimate

$$|\sin A - \sin B|^2 = 4\sin^2 \frac{A-B}{2} \cos^2 \frac{A+B}{2}$$
$$\leq 4\sin^2 \frac{A-B}{2} = 2 - 2\cos(A-B),$$

we get

$$\sum_{m=1}^{n}\left|\sin\left(m\varphi(\lambda_j^{(n)})+\theta(\lambda_j^{(n)})\right)-\sin\frac{mj\pi}{n+1}\right|^2$$
$$\leq 2n-2\sum_{m=1}^{n}\cos\left(m\left(\varphi(\lambda_j^{(n)})-\frac{\pi j}{n+1}\right)+\theta(\lambda_j^{(n)})\right).$$

To simplify the last sum, we use that

$$\begin{aligned}\sum_{m=1}^{n}\cos(m\xi+\omega) &= \frac{\sin\frac{n\xi}{2}\cos\left(\frac{(n+1)\xi}{2}+\omega\right)}{\sin\frac{\xi}{2}}\\ &= n\left(1+O(n^2\xi^2)\right)\left(1+O\left(\frac{(n+1)\xi}{2}+\omega\right)^2\right).\end{aligned}$$

In our case

$$\omega=\theta(\lambda_j^{(n)})=O\left(\sqrt{\lambda_j^{(n)}}\right)=O\left(\frac{j}{n}\right),$$
$$\xi=\varphi(\lambda_j^{(n)})-\frac{\pi j}{n+1}=-\frac{\theta(\lambda_j^{(n)})}{n+1}+O(e^{-n\delta})=O\left(\frac{j}{n^2}\right).$$

Consequently,

$$\sum_{m=1}^{n}\left|\sin\left(m\varphi(\lambda_j^{(n)})+\theta(\lambda_j^{(n)})\right)-\sin\frac{mj\pi}{n+1}\right|^2=O\left(\frac{j^2}{n}\right),$$

that is,

$$\left\|\left(\sin\left(m\varphi(\lambda_j^{(n)})+\theta(\lambda_j^{(n)})\right)-\sin\frac{mj\pi}{n+1}\right)_{m=1}^{n}\right\|_2=O\left(\frac{j}{\sqrt{n}}\right). \tag{23}$$

Combining (22) and (23)we obtain that

$$\left\|y_j^{(n)}-\sqrt{\frac{n+1}{2}}\,x_j^{(n)}\right\|_2=O\left(\frac{j}{n}\right)+O\left(\frac{j}{\sqrt{n}}\right)=O\left(\frac{j}{\sqrt{n}}\right), \tag{24}$$

which implies in particular that

$$\|y_j^{(n)}\|_2=\sqrt{\frac{n+1}{2}}\left(1+O\left(\frac{j}{n}\right)\right). \tag{25}$$

Clearly, estimates (24) and (25) yield the asserted estimate. This completes the proof of Theorem 1.4.

6 Numerical results

Given $T_n(a)$, determine the approximate eigenvalue $\lambda_{j,*}^{(n)}$ from the equation

$$(n+1)\varphi(\lambda_{j,*}^{(n)}) + \theta(\lambda_{j,*}^{(n)}) = \pi j.$$

In [2], we proposed an exponentially fast iteration method for solving this equation. Let $w_j^{(n)}(\lambda) \in \mathbf{C}^n$ be as in Section 1 and put

$$w_{j,*}^{(n)} = \frac{w_j^{(n)}(\lambda_{j,*}^{(n)})}{\|w_j^{(n)}(\lambda_{j,*}^{(n)})\|_2}.$$

We define the distance between the normalized eigenvector $v_j^{(n)}$ and the normalized vector $w_{j,*}^{(n)}$ by

$$\varrho(v_j^{(n)}, w_{j,*}^{(n)}) := \min_{\tau\in\mathbf{T}} \|\tau v_j^{(n)} - w_{j,*}^{(n)}\|_2 = \sqrt{2 - 2\langle v_j^{(n)}, w_{j,*}^{(n)}\rangle}$$

and put

$$\begin{aligned}
\Delta_*^{(n)} &= \max_{1\le j\le n} |\lambda_j^{(n)} - \lambda_{j,*}^{(n)}|,\\
\Delta_{v,w}^{(n)} &= \max_{1\le j\le n} \varrho(v_j^{(n)}, w_{j,*}^{(n)}),\\
\Delta_r^{(n)} &= \max_{1\le j\le n} \|T_n(a)w_{j,*}^{(n)}) - \lambda_{j,*}^{(n)} w_{j,*}^{(n)}\|_2.
\end{aligned}$$

The tables following below show these errors for three concrete choices of the generating function a.

For $a(t) = 8 - 5t - 5t^{-1} + t^2 + t^{-2}$ we have

	$n=10$	$n=20$	$n=50$	$n=100$	$n=150$
$\Delta_*^{(n)}$	$5.4\cdot 10^{-7}$	$1.1\cdot 10^{-11}$	$5.2\cdot 10^{-25}$	$1.7\cdot 10^{-46}$	$9.6\cdot 10^{-68}$
$\Delta_{v,w}^{(n)}$	$2.0\cdot 10^{-6}$	$1.1\cdot 10^{-10}$	$2.0\cdot 10^{-23}$	$1.9\cdot 10^{-44}$	$2.0\cdot 10^{-65}$
$\Delta_r^{(n)}$	$8.0\cdot 10^{-6}$	$2.7\cdot 10^{-10}$	$3.4\cdot 10^{-23}$	$2.2\cdot 10^{-44}$	$1.9\cdot 10^{-65}$

If $a(t) = 8 + (-4-2i)t + (-4-2i)t^{-1} + it - it^{-1}$ then

	$n=10$	$n=20$	$n=50$	$n=100$	$n=150$
$\Delta_*^{(n)}$	$3.8\cdot 10^{-8}$	$2.8\cdot 10^{-13}$	$2.9\cdot 10^{-30}$	$5.9\cdot 10^{-58}$	$1.6\cdot 10^{-85}$
$\Delta_{v,w}^{(n)}$	$1.8\cdot 10^{-7}$	$4.7\cdot 10^{-13}$	$2.0\cdot 10^{-29}$	$7.0\cdot 10^{-57}$	$2.4\cdot 10^{-84}$
$\Delta_r^{(n)}$	$5.4\cdot 10^{-7}$	$1.3\cdot 10^{-12}$	$2.7\cdot 10^{-29}$	$6.7\cdot 10^{-57}$	$1.9\cdot 10^{-84}$

In the case where $a(t) = 24 + (-12-3i)t + (-12+3i)t^{-1} + it^3 - it^{-3}$ we get

	$n=10$	$n=20$	$n=50$	$n=100$	$n=150$
$\Delta_*^{(n)}$	$6.6\cdot 10^{-6}$	$1.2\cdot 10^{-10}$	$7.6\cdot 10^{-24}$	$1.4\cdot 10^{-45}$	$3.3\cdot 10^{-67}$
$\Delta_{v,w}^{(n)}$	$1.9\cdot 10^{-6}$	$1.3\cdot 10^{-10}$	$2.0\cdot 10^{-23}$	$7.2\cdot 10^{-45}$	$2.8\cdot 10^{-66}$
$\Delta_r^{(n)}$	$2.5\cdot 10^{-5}$	$8.6\cdot 10^{-10}$	$7.3\cdot 10^{-23}$	$1.9\cdot 10^{-44}$	$5.9\cdot 10^{-66}$

References

[1] A. Böttcher and S. M. Grudsky, *Spectral Properties of Banded Toeplitz Matrices,* SIAM, Philadelphia 2005.

[2] A. Böttcher, S. M. Grudsky, and E. A. Maksimenko, *Inside the eigenvalues of certain Hermitian Toeplitz band matrices,* to appear.

[3] A. Böttcher, S. M. Grudsky, E. A. Maksimenko, and J. Unterberger, *The first order asymptotics of the extreme eigenvectors of certain Hermitian Toeplitz matrices,* Integral Equations Operator Theory, in press.

[4] A. Böttcher, S. M. Grudsky, and J. Unterberger, *Asymptotic pseudomodes of Toeplitz matrices,* Operators and Matrices **2** (2008), 525-541.

[5] A. Böttcher and B. Silbermann, *Introduction to Large Truncated Toeplitz Matrices,* Universitext, Springer-Verlag, New York 1999.

[6] C. Estatico and S. Serra Capizzano, *Superoptimal approximation for unbounded symbols,* Linear Algebra Appl. **428** (2008), 564–585.

[7] U. Grenander and G. Szegö, *Toeplitz Forms and Their Applications,* University of California Press, Berkeley 1958.

[8] C. M. Hurvich and Yi Lu, *On the complexity of the preconditioned conjugate gradient algorithm for solving Toeplitz systems with a Fisher-Hartwig singularity,* SIAM J. Matrix Anal. Appl. **27** (2005), 638–653.

[9] M. Kac, W. L. Murdock, and G. Szegö, *On the eigenvalues of certain Hermitian forms,* J. Rational Mech. Anal. **2** (1953), 767–800.

[10] A. Yu. Novosel'tsev and I. B. Simonenko, *Dependence of the asymptotics of extreme eigenvalues of truncated Toeplitz matrices on the rate of attaining an extremum by a symbol,* St. Petersburg Math. J. **16** (2005), 713–718.

[11] S. V. Parter, *Extreme eigenvalues of Toeplitz forms and applications to elliptic difference equations,* Trans. Amer. Math. Soc. **99** (1961), 153–192.

[12] S. V. Parter, *On the extreme eigenvalues of Toeplitz matrices,* Trans. Amer. Math. Soc. **100** (1961), 263–276.

[13] S. Serra Capizzano, *On the extreme spectral properties of Toeplitz matrices generated by L^1 functions with several minima/maxima,* BIT **36** (1996), 135–142.

[14] S. Serra Capizzano, *On the extreme eigenvalues of Hermitian (block) Toeplitz matrices,* Linear Algebra Appl. **270** (1998), 109–129.

[15] S. Serra Capizzano and P. Tilli, *Extreme singular values and eigenvalues of non-Hermitian block Toeplitz matrices,* J. Comput. Appl. Math. **108** (1999), 113–130.

[16] W. F. Trench, *Explicit inversion formulas for Toeplitz band matrices,* SIAM J. Algebraic Discrete Methods **6** (1985), 546–554.

[17] W. F. Trench, *Interlacement of the even and odd spectra of real symmetric Toeplitz matrices,* Linear Algebra Appl. **195** (1993), 59–68.

[18] H. Widom, *On the eigenvalues of certain Hermitian operators,* Trans. Amer. Math. Soc. **88** (1958), 491–522.

[19] N. L. Zamarashkin and E. E. Tyrtyshnikov, *Distribution of the eigenvalues and singular numbers of Toeplitz matrices under weakened requirements on the generating function,* Sb. Math. **188** (1997), 1191–1201.

[20] P. Zizler, R. A. Zuidwijk, K. F. Taylor, and S. Arimoto, *A finer aspect of eigenvalue distribution of selfadjoint band Toeplitz matrices,* SIAM J. Matrix Anal. Appl. **24** (2002), 59–67.

Albrecht Böttcher, Fakultät für Mathematik, TU Chemnitz, 09107 Chemnitz, Germany

`aboettch@mathematik.tu-chemnitz.de`

Sergei M. Grudsky, Departamento de Matemáticas, CINVESTAV del I.P.N., Apartado Postal 14-740, 07000 México, D.F., México

`grudsky@math.cinvestav.mx`

Egor A. Maksimenko, Departamento de Matemáticas, CINVESTAV del I.P.N., Apartado Postal 14-740, 07000 México, D.F., México

`emaximen@math.cinvestav.mx`