The finite section method and stable subsequences

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Abstract. The purpose of this note is to prove a sufficient and necessary criterion on the stability of a subsequence of the finite section method for a so-called band-dominated operator on $\ell^p(\mathbb{Z}^N, X)$. We hereby generalize previous results into several directions: We generalize the subsequence theorem from dimension $N = 1$ (see [11]) to arbitrary dimensions $N \geq 1$. Even for the case of the full sequence, our result is new in dimensions $N > 2$ and it corrects a mistake in the literature for $N = 2$. Finally, we allow the truncations to be taken by homothetic copies of very general starlike geometries $\Omega \in \mathbb{R}^N$ rather than convex polytopes.

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1 Introduction

Matrices. In this paper, we look at a truncation method for the approximate solution of certain operator equations $Au = b$ on the space $E := \ell^p(\mathbb{Z}^N, X)$ of functions $u : \mathbb{Z}^N \to X$ with

$$\|u\| \;=\; \left\{ \begin{array}{ll} \sqrt[p]{\sum_{k\in\mathbb{Z}^N} |u(k)|^p}, & p \in [1,\infty) \\ \sup_{k\in\mathbb{Z}^N} |u(k)|, & p = \infty \end{array} \right\} \;<\; \infty,$$

where $N \in \mathbb{N}$, $p \in [1, \infty]$ and X is an arbitrary complex Banach space. The operators A that we have in mind are bounded linear operators $E \to E$ which are induced, via

$$(Au)(i) \;=\; \sum_{j\in\mathbb{Z}^N} a_{ij}\, u(j), \qquad i \in \mathbb{Z}^N, \tag{1}$$

by a matrix $(a_{ij})_{i,j\in\mathbb{Z}^N}$ with operator entries $a_{ij} : X \to X$. Among those operators we call A a *band operator* if it is induced by a banded matrix, i.e. $a_{ij} = 0$ if $|i - j|$ is large enough, and we call A a *band-dominated operator* and write $A \in \mathrm{BDO}(E)$ if A is the limit, with respect to the operator norm induced by the norm on E, of a sequence of band operators. Also for $A \in \mathrm{BDO}(E)$, there is a unique matrix $(a_{ij})_{i,j\in\mathbb{Z}^N}$ which induces A via (1); we denote it by $[A]$.

Finite Sections. If $A \in \mathrm{BDO}(E)$ is invertible then $Au = b$ has a unique solution $u \in E$ for every right-hand side $b \in E$. An analytic computation of u, however, is in general not possible which is why one uses approximation methods. One of the most popular approximation methods is as follows: Choose a finite set $\Omega \subseteq \mathbb{Z}^N$, restrict the right-hand side b to Ω (call the restriction $\tilde b$, say) and look for a function $\tilde u : \Omega \to X$ that solves the truncated equation $\tilde A \tilde u = \tilde b$ with $[\tilde A] = (a_{ij})_{i,j\in\Omega}$ which is now an equation on a finite domain and in finitely many variables in X. Now increase Ω – for example by a concentric scaling – and watch the solutions $\tilde u$ evolve. The hope behind this procedure is that, given $Au = b$ is uniquely solvable, also $\tilde A \tilde u = \tilde b$ is uniquely solvable (at least once Ω is big enough) and the solution $\tilde u$ approximates the exact solution u in some way. This procedure is called the *finite section method*. Here is how we will formulate it:

Definition 1.1 *Let $v \in \mathbb{N}$ and $\omega_1, ..., \omega_v \in \mathbb{Z}^N$ be such that 0 is an interior point (w.r.t. $\mathbb{R}^N$) of the polytope $\Omega := \operatorname{conv}\{\omega_1, ..., \omega_v\} \subseteq \mathbb{R}^N$. Sets $\Omega \subset \mathbb{R}^N$ that can be written in this form will henceforth be referred to as valid polytopes.*

Now, for every $n \in \mathbb{N}$, put

$$\Omega_n \ := \ n\Omega \cap \mathbb{Z}^N \qquad \text{and} \qquad P_n \ := \ P_{\Omega_n}, \tag{2}$$

where, for a set $U \subseteq \mathbb{Z}^N$, by $P_U : E \to E$ we denote the operator of multiplication by the characteristic function χ_U of U. The finite section method consists in solving the equation

$$P_n A P_n \tilde{u}_n \ = \ P_n b \tag{3}$$

for large values $n \in \mathbb{N}$. The method is called *applicable* if there exists an $n_0 \in \mathbb{N}$ such that, for every $b \in E$, (3) is uniquely solvable for all $n \geq n_0$ and if the sequence $(\tilde{u}_n)$ of solutions is bounded in E and converges componentwise to the exact solution u of $Au = b$ as $n \to \infty$.

This truncation procedure is, of course, a very natural idea, and the fact that it can be performed on all infinite matrices creates the temptation to simply use it and keep fingers crossed it will work. A positive outcome, however, i.e. applicability as defined above, is in general far from guaranteed.

Example 1.2 Consider the shift operator $A = V_c : u \mapsto v$ on E with $u(k) = v(k + c)$ for every $k \in \mathbb{Z}^N$ and a fixed nonzero vector $c \in \mathbb{Z}^N$. Then V_c is invertible on E but since V_c maps functions with support in Ω_n to functions supported in $\Omega_n + c$, the truncated equation (3) is not solvable for general right-hand sides (and even if it is solvable, the solution is not unique) – no matter how big n is and how Ω is chosen. □

By [4, Corollary 1.77] (which is a consequence of [10, Theorem 6.1.3]) one has that the finite section method (3) is applicable iff A is invertible and the sequence

$$(P_n A P_n + Q_n)_{n \in \mathbb{N}} \tag{4}$$

is stable. Here we have put $Q_n := I - P_n$ and we call a sequence $(A_n)_{n \in \mathbb{N}}$ of operators $A_n : E \to E$ *stable* if there exists an $n_0 \in \mathbb{N}$ such that all operators A_n with $n \geq n_0$ are invertible and $\sup_{n \geq n_0} \|A_n^{-1}\|$ is finite. Also note that $P_n A P_n + Q_n$ is invertible on E iff $P_n A P_n$ is invertible on the image of P_n and that $\|(P_n A P_n + Q_n)^{-1}\| = \max(1, \|(P_n A P_n|_{\operatorname{im} P_n})^{-1}\|)$.

So the key condition for the applicability of (3) is the stability of the sequence (4), which, by the way, will be shown to automatically imply the other condition too: invertibility of A. In fact, one can show [3, 4, 10, 13] the following.

The sequence (4) is stable iff all operators in an associated set $\sigma^{\mathsf{stab}}_{\Omega}(A)$ are invertible and if their inverses are uniformly bounded. The elements of $\sigma^{\mathsf{stab}}_{\Omega}(A)$ are known and include A itself; the set depends, apart from the operator A, on the geometry of Ω.	(5)

Example 1.3 Let $N = 1$ and consider the operator A induced by the block diagonal matrix

$$\operatorname{diag}\left(\cdots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \cdots\right)$$

with the single 1 entry at position zero. Then $A = A^{-1}$ is invertible and, for $\Omega = [-1, 1]$, its truncations $P_n A P_n$ correspond to the finite $(2n+1) \times (2n+1)$ matrices

$$\operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \cdots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \cdots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

if n is even and to

$$\operatorname{diag}\left(0, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \cdots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \cdots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0\right)$$

if n is odd. So sequence (4) is not stable since all its entries with an odd n are non-invertible. The associated set $\sigma_\Omega^{\mathsf{stab}}(A)$ consists in this example of five operators. They are A,

$$B = \operatorname{diag}\left(\cdots, 1, 1, 1, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \cdots\right), \quad C = \operatorname{diag}\left(\cdots, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), 1, 1, 1, \cdots\right),$$
$$D = \operatorname{diag}\left(\cdots, 1, 1, 0, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \cdots\right), \quad F = \operatorname{diag}\left(\cdots, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), 0, 1, 1, \cdots\right),$$

out of which only A, B and C are invertible. $\square$

The Philosophy. As we have just seen, the finite section method cannot be expected to work for every operator A. But in some cases it is possible to "adjust" the method to the operator at hand by choosing the right geometry Ω and an appropriate subsequence of (4). The philosophy here is to give the operator A the chance to impose some of its "personality" on the (otherwise too "impersonal") method of finite sections. In the previous example, for instance, one simply has to remove all elements from the sequence (4) that correspond to an odd value of n to get a stable approximation method for A. We believe that, for a given operator A, finding the right geometry Ω and an appropriate index set $\mathcal{I} \subseteq \mathbb{N}$ such that $(P_nAP_n + Q_n)_{n\in\mathcal{I}}$ is stable is a major task in the numerical analysis of the equation $Au = b$. The following observation (6) helps to translate this task into a different, and sometimes more tractable, language.

Subsequences of Finite Sections. In [11] the following observation was made in the one-dimensional case ($N = 1$):

> An infinite subsequence $(P_nAP_n + Q_n)_{n\in\mathcal{I}}$ of (4), with index set $\mathcal{I} \subseteq \mathbb{N}$, is stable iff all operators in an associated set $\sigma_{\Omega,\mathcal{I}}^{\mathsf{stab}}(A)$ are invertible with uniformly bounded inverses.
>
> The set $\sigma_{\Omega,\mathcal{I}}^{\mathsf{stab}}(A)$ depends, in addition to A and Ω, on the index set $\mathcal{I} \subseteq \mathbb{N}$. (6)
> It holds that $\sigma_{\Omega,\mathcal{I}}^{\mathsf{stab}}(A) \subseteq \sigma_{\Omega,\mathcal{J}}^{\mathsf{stab}}(A)$ if $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathbb{N}$ and $\sigma_{\Omega,\mathbb{N}}^{\mathsf{stab}}(A) = \sigma_\Omega^{\mathsf{stab}}(A)$.
>
> We call $\sigma_{\Omega,\mathcal{I}}^{\mathsf{stab}}(A)$ the *stability spectrum of A with respect to Ω and index set $\mathcal{I} \subseteq \mathbb{N}$*.

This generalization, (6), of the old result (5) from $\mathcal{I} = \mathbb{N}$ to $\mathcal{I} \subseteq \mathbb{N}$ has two important consequences: Firstly, if the whole sequence (4) is not stable then one might be able, via the new result, to detect a stable subsequence (with index set $\mathcal{I} \subseteq \mathbb{N}$, say) and to solve (3) for $n \in \mathcal{I}$ only – thereby still approximately solving $Au = b$. Secondly, the observation (6) was used to remove the uniform boundedness condition from the same statement (6) and hence also from (5), which is the main result of [11].

If we put $\mathcal{I} = 2\mathbb{N}$ and $\mathcal{J} = 2\mathbb{N}+1$ in Example 1.3 then it turns out that $\sigma_{\Omega,\mathcal{I}}^{\mathsf{stab}}(A) = \{A, B, C\}$ while $\sigma_{\Omega,\mathcal{J}}^{\mathsf{stab}}(A) = \{A, D, F\}$, so that, by (6), the finite section subsequence corresponding to all even n is stable. Of course this just confirms what we already observed directly in Example 1.3 but there are, however, examples in which the detection of a less obvious stable subsequence of (4) is possible via (6).

What's new? The purpose of this note is to generalize statement (6) from dimension $N = 1$ (see [11]) to arbitrary dimensions $N \geq 1$. Even for the case $\mathcal{I} = \mathbb{N}$ of the full sequence, our result is new in dimensions $N > 2$ (and it corrects an error in the literature for $N = 2$). Another direction of generalization is that we can go away from truncations (2) by homothetic copies of a convex polytope $\Omega \in \mathbb{R}^N$ with integer vertices and pass to more general geometries instead.

The question of the uniform boundedness condition in (6), however, is much more subtle if $N > 1$ than it was in [11] for $N = 1$. We will say a bit about this in Section 6.

2 Preliminaries

Let $E = \ell^p(\mathbb{Z}^N, X)$, $A \in \mathrm{BDO}(E)$ and $\Omega = \mathrm{conv}\{\omega_1, ..., \omega_v\} \subset \mathbb{R}^N$ a valid polytope as in Definition 1.1. For an infinite index set $\mathcal{I} = \{n_1, n_2, ...\} \subseteq \mathbb{N}$, we want to study the stability of the operator sequence

$$(P_n A P_n + Q_n)_{n\in\mathcal{I}} \;=\; (P_{n_i} A P_{n_i} + Q_{n_i})_{i=1}^{\infty}, \tag{7}$$

where we suppose that $n_1, n_2, ...$ is a strictly monotonous enumeration of $\mathcal{I}$. For the study of this sequence as one item, we will assemble it to a single operator. To do this, let

$$A_i \;:=\; \begin{cases} P_{n_i} A P_{n_i} + Q_{n_i}, & i \in \mathbb{N}, \\ I, & i \in \mathbb{Z} \setminus \mathbb{N}, \end{cases} \tag{8}$$

put $E' := \ell^p(\mathbb{Z}^{N+1}, X)$, thought of as $\ell^p(\mathbb{Z}, E)$, and write $\oplus A_i$ for the map $u \mapsto v$ on E' with

$$v(j,i) \;=\; (A_i\, u(\cdot, i))(j), \qquad j \in \mathbb{Z}^N,\ i \in \mathbb{Z}. \tag{9}$$

In other words, we think of $u \in E'$ as decomposed into layers $u(\cdot, i) \in E$, $i \in \mathbb{Z}$, and let each A_i act on the $i-$th layer of u. We will therefore refer to A_i as the $i-$th layer of $\oplus A_i$. One can show that then $\oplus A_i \in \mathrm{BDO}(E')$.

A key argument in [3, 4, 10, 13] is that the stability of (7) is equivalent to $\oplus A_i$ being invertible at infinity. Here we say that an operator $B \in \mathrm{BDO}(E')$ is *invertible at infinity* if there exist $C, D \in \mathrm{BDO}(E')$ and an $m \in \mathbb{N}$ such that $CB\Theta_m = \Theta_m = \Theta_m BD$ holds, where Θ_m is the operator of multiplication by the characteristic function of $\mathbb{Z}^{N+1} \setminus \{-m, ..., m\}^{N+1}$.

So it remains to study invertibility at infinity of $\oplus A_i$. This is done in terms of so-called limit operators. The idea is to reflect the behaviour of an operator $B \in \mathrm{BDO}(E')$ at infinity by a family of operators on E' and to evaluate this family. To do this, we need two notations. Firstly, for $B, B_1, B_2, ... \in \mathrm{BDO}(E')$, we write $B = \mathcal{P}'\text{-}\lim B_n$ if $[B_n]$ converges entrywise (in the norm of $L(X)$) to $[B]$ as $n \to \infty$ and if $\sup_n \|B_n\| < \infty$. Secondly, for $\alpha \in \mathbb{Z}^{N+1}$, let $V'_\alpha : E' \to E'$ denote the shift operator with $(V'_\alpha u)(k) = u(k - \alpha)$ for all $k \in \mathbb{Z}^{N+1}$ and $u \in E'$.

If $B \in \mathrm{BDO}(E')$, $h = (h(1), h(2), ...) \subseteq \mathbb{Z}^{N+1}$ is a sequence with $|h(n)| \to \infty$ and the operator sequence $V'_{-h(n)} B V'_{h(n)}$ is $\mathcal{P}'-$convergent as $n \to \infty$ then its limit will be denoted by B_h and is called *limit operator of* B *w.r.t. the sequence* h. In a completely analogous fashion, one defines limit operators in $\mathrm{BDO}(E)$. To distinguish between operators on E' and on E we write $\mathcal{P}$-lim and V_α with $\alpha \in \mathbb{Z}^N$ if we are in the E setting. Different sequences h generally lead to different limit operators and often the sequence $V_{-h(n)} B V_{h(n)}$ does not $\mathcal{P}-$converge at all. We will call $B \in \mathrm{BDO}(E)$ a *rich* operator if every sequence $h = (h(1), h(2), ...) \subseteq \mathbb{Z}^N$ with $|h(n)| \to \infty$ has a subsequence g such that the limit operator B_g exists.

As a final preparation, we turn our attention to the geometry of Ω. Let $\Gamma := \partial\Omega$ be the boundary of Ω and, for every $n \in \mathbb{N}$, put

$$\Gamma_n \;:=\; n\Gamma \cap \mathbb{Z}^N \qquad \text{and then let} \qquad \Gamma_{\mathcal{I}} \;:=\; \bigcup_{n\in\mathcal{I}} \Gamma_n. \tag{10}$$

For a sequence $h = (h(1), h(2), ...) \subseteq \Gamma_{\mathcal{I}}$, say $h(k) \in \Gamma_{m_k}$ for some $m_k \in \mathcal{I}$, and a set $S \subseteq \mathbb{Z}^N$, we call S the *geometric limit of* Ω *w.r.t.* h and write $S = \Omega_h$ if, for every $m \in \mathbb{N}$, there exists a $k_0 \in \mathbb{N}$ such that

$$(\Omega_{m_k} - h(k)) \cap \{-m, ..., m\}^N \;=\; S \cap \{-m, ..., m\}^N, \qquad k \geq k_0.$$

Note that in this case $V_{-h(k)} P_{m_k} V_{h(k)}$ is $\mathcal{P}-$convergent to P_S as $k \to \infty$. For a polytope Ω, the only candidates for the geometric limit S w.r.t a sequence $h \subseteq \Gamma_{\mathcal{I}}$ are intersections of finitely many half spaces and $\mathbb{Z}^N$ (discrete half spaces, edges, corners, etc.).

3 The stability theorem for subsequences

Given a rich operator $A \in \mathrm{BDO}(E)$ on $E = \ell^p(\mathbb{Z}^N, X)$ with $p \in [1, \infty]$, $N \in \mathbb{N}$ and a complex Banach space X, a valid polytope $\Omega \in \mathbb{R}^N$, and an index set $\mathcal{I} = \{n_1, n_2, ...\} \subseteq \mathbb{N}$ with $n_1 < n_2 < \cdots$, we put

$$\mathcal{H}_{\Omega,\mathcal{I}}(A) \;:=\; \{\, h = (h(1), h(2), ...) \,:\, h(k) \in \Gamma_{\mathcal{I}} \;\forall k,\; |h(k)| \to \infty,\; A_h \text{ exists},\; \Omega_h \text{ exists} \,\}$$

and

$$\sigma^{\mathsf{stab}}_{\Omega,\mathcal{I}}(A) \;:=\; \{A\} \cup \{P_{\Omega_h} A_h P_{\Omega_h} + Q_{\Omega_h} \,:\, h \in \mathcal{H}_{\Omega,\mathcal{I}}(A)\}. \tag{11}$$

Then the following theorem holds.

Theorem 3.1 *Under the conditions mentioned above, the following are equivalent.*

(i) *The sequence* $(P_{n_i} A P_{n_i} + Q_{n_i})_{i=1}^{\infty}$ *is stable.*

(ii) *The operator* $\oplus A_i$*, with* A_i *as in (8), is invertible at infinity.*

(iii) *All operators in* $\sigma^{\mathsf{stab}}_{\Omega,\mathcal{I}}(A)$ *are invertible and their inverses are uniformly bounded.*

For dimension $N = 1$, our statement coincides with a two-sided version of [11, Theorem 3]. As such it generalizes [9, Theorem 3] (also see [10, Theorem 6.2.2], [4, Theorem 4.2] and [13, Theorem 2.7]) from the full sequence $\mathcal{I} = \mathbb{N}$ to arbitrary infinite subsequences with index set $\mathcal{I} \subseteq \mathbb{N}$. For $N = 2$, our Theorem 3.1, together with (11), corrects another version of the stability spectrum (see (16) and Example 4.1 below) that was previously suggested in the literature (see [9, 10]) for $\mathcal{I} = \mathbb{N}$. Moreover, our result demonstrates how to deal with subsequences $\mathcal{I} \subseteq \mathbb{N}$ by restricting consideration to sequences $h = (h(1), h(2), ...)$ with values in the set $\Gamma_{\mathcal{I}} = \cup_{n \in \mathcal{I}} \Gamma_n$. For dimensions $N > 2$, to our knowledge, the result is new – even in the case $\mathcal{I} = \mathbb{N}$.

Proof of Theorem 3.1. To simplify the following, we put $n_i := 0$ for all $i \in \mathbb{Z} \setminus \mathbb{N}$ which has the effect that (A_i) from (8) can be written as $P_{n_i} A P_{n_i} + Q_{n_i}$ for all $i \in \mathbb{Z}$ with $P_0 := 0$ and $Q_0 := I$.

Now we start by reformulating (ii) in terms of limit operators of $\oplus A_i$. By [4, Proposition 2.22 b)] and $A \in \mathrm{BDO}(E)$ it follows that $\oplus A_i \in \mathrm{BDO}(E')$. Since $\oplus P_{n_i}$ and $\oplus A$ are rich if A is rich, we have that $\oplus A_i = (\oplus P_{n_i})(\oplus A)(\oplus P_{n_i}) + I' - (\oplus P_{n_i})$ is rich with I' denoting the identity operator on E'. Consequently, [4, Theorem 1] is applicable and shows that $\oplus A_i$ is invertible at infinity iff all its limit operators are invertible and their inverses are uniformly bounded. (In case $p = \infty$, it is required in [4, Theorem 1] that the operator has a so-called preadjoint – an operator whose adjoint it is. Note that this requirement recently turned out to be unnecessary [1, Theorem 6.28(iii)].)

The equivalence of (i) and (ii) follows from [4, Theorem 2.28].

$(ii) \Rightarrow (iii)$: Let $\oplus A_i$ be invertible at infinity. From the discussion at the beginning of the proof we know that all limit operators of $\oplus A_i$ are invertible and that there is a uniform upper bound $C > 0$ on the norms of their inverses. We use this to prove (iii). Firstly, take $h = (h(1), h(2), ...) \subseteq \mathbb{Z}^{N+1} = \mathbb{Z}^N \times \mathbb{Z}$ with $h(m) = (0, m)$ for $m \in \mathbb{N}$. Then, since 0 is an interior point of Ω, it is easy to see that $(\oplus A_i)_h = \oplus A$. From this and the invertibility of all limit operators of $\oplus A_i$ (with uniform bound C on the inverses) we get invertibility of A and $\|A^{-1}\| = \|(\oplus A)^{-1}\| \le C$. Secondly, take an arbitrary $\alpha = (\alpha(1), \alpha(2), ...) \in \mathcal{H}_{\Omega,\mathcal{I}}(A)$ with $\alpha(m) \in \Gamma_{n_{\beta(m)}}$ for some $\beta(m) \in \mathbb{N}$ and put $h(m) = (\alpha(m), \beta(m))$ for every $m \in \mathbb{N}$. Then also the following limit operator of $\oplus A_i$ is invertible and the norm of its inverse is bounded by C:

$$
\begin{aligned}
\mathcal{P}'\text{-}\lim\ V'_{-h(m)}(\oplus A_i)V'_{h(m)} &= \mathcal{P}'\text{-}\lim\ V'_{(-\alpha(m),-\beta(m))}(\oplus A_i)V'_{(\alpha(m),\beta(m))} \\
&= \mathcal{P}'\text{-}\lim_{m\to+\infty} \bigoplus_{i\in\mathbb{Z}} \big(V_{-\alpha(m)}\, A_{\beta(m)+i}\, V_{\alpha(m)}\big) \;=\; \oplus_i\ \mathcal{P}\text{-}\lim_{m\to+\infty}\ V_{-\alpha(m)}\, A_{\beta(m)+i}\, V_{\alpha(m)} \\
&= \oplus_i\ \mathcal{P}\text{-}\lim_{m\to+\infty}\ V_{-\alpha(m)}\,(P_{n_{\beta(m)+i}} A P_{n_{\beta(m)+i}} + Q_{n_{\beta(m)+i}})\, V_{\alpha(m)} \qquad (12) \\
&= \oplus_i\ \mathcal{P}\text{-}\lim_{m\to+\infty} \Big((V_{-\alpha(m)} P_{n_{\beta(m)+i}} V_{\alpha(m)})(V_{-\alpha(m)} A V_{\alpha(m)})(V_{-\alpha(m)} P_{n_{\beta(m)+i}} V_{\alpha(m)}) \\
&\qquad\qquad + \ V_{-\alpha(m)} Q_{n_{\beta(m)+i}} V_{\alpha(m)}\Big)
\end{aligned}
$$

So in particular, its $i = 0-$th layer is invertible with its inverse bounded by C. But this operator is equal to $P_{\Omega_\alpha} A_\alpha P_{\Omega_\alpha} + Q_{\Omega_\alpha}$ by the compatibility of $\mathcal{P}$-lim with addition and composition and by our assumption $\alpha \in \mathcal{H}_{\Omega,\mathcal{I}}(A)$.

$(iii) \Rightarrow (ii)$: Suppose all operators in $\sigma^{\mathsf{stab}}_{\Omega,\mathcal{I}}(A)$ are invertible and their inverses are bounded by a constant $C > 1$. We will show that the same holds for all limit operators of $\oplus A_i$ which implies (ii) by the discussion at the beginning of this proof. So let L be an arbitrary limit operator of $\oplus A_i$, say w.r.t. the sequence $h = (h(1), h(2), ...) \in \mathbb{Z}^{N+1}$ with $h(m) = (\alpha(m), \beta(m))$ where $\alpha(m) \in \mathbb{Z}^N$ and $\beta(m) \in \mathbb{Z}$ for all $m \in \mathbb{N}$ and $|h(m)| \to \infty$. To understand the operator L, we will pass to a suitable subsequence of h which, clearly, does not change the limit operator L. By passing to a subsequence of h, it can be arranged that one of the following four cases holds.

Case 1. $\beta(m) \not\to +\infty$. Then we can choose an infinite subset $M \subseteq \mathbb{N}$ such that $\beta|_M$ either tends to $-\infty$ or is bounded. In either case it is easy to see that $L = I'$ and $\|L^{-1}\| = 1 < C$.

Case 2. $\beta(m) \to +\infty$ and $\operatorname{dist}(\alpha(m), \Omega_{n_{\beta(m)}}) \to \infty$. Also then, clearly, $L = I'$.

Case 3. $\beta(m) \to +\infty$, $\alpha(m) \in \Omega_{n_{\beta(m)}}$ for all m and $\operatorname{dist}(\alpha(m), \Gamma_{n_{\beta(m)}}) \to \infty$. Note that under these conditions (also see (13) and Remark 3.2 below), $\operatorname{dist}(\alpha(m), n_{\beta(m)}\Gamma) \to \infty$ as $m \to \infty$, whence $\mathcal{P}\text{-}\lim V_{-\alpha(m)} P_m V_{\alpha(m)} = I$. Now consider these two subcases:

Case 3.1. If $|\alpha(m)| \to \infty$, choose an infinite subset M of $\mathbb{N}$ such that A has a limit operator w.r.t. the remaining subsequence $\alpha|_M$, for simplicity again denoted by α, which is possible since A is rich. Then $L = \oplus_i A_\alpha$, which is invertible since A_α is invertible by the invertibility of A. Also $\|L^{-1}\| = \|A_\alpha^{-1}\| \le \|A^{-1}\| \le C$.

Case 3.2. If $|\alpha(m)| \not\to \infty$ then α has a bounded and therefore even a constant subsequence. So take $M \subseteq \mathbb{N}$ such that $\alpha|_M \equiv: d \in \mathbb{Z}^N$. Then $L = \oplus_i (V_{-d} A V_d)$ is invertible since A is invertible, and $\|L^{-1}\| = \|A^{-1}\| \le C$.

Case 4. $\beta(m) \to +\infty$ and $\operatorname{dist}(\alpha(m), \Gamma_{n_{\beta(m)}})$ remains bounded. For $m \in \mathbb{N}$ and $i \in \mathbb{Z}$, put

$$
\gamma^{(i)}(m) \;:=\; \operatorname{argmin}\{\, |\alpha(m) - \gamma| \;:\; \gamma \in \Gamma_{n_{\beta(m)+i}} \} \qquad \text{and} \qquad \delta^{(i)}(m) \;:=\; \alpha(m) - \gamma^{(i)}(m).
$$

By our condition, $\delta^{(0)}$ is bounded in $\mathbb{Z}^N$. Choose $M \subseteq \mathbb{N}$ such that $\delta^{(0)}|_M$ is constant, say equal to $\delta^{(0)}_\infty \in \mathbb{Z}^N$. For every $i \in \mathbb{Z} \setminus \{0\}$, $\delta^{(i)}|_M$ either tends to infinity (in absolute value) or it has a constant subsequence. By a simple diagonal construction, we can pass to a subset of M (for simplicity again denoted by M) such that, for every $i \in \mathbb{Z}$, $\delta^{(i)}|_M$ either tends to infinity or is constant, say equal to $\delta^{(i)}_\infty \in \mathbb{Z}^N$. The set of all $i \in \mathbb{Z}$ for which $\delta^{(i)}|_M$ is constant will be denoted by $\mathbb{Z}_{\mathsf{finite}}$; otherwise, i.e. if $|\delta^{(i)}|_M| \to \infty$, we will write $i \in \mathbb{Z}^+_\infty$ if $\alpha(m) \in \Omega_{n_{\beta(m)+i}}$ as $m \to \infty$ and $i \in \mathbb{Z}^-_\infty$ if $\alpha(m) \notin \Omega_{n_{\beta(m)+i}}$ as $m \to \infty$ (note that it can be arranged in the choice of the subsequence above that $\alpha(m)$ is either $\in$ or $\notin$ of $\Omega_{n_{\beta(m)+i}}$ for all $m > m_0$, say). Finally, again by a diagonal procedure, pass to an infinite subset of M, again denoted by M, such that, for every $i \in \mathbb{Z}$, the geometric limit $\Omega_{\gamma^{(i)}|_M}$ exists (see the construction in the proof of [8, Proposition

5] or [10, Theorem 2.1.16]) and the limit operators $A_{\gamma^{(i)}|_M}$ and $A_{\alpha|_M}$ exist (possible since A is rich). Abbreviating $\alpha|_M$, $\gamma^{(i)}|_M$ and $\delta^{(i)}|_M$ by α, $\gamma^{(i)}$ and $\delta^{(i)}$, respectively, and repeating the previously performed computations up to line (12), we get that $L = \oplus L_i$, where the $i-$th layer of L is

$$\begin{aligned}
L_i &= \mathcal{P}\text{-}\lim_{m\to+\infty} V_{-\alpha(m)} \left(P_{n_{\beta(m)+i}} A P_{n_{\beta(m)+i}} + Q_{n_{\beta(m)+i}}\right) V_{\alpha(m)} \\
&= \mathcal{P}\text{-}\lim_{m\to+\infty} V_{-\delta^{(i)}(m)} V_{-\gamma^{(i)}(m)} \left(P_{n_{\beta(m)+i}} A P_{n_{\beta(m)+i}} + Q_{n_{\beta(m)+i}}\right) V_{\gamma^{(i)}(m)} V_{\delta^{(i)}(m)} \\
&= \mathcal{P}\text{-}\lim_{m\to+\infty} V_{-\delta^{(i)}_\infty} \Big(P_{\Omega_{n_{\beta(m)+i}}-\gamma^{(i)}(m)} V_{-\gamma^{(i)}(m)} A V_{\gamma^{(i)}(m)} P_{\Omega_{n_{\beta(m)+i}}-\gamma^{(i)}(m)} \\
&\qquad\qquad + \; Q_{\Omega_{n_{\beta(m)+i}}-\gamma^{(i)}(m)}\Big) V_{\delta^{(i)}_\infty} \\
&= V_{-\delta^{(i)}_\infty} \left(P_{\Omega_{\gamma^{(i)}}} A_{\gamma^{(i)}} P_{\Omega_{\gamma^{(i)}}} \; + \; Q_{\Omega_{\gamma^{(i)}}}\right) V_{\delta^{(i)}_\infty}
\end{aligned}$$

if $i \in \mathbb{Z}_{\mathsf{finite}}$ and

$$\begin{aligned}
L_i &= \mathcal{P}\text{-}\lim_{m\to+\infty} V_{-\alpha(m)} \left(P_{n_{\beta(m)+i}} A P_{n_{\beta(m)+i}} + Q_{n_{\beta(m)+i}}\right) V_{\alpha(m)} \\
&= \mathcal{P}\text{-}\lim_{m\to+\infty} \left(P_{\Omega_{n_{\beta(m)+i}}-\alpha(m)} V_{-\alpha(m)} A V_{\alpha(m)} P_{\Omega_{n_{\beta(m)+i}}-\alpha(m)} \; + \; Q_{\Omega_{n_{\beta(m)+i}}-\alpha(m)}\right) \\
&= \begin{cases} A_\alpha, & i \in \mathbb{Z}^+_\infty, \\ I, & i \in \mathbb{Z}^-_\infty \end{cases}
\end{aligned}$$

if $i \in \mathbb{Z} \setminus \mathbb{Z}_{\mathsf{finite}}$. In either case, L_i is invertible and $\|L_i^{-1}\| \le C$ by (iii). Consequently, L is invertible and $\|L^{-1}\| = \sup_i \|L_i^{-1}\| \le C$. ■

Remark 3.2 In case 3 of the proof we used the implication

$$\operatorname{dist}(x_n, \Gamma_n) \to \infty \qquad \Longrightarrow \qquad \operatorname{dist}(x_n, n\Gamma) \to \infty \tag{13}$$

for arbitrary points $x_n \in \mathbb{Z}^N$. It is easy to see that (13) is equivalent to

$$\operatorname{dist}(x_n, \Gamma_n) \;\le\; \operatorname{dist}(x_n, n\Gamma) + \delta \tag{14}$$

with a global finite constant $\delta := \max_{\gamma\in n\Gamma} \operatorname{dist}(\gamma, \Gamma_n)$ independent of n, so that the $\delta-$neighbourhood of every $\gamma \in n\Gamma$ contains a point from Γ_n (that is, Γ_n is "relatively dense" in $n\Gamma$). For convex polytopes Ω with vertices in $\mathbb{Z}^N$, it is clear that (14) and hence (13) holds – with a constant δ that is the maximum of the respective constants for the finitely many facets of Ω. □

Definition 3.3 *We will say that $\Omega \subset \mathbb{R}^N$ is a valid starlike set if Ω is bounded, nonempty and has the property that, for every $x \in \Omega$ and $\alpha \in [0, 1)$, αx is an interior point of Ω.*

So in particular, 0 is an interior point of every valid starlike set. Moreover, all bounded convex sets $\Omega \subset \mathbb{R}^N$ with interior point 0 are valid starlike sets. We claim that we can prove a version of Theorem 3.1 in the much more general setting of a valid starlike set Ω. Our reason for this choice of geometry (as opposed to convex polytopes with integer vertices) is, of course, more generality (including e.g. the ball) but at the same time still to make sure that the boundaries of $m\Omega$ and $n\Omega$ are disjoint if $m \ne n$. However, for valid starlike sets, the implication (13) is in general not true:

Example 3.4 For example, let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, |x_1|^3 + |x_2|^3 \le 1\}$ and put $\Gamma := \partial\Omega$ and $\Gamma_n := (n\Gamma) \cap \mathbb{Z}^N$ for $n = 1, 2, \ldots$. Then it is well-known (this is a result of Euler and of course a special case of Fermat's Last Theorem) that every Γ_n only consists of the four points $(\pm n, 0)$ and $(0, \pm n)$, so that (13) and (14) clearly fail. □

The workaround is to use “fat boundaries”: If we replace the first definition in (10) by

$$\Gamma_n \ := \ (n\Gamma + H) \cap \mathbb{Z}^N \qquad \text{with} \qquad H = (-1/2\,,\,1/2\,]^N \tag{15}$$

then (13) and (14) always hold with $\delta = 1$ (distances measured in the $|\cdot|_\infty$ metric). As the rest of the proof of Theorem 3.1 carries over to valid starlike sets, we get the following generalization.

Theorem 3.5 *Let $A \in \mathrm{BDO}(E)$ be rich, $\Omega \subset \mathbb{R}^N$ a valid starlike set, and $\mathcal{I} = \{n_1, n_2, ...\} \subseteq \mathbb{N}$ an infinite index set with $n_1 < n_2 < \cdots$. Then, with $\Gamma_{\mathcal{I}}$, $\mathcal{H}_{\Omega,\mathcal{I}}(A)$ and $\sigma^{\mathsf{stab}}_{\Omega,\mathcal{I}}(A)$ defined as before but now with Γ_n given by (15) for every $n \in \mathbb{N}$, the following are equivalent.*

(i) *The sequence $(P_{n_i} A P_{n_i} + Q_{n_i})_{i=1}^\infty$ is stable.*

(ii) *The operator $\oplus A_i$, with A_i as in (8), is invertible at infinity.*

(iii) *All operators in $\sigma^{\mathsf{stab}}_{\Omega,\mathcal{I}}(A)$ are invertible and their inverses are uniformly bounded.*

Unlike for the polytopes in Theorem 3.1, for valid starlike sets $\Omega \subset \mathbb{R}^N$, there can be an infinite amount of different geometric limits Ω_h. For example, if $N = 2$ and Ω is the unit disk $\{(x_1, x_2) : |x_1|^2 + |x_2|^2 \leq 1\}$ in $\mathbb{R}^2$, then all discrete half planes $\{(x_1, x_2) \in \mathbb{Z}^2 : ax_1 + bx_2 < 0\}$ with $(a, b) \in \mathbb{R}^2 \setminus \{0\}$ occur as geometric limits, but also the same sets with one additional point (x_1, x_2) with $ax_1 + bx_2 = 0$ (e.g. look at $h(n) = (c, n)$ for fixed $c \in \mathbb{Z}$ and all $n \in \mathbb{N}$ – note that $h(n) \in \Gamma_n$ for all sufficiently large n – to see this effect for the case $(a, b) = (0, 1)$) are geometric limits of the disk Ω. In Example 3.4 the same discrete half planes occur but only those with a fully vertical or horizontal ascent appear again with an additional point.

4 Examples

As a particularly illustrative and not too difficult class of examples, we will look at operators that are induced by an adjacency matrix. Therefore, let $\mathcal{E}$ denote a set of pairwise disjoint doubletons $\{i, j\}$ with $i, j \in \mathbb{Z}^N$, $i \neq j$, and put

$$a_{ij} \ := \ \begin{cases} I_X, & \text{if } \{i,j\} \in \mathcal{E} \text{ or } i = j \notin \bigcup\limits_{e \in \mathcal{E}} e, \\ 0_X, & \text{otherwise,} \end{cases}$$

for all $i, j \in \mathbb{Z}^N$, where I_X and 0_X stand for the identity and zero operator, respectively, on the Banach space X at hand. Then $(a_{ij})_{i,j \in \mathbb{Z}^N}$ is the *extended adjacency matrix* of the undirected graph $\mathcal{G} = (\mathbb{Z}^N, \mathcal{E})$ with vertex set $\mathbb{Z}^N$ and edges $\mathcal{E}$. We write $Adj(\mathcal{G})$ for the operator that is induced by this matrix (a_{ij}) and note that $Adj(\mathcal{G})$ is band-dominated iff $b := \sup_{\{i,j\} \in \mathcal{E}} |i - j|$ is finite, in which case $Adj(\mathcal{G})$ is even a band operator with band-width b.

If applied to an element $u \in E = \ell^p(\mathbb{Z}^N, X)$, the operator $Adj(\mathcal{G})$ “swaps” the values $u(i)$ and $u(j)$ around if $\{i, j\}$ is an edge of $\mathcal{G}$, and it leaves all values $u(k)$ untouched for which $k \in \mathbb{Z}^N$ is not part of an edge of $\mathcal{G}$. From this it is obvious that $\|Adj(\mathcal{G})\| = 1$ and that $Adj(\mathcal{G})$ is invertible and coincides with its inverse. Moreover, it is clear that, for $n \in \mathbb{N}$, the n-th finite section $P_n Adj(\mathcal{G}) P_n + Q_n$ is invertible iff each edge $e \in \mathcal{E}$ has either both or no vertices in $\Omega_n = n\Omega \cap \mathbb{Z}^N$. In the latter case, $P_n Adj(\mathcal{G}) P_n + Q_n$ equals $Adj(\mathcal{G}_n)$, where $\mathcal{G}_n = (\mathbb{Z}^N, \mathcal{E} \cap \Omega_n^2)$, is again its own inverse and has norm 1. So we get that, for $A = Adj(\mathcal{G})$, the sequence (7) is stable iff, for all sufficiently large $n \in \mathcal{I}$, each edge $e \in \mathcal{E}$ has either both or no vertices in Ω_n.

Note that Example 1.3 was already of the form $A = Adj(\mathcal{G})$, namely with $N = 1$ and

$$\mathcal{E} \ = \ \Big\{ ..., \{-4, -3\}, \{-2, -1\}, \{1, 2\}, \{3, 4\}, ... \Big\}.$$

Here Ω_n separates the two vertices of the edge $\{-n-1, -n\}$ and also of $\{n, n+1\}$ if n is odd.

We continue with two examples demonstrating that two particular sets of operators that are closely related to $\sigma_\Omega^{\mathsf{stab}}(A)$ – and that have, in the past, been suggested to replace (11) in the $N = 2$, $\mathcal{I} = \mathbb{N}$ version of Theorem 3.1 – are actually not stability spectra (meaning that Theorem 3.1 is incorrect for $\mathcal{I} = \mathbb{N}$ with $\sigma_\Omega^{\mathsf{stab}}(A)$ replaced by any of them) if $N > 1$. These two "non-replacements" for $\sigma_\Omega^{\mathsf{stab}}(A)$ are

$$\{A\} \;\cup\; \bigcup_{x\in\Gamma}\{P_{\Omega_x}BP_{\Omega_x} + Q_{\Omega_x} \;:\; B \in \sigma_x^{\mathsf{op}}(A)\} \tag{16}$$

and

$$\{A\} \;\cup\; \bigcup_{x\in\Gamma}\{P_{\Omega_x}BP_{\Omega_x} + Q_{\Omega_x} \;:\; B \in \sigma_{x,\mathsf{ray}}^{\mathsf{op}}(A)\}, \tag{17}$$

where $\Gamma = \partial\Omega$ and, for every $x \in \Gamma$, $\Omega_x \subseteq \mathbb{Z}^N$ is the limit of $n(\Omega - x) \cap \mathbb{Z}^N$ as $n \to \infty$ in the sense that, for each $m \in \mathbb{N}$,

$$n(\Omega - x) \cap \{-m, ..., m\}^N \;=\; \Omega_x \cap \{-m, ..., m\}^N$$

for all sufficiently large $n \in \mathbb{N}$. Finally, $\sigma_x^{\mathsf{op}}(A)$ is the set of all limit operators A_h of A with respect to sequences $h = (h(1), h(2), ...) \subseteq \mathbb{Z}^N$ going to infinity in the direction x, i.e. $h(n)/|h(n)| \to x/|x|$, and $\sigma_{x,\mathsf{ray}}^{\mathsf{op}}$ is the set of all limit operators A_h with respect to sequences of the form $h = ([m_1\, x], [m_2\, x], ...) \subseteq \mathbb{Z}^N$ where (m_n) is an unbounded monotonously increasing sequence of positive reals and $[\,\cdot\,]$ means componentwise rounding to the nearest integer.

Example 4.1 Take $N = 2$, $\Omega = [-1, 1]^2$ and let $A = Adj(\mathcal{G})$ with $\mathcal{G} = (\mathbb{Z}^2, \mathcal{E})$ and

$$\mathcal{E} = \Big\{\{\,(k^2 - k - 1, k^2)\,,\,(k^2 - k, k^2)\,\} \;:\; k = 1, 2, ...\Big\}.$$

Then, with respect to $h = (h(1), h(2), ...)$ with $h(k) = (k^2 - k - 1, k^2) \in \mathbb{Z}^2$, the limit operator of A exists and is equal to $B = Adj(\mathcal{G}')$, where $\mathcal{G}' = \Big(\mathbb{Z}^2, \Big\{\{(0, 0), (1, 0)\}\Big\}\Big)$. Since $h(k)/|h(k)| \to x/|x|$ with $x = (1, 1)$, we have that $B \in \sigma_x^{\mathsf{op}}(A)$. But $\Omega_x = \{..., -1, 0\}^2$ separates $(0, 0)$ from $(1, 0)$ so that $P_{\Omega_x}BP_{\Omega_x} + Q_{\Omega_x} \in (16)$ is not invertible. However, the whole finite section sequence (4) is stable since all edges $e \in \mathcal{E}$ have either both or no points in Ω_n, so that $P_nAP_n + Q_n = Adj(\mathcal{G}_n)$ with $\mathcal{G}_n = (\mathbb{Z}^2, \mathcal{E} \cap \Omega_n^2)$ for every $n \in \mathbb{N}$. So (16) is not a valid replacement of (11) as stability spectrum.

Note that the element of (11) that corresponds to the limit operator $B = A_h$ of A is $P_{\Omega_h}BP_{\Omega_h} + Q_{\Omega_h}$ with $\Omega_h = \mathbb{Z} \times \{..., -1, 0\}$ instead of $\{..., -1, 0\}^2$, which is again equal to B (since both $(0, 0)$ and $(1, 0)$ are in Ω_h) and hence invertible. □

Similarly, we can rule out (17) as stability spectrum by the following example:

Example 4.2 Again take $N = 2$, $\Omega = [-1, 1]^2$ and let $A = Adj(\mathcal{G})$ with $\mathcal{G} = (\mathbb{Z}^2, \mathcal{E})$ and

$$\mathcal{E} = \Big\{\{\,(k^2 - k, k^2)\,,\,(k^2 - k, k^2 + 1)\,\} \;:\; k = 1, 2, ...\Big\}.$$

Then, with respect to $h = (h(1), h(2), ...)$ with $h(k) = (k^2 - k, k^2) \in \mathbb{Z}^2$, the limit operator of A exists and is equal to $B = Adj(\mathcal{G}')$, where $\mathcal{G}' = \Big(\mathbb{Z}^N, \Big\{\{(0, 0), (0, 1)\}\Big\}\Big)$.

Again $B \in \sigma_x^{\mathsf{op}}(A)$ with $x = (1, 1)$. But $B \notin \sigma_{x,\mathsf{ray}}^{\mathsf{op}}(A)$ neither is B in $\sigma_{y,\mathsf{ray}}^{\mathsf{op}}(A)$ for any other $y \in \Gamma$! In fact, it holds that $\sigma_{y,\mathsf{ray}}^{\mathsf{op}}(A) = \{I\}$ for all $y \in \Gamma$, whence (17) is elementwise invertible with uniformly bounded inverses. However, the finite section sequence (4) is not stable since Ω_n separates $(k^2 - k, k^2)$ from $(k^2 - k, k^2 + 1)$ if $n = k^2$. So also (17) is not a valid replacement of (11) as stability spectrum.

Note that, for $\mathcal{I} = \mathbb{N}$, (11) contains the operator $P_{\Omega_h} B P_{\Omega_h} + Q_{\Omega_h}$ with $\Omega_h = \mathbb{Z} \times \{..., -1, 0\}$, which is non-invertible since Ω_h separates $(0,0)$ from $(0,1)$. This operator is however removed from (11) if we remove all (sufficiently large) square numbers from $\mathcal{I}$, which matches our direct observation that $P_n A P_n + Q_n$ is non-invertible iff n is a square number. □

It is clear that Examples 4.1 and 4.2 can easily be heaved to dimensions $N > 2$. Let us look at another example, for simplicity also in dimension $N = 2$.

Example 4.3 We look at $A = Adj(\mathcal{G})$ for $\mathcal{G} = (\mathbb{Z}^2, \mathcal{E})$, where

$$\mathcal{E} \;=\; \Big\{ \{ (k,1) \,,\, (k+1,0) \} \;:\; k = 1, 2, ... \Big\}.$$

It is not hard to see that every limit operator of A is either the identity operator I or the operator $B = Adj(\mathcal{G}')$ for $\mathcal{G}' = (\mathbb{Z}^2, \mathcal{E}')$, where

$$\mathcal{E}' \;=\; \Big\{ \{ (k,1) \,,\, (k+1,0) \} \;:\; k \in \mathbb{Z} \Big\},$$

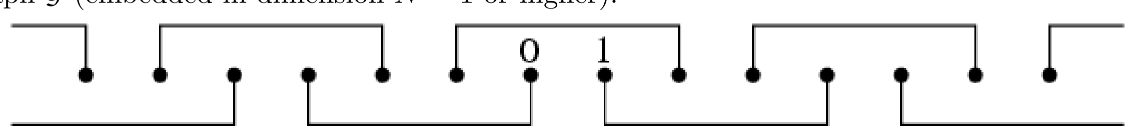
or it is a translate of B. Looking at B and noting that $B = A_h$ for all sequences $h = (h(1), h(2), ...)$ with $h(k) = (m_k, 0)$ and $m_k \to +\infty$, we can say how Ω has to look locally at the intersection z of its boundary Γ with the positive x-axis in order for the finite section method to be stable. Here the upward tangent of Γ at z has to enclose an angle $\alpha \in (90^o, 135^o]$ with the positively directed x-axis. So, for example, the finite section sequence is stable if Ω is the square $\mathrm{conv}\{(1,0),(0,1),(-1,0),(0,-1)\}$ or the triangle $\mathrm{conv}\{(0,2),(2,-2),(-2,-2)\}$, whereas it does not even have a stable subsequence if Ω is the square $[-1,1]^2$. □

The next example is closely related to Example 1.3.

Example 4.4 **a)** Let $A = Adj(\mathcal{G})$ where $\mathcal{G} = (\mathbb{Z}, \mathcal{E})$ is the following infinite graph:

Then, no matter how we choose $\Omega = [a, b]$ with integers $a < 0 < b$, the finite section method does not even have a stable subsequence. A workaround would be to take $\Omega = [-1, 1)$ (which is not a valid polytope in our sense but a valid starlike set) or to increase the dimension to $N = 2$, where we place the edges $\mathcal{E}$ along the x-axis and put $\Omega = \mathrm{conv}\{(-1,0),(1,1),(0,-1)\}$, for example. In the latter case, the finite section subsequence corresponding to $\mathcal{I} = 4\mathbb{N}+1$ turns out to be stable.

b) In contrast to a), there is no workaround whatsoever if $A = Adj(\mathcal{G})$ with the following graph $\mathcal{G}$ (embedded in dimension $N = 1$ or higher):



For every valid polytope or starlike set Ω and every $n \in \mathbb{N}$, the set Ω_n separates the endpoints of at least two edges of $\mathcal{G}$ so that $P_n A P_n + Q_n$ is non-invertible. □

For any dimension $N \in \mathbb{N}$, any valid set $\Omega \in \mathbb{R}^N$ and any given sequence $n_1 < n_2 < \cdots$ of naturals, one can construct a graph $\mathcal{G}$ in the style[1] of Example 4.4 b) such that $(P_n A P_n + Q_n)_{n \in \mathcal{I}}$ is stable iff $\mathcal{I}$ is a subset of $\{n_1, n_2, ...\}$.

[1] The idea is to take the graph from Example 4.4 b) and to place "gaps" between $a_i := \lceil a n_i \rceil$ and $a_i - 1$ and between $b_i := \lfloor b n_i \rfloor$ and $b_i + 1$ for $i = 1, 2, ...$, where $a < 0$ and $b > 0$ are the unique intersection points of $\Gamma = \partial\Omega$ with the x-axis and $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ stand for rounding up and down to the next integer, respectively.

5 Some specialities in the case $N = 1$

In this section we let $N = 1$. Not surprisingly, our results are most complete in this case, where we can sharpen and extend much of what was said previously. This is clearly due to the simple geometry of this setting: Firstly, to infinity there are only two ways to go: right or left, and secondly, all valid starlike sets Ω are intervals (open, closed or semi-open) from a to b with reals $a < 0 < b$ (for valid polytopes, the interval is closed and a, b are integers) so that there are only two possibilities for the set Ω_h in (11), namely $\{0, 1, ...\}$ and $\{..., -1, 0\}$.

The first result is from [11]. We include it here for completeness and because it highlights another important benefit from extending the original stability theorem (5) to subsequences.

Proposition 5.1 *The uniform boundedness condition in Theorems 3.1 (iii) and 3.5 (iii) is redundant if $N = 1$.*

We give the proof later in Section 6 as a special case of Lemma 6.1, where we discuss possible extensions to $N \geq 2$. Next we show that, in dimension $N = 1$, if the full finite section sequence (4) is stable for one valid polytope Ω (i.e. interval $[a, b]$ with integers $a < 0 < b$) then (4) has a stable subsequence for all valid polytopes Ω. So conversely, if there exists a valid polytope Ω for which (4) has no stable subsequence then there is no valid polytope Ω for which the whole sequence (4) is stable.

Proposition 5.2 *Let $E = \ell^p(\mathbb{Z}, X)$ with $p \in [1, \infty]$ and a Banach space X, let $A \in \mathrm{BDO}(E)$ be a rich operator, and take integers $a < 0 < b$. If the full finite section method $(P_nAP_n + Q_n)_{n\in\mathbb{N}}$ is stable for $\Omega = [a, b]$ then, for all integers $a' < 0 < b'$, there exists an infinite index set $\mathcal{I} \subseteq \mathbb{N}$ such that the finite section subsequence $(P_nAP_n + Q_n)_{n\in\mathcal{I}}$ is stable for $\Omega = [a', b']$.*

Proof. Let (4) be stable for $\Omega = \Omega^{(1)} = [a, b]$. Putting $\Omega^{(2)} := [a', b']$ for two arbitrary integers $a' < 0 < b'$, $\Gamma^{(i)} := \partial\Omega^{(i)}$ for $i = 1, 2$, and $\mathcal{I} := \{-ab, -2ab, -3ab, ...\} \subseteq \mathbb{N}$, it is easy to see that

$$\begin{aligned} \Gamma^{(2)}_{\mathcal{I}} &= \bigcup_{n\in\mathcal{I}} n\Gamma^{(2)} = \{na', nb' : n \in \mathcal{I}\} = \{-maba', -mabb' : m \in \mathbb{N}\} \\ &\subseteq \{nab, -nab : n \in \mathbb{N}\} \subseteq \{ka, kb : k \in \mathbb{N}\} = \bigcup_{k\in\mathbb{N}} k\Gamma^{(1)} = \Gamma^{(1)}_{\mathbb{N}} \end{aligned}$$

and consequently $\mathcal{H}_{\Omega^{(2)},\mathcal{I}}(A) \subseteq \mathcal{H}_{\Omega^{(1)},\mathbb{N}}(A)$. Since, moreover, for both choices of Ω and all sequences $h = (h(1), h(2), ...)$ with values in, respectively, $\Gamma^{(1)}_{\mathbb{N}}$ or $\Gamma^{(2)}_{\mathcal{I}}$, it holds that

$$\Omega_h = \begin{cases} \{0, 1, 2, ...\} & \text{if } h(k) \to -\infty, \\ \{..., -2, -1, 0\} & \text{if } h(k) \to +\infty, \end{cases}$$

we get that $\sigma^{\mathsf{stab}}_{\Omega^{(2)},\mathcal{I}}(A) \subseteq \sigma^{\mathsf{stab}}_{\Omega^{(1)},\mathbb{N}}(A)$. Using Theorem 3.1 (twice), we get that $(P_nAP_n + Q_n)_{n\in\mathcal{I}}$ is stable for $\Omega = \Omega^{(2)}$. ■

Note that it is not true that if (4) has a stable **sub**sequence for one valid polytope Ω then (4) has a stable subsequence for all valid polytopes Ω. For example, (7) is stable for $A = Adj(\mathcal{G})$ with $\mathcal{G} = (\mathbb{Z}, \mathcal{E})$ and

$$\mathcal{E} = \Big\{ \{-2(2k + 1), -2(2k - 1) - 1\}, \{3(2k - 1) + 1, 3(2k + 1)\} \ : \ k \in \mathbb{N} \Big\}$$

if one takes $\Omega = [-2, 3]$ and $\mathcal{I} = 2\mathbb{N} + 1$ but there is no stable subsequence of (4) for $\Omega = [-1, 1]$.

Example 4.4 b) has shown that, for some operators, the finite section method cannot be "adjusted" via choosing Ω and $\mathcal{I}$ to become stable. We now give a necessary criterion for the existence of an index set $\mathcal{I} \subseteq \mathbb{N}$ such that (7) is stable.

Proposition 5.3 *Let $E = \ell^p(\mathbb{Z}, \mathbb{C})$ with $p \in [1, \infty]$ and $A \in \mathrm{BDO}(E)$. For the existence of a valid starlike set Ω and an infinite index set $\mathcal{I} \subseteq \mathbb{N}$ such that the sequence $(P_n A P_n + Q_n)_{n \in \mathcal{I}}$ is stable it is necessary, but not sufficient, that A is invertible and* $\mathrm{ind}_+(A) = 0$.

Here we denote by $\mathrm{ind}_+(A)$ the Fredholm index of $PAP+Q$ and by $\mathrm{ind}_-(A)$ the Fredholm index of $QAQ + P$, where we abbreviate $P_{\mathbb{N}} =: P$ and $I - P_{\mathbb{N}} =: Q$. From

$$\begin{aligned} A &= PAP + QAQ + PAQ + QAP \\ &= (PAP + Q)(QAQ + P) + PAQ + QAP \end{aligned}$$

and the compactness of PAQ and QAP (both are of finite rank if A is a band operator), it follows that the Fredholm index of A, $\mathrm{ind}(A)$, is equal to the sum of $\mathrm{ind}_+(A)$ and $\mathrm{ind}_-(A)$.

Proof of Proposition 5.3. That invertibility of A and $\mathrm{ind}_+(A) = 0$ are not enough for the existence of an Ω and an index set $\mathcal{I} \subseteq \mathbb{N}$ such that (7) is stable can be seen in Example 4.4 b) (note that $\mathrm{ind}_+(A) = 0$ there since the adjacency matrix of an undirected graph $\mathcal{G}$, and hence also $P\,Adj(\mathcal{G})\,P + Q$, is symmetric).

Now suppose Ω is a valid starlike set (i.e. an interval from $a < 0$ to $b > 0$) and an index set $\mathcal{I} \subseteq \mathbb{N}$ is found such that (7) is stable. Then, by Theorem 3.5, we have that A is invertible and $QA_hQ + P$ is invertible for all $h \in \mathcal{H}_{\Omega,\mathcal{I}}(A)$ tending to $+\infty$ (note that A is automatically rich if $X = \mathbb{C}$, see e.g. [4, Corollary 3.24]). If we denote by $\sigma_+^{\mathsf{op}}(A)$ the collection of all limit operators A_h of A with a sequence $h = (h(1), h(2), ...)$ tending to $+\infty$ then the latter clearly implies that $QBQ + P$ is invertible for some operator $B \in \sigma_+^{\mathsf{op}}(A)$, whence, $\mathrm{ind}_-(B) = 0$. Since A is Fredholm (even invertible), all its limit operators (including B) are invertible [8], so that also $\mathrm{ind}_+(B) = 0$ holds since $\mathrm{ind}_+(B) + \mathrm{ind}_-(B) = \mathrm{ind}(B) = 0$. By [12, Theorem 2.3] (which also holds in case $p = \infty$, by [5, Theorem 1.2]) we get that not only B but all operators in $\sigma_+^{\mathsf{op}}(A)$ have plus-index zero, and even more: $\mathrm{ind}_+(A) = 0$. (Analogously, all operators in $\sigma_-^{\mathsf{op}}(A)$ and A itself have minus-index zero, but the latter also follows from $\mathrm{ind}_+(A) = 0$ and $\mathrm{ind}(A) = 0$). ■

In [6] (see [12] for $p \neq 2$) we have shown that, under the additional condition that all diagonals of $[A]$ are slowly oscillating, invertibility of A and $\mathrm{ind}_+(A) = 0$ are even sufficient for the stability of the full finite section sequence (4) for all valid Ω. Here we call a sequence $(b_k)_{k \in \mathbb{Z}}$ *slowly oscillating* if $b_{k+1} - b_k \to 0$ as $k \to \pm\infty$.

By Proposition 5.3, for an invertible operator A with $\kappa := \mathrm{ind}_+(A) \neq 0$, there is no valid Ω and no index set $\mathcal{I} \subseteq \mathbb{N}$ for which (7) is stable. This problem of a nonzero plus-index κ can be overcome as follows: Instead of solving $Au = b$, one looks at $V_\kappa A u = V_\kappa b$ with V_κ as in Example 1.2. Since V_κ is invertible, these two equations are equivalent. Moreover, we have that also $A' := V_\kappa A$ is invertible and

$$\mathrm{ind}_+(A') \;=\; \mathrm{ind}_+(V_\kappa A) \;=\; \mathrm{ind}_+(V_\kappa) + \mathrm{ind}_+(A) \;=\; -\kappa + \kappa \;=\; 0.$$

This preconditioning-type procedure of shifting all matrix entries (incl. the right hand side b) down by κ rows is reminiscent of Gohberg's statement that, in a two-sided infinite matrix, "it is every diagonal's right to claim to be the main one" (see page 51 in [2] and the discussion there). Our computations show that, from the perspective of the finite section method, there is one diagonal that deserves being the main diagonal a bit more than the others.

6 On the uniform boundedness condition in statement (iii)

Of course, it would be desirable to remove the condition on the uniform boundedness of the inverses from statement (iii) of our Theorems 3.1 and 3.5 for arbitrary dimensions N. For $N \geq 2$ this is a much more delicate problem than for $N = 1$.

What clearly can be said by looking at the proof of Theorem 3.1 is that the uniform boundedness condition (UBC) can be removed from statement (iii) in all cases where it can be removed from Theorem 1 of [4]. This is known to be the case if one of the following holds

- $p \in \{1, \infty\}$ (see [4, Theorem 3.109] or [1, Theorem 6.28]),
- $[A]$ has slowly oscillating diagonals (see [7, Theorem 7.2] or [10, Theorem 2.4.27]),
- A is contained in the Wiener algebra $\mathcal{W}$ (see [10, Theorem 2.5.7] or [1, Theorem 6.40]).

Here, by a diagonal of $[A] = (a_{ij})_{i,j\in\mathbb{Z}^N}$ we mean a sequence $(a_{j+k,j})_{j\in\mathbb{Z}^N}$ with $k \in \mathbb{Z}^N$, and a sequence $(b_j)_{j\in\mathbb{Z}^N}$ of operators on X is called *slowly oscillating* if $\|b_{j+d} - b_j\|_{L(X)} \to 0$ as $|j| \to \infty$ for all $d \in \{-1, 0, 1\}^N$. Moreover, the term *Wiener algebra* stands for the completion $\mathcal{W}$ of the set of all band operators with respect to the norm

$$\|A\|_{\mathcal{W}} := \sum_{k\in\mathbb{Z}^N} \sup_{j\in\mathbb{Z}^N} \|a_{j+k,j}\|_{L(X)}.$$

So in particular, we can remove the UBC from Theorems 3.1 and 3.5 (iii) if A is a band operator.

For a more general removal of the UBC in dimension $N \geq 2$, we try to generalize the approach that has worked successfully for $N = 1$ in [11]. Therefore, given a rich operator $A \in \mathrm{BDO}(E)$ and a valid starlike set $\Omega \subset \mathbb{R}^N$, we will call the infinite index set $\mathcal{I} \subseteq \mathbb{N}$ *sufficient w.r.t. A and* Ω, and write $\mathcal{I} \in \mathrm{suff}(A, \Omega)$, if $\sigma^{\mathsf{stab}}_{\Omega,\mathcal{I}}(A)$ is either uniformly invertible or not elementwise invertible, i.e. it holds that elementwise invertibility of $\sigma^{\mathsf{stab}}_{\Omega,\mathcal{I}}(A)$ implies its uniform invertibility.

Here, as usual, we call a set of operators elementwise invertible if all its elements are invertible, and we call it uniformly invertible if, in addition, the inverses are uniformly bounded. In what follows, when we use the letters $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$ for subsets of $\mathbb{N}$, we always mean infinite subsets.

In Example 1.3, one has that for every $\mathcal{I} \subset \mathbb{N}$ and every valid Ω, the set $\sigma^{\mathsf{stab}}_{\Omega,\mathcal{I}}(A)$ is finite so that, clearly, $\mathcal{I} \in \mathrm{suff}(A, \Omega)$. In general, the following lemma holds.

Lemma 6.1 *Let $A \in \mathrm{BDO}(E)$ be a rich operator, $\Omega \subset \mathbb{R}^N$ be a valid starlike set, and take $\mathcal{I} \subseteq \mathbb{N}$. If every $\mathcal{J} \subseteq \mathcal{I}$ has a subset $\mathcal{K} \subseteq \mathcal{J}$ with $\mathcal{K} \in \mathrm{suff}(A, \Omega)$ then $\mathcal{I} \in \mathrm{suff}(A, \Omega)$.*

Proof. Contrarily to what we claim suppose $\mathcal{I} \notin \mathrm{suff}(A, \Omega)$. Then $\sigma^{\mathsf{stab}}_{\Omega,\mathcal{I}}(A)$ is elementwise but not uniformly invertible. By Theorem 3.5, $(P_nAP_n + Q_n)_{n\in\mathcal{I}}$ is not stable. So there is a subset $\mathcal{J} = \{m_1, m_2, ...\} \subseteq \mathcal{I}$ with $\|(P_{m_j}AP_{m_j} + Q_{m_j})^{-1}\| \geq j$ for $j = 1, 2, ...$ with the convention that $\|B^{-1}\| = \infty$ if B is not invertible. Hence, $(P_nAP_n + Q_n)_{n\in\mathcal{J}}$ has no stable subsequence.

By our assumption, the index set $\mathcal{J} \subseteq \mathcal{I}$ has a subset $\mathcal{K} \in \mathrm{suff}(A, \Omega)$. Since $\sigma^{\mathsf{stab}}_{\Omega,\mathcal{K}}(A) \subseteq \sigma^{\mathsf{stab}}_{\Omega,\mathcal{I}}(A)$ and all elements of the latter are invertible, we have that $\sigma^{\mathsf{stab}}_{\Omega,\mathcal{K}}(A)$ is elementwise and hence uniformly invertible. By Theorem 3.5 again, $(P_nAP_n + Q_n)_{n\in\mathcal{K}}$ is stable. But this contradicts the fact that $\mathcal{K} \subseteq \mathcal{J}$ and $(P_nAP_n + Q_n)_{n\in\mathcal{J}}$ has no stable subsequence. ∎

Lemma 6.1 reduces the problem of showing that $\mathcal{I}$ is sufficient to showing that every subset of $\mathcal{I}$ has a sufficient subset $\mathcal{K}$. The new problem is about how to choose $\mathcal{K}$; that is, one has to single out a subset $\mathcal{K} \subseteq \mathcal{J} \subseteq \mathcal{I}$ such that $\sigma^{\mathsf{stab}}_{\Omega,\mathcal{K}}(A)$ is as small as possible (ideally finite or compact in some sense) in order to be uniformly invertible if elementwise invertible. This is exactly what one does in case $N = 1$ (see [11, Theorem 6]):

Proof of Proposition 5.1. Let $\mathcal{I} \subseteq \mathbb{N}$ and $\mathcal{J} \subseteq \mathcal{I}$ be arbitrary and let Ω be the interval (open, closed or semi-open) from $a < 0$ to $b > 0$. Since A is rich there is a subset $\mathcal{K} = \{k_1, k_2, ...\} \subseteq \mathcal{J}$ such that both limit operators $B = A_h$ and $C = A_g$ exist, where $h = ([k_1\, a], [k_2\, a], ...)$ tends to $-\infty$ and $g = ([k_1\, b], [k_2\, b], ...)$ to $+\infty$, and hence

$$\sigma^{\mathsf{stab}}_{\Omega,\mathcal{K}}(A) \;=\; \{\ A\ ,\ P_{\mathbb{N}_0} B P_{\mathbb{N}_0} + Q_{\mathbb{N}_0}\ ,\ P_{-\mathbb{N}_0} C P_{-\mathbb{N}_0} + Q_{-\mathbb{N}_0}\ \} \tag{18}$$

is finite, where $\mathbb{N}_0 = \{0, 1, 2, ...\}$ and $-\mathbb{N}_0 = \{..., -2, -1, 0\}$. So $\mathcal{K}$ is sufficient w.r.t. A and Ω and thus, by Lemma 6.1, $\mathcal{I}$ is sufficient. ■

For $N \geq 2$, a strategy might be to look at the partially ordered set of all $\sigma^{\mathsf{stab}}_{\Omega,\mathcal{K}}(A)$ with $\mathcal{K} \subseteq \mathcal{J}$, ordered by inclusion, and to look for minimal elements. In case $N = 1$, these minimal elements consist of only three operators: A itself and one operator associated with each "direction" leading to infinity, as in (18). How can we capture this notion of "direction" in dimensions $N \geq 2$? In Example 4.2 we have seen that, for our purposes, it is not enough to associate a "direction" with each straight line from the origin to infinity; instead it seems one has to look at what are called admissible domains in the first symbol calculus of Rabinovich, Roch and Silbermann [8, 10] or, alternatively, at the Stone-Čech boundary of $\mathbb{Z}^N$ as in the second symbol calculus by the same authors. But this shall be the subject of a later investigation.

We close with a particular situation in $N \geq 2$ where the UBC can be easily removed. Suppose Ω is a valid polytope and the rich operator $A \in \mathrm{BDO}(E)$ is such that, for every $x \in \Gamma = \partial\Omega$ and every sequence $h = (h(1), h(2), ...) \subset \mathbb{Z}^N$ with $h(n)/|h(n)| \to x/|x|$, the limit operator A_h (if it exists) only depends on x – let us write A_x for A_h then – and that the mapping $x \mapsto A_x$ is continuous in the operator norm. Then, without having recourse to subsequences and Lemma 6.1, every $\mathcal{I} \subseteq \mathbb{N}$ is sufficient. This is because, in accordance with the geometry of Γ, $\sigma^{\mathsf{stab}}_{\Omega,\mathcal{I}}(A)$ splits into finitely many sets, each of them being the image of the continuous function $x \mapsto P_{\Omega_h} A_x P_{\Omega_h} + Q_{\Omega_h}$ over a compact set (the corresponding face of Ω, including its boundary). However, one can show that if A_h only depends on $\lim h(n)/|h(n)|$ then the diagonals of $[A]$ are convergent and hence slowly oscillating so that the case is already settled by what we said earlier.

References

[1] S. N. Chandler-Wilde and M. Lindner: *Limit Operators, Collective Compactness, and the Spectral Theory of Infinite Matrices*, in publication (also see TU Chemnitz Preprint 7, 2008).

[2] R. Hagen, S. Roch and B. Silbermann: *C^*-Algebras and Numerical Analysis*, Marcel Dekker, Inc., New York, Basel, 2001.

[3] M. Lindner: The finite section method in the space of essentially bounded functions: An approach using limit operators, *Numer. Func. Anal. & Optim.* **24** (2003) no. 7&8, 863–893.

[4] M. Lindner: *Infinite Matrices and their Finite Sections: An Introduction to the Limit Operator Method*, Frontiers in Mathematics, Birkhäuser 2006.

[5] M. Lindner: Fredholmness and index of operators in the Wiener algebra are independent of the underlying space, *Operators and Matrices* **2** (2008), 297–306.

[6] M. Lindner, V. S. Rabinovich and S. Roch: Finite sections of band operators with slowly oscillating coefficients, *Linear Algebra and Applications* **390** (2004), 19–26.

[7] V. S. Rabinovich and S. Roch: Local theory of the Fredholmness of band-dominated operators with slowly oscillating coefficients, *Toeplitz matrices and singular integral equations (Pobershau, 2001)*, 267–291, *Oper. Theory Adv. Appl.*, 135, Birkhäuser, Basel, 2002.

[8] V. S. Rabinovich, S. Roch and B. Silbermann: Fredholm Theory and Finite Section Method for Band-dominated operators, *Integral Equations Operator Theory* **30** (1998), no. 4, 452–495.

[9] V. S. Rabinovich, S. Roch and B. Silbermann: Algebras of approximation sequences: Finite sections of band-dominated operators, *Acta Appl. Math.* **65** (2001), 315–332.

[10] V. S. Rabinovich, S. Roch and B. Silbermann: *Limit Operators and Their Applications in Operator Theory*, Birkhäuser 2004.

[11] V. S. Rabinovich, S. Roch and B. Silbermann: On finite sections of band-dominated operators, *TU Darmstadt Preprint Nr. 2486*, 2006.

[12] S. Roch: Band-dominated operators on ℓ^p-spaces: Fredholm indices and finite sections, *Acta Sci. Math.* **70** (2004), no. 3–4, 783–797.

[13] S. Roch: *Finite sections of band-dominated operators*, Memoirs AMS, Vol. 191, Nr. 895, 2008.

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