A note on the spectrum of bi-infinite bi-diagonal random matrices

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May 16, 2008

Abstract. The purpose of this note is to demonstrate the use of the results from [5, 6] for the explicit computation of the spectrum of two-sided infinite matrices with random diagonals. Here we consider the case of two random diagonals, one of them the main diagonal. Our result is a generalization of [24, Theorem 8.1] by Trefethen, Contedini and Embree from the case of one random and one constant diagonal to the case of two random diagonals.

Mathematics subject classification (2000): Primary 47B80; Secondary 47A10, 47B36.
Keywords: random matrix, spectral theory, Jacobi matrix, operators on ℓ^p.

1 Introduction, Preliminaries, Result

The problem. Given two compact sets Σ and $\mathcal{T}$ in the complex plane, we study the spectrum of the two-sided infinite matrix

$$A = \begin{pmatrix} \ddots & \ddots & & & & & \\ & \sigma_{-2} & \tau_{-2} & & & & \\ & & \sigma_{-1} & \tau_{-1} & & & \\ & & & \sigma_0 & \tau_0 & & \\ & & & & \sigma_1 & \tau_1 & \\ & & & & & \sigma_2 & \ddots \\ & & & & & & \ddots \end{pmatrix}, \tag{1}$$

considered as an operator on $\ell^p(\mathbb{Z})$, where $\sigma_k \in \Sigma$ and $\tau_k \in \mathcal{T}$ are independent samples from a random distribution on Σ and $\mathcal{T}$, respectively. Here we will suppose that, for all $\varepsilon > 0$, $k \in \mathbb{Z}$, $\sigma \in \Sigma$ and $\tau \in \mathcal{T}$, the probabilities of $|\sigma_k - \sigma| < \varepsilon$ and $|\tau_k - \tau| < \varepsilon$ are both nonzero.

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As usual, *spectrum* $\operatorname{spec} A$ and *essential spectrum* $\operatorname{spec}_{\mathrm{ess}} A$ of A, considered as a bounded linear operator on $\ell^p(\mathbb{Z})$, are the sets of all $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is, respectively, not invertible or not a Fredholm operator. It is well-known (see e.g. [16, 19]) that neither $\operatorname{spec} A$ nor $\operatorname{spec}_{\mathrm{ess}} A$ depend on the choice of $p \in [1, \infty]$ if A, as in our case, is a banded matrix.

Matrices like (1) and the question about their spectrum originate from problems in quantum mechanics. For example, they appear as Hamiltonians of asymmetric randomly hopping quantum particles, where, in the special case when $\Sigma = \{-1, 1\}$ and $\mathcal{T} = \{1\}$, (1) is called "one-way model" by Brézin, Feinberg and Zee [2, 13, 14].

The result. For $\varepsilon \geq 0$, put

$$\Sigma_{\cup}^{\varepsilon} := \bigcup_{\sigma \in \Sigma} \overline{U}_{\varepsilon}(\sigma) \qquad \text{and} \qquad \Sigma_{\cap}^{\varepsilon} := \bigcap_{\sigma \in \Sigma} U_{\varepsilon}(\sigma)$$

with $U_{\varepsilon}(\sigma) = \{\lambda \in \mathbb{C} : |\lambda - \sigma| < \varepsilon\}$ and $\overline{U}_{\varepsilon}(\sigma) = \{\lambda \in \mathbb{C} : |\lambda - \sigma| \leq \varepsilon\}$ denoting the open and the closed ε-neighbourhood of σ in $\mathbb{C}$, respectively. Then our result reads as follows:

Theorem 1.1 *If A is the random matrix shown in (1) then, with probability 1,*

$$\operatorname{spec} A = \operatorname{spec}_{\mathrm{ess}} A = \Sigma_{\cup}^{T} \setminus \Sigma_{\cap}^{t} ,$$

where $T = \max\{|\tau| : \tau \in \mathcal{T}\}$ and $t = \min\{|\tau| : \tau \in \mathcal{T}\}$.

We hereby generalize Theorem 8.1 of Trefethen, Contedini and Embree's paper [24] (and see [25, Section VIII]) where $\mathcal{T} = \{1\}$ and therefore $T = t = 1$. If we put $\Sigma = \{\sigma\}$ and $\mathcal{T} = \{\tau\}$ with $\sigma, \tau \in \mathbb{C}$ fixed then (1) is a Laurent matrix with two constant diagonals of value σ and τ, and Theorem 1.1 resembles the well-known fact (see e.g. [1]) that $\operatorname{spec} A = \operatorname{spec}_{\mathrm{ess}} A$ is the circle of radius $T = t = |\tau|$ around σ. If again, $\Sigma = \{\sigma\}$ is a singleton and $\mathcal{T}$ consists of at least two points with different moduli $|\tau| \in [t, T]$ then letting $t = \min |\tau| \to 0$ in Theorem 1.1 demonstrates what is called "disk-annulus transition" in [11, 12].

Another observation is that, if $\Sigma, \mathcal{T} \subset \mathbb{C}$ are compact sets and $t = \operatorname{dist}(\mathcal{T}, 0)$ is small enough for $\Sigma_{\cap}^{t} = \varnothing$ (e.g. when $t \in [0, \operatorname{diam} \Sigma/2]$) then we get that $\operatorname{spec} A = \Sigma_{\cup}^{T}$ which coincides with the ε-pseudospectrum, for $\varepsilon = T$, of the diagonal matrix that results from (1) by deleting the 1st superdiagonal.

We would also like to mention that, as expected for a non-symmetric matrix, the spectrum of A differs (unless $\mathcal{T} = \{0\}$, i.e. the symmetric case) from the limit as $n \to \infty$ of the spectra of its n-by-n finite sections which obviously is Σ.

Our approach. Our tool for computing $\operatorname{spec} A$ and $\operatorname{spec}_{\text{ess}} A$ for (1) is the method of so-called limit operators [6, 18, 23], where A is studied in terms of a family of infinite matrices that represents the behaviour of A at infinity. More precisely, we say that the operator induced by the matrix $B = (b_{ij})_{i,j\in\mathbb{Z}}$ is a *limit operator* of the operator induced by the banded matrix $A = (a_{ij})_{i,j\in\mathbb{Z}}$ if, for a sequence $h(1), h(2), \ldots$ of integers with $|h(k)| \to \infty$, it holds that

$$a_{i+h(k),j+h(k)} \ \to \ b_{ij} \qquad \text{as} \qquad k \to \infty$$

for all $i, j \in \mathbb{Z}$. The set of all limit operators of A is denoted by $\sigma^{\mathsf{op}}(A)$. Combining the main theorem on limit operators (going back in this simple form to [17, 22]) with recent results of Chandler-Wilde and the author [5], one gets that, if A is a two-sided infinite banded matrix with bounded diagonals, then

$$\operatorname{spec}_{\text{ess}} A \ = \bigcup_{B\in\sigma^{\mathsf{op}}(A)} \operatorname{spec} B \ = \bigcup_{B\in\sigma^{\mathsf{op}}(A)} \operatorname{spec}^{\infty}_{\text{point}} B, \tag{2}$$

where $\operatorname{spec}^{\infty}_{\text{point}} B$ is the point spectrum (set of eigenvalues) of B as operator on $\ell^\infty(\mathbb{Z})$.

If A is our random matrix (1) then it is easy to see (the argument is sometimes called "the Infinite Monkey Theorem" and it follows from the 2nd Borel Cantelli Lemma, see [3, Theorem 8.16] or [7, Theorem 4.2.4]) that, with probability 1, the function $k \mapsto (\sigma_k, \tau_k)$ is a pseudo-ergodic mapping $\mathbb{Z} \to \Sigma \times \mathcal{T}$ in the sense of Davies [9], in which case we call the matrix A *pseudo-ergodic*. This, however, is equivalent (see e.g. [8, Lemma 6], [18, Corollary 3.70] or [6, Theorem 7.6]) to the following fact:

$$\begin{array}{c}\sigma^{\mathsf{op}}(A) \text{ consists of } \textbf{all} \text{ matrices of the form (1)} \\ \text{with } \sigma_k \in \Sigma \text{ and } \tau_k \in \mathcal{T} \text{ for all } k \in \mathbb{Z}.\end{array} \tag{3}$$

So in particular, $A \in \sigma^{\mathsf{op}}(A)$, which shows that, by (2), $\operatorname{spec} A \subset \operatorname{spec}_{\text{ess}} A$ and hence

$$\operatorname{spec} A \ = \ \operatorname{spec}_{\text{ess}} A \ = \bigcup_{B\in\sigma^{\mathsf{op}}(A)} \operatorname{spec} B \ = \bigcup_{B\in\sigma^{\mathsf{op}}(A)} \operatorname{spec}^{\infty}_{\text{point}} B. \tag{4}$$

The proof of Theorem 1.1 now rests on a combination of (3) and (4).

Limit operator ideas, the "Infinite Monkey" argument and the validity of the first two "=" signs in (4) are not new in the spectral theory of random matrices (see e.g. [4, 8, 9, 15, 21]) but what seems to be new here is the third "=" sign in (4), due to [5] (or [6, Theorem 7.6]), and hence the possibility of the simple proof that is presented here.

Acknowledgements. I would like to thank Simon Chandler-Wilde for his continuous support and Estelle Basor for the inspiration to study infinite random matrices of this and similar type. This research was financially supported by Marie-Curie Grants MEIF-CT-2005-009758 and PERG02-GA-2007-224761.

2 Proof of Theorem 1.1

Let A be the random matrix (1) with samples $\sigma_k \in \Sigma$ and $\tau_k \in \mathcal{T}$ from the probability distributions on the compact sets $\Sigma, \mathcal{T} \subset \mathbb{C}$ as described above, and put $T = \max\{|\tau| : \tau \in \mathcal{T}\}$ and $t = \min\{|\tau| : \tau \in \mathcal{T}\}$.

For the calculation of the point spectra in (4), take $\lambda \in \mathbb{C}$ and let $B \in \sigma^{\mathsf{op}}(A)$, i.e. B is of the form (1) with $\sigma_k \in \Sigma$ and $\tau_k \in \mathcal{T}$ for all $k \in \mathbb{Z}$, by (3). If $x : \mathbb{Z} \to \mathbb{C}$ is a nontrivial solution of $Bx = \lambda x$ then $x(n_0) \neq 0$ for some $n_0 \in \mathbb{Z}$, w.l.o.g. let $x(n_0) = 1$, and

$$\tau_k \; x(k+1) \;=\; (\lambda - \sigma_k) \; x(k) \qquad \text{for all} \qquad k \in \mathbb{Z}. \tag{5}$$

Case 1: $0 \notin \mathcal{T}$, i.e. $t > 0$.

Note that, by (5), $\lambda \neq \sigma_k$ for all $k < n_0$ since otherwise $x(n_0) = 0$ (recall that $\tau_k \neq 0$ for all k). As a consequence we get that

$$x(n) \;=\; \begin{cases} \prod\limits_{k=n_0}^{n-1} \frac{\lambda - \sigma_k}{\tau_k}, & n \geq n_0, \\ \prod\limits_{k=n}^{n_0-1} \frac{\tau_k}{\lambda - \sigma_k}, & n < n_0 \end{cases} \tag{6}$$

for every $n \in \mathbb{Z}$.

Clearly, if $\lambda \notin \Sigma_\cup^T$ then $|\lambda - \sigma| > T \geq |\tau|$ for all $\sigma \in \Sigma$ and $\tau \in \mathcal{T}$ and hence, for every nontrivial eigenvector x of B, we have that $|x(n)| \to \infty$ in (6) as $n \to +\infty$ since $|\frac{\lambda - \sigma_k}{\tau_k}| > 1$ for all $k \in \mathbb{Z}$, regardless of the particular entries σ_k and τ_k of B.

Similarly, if $\lambda \in \Sigma_\cap^t$ then $|\lambda - \sigma| < t \leq |\tau|$ for all $\sigma \in \Sigma$ and $\tau \in \mathcal{T}$ and hence, for every nontrivial eigenvector x of B, $|x(n)| \to \infty$ in (6) as $n \to -\infty$ since $|\frac{\tau_k}{\lambda - \sigma_k}| > 1$ for all $k \in \mathbb{Z}$, regardless of the particular entries σ_k and τ_k of B. (Note that n_0 in (6) depends on B and λ.)

So in both cases, $Bx = \lambda x$ has no nontrivial solution $x \in \ell^\infty(\mathbb{Z})$, so $\lambda \notin \mathrm{spec}_{\mathrm{point}}^\infty B$ for all $B \in \sigma^{\mathsf{op}}(A)$ and hence, by (4), $\lambda \notin \mathrm{spec}\, A$. Now it remains to look at $\lambda \in \Sigma_\cup^T \setminus \Sigma_\cap^t$. In this case, let $\sigma^*, \sigma_* \in \Sigma$ and $\tau^*, \tau_* \in \mathcal{T}$ be such that $|\lambda - \sigma_*| \leq |\tau^*|$ and $|\lambda - \sigma^*| \geq |\tau_*|$, which is possible by the choice of λ. Now consider

$$B \;=\; \begin{pmatrix} \ddots & \ddots & & & & & \\ & \sigma^* & \tau_* & & & & \\ & & \sigma^* & \tau_* & & & \\ & & & \sigma_* & \tau^* & & \\ & & & & \sigma_* & \tau^* & \\ & & & & & \sigma_* & \ddots \\ & & & & & & \ddots \end{pmatrix} \;\in\; \sigma^{\mathsf{op}}(A)$$

with σ_* and τ^* in row $0,1,2,\ldots$ and σ^* and τ_* in row $-1,-2,\ldots$ to see that

$$x = \left(\cdots, \left(\frac{\tau_*}{\lambda-\sigma^*}\right)^2, \left(\frac{\tau_*}{\lambda-\sigma^*}\right)^1, 1, \left(\frac{\lambda-\sigma_*}{\tau^*}\right)^1, \left(\frac{\lambda-\sigma_*}{\tau^*}\right)^2, \cdots\right)^\top,$$

with the 1 at position $n_0 = 0$, is an eigenvector in $\ell^\infty(\mathbb{Z})$ of B w.r.t. λ. So we have that $\lambda \in \operatorname{spec}_{\text{point}}^\infty B \subset \operatorname{spec} A$, by (4).

Summarizing, we see that the formula in Theorem 1.1 holds in Case 1.

Case 2: $0 \in \mathcal{T}$ with $0 = t < T$, i.e. $\mathcal{T}$ has points other than 0.

Suppose $\lambda \notin \Sigma_\cup^T$. Then $\lambda \neq \sigma_k$ for all $k \in \mathbb{Z}$ and, by (5), $\tau_k \neq 0$ for all $k \geq n_0$ since otherwise $x(n_0) = 0$. So again, (6) holds for all $n \in \mathbb{Z}$. But from $|\lambda - \sigma| > T \geq |\tau|$ for all $\sigma \in \Sigma$ and $\tau \in \mathcal{T}$ we again get that $|x(n)| \to \infty$ in (6) as $n \to +\infty$ since $|\frac{\lambda-\sigma_k}{\tau_k}| > 1$ for all $k \in \mathbb{Z}$, regardless of the particular entries σ_k and τ_k of B.

Now suppose $\lambda \in \Sigma_\cup^T$. Then fix $\tau \in \mathcal{T}$ with maximal modulus, i.e. $|\tau| = T > 0$ and take a $\sigma \in \Sigma$ with $|\lambda - \sigma| \leq T = |\tau|$. Now

$$B = \begin{pmatrix} \ddots & \ddots & & & & \\ & \sigma & 0 & & & \\ & & \sigma & 0 & & \\ & & & \sigma & \tau & \\ & & & & \sigma & \tau \\ & & & & & \sigma & \ddots \\ & & & & & & \ddots \end{pmatrix} \in \sigma^{\mathsf{op}}(A)$$

with τ in row $0,1,2,\ldots$ and 0 in row $-1,-2,\ldots$ has

$$x = \left(\cdots, 0, 0, 1, \left(\frac{\lambda-\sigma}{\tau}\right)^1, \left(\frac{\lambda-\sigma}{\tau}\right)^2, \cdots\right)^\top \in \ell^\infty(\mathbb{Z}),$$

with the 1 at position $n_0 = 0$, as its eigenvector w.r.t. λ. So $\lambda \in \operatorname{spec}_{\text{point}}^\infty B \subset \operatorname{spec} A$, by (4).

So in Case 2 we get $\operatorname{spec} A = \Sigma_\cup^T$. But the latter is equal to $\Sigma_\cup^T \setminus \Sigma_\cap^t$ since $t = 0$ and $\Sigma_\cap^0 = \varnothing$.

Case 3: $0 \in \mathcal{T}$ with $0 = t = T$, i.e. $\mathcal{T} = \{0\}$.

In this trivial case, A is a diagonal matrix, so that, with probability 1, $\operatorname{spec} A = \Sigma$. But $\Sigma = \Sigma_\cup^T \setminus \Sigma_\cap^t$ if $T = t = 0$. ■

3 An a-posteriori experiment: Is it enough to look at periodic limit operators?

Recall formula (4) for the spectrum of a bi-infinite, pseudo-ergodic and banded matrix A. Generally it is difficult to evaluate the rightmost term in (4) since the index set $\sigma^{\mathsf{op}}(A)$ of this union is a very large set and the point spectrum of most operators $B \in \sigma^{\mathsf{op}}(A)$ is difficult to determine. An approach which has been used by Davies and co-workers (see e.g. [9, 10, 20] and references therein) for studying the spectrum of such an operator A is to look at a large amount of *periodic* limit operators B of A. More precisely, one looks at the subsets

$$\operatorname{spec}^n_{\mathrm{per}} A \;:=\; \bigcup_{B\in\mathcal{P}_n(A)} \operatorname{spec}^\infty_{\mathrm{point}} B \qquad \text{of} \qquad \operatorname{spec} A \;=\; \bigcup_{B\in\sigma^{\mathsf{op}}(A)} \operatorname{spec}^\infty_{\mathrm{point}} B$$

for large values of $n \in \mathbb{N}$, where $\mathcal{P}_n(A) \subset \sigma^{\mathsf{op}}(A)$ denotes the set of all limit operators of A with n-periodic diagonals. For $B \in \mathcal{P}_n(A)$, spectrum and ℓ^∞ point spectrum coincide (see e.g. [6, Theorem 6.7]) and its computation reduces to the computation of the spectra of certain finite matrices by treating B as a block Laurent matrix with n-by-n block entries (see e.g. [1, 10, 20]).

An interesting question is under what circumstances does it hold that the left-hand side of the inclusion

$$\operatorname{spec}_{\mathrm{per}} A \;:=\; \bigcup_{n=1}^{\infty} \operatorname{spec}^n_{\mathrm{per}} A \;\subset\; \operatorname{spec} A \tag{7}$$

is dense in the right-hand side. In this section we illustrate that, even when the pseudo-ergodic operator A is non-normal, it can hold that the closure of the left-hand side of (7) is equal to the spectrum of A.

To do this, we will look at Brézin, Feinberg and Zee's "one-way model" (1), where $\Sigma = \{-1, 1\}$ and $\mathcal{T} = \{1\}$; that is,

$$A = \begin{pmatrix} \ddots & \ddots & & & & \\ & \sigma_{-1} & 1 & & & \\ & & \sigma_0 & 1 & & \\ & & & \sigma_1 & 1 & \\ & & & & \sigma_2 & \ddots \\ & & & & & \ddots \end{pmatrix} \tag{8}$$

with σ_k randomly chosen from $\Sigma = \{-1, 1\}$. The spectrum of A is explicitly known due to [24] or from our Theorem 1.1: It is the union of the two disks of radius 1 centered at 1 and -1 (see Figure 3.1).

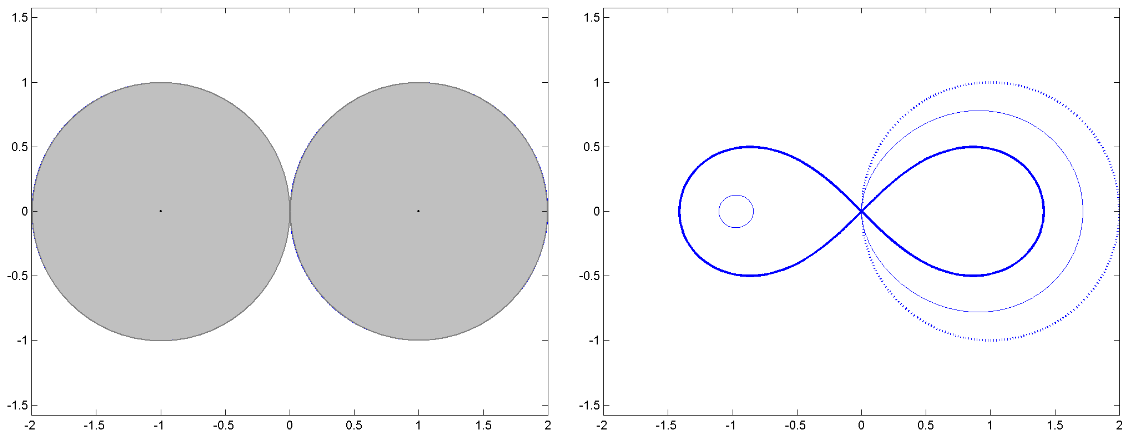

Figure 3.1: The left image shows the spectrum of the infinite random matrix (8). The right image shows the point spectra (solutions λ of (10)) corresponding to ratio $r = 0.5$ (lemniscate, bold), $r = 0.75$ (thin) and $r = 1$ (dotted).

Now take $n \in \mathbb{N}$ and $B \in \mathcal{P}_n(A)$, i.e. B is of the form (8), where we choose $\sigma_1, ..., \sigma_n \in \{-1, 1\}$ and let $\sigma_{k+n} = \sigma_k$ for all $k \in \mathbb{Z}$. Let m denote the number of 1's in $\sigma_1, ..., \sigma_n$ so that the remaining $n - m$ entries are equal to -1. Now we are in the situation of Case 1 ($0 \notin \mathcal{T}$) in our proof of Theorem 1.1. So take a $\lambda \in \mathbb{C}$ and look at a nontrivial solution x of $Bx = \lambda x$. Looking at (6) and taking into account $\tau_k = 1$ $\forall k$ and the periodicity of the σ_k-sequence, we get that $x \in \ell^\infty(\mathbb{Z})$ iff

$$|\lambda - 1|^m \; |\lambda + 1|^{n-m} \;=\; |\lambda - \sigma_1| \cdots |\lambda - \sigma_n| \;=\; 1. \tag{9}$$

Indeed, $|x(n)|$ from (6) remains bounded for $n \to +\infty$ iff the left-hand side of (9) is ≤ 1, and it remains bounded for $n \to -\infty$ iff the left-hand side of (9) is ≥ 1 (also cf. [14]).

So we have that $\lambda \in \mathrm{spec}^\infty_{\mathrm{point}} B$ iff (9) holds. Taking n-th roots in (9), we get the slightly more convenient formula

$$|\lambda - 1|^r \; |\lambda + 1|^{1-r} \;=\; 1, \tag{10}$$

where $r = m/n$ is the ratio of 1's among all entries σ_k in a period of length n. The set $\mathrm{spec}_{\mathrm{per}} A$, as defined in (7), is hence equal to the set of all solutions λ of (10) with a rational ratio $r = m/n \in [0, 1]$.

For example, if $r = 0.5$, i.e. if n is even and $m = n/2$ then (10) is equivalent to $|\lambda - 1| \cdot |\lambda + 1| = 1$, which is the equation of the lemniscate with focal points -1 and 1 (see Figures 3.1 and 3.2, and cf. [24, Figures 2.1 and 3.1(b)] and [14, Figure 2]). By the same argument, it can be shown that the same lemniscate is the point spectrum not only of all periodic matrices (8) with an equal share of 1's and -1's per period but also for the much larger class of all matrices of the form (8) for which the ratio of 1's within $\sigma_{-k}, ..., \sigma_k$ tends to 0.5 as $k \to \infty$ – which is what one expects from a random matrix if the probability is distributed equally on $\Sigma = \{-1, 1\}$.

For $r=0$ and $r=1$, (10) is the equation of the circle with radius 1 around -1 and 1, respectively. For every $r \in (0.5,1)$, the solutions of (10) form two closed curves: one curve lies inside the left loop of the lemniscate, and the second curve lies inside the radius 1 circle around 1 but outside the right loop of the lemniscate (see the right image of Figure 3.1, also cf. the resolvent level plots in [24, Figure 2.1]).

It is easy to see that every point $\lambda \in \overline{U}_1(-1) \cup \overline{U}_1(1)$, with the only two exceptions $\lambda=-1$ and $\lambda=1$, solves (10) for a particular value of $r \in [0,1]$, namely for

$$r \;=\; \frac{1}{1-\log_{|\lambda+1|}|\lambda-1|} \tag{11}$$

(the origin $\lambda=0$, for which this formula is not applicable, is a solution of (10) for every $r \in [0,1]$ and every point on the circle $|\lambda+1|=1$ is the solution of (10) for $r=0$), and that no λ outside these two disks solves (10) for any value of $r \in [0,1]$.

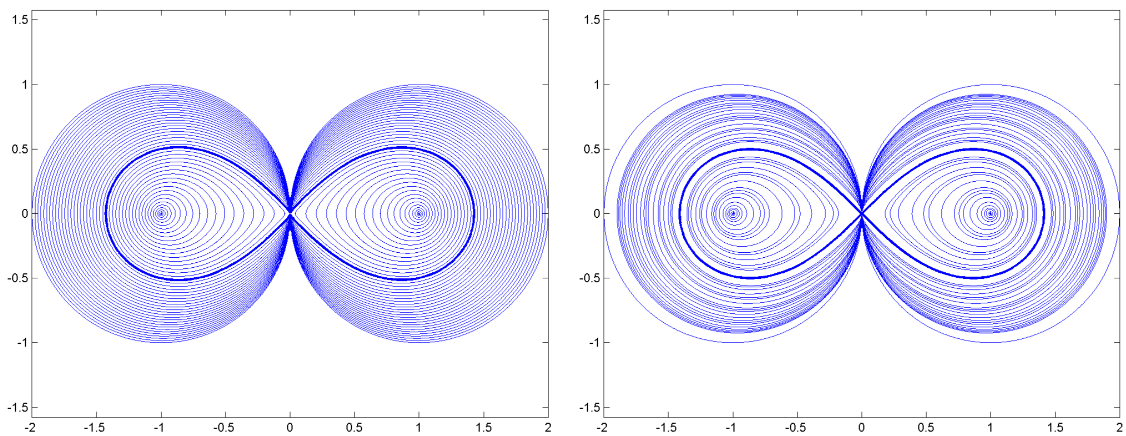

Figure 3.2: The left image shows the point spectra (solutions of (10)) corresponding to ratios $r=0,\ 0.02,\ \ldots,0.98,\ 1$ with $r=0.5$ highlighted (bold). So what we see on the left is $\mathrm{spec}^{50}_{\mathrm{per}}\, A$. What we see on the right is the union $\cup_{n=1}^{12}\mathrm{spec}^{n}_{\mathrm{per}}\, A$.

From (11) it is not hard to see that the set of all $\lambda \in \overline{U}_1(-1) \cup \overline{U}_1(1) = \mathrm{spec}\, A$ for which (11) is rational is a dense subset of $\mathrm{spec}\, A$. So here we have that the left-hand side of (7) is indeed dense in the right-hand side, i.e.

$$\mathrm{spec}\, A \;=\; \mathrm{clos}\left(\mathrm{spec}_{\mathrm{per}}\, A\right). \tag{12}$$

In this sense, for the determination of the spectrum of A, it is indeed enough to look at the periodic limit operators of A. We have tried to illustrate (12) in Figure 3.2.

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