Bezoutians

Georg Heinig and Karla Rost

Mathematics Subject Classification (2000). Primary 15-01; Secondary 15A09, 15A23, 65F05.

Keywords. Bezoutian, Toeplitz matrix inverses, Hankel matrix inverses, resultant matrix, Toeplitz-plus-Hankel matrix inverses.

Foreword

In the present paper we consider classes of matrices the entries of which are in a given field $\mathbb{F}$. These matrices have a special structure, they are *Bezoutians*. Historically, Bezoutians were at first introduced in connection with the elimination for the solution of systems of nonlinear algebraic equations and in connection with root localization problems. Only much later their importance for Hankel and Toeplitz matrix inversion became clear.

We will introduce three kinds of Bezoutians: Toeplitz Bezoutians, Hankel Bezoutians, and Toeplitz-plus-Hankel Bezoutians. The classes of Toeplitz and Hankel Bezoutians are related to Toeplitz and Hankel matrices in two ways. First, the inverses of Toeplitz and Hankel matrices are Toeplitz and Hankel Bezoutians, respectively. Furthermore, in case where $\mathbb{F} = \mathbb{C}$, Hermitian Toeplitz and Hankel Bezoutians are congruent to Toeplitz and Hankel matrices. The class of Toeplitz-plus-Hankel Bezoutians includes the inverses of Toeplitz-plus-Hankel matrices. Instead of a summary of the content we will offer the table of contents at the end of this foreword.

The present paper is not a usual paper. It originated from the draft of one chapter of a textbook on structured matrices planned by both authors. This textbook for graduate students was intended to range from the basics for beginners up to recent investigations. At the beginning of 2005 the outlines for the first three chapters were ready and parts of the text were in an acceptable form when Georg Heinig, the head of this project, unexpectedly died of a heart attack on May 10, 2005. We have lost one of the top experts in the field of structured matrices. His death reveals a gap we cannot overcome. This is the tragedy of the planned book and also of the present paper.

In the last period of our cooperation that had lasted 30 years we mainly worked on the third chapter of the textbook, which was dedicated to Bezoutians, so that I think that this part of the book was perhaps the favorite "child" of Georg.

Thus I felt obliged to continue and complete this text to achieve a selfcontained, improved version which can be published separately. I started with a preliminary section to make the presentation more selfcontained. Then I corrected and completed the other sections. Since the Toeplitz-plus-Hankel case was not included, I added main results concerning this case in Sections 11 and 12. Moreover, I finished, as planned, with exercises – part of which were already discussed with Georg – and then I make some short historical notes and provide hints to literature pursuing and accentuating the topic in different directions.

I hope I was able to do all these things in such a way that Georg would not be ashamed. In fact it is a hard burden of responsibility for me, in particular, since Georg was an outstanding mathematician with excellent abilities in teaching and in writing papers.

A further reason for this paper is that the topic of Bezoutians is very nice, interesting and important with a lot of connections and applications. In the last few years, one can even observe a revival of the interest in Bezoutians, mainly motivated by their importance in many modern fields such as numerical computing and control theory. Thus it would be very useful to have an introductional paper into this topic, where a lot of properties and relations are systematically collected and explained.

I neither intend to quote a huge number of relevant papers nor to mention all corresponding generalizations and applications. What I try and do is to appreciate Georg's contributions, since his legacy concerning this topic is enormously.

In 1971 Georg started his PhD studies at the State University of Moldavia in Kishinev under the supervision of Israel Gohberg. By this time he was irretrievably cast into the realm of structured matrices, in particular of Toeplitz and Hankel matrices as well as Bezoutians. His very early joint papers with I. Gohberg [12], [13] dealt with the inversion of finite Toeplitz matrices, the papers [14], [15], [16], [17], [18], [19], [20], [21] were dedicated to Bezoutians and resultant matrices mainly for operator-valued polynomials or to continual analogs of resultants and Bezoutians. The main results of these papers were milestones of the research in this field.

In 1975 Georg waked up my interests in the topic of Toeplitz and Hankel matrices and their inverses. Thus, in 1981, I wrote a large part of my PhD thesis on the method of UV-reduction for inverting structured matrices under his excellent supervision. When we started our work on the book "Algebraic Methods for Toeplitz-like Matrices and Operators" [32], [33] he introduced me, in particular, into the wonderful world of Bezoutians. In this book, Section 2 of Part I is dedicated to Bezoutians and resultant matrices. Some of the results presented there are, of course, also offered here but, as the result of new thoughts about the matter, from another point of view.

Moreover, in Subsection 2.2, Part II of [33] we present first ideas and results concerning matrices which are the sum of a Toeplitz and a Hankel matrix (briefly T+H matrices). In my opinion, one of our most important joint result is that in 1986 we discovered a Bezoutian structure also for inverses of T+H matrices (see [34]). This was the starting point of a long interesting and fruitful joint work on these special cases of structured matrices In fact, until now I feel a motivation given by Georg to deal with the T+H case (see [48], [64]).

Beginning with the joint paper [37] we wrote a number of papers on matrix representations for T+H matrices and their inverses which allow fast matrix-vector multiplication (see e.g. [39], [38], [40], [42], [43]).

Then we dealt with the problem how to connect the Toeplitz or T+H structure of matrices with possibly additional symmetries in order to reduce the number of parameters involved in these formulas or in the corresponding algorithms. Georg's paper [22] showed that splitting ideas in the spirit of Delsarte and Genin were very promising. The splitting approaches of our joint papers [44], [45], [46] differ from those of [22].

(Note that in [34] the concept of ω-structured matrices was introduced as a generalization of matrices possessing a Toeplitz, Hankel, or T+H structure. This class of and further investigated in [35], [36]. But these considerations are not included in this paper.)

I was only one of a large number of Georg's coauthors and pupils. In particular, Uwe Jungnickel wrote his PhD thesis under Georg's supervision in 1986. In their joint papers [27], [28] they considered Routh-Hurwitz or Schur-Cohn problems of counting the roots of a given polynomial in a half plane or in a circle. They investigated Hankel matrices generated by the Markov parameters of rational functions and their importance for partial realization and Pade approximation in [30], [29]. They investigated the connection of Bezoutians and resultant matrices for the solution of matrix equations in [26], [31].

In Section 7 of Part I of [33] some first results concerning the Bezoutian structure of generalized inverses are presented. Georg continued this investigation together with his student Frank Hellinger. Their results published in [23], [25], [24] have found perpetual interest by a large community. Since they go beyond the scope of the present paper they are not included.

It is not possible to recognize the full extent and importance of Georg's work concerning Bezoutians. I beg your pardon for all I will forget to mention or I will not appreciate to the due extend.

Karla Rost

Contents

1. Preliminaries

1. Notations. Throughout the paper, $\mathbb{F}$ will denote an arbitrary field. In some sections we restrict ourselves to the case that $\mathbb{F}=\mathbb{C}$ or $\mathbb{R}$, the fields of complex or real numbers, respectively. By $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ the standard basis of $\mathbb{F}^n$ is denoted. Furthermore, $\mathbf{0}_k$ will stand for a zero vector of length k. If there is no danger of misunderstanding we will omit the subscript k.

As usual, an element of the vector space $\mathbb{F}^n$ will be identified with the corresponding $n\times 1$ (column) matrix. That means

$$(x_i)_{i=1}^n=(x_1,\dots,x_n)=\begin{bmatrix} x_1\\ \vdots \\ x_n\end{bmatrix}.$$

In all what follows we denote by $\ell_n(t)$, $t\in\mathbb{F}$, the vector

$$\ell_n(t)=(1,t,t^2,\dots,t^{n-1}). \tag{1.1}$$

The Bezoutian concept is convenient introduced in polynomial language. First we introduce "polynomial language" for vectors. For $\mathbf{x}=(x_i)_{i=1}^n\in\mathbb{F}^n$, we consider the polynomial

$$\mathbf{x}(t)=\ell_n(t)^T\mathbf{x}=\sum_{k=1}^n x_k t^{k-1}\in\mathbb{F}^n(t)$$

and call it *generating polynomial* of $\mathbf{x}$. Polynomial language for matrices means that we introduce the *generating polynomial of an $m\times n$ matrix $A=[\,a_{ij}\,]_{i=1,j=1}^{m\ \ n}\in\mathbb{F}^{m\times n}$* as the bivariate polynomial

$$A(t,s)=\ell_m(t)^T A\ell_n(s)=\sum_{i=1}^m\sum_{j=1}^n a_{ij}\,t^{i-1}s^{j-1}.$$

At several places in this paper we will exploit symmetry properties of matrices. Besides symmetry, skewsymmetry and Hermitian symmetry in the usual sense we also deal with persymmetry and centrosymmetry. To be more precise we introduce some notations. Let J_n be the matrix of the flip operator in $\mathbb{F}^n$ mapping $(x_1,x_2,\dots,x_n)$ to $(x_n,x_{n-1},\dots,x_1)$,

$$J_n=\begin{bmatrix} 0 & & 1\\ & \iddots & \\ 1 & & 0\end{bmatrix}. \tag{1.2}$$

For a vector $\mathbf{x}\in\mathbb{F}^n$ we denote by $\mathbf{x}^J$ the vector $J_n\mathbf{x}$ and, in case $\mathbb{F}=\mathbb{C}$, by $\mathbf{x}^\#$ the vector $J_n\overline{\mathbf{x}}$, where $\overline{\mathbf{x}}$ is the vector with the conjugate complex entries,

$$\mathbf{x}^J=J_n\mathbf{x}\quad\text{and}\quad \mathbf{x}^\#=J_n\overline{\mathbf{x}}.$$

In polynomial language the latter looks like

$$\mathbf{x}^J(t)=\mathbf{x}(t^{-1})t^{n-1},\quad \mathbf{x}^\#(t)=\overline{\mathbf{x}}(t^{-1})t^{n-1}.$$

A vector is called *symmetric* if $\mathbf{x}^J=\mathbf{x}$, *skewsymmetric* if $\mathbf{x}^J=-\mathbf{x}$, and *conjugate symmetric* if $\mathbf{x}^\#=\mathbf{x}$. Let $\mathbb{F}^n_+(\mathbb{F}^n_-)$ denote the subspace of all symmetric (skewsymmetric) vectors of $\mathbb{F}^n$, and let $P_\pm$ be the matrices

$$P_\pm=\frac{1}{2}(I_n\pm J_n). \tag{1.3}$$

These matrices are projections onto $\mathbb{F}^n_\pm$ and

$$P_+ + P_- = I_n,\quad P_+ - P_- = J_n.$$

For an $n \times n$ matrix A, we denote

$$A^J = J_n A J_n \quad \text{and} \quad A^{\#} = J_n \overline{A} J_n \,,$$

where $\overline{A}$ is the matrix with the conjugate complex entries. An $n \times n$ matrix A is called *persymmetric* if $A^J = A^T$. The matrix A is called *centrosymmetric* if $A^J = A$. It is called *centro-skewsymmetric* if $A^J = -A$ and *centro-Hermitian* if $A^{\#} = A$.

2. Sylvester's inertia law. Assume that $\mathbb{F} = \mathbb{C}$. Let A be an Hermitian $n \times n$ matrix. The triple of integers

$$\operatorname{In} A = (p_+, p_-, p_0)$$

in which p_+ is the number of positive, p_- the number of negative, and p_0 the number of zero eigenvalues, counting multiplicities, is called the *inertia of* A. Clearly $p_+ + p_- + p_0 = n$. The integer

$$\operatorname{sgn} A = p_+ - p_-$$

is called the *signature of* A. Note that $p_- + p_+$ is the rank of A, so that rank and signature of an Hermitian matrix determine its inertia.

Two Hermitian $n \times n$ matrices A and B are called *congruent* if there is a nonsingular matrix C such that $B = C^*AC$, where C^* denotes the conjugate transpose of C. The following is *Sylvester's inertia law*, which will frequently be applied in this paper.

Theorem 1.1. *Congruent matrices have the same inertia.*

We will often apply the following version of Sylvester's inertia law.

Corollary 1.2. *Let A be an Hermitian $m \times m$ matrix and C an $m \times n$ matrix with $m \leq n$ and* $\operatorname{rank} C = m$. *Then the signatures of A and C^*AC coincide.*

To see that Corollary 1.2 follows from Theorem 1.1 we extend C to a nonsingular $n \times n$ matrix $\widetilde{C}$ by adding rows at the bottom. Then $C^*AC = \widetilde{C}^*\widetilde{A}\widetilde{C}$, where $\widetilde{A}$ is the extension of A by $n-m$ zero columns and zero rows on the right and at the bottom, respectively. This means that C^*AC is congruent to $\widetilde{A}$, and thus $\operatorname{sgn} C^*AC = \operatorname{sgn} \widetilde{A} = \operatorname{sgn} A$.

3. Toeplitz, Hankel, and Toeplitz-plus-Hankel matrices. Let $\mathcal{T}_{mn}$ be the subspace of $\mathbb{F}^{m \times n}$ consisting of all $m \times n$ Toeplitz matrices

$$T_{mn}(\mathbf{a}) = [\, a_{i-j} \,]_{i=1,j=1}^{m\quad n}, \quad \mathbf{a} = (a_i)_{i=1-n}^{m-1} \in \mathbb{F}^{m+n-1}.$$

The subspace of all $m \times n$ Hankel matrices

$$H_{mn}(\mathbf{s}) = [\, s_{i+j-1} \,]_{i=1,j=1}^{m\quad n}, \quad \mathbf{s} = (s_i)_{i=1}^{m+n-1} \in \mathbb{F}^{m+n-1}$$

is denoted by $\mathcal{H}_{mn}$. The dimension of these subspaces is $m+n-1$. The intersection $\mathcal{T}_{mn} \cap \mathcal{H}_{mn}$ consists of all *chess-board matrices*,

$$B = \begin{bmatrix} c & b & c & \cdots \\ b & c & b & \cdots \\ c & b & c & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \qquad (c, b \in \mathbb{F}) \tag{1.4}$$

which form a two-dimensional subspace of $\mathbb{F}^{m \times n}$. The subspace of all $m \times n$ matrices R_{mn} which are the sum of a Toeplitz and a Hankel matrix (briefly *T+H matrices*)

$$R_{mn} = T_{mn}(\mathbf{a}) + H_{mn}(\mathbf{s})$$

is $2(m+n-2)$ dimensional. Since for an $m\times n$ Hankel matrix H_{mn} the matrix $H_{mn}J_n$ is Toeplitz any T+H matrix can be represented in the form

$$R_{mn} = T_{mn}(\mathbf{a}) + T_{mn}(\mathbf{b})J_n \qquad (\mathbf{a},\mathbf{b} \in \mathbb{F}^{m+n-1}). \tag{1.5}$$

From this another representation is derived involving the projections $P_\pm$ introduced in (1.3),

$$R_{mn} = T_{mn}(\mathbf{c})P_+ + T_{mn}(\mathbf{d})P_- \tag{1.6}$$

with $\mathbf{c} = \mathbf{a}+\mathbf{b}, \mathbf{d} = \mathbf{a}-\mathbf{b}$. Obviously, all these representations are not unique (see Exercises 15 and 16).

4. Quasi-Toeplitz matrices, quasi-Hankel matrices, and quasi-T+H matrices. We consider the transformation ∇_+ in the space of $n \times n$ matrices defined by

$$\nabla_+(A) = A - S_n A S_n^T , \tag{1.7}$$

where S_n is the forward shift in $\mathbb{F}^n$ mapping $(x_1, x_2, \ldots, x_n)$ to $(0, x_1, \ldots, x_{n-1})$,

$$S_n = \begin{bmatrix} 0 & 0 & & 0 \\ 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix} . \tag{1.8}$$

It can easily be checked that this transformation is one-to-one. The transformation ∇_+ is called *shift displacement operator*. For a Toeplitz matrix $T_n = [\, a_{i-j}\,]_{i,j=1}^n$ we have, obviously,

$$\nabla_+(T_n) = \begin{bmatrix} a_0' & 1 \\ a_1 & 0 \\ \vdots & \vdots \\ a_{n-1} & 0 \end{bmatrix} \begin{bmatrix} a_0' & a_{-1} & \ldots & a_{1-n} \\ 1 & 0 & \ldots & 0 \end{bmatrix} ,$$

where $a_0' = \frac{1}{2}\, a_0$. In particular, the rank of $\nabla_+(T_n)$ equals 2, unless T_n is triangular. In the latter case the rank of $\nabla_+(T_n)$ equals 1, unless $T_n = O$.

Notice that if T_n is Hermitian, then $\nabla_+(T_n)$ is also Hermitian, and the signature of $\nabla_+(T_n)$ equals zero, unless T_n is diagonal. (Obviously, T_n diagonal means $T_n = a_0 I_n$ and $\operatorname{sgn}(\nabla_+(T_n))$ equals the signum of a_0.)

Moreover, a matrix A is Toeplitz if and only if the $(n-1) \times (n-1)$ submatrix in the lower right corner of $\nabla_+(A)$ is the zero matrix. An $n \times n$ matrix A is called *quasi-Toeplitz* if $\operatorname{rank} \nabla_+(A) \le 2$.

Clearly, Toeplitz matrices are also quasi-Toeplitz, but not vice versa. The following proposition gives a complete description of quasi-Toeplitz matrices. Since the proof is an elementary calculation, we leave it to the reader.

Proposition 1.3. *Suppose that* $\nabla_+(A) = \mathbf{g}_+\mathbf{g}_-^T - \mathbf{h}_+\mathbf{h}_-^T$, $\mathbf{g}_\pm = (g_i^\pm)_{i=1}^n$, $\mathbf{h}_\pm = (h_i^\pm)_{i=1}^n$. *Then* A *can be represented as the sum of* 2 *products of triangular Toeplitz matrices,*

$$A = \begin{bmatrix} g_1^+ & & 0 \\ \vdots & \ddots & \\ g_n^+ & \ldots & g_1^+ \end{bmatrix} \begin{bmatrix} g_1^- & \ldots & g_n^- \\ & \ddots & \vdots \\ 0 & & g_1^- \end{bmatrix} - \begin{bmatrix} h_1^+ & & 0 \\ \vdots & \ddots & \\ h_n^+ & \ldots & h_1^+ \end{bmatrix} \begin{bmatrix} h_1^- & \ldots & h_n^- \\ & \ddots & \vdots \\ 0 & & h_1^- \end{bmatrix} . \tag{1.9}$$

Conversely, if A *is given by* (1.9), *then* $\nabla_+(A) = \mathbf{g}_+\mathbf{g}_-^T - \mathbf{h}_+\mathbf{h}_-^T$.

Analogously, we consider the transformation $\nabla^+ : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ defined by

$$\nabla^+(A) = S_n A - A S_n^T . \tag{1.10}$$

A matrix A is Hankel if and only if the $(n-1)\times(n-1)$ submatrix in the lower right corner of $\nabla^+(A)$ is the zero matrix. We call a matrix A *quasi-Hankel* if $\operatorname{rank}\nabla^+(A) \le 2$. A similar representation to (1.9) can be obtained.

Let W_n be the matrix $W_n = S_n + S_n^T$, and let $\nabla : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ be defiend by

$$\nabla(A) = AW_n - W_n A . \tag{1.11}$$

Proposition 1.4. *A matrix A is a T+H matrix if and only if the $(n-2)\times(n-2)$ submatrix in the center of $\nabla(A)$ is the zero matrix.*

We call a matrix A *quasi-T+H* if $\operatorname{rank}\nabla(A) \le 4$. T+H matrices are also quasi-T+H, but not vice versa.

5. Möbius transformations. The flip operator J_n introduced in (1.2) is a special case of a class of operators which will be described in this subsection. Let $\phi = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ be a nonsingular 2×2 matrix with entries from $\mathbb{F}$. We associate ϕ with the linear fractional function

$$\phi(t) = \frac{at+b}{ct+d} .$$

Despite we use the name "function", $\phi(t)$ is understood here in a formal sense, i.e. t is considered as an abstract variable. In the case where $\mathbb{F} = \mathbb{C}$, $\phi(t)$ can be seen as a function mapping the Riemann sphere onto itself. These linear fractional functions form a group $\mathcal{M}$ with respect to composition. This group is *isomorphic*, modulo multiples of I_2, to the group $\mathrm{GL}(\mathbb{F}^2)$ of nonsingular 2×2 matrices. The latter means that if $\phi = \phi_1\phi_2$, then $\phi(t) = \phi_2(\phi_1(t))$, and $\phi(t) = t$ if and only if $\phi = \alpha I_2$ for some $\alpha \in \mathbb{F}$.

We will make use of the fact that the group $\mathrm{GL}(\mathbb{F}^2)$ is generated by matrices of the form

$$\text{(a)}\ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\ (a \neq 0), \quad \text{(b)}\ \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}, \quad \text{(c)}\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{1.12}$$

For $\phi \in \mathrm{GL}(\mathbb{F}^2)$ and a natural number n, let $K_n(\phi)$ denote the operator defined by

$$K_n(\phi)\mathbf{x}(t) = \mathbf{x}(\phi(t))(ct+d)^{n-1}$$

for $\mathbf{x}(t) \in \mathbb{F}^n(t)$. An operator of this form will be called *Möbius transformation.* It is easily checked that $K_n(\phi)$ maps $\mathbb{F}^n(t)$ into itself and is linear. In the special cases (1.12) we have

$$\text{(a)}\ K_n(\phi)\mathbf{x}(t) = \mathbf{x}(at),\ \text{(b)}\ K_n(\phi)\mathbf{x}(t) = \mathbf{x}(t+b),\ \text{(c)}\ K_n(\phi)\mathbf{x}(t) = \mathbf{x}(t^{-1})t^{n-1}. \tag{1.13}$$

The matrix representations of these transformations (called Möbius matrices) with repsect to the standard basis in $\mathbb{F}^n(t)$ are

$$\text{(a)}\ K_n(\phi) = \operatorname{diag}(a^j)_{j=0}^{n-1},\ \text{(b)}\ K_n(\phi) = \left[\binom{k}{j} b^{k-j} \right]_{j,k=0}^{n-1},\ \text{(c)}\ K_n(\phi) = J_n. \tag{1.14}$$

Furthermore, the following is true.

Proposition 1.5. *If $\phi_1, \phi_2 \in \mathrm{GL}(\mathbb{F}^2)$ and $\phi = \phi_1\phi_2$, then $K_n(\phi) = K_n(\phi_1)K_n(\phi_2)$.*

It is sufficient to prove the proposition for the special matrices (1.12). We leave this to the reader.

According to Proposition 1.5 the Möbius transformations are all invertible and form a subgroup of the group of invertible linear operators on $\mathbb{F}^n$. Furthermore, $K_n(\phi)^{-1} = K_n(\phi^{-1})$. Möbius matrices are mostly used in connection with matrix transformations of the form

$$A \mapsto K_n(\psi)^T A K_n(\phi) \tag{1.15}$$

for fixed $\phi, \psi \in \mathrm{GL}(\mathbb{F}^2)$. In the literature special cases of such transformations are called *Frobenius-Fischer transformations*. We will use this name for all transformations of this form. Some Frobenius-Fischer transformations are mappings inside a class of structured matrices, other build a bridge between different classes. We discuss here the situation with Hankel and Toeplitz matrices.

Let $\mathcal{H}_n$ denote the class of $n \times n$ Hankel matrices $H_n(\mathbf{s}) = [\, s_{i+j-1} \,]_{i,j=1}^n$, where $\mathbf{s} = (s_i)_{i=1}^{2n-1} \in \mathbb{F}^{2n-1}$. The following proposition describes Frobenius-Fischer transformations that map $\mathcal{H}_n$ into itself.

Proposition 1.6. *For $\phi \in \mathrm{GL}(\mathbb{F}^2)$ and $\mathbf{s} \in \mathbb{F}^{2n-1}$, the equality*

$$K_n(\phi)^T H_n(\mathbf{s}) K_n(\phi) = H_n(\widetilde{\mathbf{s}})$$

with $\widetilde{\mathbf{s}} = K_{2n-1}(\phi)^T \mathbf{s}$ is satisfied.

Proof. It suffices to prove the proposition for the special cases (1.12). The cases (a) and (c) are obvious. Let ϕ be now of the form (b), $K_n(\phi)^T H_n(\mathbf{s}) K_n(\phi) = [\, g_{ij} \,]_{i,j=0}^{n-1}$. Then

$$g_{ij} = \sum_{k=0}^{i} \sum_{l=0}^{j} \binom{i}{k}\binom{j}{l} b^{i+j-k-l} s_{k+l+1} = \sum_{r=0}^{i+j} \binom{i+j}{r} b^{i+j-r} s_{r+1} \,.$$

This implies the assertion. ■

Now we consider besides Hankel also Toeplitz matrices $T_n = [\, a_{i-j} \,]_{i,j=1}^n$. The class of $n \times n$ Toeplitz matrices will be denoted by $\mathcal{T}_n$. Obviously, T_n is Toeplitz if and only if $J_n T_n$ is Hankel. Remember that J_n is the special Möbius matrix $K_n(J_2)$. Thus modifications of Propositions 1.6 can be stated about Frobenius-Fischer transformations transforming Toeplitz into Hankel, Hankel into Toeplitz and Toeplitz into Toeplitz matrices. In particular, we have the following.

Corollary 1.7. *For $\phi \in \mathrm{GL}(\mathbb{F}^2)$, the transformation $A \mapsto K_n(\psi)^T A\, K_n(\phi)$ maps*

1. *$\mathcal{H}_n$ into $\mathcal{H}_n$ if $\psi = \phi$,* 2. *$\mathcal{T}_n$ into $\mathcal{H}_n$ if $\psi = J_2\phi$,*
3. *$\mathcal{H}_n$ into $\mathcal{T}_n$ if $\psi = \phi J_2$,* 4. *$\mathcal{T}_n$ into $\mathcal{T}_n$ if $\psi = J_2 \phi J_2$.*

In the case $\mathbb{F} = \mathbb{C}$ we are in particular interested in congruence transformations, i.e. transformations that preserve Hermitian symmetry. For this we have to check under which condition $K_n(\psi)^T = K_n^*(\phi)$. In terms of the matrix $\phi = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ this condition is equivalent to

1. a, b, c, d real, 2. $a = \overline{b}\,,\ c = \overline{d}\,,$ 3. $a = \overline{c}\,,\ b = \overline{d}\,,$ 4. $a = \overline{d}\,,\ b = \overline{c}$

in the cases of Corollary 1.7. In terms of the linear fractional function $\phi(t) = \dfrac{at+b}{ct+d}$ this means that

1. $\phi(t)$ maps $\mathbb{R}$ to $\mathbb{R}$, 2. $\phi(t)$ maps $\mathbb{T}$ to $\mathbb{R}$, 3. $\phi(t)$ maps $\mathbb{R}$ to $\mathbb{T}$, 4. $\phi(t)$ maps $\mathbb{T}$ to $\mathbb{T}$,

where $\mathbb{T}$ denotes the unit circle. For transformations with this property the inertia of the matrix remains invariant by Sylvester's inertia law.

2. Definitions and Properties for the Hankel and Toeplitz Case

1. Hankel Bezoutians. Let $\mathbf{u}(t), \mathbf{v}(t) \in \mathbb{F}^{n+1}(t)$ be two polynomials. The *Hankel Bezoutian* or briefly *H-Bezoutian* of $\mathbf{u}(t)$ and $\mathbf{v}(t)$ is, by definition, the $n \times n$ matrix $B = \mathrm{Bez}_H(\mathbf{u}, \mathbf{v})$ with the generating polynomial

$$B(t,s) = \frac{\mathbf{u}(t)\mathbf{v}(s) - \mathbf{v}(t)\mathbf{u}(s)}{t-s}\,.$$

We will also say that B is the H-Bezoutian of the vectors $\mathbf{u}$ and $\mathbf{v}$. It is easily seen that $B(t,s)$ is really a polynomial in t and s. A simple argumentation for this is as follows. We fix $s = s_0$. Then the numerator is a polynomial in t vanishing at $t = s_0$. Hence we obtain a polynomial after dividing the numerator by $t - s_0$. Thus $B(t, s_0)$ is a polynomial in t. Analogously $B(t_0, s)$ is a polynomial in s for fixed $t = t_0$, and so the claim is proved.

For example, if $\mathbf{u}(t) = t - a$ and $\mathbf{v}(t) = t - b$, then we have

$$B(t,s) = \frac{(t-a)(s-b) - (t-b)(s-a)}{t-s} = a - b\,.$$

Thus $\mathrm{Bez}_H(\mathbf{u}, \mathbf{v}) = a - b$.

As a second example we consider the flip matrix J_n introduced in (1.2). The generating function of J_n is

$$J_n(t) = \frac{t^n - s^n}{t-s}\,.$$

Thus J_n is the H-Bezoutian of the polynomials t^n and 1 or, in other words, of the last and first unit vectors $\mathbf{e}_{n+1}$ and $\mathbf{e}_1$.

As a more general example, let us compute the H-Bezoutian B_k of a general polynomial $\mathbf{u}(t) \in \mathbb{F}^{n+1}(t)$ and $\mathbf{e}_k(t) = t^{k-1}$, $k = 1, \dots, n+1$. Suppose that $\mathbf{u}(t) = \sum_{i=1}^{n+1} u_i t^{i-1}$. Then we have

$$\begin{aligned} B_k(t,s) &= \sum_{i=1}^{n+1} u_i \frac{t^{i-1}s^{k-1} - t^{k-1}s^{i-1}}{t-s} \\ &= \sum_{i=k+1}^{n+1} u_i \frac{t^{i-k} - s^{i-k}}{t-s} t^{k-1}s^{k-1} - \sum_{i=1}^{k-1} u_i \frac{t^{k-i} - s^{k-i}}{t-s} t^{i-1}s^{i-1}\,. \end{aligned}$$

In matrix language this means that

$$\mathrm{Bez}_H(\mathbf{u}, \mathbf{e}_k) = \begin{bmatrix} & & -u_1 & & & \\ & \iddots & \vdots & & O & \\ -u_1 & \dots & -u_{k-1} & & & \\ & & & u_{k+1} & \dots & u_{n+1} \\ & O & & \vdots & \iddots & \\ & & & u_{n+1} & & \end{bmatrix}\,. \tag{2.1}$$

The case $k = 1$ is of particular importance. For this reason we introduce the notation

$$B(\mathbf{u}) = \mathrm{Bez}_H(\mathbf{u}, \mathbf{e}_1) = \begin{bmatrix} u_2 & \dots & u_{n+1} \\ \vdots & \iddots & \\ u_{n+1} & & \end{bmatrix}\,. \tag{2.2}$$

H-Bezoutians are obviously symmetric matrices. They are skewsymmetric with respect to the arguments, i.e.

$$\mathrm{Bez}_H(\mathbf{u},\mathbf{v}) = -\mathrm{Bez}_H(\mathbf{v},\mathbf{u})\,.$$

Furthermore, $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ is linear in each argument. That means that, for $c_1, c_2 \in \mathbb{F}$,

$$\mathrm{Bez}_H(c_1\mathbf{u}_1 + c_2\mathbf{u}_2, \mathbf{v}) = c_1\mathrm{Bez}_H(\mathbf{u}_1,\mathbf{v}) + c_2\mathrm{Bez}_H(\mathbf{u}_2,\mathbf{v})\,.$$

To present a product rule for H-Bezoutians we need the matrix of the multiplication operator which is introduced as follows. Let $\mathbf{a}(t) = \sum_{k=0}^{m} a_k t^k \in \mathbb{F}^{m+1}(t)$. For $n = 1, 2, \ldots$, define a linear operator $M_n(\mathbf{a}) : \mathbb{F}^n \longrightarrow \mathbb{F}^{m+n}$ by

$$(M_n(\mathbf{a})\mathbf{x})(t) = \mathbf{a}(t)\mathbf{x}(t)\,.$$

The matrix of this operator with respect to the standard bases is the $(m+n) \times n$ matrix

$$M_n(\mathbf{a}) = \left.\left[\begin{array}{cccc} a_0 & & & \\ a_1 & a_0 & & \\ \vdots & a_1 & \ddots & \\ a_m & \vdots & \ddots & a_0 \\ & a_m & & a_1 \\ & & \ddots & \vdots \\ & & & a_m \end{array}\right]\right\} m+n\,. \tag{2.3}$$

Moreover, we need the following matrix. Let $\mathbf{a}(t) = \sum_{k=0}^{m} a_k t^k$ and $\mathbf{b}(t) = \sum_{k=0}^{n} b_k t^k$ be two given polynomials. Then

$$(\mathbf{x}(t), \mathbf{y}(t)) \mapsto \mathbf{a}(t)\mathbf{x}(t) + \mathbf{b}(t)\mathbf{y}(t), \quad \mathbf{x}(t) \in \mathbb{F}^n(t),\ \mathbf{y}(t) \in \mathbb{F}^m(t)$$

is a linear operator from the direct product $\mathbb{F}^n(t) \otimes \mathbb{F}^m(t)$ to $\mathbb{F}^{m+n}(t)$. The matrix of this operator with respect to the standard bases is given by $[\, M_n(\mathbf{a}) \;\; M_m(\mathbf{b})\,]$. (Here we identify $(\mathbf{x}(t), \mathbf{y}(t))$ with $\mathbf{x}(t) + t^n\mathbf{y}(t)$.) The transpose of this matrix is called the *resultant matrix* (or *Sylvester matrix*) of $\mathbf{a}(t)$ and $\mathbf{b}(t)$ (or of the vectros $\mathbf{a}$ and $\mathbf{b}$) and is denoted by $\mathrm{Res}(\mathbf{a},\mathbf{b})$,

$$\mathrm{Res}(\mathbf{a},\mathbf{b}) = \begin{bmatrix} M_n(\mathbf{a})^T \\ M_m(\mathbf{b})^T \end{bmatrix}. \tag{2.4}$$

If we assume that $a_m \neq 0$ or $b_n \neq 0$ then $\mathrm{Res}(\mathbf{a},\mathbf{b})$ is nonsingular if and only if $\mathbf{a}(t)$ and $\mathbf{b}(t)$ are coprime (cf. Exercise 3).

Proposition 2.1. *Let* $\mathbf{u},\mathbf{v} \in \mathbb{F}^{n+1}$, $\mathbf{u}(t) = \mathbf{u}_1(t)\mathbf{u}_2(t)$, $\mathbf{v}(t) = \mathbf{v}_1(t)\mathbf{v}_2(t)$, *where* $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{F}^{n_i+1}$ $(i = 1,2)$ *and* $n_1 + n_2 = n - 1$. *Then*

$$\mathrm{Bez}_H(\mathbf{u},\mathbf{v}) = \mathrm{Res}\,(\mathbf{u}_2,\mathbf{v}_1)^T \begin{bmatrix} \mathrm{Bez}_H(\mathbf{u}_1,\mathbf{v}_1) & O \\ O & \mathrm{Bez}_H(\mathbf{u}_2,\mathbf{v}_2) \end{bmatrix} \mathrm{Res}\,(\mathbf{v}_2,\mathbf{u}_1)\,. \tag{2.5}$$

Proof. Let $B = \mathrm{Bez}_H(\mathbf{u},\mathbf{v})$. Then, $B(t,s)$ has the representation

$$\mathbf{u}_2(t)\,\frac{\mathbf{u}_1(t)\mathbf{v}_1(s) - \mathbf{v}_1(t)\mathbf{u}_1(s)}{t-s}\,\mathbf{v}_2(s) + \mathbf{v}_1(t)\,\frac{\mathbf{u}_2(t)\mathbf{v}_2(s) - \mathbf{v}_2(t)\mathbf{u}_2(s)}{t-s}\,\mathbf{u}_1(s)\,.$$

In matrix language this means

$$B = M_{n_1}(\mathbf{u}_2)\mathrm{Bez}_H(\mathbf{u}_1,\mathbf{v}_1)M_{n_1}(\mathbf{v}_2)^T + M_{n_2}(\mathbf{v}_1)\mathrm{Bez}_H(\mathbf{u}_2,\mathbf{v}_2)M_{n_2}(\mathbf{u}_1)^T\,.$$

From this relation the assertion is immediate. ■

2. The transformation ∇_H. Next we clarify what means for a matrix to be an H-Bezoutian in matrix language. For this we introduce the transformation ∇_H transforming an $n\times n$ matrix $A=[\,a_{ij}\,]_{i,j=1}^n$ into a $(n+1)\times(n+1)$ matrix according to

$$\nabla_H A = [\,a_{i-1,j}-a_{i,j-1}\,]_{i,j=1}^{n+1}\,.$$

Here we set $a_{ij}=0$ if one of the integers i or j is not in the set $\{1,2,\dots,n\}$. We have

$$\nabla_H A = \begin{bmatrix} S_nA-AS_n^T & * \\ * & * \end{bmatrix} = \begin{bmatrix} * & * \\ * & AS_n-S_n^TA \end{bmatrix}, \tag{2.6}$$

where S_n is defined in (1.8). Comparing the coefficients it is easy to verify that

$$(\nabla_H A)(t,s)=(t-s)A(t,s)\,.$$

Hence the Bezoutians $B=\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ can be characterized with the help of ∇_H by

$$\nabla_H B = [\,\mathbf{u}\ \ \mathbf{v}\,]\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}[\,\mathbf{u}\ \ \mathbf{v}\,]^T\,. \tag{2.7}$$

In particular, the rank of ∇B_H is equal to 2, unless $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent. In the latter case the H-Bezoutian is the zero matrix. The representation (2.6) shows that the transformation ∇^+ introduced in (1.10) is a restriction of ∇_H . Thus, H-Bezoutians are quasi-Hankel matrices.

3. Uniqueness. Different pairs of polynomials may produce the same H-Bezoutian. However, from (2.7) one can conclude that if $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})=\mathrm{Bez}_H(\mathbf{u}_1,\mathbf{v}_1)\neq O$, then $\mathrm{span}\{\mathbf{u},\mathbf{v}\}=\mathrm{span}\{\mathbf{u}_1,\mathbf{v}_1\}$. In the latter case there is a nonsingular 2×2 matrix φ such that

$$[\,\mathbf{u}_1\ \ \mathbf{v}_1\,]=[\,\mathbf{u}\ \ \mathbf{v}\,]\,\varphi\,. \tag{2.8}$$

Lemma 2.2. *Let $\mathbf{u},\mathbf{v},\mathbf{u}_1,\mathbf{v}_1\in\mathbb{F}^{n+1}$ related via (2.8). Then*

$$\mathrm{Bez}_H\,(\mathbf{u}_1,\mathbf{v}_1)=(\det\varphi)\ \mathrm{Bez}_H\,(\mathbf{u},\mathbf{v})\,.$$

Proof. Suppose that $\varphi=\begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $B_1=\mathrm{Bez}_H\,(\mathbf{u}_1,\mathbf{v}_1)$. Then

$$\begin{aligned}(t-s)B_1(t,s) &= (a\mathbf{u}(t)+b\mathbf{v}(t))(c\mathbf{u}(s)+d\mathbf{v}(s))-(c\mathbf{u}(t)+d\mathbf{v}(t))(a\mathbf{u}(s)+b\mathbf{v}(s))\\ &= (ad-bc)(\mathbf{u}(t)\mathbf{v}(s)-\mathbf{v}(t)\mathbf{u}(s))\,,\end{aligned}$$

which proves the lemma. ■

Corollary 2.3. *The H-Bezoutians $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})\neq O$ and $\mathrm{Bez}_H(\mathbf{u}_1,\mathbf{v}_1)$ coincide if and only if the vectors $\mathbf{u},\mathbf{v}$ and $\mathbf{u}_1$, $\mathbf{v}_1$ are related via (2.8) with $\det\varphi=1$.*

From Corollary 2.3 we can conclude that the H-Bezoutian $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ is equal to a H-Bezoutian $\mathrm{Bez}_H(\widetilde{\mathbf{u}},\widetilde{\mathbf{v}})$ in which the last coefficient of $\widetilde{\mathbf{v}}$ vanishes, i.e. $\widetilde{\mathbf{v}}(t)\in\mathbb{F}^n(t)$.

4. Quasi-H-Bezoutians. A matrix B is called *quasi-H-Bezoutian* if $\mathrm{rank}\,\nabla_H B\leq 2$. We give a general representation of quasi-H-Bezoutians that is also important for H-Bezoutians.

Proposition 2.4. *A quasi-H-Bezoutian $B\neq O$ of order n admits a representation*

$$B=M_r(\mathbf{p})\,\mathrm{Bez}_H(\mathbf{u},\mathbf{v})\,M_r(\mathbf{q})^T\,, \tag{2.9}$$

where $\mathbf{u}(t),\mathbf{v}(t)\in\mathbb{F}^{r+1}(t)$ are coprime and $r\leq n$. Here $M_r(\,\cdot\,)$ is defined in (2.3).

Proof. For B is a quasi-H-Bezoutian, there exist $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{F}^{n+1}$ such that

$$(t-s)B(t,s) = \mathbf{a}(t)\mathbf{d}(s) - \mathbf{b}(t)\mathbf{c}(s)\,.$$

Since for $t = s$ the left-hand side vanishes, we have $\mathbf{a}(t)\mathbf{d}(t) = \mathbf{b}(t)\mathbf{c}(t)$. Let $\mathbf{p}(t)$ be the greatest common divisor of $\mathbf{a}(t)$ and $\mathbf{b}(t)$ and $\mathbf{q}(t)$ the greatest common divisor of $\mathbf{c}(t)$ and $\mathbf{d}(t)$. Then $\mathbf{a}(t) = \mathbf{p}(t)\mathbf{u}(t)$ and $\mathbf{b}(t) = \mathbf{p}(t)\mathbf{v}(t)$ for some coprime $\mathbf{u}(t), \mathbf{v}(t) \in \mathbb{F}^{r+1}(t)$ $(r \leq n)$. Furthermore, $\mathbf{c}(t) = \mathbf{q}(t)\mathbf{u}_1(t)$ and $\mathbf{d}(t) = \mathbf{q}(t)\mathbf{v}_1(t)$ for some coprime $\mathbf{u}_1(t)$, $\mathbf{v}_1(t) \in \mathbb{F}^{r_1+1}(t)$ $(r_1 \leq n)$. Since

$$\frac{\mathbf{a}(t)}{\mathbf{b}(t)} = \frac{\mathbf{u}(t)}{\mathbf{v}(t)} = \frac{\mathbf{c}(t)}{\mathbf{d}(t)} = \frac{\mathbf{u}_1(t)}{\mathbf{v}_1(t)}\,,$$

we conclude that, for some $\gamma \neq 0$, $\mathbf{u}_1 = \gamma\mathbf{u}$, $\mathbf{v}_1 = \gamma\mathbf{v}$, and $r = r_1$. Now we have

$$\mathbf{a}(t)\mathbf{d}(s) - \mathbf{b}(t)\mathbf{c}(s) = \gamma\,\mathbf{p}(t)(\mathbf{u}(t)\mathbf{v}(s) - \mathbf{v}(t)\mathbf{u}(s))\mathbf{q}(s)\,.$$

We can replace $\gamma\,\mathbf{p}$ by $\mathbf{p}$. Now it remains to translate this into matrix language to obtain (2.9). ■

The matrix on the right-hand side of (2.9) has rank r at most. Hence if $r < n$, then B is singular. This leads to the following somehow surprising conclusion.

Corollary 2.5. *Any nonsingular quasi-H-Bezoutian is an H-Bezoutian of two coprime polynomials.*

Later we will show that, vice versa, the H-Bezoutian of two coprime polynomials is nonsingular (cf. Corollary 3.4).

If the quasi-H-Bezoutian is symmetric, then in (2.9) we must have $\mathbf{q} = \mathbf{p}$, since the middle factor is symmetric. This implies the following.

Corollary 2.6. *Any symmetric quasi-H-Bezoutian is an H-Bezoutian*

$$B = \mathrm{Bez}_H(\mathbf{a}, \mathbf{b})\,.$$

In particular, (2.9) *can be written in the form*

$$B = M_r(\mathbf{p})\,\mathrm{Bez}_H(\mathbf{u}, \mathbf{v})\,M_r(\mathbf{p})^T\,, \tag{2.10}$$

where $\mathbf{p}(t)$ *is the greatest common divisor of* $\mathbf{a}(t)$ *and* $\mathbf{b}(t)$.

5. Frobenius-Fischer transformations. We show now that Frobenius-Fischer transformations introduced in Section 1 transform the class of H-Bezoutians into itself. In particular, the following result is the Bezoutian counterpart of Proposition 1.6.

Theorem 2.7. *For any* $\varphi \in \mathrm{GL}(\mathbb{F}^2)$*, the transformation*

$$B \mapsto K_n(\varphi)BK_n(\varphi)^T$$

maps H-Bezoutians into H-Bezoutians. Moreover

$$K_n(\varphi)\mathrm{Bez}_H(\mathbf{u}, \mathbf{v})K_n(\varphi)^T = \frac{1}{\det\varphi}\,\mathrm{Bez}_H(\widetilde{\mathbf{u}}, \widetilde{\mathbf{v}})\,, \tag{2.11}$$

where $\widetilde{\mathbf{u}} = K_{n+1}(\varphi)\mathbf{u}$ *and* $\widetilde{\mathbf{v}} = K_{n+1}(\varphi)\mathbf{v}$.

Proof. Let $B = \mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ and $\widetilde{B} = \mathrm{Bez}_H(\widetilde{\mathbf{u}},\widetilde{\mathbf{v}})$. It is sufficient to prove the theorem for the matrices (1.12) that generate $\mathrm{GL}(\mathbb{F}^2)$. If $\varphi = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$, then

$$\widetilde{B}(t,s) = a\,\frac{\mathbf{u}(at)\mathbf{v}(as) - \mathbf{v}(at)\mathbf{u}(as)}{at - as} = a\,B(at,as) = a\,(K_n(\varphi)BK_n(\varphi)^T)(t,s)\,,$$

which is equivalent to (2.11). If $\varphi = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$, then

$$\begin{aligned}\widetilde{B}(t,s) &= \frac{\mathbf{u}(t+b)\mathbf{v}(s+b) - \mathbf{v}(t+b)\mathbf{u}(s+b)}{t-s} = B(t+b,s+b)\\ &= (K_n(\varphi)BK_n(\varphi)^T)(t,s)\,,\end{aligned}$$

which is equivalent to (2.11). Finally, let $\varphi = J_2$. Then

$$\begin{aligned}\widetilde{B}(t,s) &= \frac{\mathbf{u}(t^{-1})t^n\mathbf{v}(s^{-1})s^n - \mathbf{v}(t^{-1})t^n\mathbf{u}(s^{-1})s^n}{t-s} = -B(t^{-1},s^{-1})(ts)^{n-1}\\ &= -B^J(t,s)\,,\end{aligned}$$

which is again equivalent to (2.11). ■

6. Splitting of H-Bezoutians. In some applications, like stability tests for real polynomials, $\mathbf{u}(t)$ and $\mathbf{v}(t)$ have the special form

$$\mathbf{u}(t) = \mathbf{a}(t^2) \quad \text{and} \quad \mathbf{v}(t) = t\,\mathbf{b}(t^2). \tag{2.12}$$

That means $\mathbf{u}(t)$ has only even powers and $\mathbf{v}(t)$ only odd powers. In this case we have for the generating polynomial of $B = \mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ after multiplying numerator and denominator by $(t+s)$

$$\begin{aligned}B(t,s) &= \frac{ts(\mathbf{a}(t^2)\mathbf{b}(s^2) - \mathbf{b}(t^2)\mathbf{a}(s^2)) + \mathbf{a}(t^2)s^2\mathbf{b}(s^2) - t^2\mathbf{b}(t^2)\mathbf{a}(s^2)}{t^2 - s^2}\\ &= tsB_1(t^2,s^2) + B_0(t^2,s^2)\,,\end{aligned}$$

where $B_1 = \mathrm{Bez}_H(\mathbf{a},\mathbf{b})$ and $B_0 = \mathrm{Bez}_H(\mathbf{a},t\mathbf{b})$. To translate this into matrix language we introduce the matrix Σ_n of the even-odd shuffle operator:

$$\Sigma_n(x_i)_{i=1}^n = (x_1,x_3,\ldots,x_2,x_4,\ldots)\,.$$

Proposition 2.8. *Let $\mathbf{u}(t) \in \mathbb{F}^n(t)$ and $\mathbf{v}(t) \in \mathbb{F}^n(t)$ be given by (2.12). Then*

$$\Sigma_n^T \mathrm{Bez}_H(\mathbf{u},\mathbf{v})\Sigma_n = \begin{bmatrix} \mathrm{Bez}_H(\mathbf{a},t\,\mathbf{b}) & O \\ O & \mathrm{Bez}_H(\mathbf{a},\mathbf{b}) \end{bmatrix}.$$

7. Toeplitz Bezoutians. We introduce the Toeplitz analogue of the H-Bezoutian. The *Toeplitz Bezoutian* or briefly *T-Bezoutian* of the two polynomials $\mathbf{u}(t) \in \mathbb{F}^{n+1}(t)$ and $\mathbf{v}(t) \in \mathbb{F}^{n+1}(t)$ is, by definition, the matrix $B = \mathrm{Bez}_T(\mathbf{u},\mathbf{v})$ with the generating polynomial

$$B(t,s) = \frac{\mathbf{u}(t)\mathbf{v}^J(s) - \mathbf{v}(t)\mathbf{u}^J(s)}{1 - ts}\,.$$

Like for H-Bezoutians, it is easily checked that $B(t,s)$ is really a polynomial in t and s. If, for example, the polynomials $\mathbf{u}(t) = t - a$ and $\mathbf{v}(t) = t - b$ of $\mathbb{F}^{n+1}(t)$ are given, then for $n = 1$

$$B(t,s) = \frac{(t-a)(1-bs) - (t-b)(1-as)}{1 - ts} = b - a\,.$$

Hence $\mathrm{Bez}_T(\mathbf{u},\mathbf{v}) = b-a$. But in case $n>1$ we have

$$\mathrm{Bez}_T(\mathbf{u},\mathbf{v}) = \begin{bmatrix} O & b-a \\ O & O \end{bmatrix}.$$

We state that the definition of a T-Bezoutian of two polynomials depends, in contrast to the H-Bezoutian, essentially on the integer n. That means if we consider $\mathbf{u}(t)$ and $\mathbf{v}(t)$ as elements of $\mathbb{F}^{N+1}(t)$ for $N>n$, then we will have a different T-Bezoutian. Indeed, let B_N denote the T-Bezoutian in this sense. Then we obtain

$$B_N(t,s) = \frac{\mathbf{u}(t)\mathbf{v}(s^{-1})s^N - \mathbf{v}(t)\mathbf{u}(s^{-1})s^N}{1-ts} = B(t,s)s^{N-n},$$

where B is the T-Bezoutian of $\mathbf{u}$ and $\mathbf{v}$ in the original sense. Thus, B_N is of the form

$$B_N = \begin{bmatrix} O & B \\ O & O \end{bmatrix}.$$

If $t=0$ is a common zero of $\mathbf{u}(t)$ and $\mathbf{v}(t)$, $\mathbf{u}(t)=t^r\mathbf{u}_0(t)$, $\mathbf{v}(t)=t^r\mathbf{v}_0(t)$ $(r>0)$, then B is of the form

$$B = \begin{bmatrix} O & O \\ B_0 & O \end{bmatrix}$$

where B_0 is the $(n-r)\times(n-r)$ T-Bezoutian of $\mathbf{u}_0$ and $\mathbf{v}_0$.

As an example, we compute the T-Bezoutian of a polynomial and a power of t. Let $B_{(k)} = \mathrm{Bez}_T(\mathbf{u},\mathbf{e}_k)$ and $\mathbf{u}=(u_i)_{i=1}^{n+1}$. Then

$$\begin{aligned} B_{(k)}(t,s) &= \sum_{i=1}^{n+1} u_i \, \frac{t^{i-1}s^{n-k+1} - t^{k-1}s^{n-i+1}}{1-ts} \\ &= \sum_{i=1}^{k-1} u_i \, t^{i-1}s^{n-k+1} \, \frac{1-(ts)^{k-i}}{1-ts} + \sum_{i=k+1}^{n+1} u_i \, t^{k-1}s^{n-i+1} \, \frac{(ts)^{i-k}-1}{1-ts}. \end{aligned}$$

We take into account that $\dfrac{1-(ts)^k}{1-ts}$ is the generating polynomial of I_k and obtain in matrix form

$$\mathrm{Bez}_T(\mathbf{u},\mathbf{e}_k) = \begin{bmatrix} & & & u_1 & & \\ & O & & \vdots & \ddots & \\ & & & u_{k-1} & \cdots & u_1 \\ -u_{n+1} & \cdots & -u_{k+1} & & & \\ & \ddots & \vdots & & O & \\ & & -u_{n+1} & & & \end{bmatrix}. \tag{2.13}$$

For the special cases $k=1$ and $k=n+1$ we introduce the notations

$$B_+(\mathbf{u}) = -\begin{bmatrix} u_{n+1} & \cdots & u_2 \\ & \ddots & \vdots \\ & & u_{n+1} \end{bmatrix}, \quad B_-(\mathbf{u}) = \begin{bmatrix} u_1 & & \\ \vdots & \ddots & \\ u_n & \cdots & u_1 \end{bmatrix}. \tag{2.14}$$

Obviously, the T-Bezoutian is, like the H-Bezoutian, linear in its arguments. Furthermore, it is skewsymmetric with respect to the arguments, i.e.

$$\mathrm{Bez}_T(\mathbf{u},\mathbf{v}) = -\mathrm{Bez}_T(\mathbf{v},\mathbf{u}).$$

Moreover, we have

$$\mathrm{Bez}_T(\mathbf{u},\mathbf{v})^T = -\mathrm{Bez}_T(\mathbf{u}^J,\mathbf{v}^J) \tag{2.15}$$

and

$$\begin{aligned}\mathrm{Bez}_T(\mathbf{u},\mathbf{v})^J(t,s) &= \frac{\mathbf{u}(t^{-1})\mathbf{v}(s)s^{-n} - \mathbf{v}(t^{-1})\mathbf{u}(s)s^{-n}}{1-t^{-1}s^{-1}}\, t^{n-1}s^{n-1} \\ &= \frac{\mathbf{u}(t^{-1})t^n\mathbf{v}(s) - \mathbf{v}(t^{-1})t^n\mathbf{u}(s)}{ts-1},\end{aligned}$$

which means that

$$\mathrm{Bez}_T(\mathbf{u},\mathbf{v})^J = -\mathrm{Bez}_T(\mathbf{u}^J,\mathbf{v}^J)\,. \tag{2.16}$$

From (2.15) and (2.16) we conclude that

$$\mathrm{Bez}_T(\mathbf{u},\mathbf{v})^T = \mathrm{Bez}_T(\mathbf{u},\mathbf{v})^J\,.$$

Hence T-Bezoutians are, like Toeplitz matrices, persymmetric.

The discussion about uniqueness of H-Bezoutians in Section 2,1 can be immediately transferred to T-Bezoutians. In fact, there is a Toeplitz analogue of Lemma 2.2.

Lemma 2.9. *If* $[\,\mathbf{u}_1\;\;\mathbf{v}_1\,] = [\,\mathbf{u}\;\;\mathbf{v}\,]\,\varphi$ *for some* 2×2 *matrix* φ, *then*

$$\mathrm{Bez}_T(\mathbf{u}_1,\mathbf{v}_1) = (\det\varphi)\,\mathrm{Bez}_T(\mathbf{u},\mathbf{v})\,.$$

Hence the following is true.

Corollary 2.10. *The T-Bezoutians* $\mathrm{Bez}_T(\mathbf{u},\mathbf{v}) \neq O$ *and* $\mathrm{Bez}_T(\mathbf{u}_1,\mathbf{v}_1)$ *coincide if and only if*

$$[\,\mathbf{u}_1\;\;\mathbf{v}_1\,] = [\,\mathbf{u}\;\;\mathbf{v}\,]\,\varphi$$

for some matrix φ *with* $\det\varphi = 1$.

8. The transformation ∇_T. The Toeplitz analogue of the transformation ∇_H is the transformation ∇_T transforming an $n\times n$ matrix $A = [\,a_{ij}\,]_{i,j=1}^n$ into a $(n+1)\times(n+1)$ matrix according to

$$\nabla_T A = [\,a_{ij} - a_{i-1,j-1}\,]_{i,j=1}^{n+1}\,.$$

Here we set $a_{ij} = 0$ if one of the integers i or j is not in the set $\{1,2,\dots,n\}$. Obviously,

$$\nabla_T A = \begin{bmatrix} A - S_nAS_n^T & * \\ * & * \end{bmatrix} = \begin{bmatrix} * & * \\ * & S_n^TAS_n - A \end{bmatrix}. \tag{2.17}$$

In polynomial language the transformation ∇_T is given by

$$(\nabla_T A)(t,s) = (1-ts)A(t,s)\,.$$

That means the T-Bezoutian $B = \mathrm{Bez}_T(\mathbf{u},\mathbf{v})$ is characterized by

$$\nabla_T B = [\,\mathbf{u}\;\;\mathbf{v}\,]\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}[\,\mathbf{u}^J\;\;\mathbf{v}^J\,]^T\,.$$

Taking into account (2.16) we observe that $(\nabla_T B)^J = -\nabla_T B^J$.

The representation (2.17) shows that the transformation ∇_+ introduced in (1.7) is a restriction of ∇_T. In particular, we conclude that T-Bezoutians are quasi-Toeplitz matrices. Furthermore, if B is a T-Bezoutian, then B^J is also a quasi-Toeplitz matrix.

9. Symmetric and skewsymmetric T-Bezoutians. We discuss now how symmetric and skewsymmetric T-Bezoutians can be characterized. First we observe that (2.16) implies that $\mathrm{Bez}_T(\mathbf{u},\mathbf{v})$ is symmetric if one of the vectors $\mathbf{u}$ or $\mathbf{v}$ is symmetric and the other one is skewsymmetric.

Furthermore, $\mathrm{Bez}_T(\mathbf{u},\mathbf{v})$ is skewsymmetric if both vectors $\mathbf{u}$ and $\mathbf{v}$ are symmetric or both are skewsymmetric. We show that the converse is also true. For simplicity of notation we write $B(\mathbf{u},\mathbf{v})$ instead of $\mathrm{Bez}_T(\mathbf{u},\mathbf{v})$.

Let $\mathbf{u},\mathbf{v} \in \mathbb{F}^{n+1}$ be any vectors, $\mathbf{u} = \mathbf{u}_+ + \mathbf{u}_-$ and $\mathbf{v} = \mathbf{v}_+ + \mathbf{v}_-$, where $\mathbf{u}_+,\mathbf{v}_+$ are symmetric and $\mathbf{u}_-,\mathbf{v}_-$ are skewsymmetric. Then $B(\mathbf{u},\mathbf{v}) = B_+ + B_-$, where

$$B_+ = B(\mathbf{u}_+,\mathbf{v}_-) + B(\mathbf{u}_-,\mathbf{v}_+)\,, \quad B_- = B(\mathbf{u}_+,\mathbf{v}_+) + B(\mathbf{u}_-,\mathbf{v}_-)\,,$$

B_+ is symmetric, and B_- is skewsymmetric. Suppose that $B = B(\mathbf{u},\mathbf{v})$ is symmetric. Then $B_- = O$. Hence

$$B(\mathbf{u}_+,\mathbf{v}_+) = B(\mathbf{v}_-,\mathbf{u}_-)\,.$$

Since the vectors $\mathbf{u}_+$ and $\mathbf{v}_+$ cannot be linear combinations of $\mathbf{u}_-$ and $\mathbf{v}_-$ from Corollary 2.10 it becomes clear that

$$B(\mathbf{u}_+,\mathbf{v}_+) = B(\mathbf{v}_-,\mathbf{u}_-) = O\,.$$

Thus $\mathbf{v}_\pm = \alpha_\pm \mathbf{u}_\pm$ for some $\alpha_\pm \in \mathbb{F}$ or $B(\mathbf{u},\mathbf{v}) = O$. We conclude that

$$B = B_+ = B((\alpha_- - \alpha_+)\mathbf{u}_+,\mathbf{u}_-)\,.$$

That means that B is the T-Bezoutian of a symmetric and a skewsymmetric vector.

Suppose now that $B = B(\mathbf{u},\mathbf{v}) \neq O$ is skewsymmetric. Then $B_+ = O$. Hence

$$B(\mathbf{u}_+,\mathbf{v}_-) = B(\mathbf{v}_+,\mathbf{u}_-)\,.$$

From Corollary 2.10 and the symmetry properties of the vectors we conclude that either $\{\mathbf{u}_+,\mathbf{v}_+\}$ as well as $\{\mathbf{u}_-,\mathbf{v}_-\}$ are linearly dependent or

$$B(\mathbf{u}_+,\mathbf{v}_-) = B(\mathbf{v}_+,\mathbf{u}_-) = O\,.$$

In the former case we would have $B_- = O$, so we have the latter case. Using again the symmetry properties of the vectors we find that either $\mathbf{u}_- = \mathbf{v}_- = \mathbf{0}$ or $\mathbf{u}_+ = \mathbf{v}_+ = \mathbf{0}$. That means that B is the Bezoutian of two symmetric or two skewsymmetric vectors. Let us summarize.

Proposition 2.11. *A T-Bezoutian is symmetric if and only if it is the T-Bezoutian of a symmetric and a skewsymmetric vector. A T-Bezoutian is skewsymmetric if and only if it is the T-Bezoutian of two symmetric vectors or two skewsymmetric vectors.*

Note that the T-Bezoutian $B(\mathbf{u},\mathbf{v})$ of two skewsymmetric vectors cannot be nonsingular. In fact, in this case we have $\mathbf{u}(1) = \mathbf{v}(1) = 0$ such that $\mathbf{u}(t) = (t-1)\mathbf{u}_1(t)$ and $\mathbf{v}(t) = (t-1)\mathbf{v}_1(t)$. Then $\mathbf{u}_1$ and $\mathbf{v}_1$ are symmetric, and as in Proposition 2.4 we obtain

$$B(\mathbf{u},\mathbf{v}) = M_{n-1}(t-1)\mathrm{Bez}_T(\mathbf{u}_1,\mathbf{v}_1)M_{n-1}(t-1)^T\,.$$

Thus $B(\mathbf{u},\mathbf{v})$ has rank $n-1$ at most.

There is an alternative representation for symmetric T-Bezoutians, which has no skewsymmetric counterpart. Suppose that $B = B(\mathbf{u}_+,\mathbf{u}_-)$. We set $\mathbf{v} = -\frac{1}{2}\,\mathbf{u}_+ + \mathbf{u}_-$. Then

$$B(\mathbf{v},\mathbf{v}^J) = B\,.$$

On the other hand, $B(\mathbf{v},\mathbf{v}^J)$ is symmetric for any vector $\mathbf{v} \in \mathbb{F}^{n+1}$. Thus the following is true.

Corollary 2.12. *A T-Bezoutian B is symmetric if and only if it can be represented in the form $B = \mathrm{Bez}_T(\mathbf{v},\mathbf{v}^J)$ for some $\mathbf{v} \in \mathbb{F}^{n+1}$.*

10. Hermitian T-Bezoutians. Now we characterize Hermitian T-Bezoutians. Suppose that the vectors $\mathbf{u}_+, \mathbf{u}_- \in \mathbb{C}^{n+1}$ are conjugate-symmetric. Then we conclude from (2.15) that the matrix $\mathrm{i}B(\mathbf{u}_+, \mathbf{u}_-)$ is Hermitian. Conversely, let $B = B(\mathbf{u}, \mathbf{v})$ be Hermitian, $\mathbf{u} = \mathbf{u}_+ + \mathrm{i}\mathbf{u}_-$ and $\mathbf{v} = \mathbf{v}_+ + \mathrm{i}\mathbf{v}_-$, where $\mathbf{u}_\pm, \mathbf{v}_\pm$ are conjugate-symmetric. Then $B(\mathbf{u}, \mathbf{v}) = B_+ + \mathrm{i}B_-$, where

$$B_+ = \mathrm{i}(B(\mathbf{u}_+, \mathbf{v}_-) + B(\mathbf{u}_-, \mathbf{v}_+)) , \quad B_- = \mathrm{i}(B(\mathbf{u}_-, \mathbf{v}_-) - B(\mathbf{u}_+, \mathbf{v}_+)) .$$

The matrices B_+ and $\mathrm{i}B_-$ are Hermitian. Since $B = B(\mathbf{u}, \mathbf{v})$ is assumed to be Hermitian, we have $B_- = O$, which means

$$B(\mathbf{u}_+, \mathbf{v}_+) = B(\mathbf{u}_-, \mathbf{v}_-) .$$

Using Corollary 2.10 we conclude that

$$[\,\mathbf{u}_- \;\; \mathbf{v}_-\,] = [\,\mathbf{u}_+ \;\; \mathbf{v}_+\,] \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

for some a, b, c, d with $ad - bc = 1$. Since all vectors under consideration are conjugate-symmetric, these numbers must be real. We obtain after elementary calculations

$$B(\mathbf{u}, \mathbf{v}) = \mathrm{i}B(\mathbf{u}_+, -(a+d)\mathbf{v}_+) = B(\mathbf{u}_+, -(a+d)\mathrm{i}\mathbf{v}_+) .$$

Thus, B is the Bezoutian of a conjugate-symmetric and a conjugate-skewsymmetric vector. Let us summarize.

Proposition 2.13. *A T-Bezoutian B is Hermitian if and only if it is of the form*

$$B = \mathrm{iBez}_T(\mathbf{u}_+, \mathbf{u}_-)$$

for conjugate-symmetric vectors $\mathbf{u}_+$ and $\mathbf{u}_-$.

As for symmetric T-Bezoutians, we have an alternative form. Suppose that $B = \mathrm{i}B(\mathbf{u}_+, \mathbf{u}_-)$ and set $\mathbf{v} = -\frac{1}{2}\,\mathbf{u}_+ + \mathrm{i}\mathbf{u}_-$. Then $B(\mathbf{v}, \mathbf{v}^\#) = B$. Since, on the other hand, the matrix $B(\mathbf{v}, \mathbf{v}^\#)$ is Hermitian for any vector $\mathbf{v} \in \mathbb{F}^{n+1}$, which is easily checked, the following is true.

Corollary 2.14. *A T-Bezoutian B is Hermitian if and only if it can be represented in the form $B = \mathrm{Bez}_T(\mathbf{v}, \mathbf{v}^\#)$ for some $\mathbf{v} \in \mathbb{C}^{n+1}$.*

11. Splitting of symmetric T-Bezoutians. It was mentioned in Section 2,7 that T-Bezoutians are persymmetric. Hence a symmetric T-Bezoutian B is also centrosymmetric. That means that the subspaces of symmetric or skewsymmetric vectors $\mathbb{F}^n_\pm$ are invariant under B. We show that the restrictions of a symmetric T-Bezoutian to $\mathbb{F}^n_\pm$ can be characterized by another kind of Bezoutians which is introduced next.

Let $\mathbf{p}, \mathbf{q} \in \mathbb{F}^{n+2}$ be either both symmetric or both skewsymmetric. Then

$$B_{\mathrm{split}}(t, s) = \frac{\mathbf{p}(t)\mathbf{q}(s) - \mathbf{q}(t)\mathbf{p}(s)}{(t-s)(ts-1)}$$

is a polynomial in t and s. The $n \times n$ matrix with the generating polynomial $B_{\mathrm{split}}(t, s)$ will be called *split Bezoutian* of $\mathbf{p}(t)$ and $\mathbf{q}(t)$ and denoted by

$$\mathrm{Bez}_{\mathrm{split}}(\mathbf{p}, \mathbf{q}) .$$

Obviously, $\mathrm{Bez}_{\mathrm{split}}(\mathbf{p}, \mathbf{q})$ is a symmetric and centrosymmetric matrix. If $\mathbf{p}$ and $\mathbf{q}$ are symmetric, then we will speak about a *split Bezoutian of (+)-type* and if these vectors are skewsymmetric about a *split Bezoutian of (−)-type*. Instead of B_{split} we write B_+ or B_-, respectively.

The columns and rows of a split Bezoutian of (+)-type are all symmmetric and of a split Bezoutian of (−)-type are all skewsymmetric, so that its rank is at most $\frac{1}{2}(n+1)$ in the (+)

case and $\frac{1}{2}n$ in the $(-)$ case. As an example we consider the case $\mathbf{p}(t) = t^{2k} + 1 \in \mathbb{F}^{2k+1}(t)$ and $\mathbf{q}(t) = t^k \in \mathbb{F}^{2k+1}(t)$. In this case

$$\begin{aligned} B_+(t,s) &= \frac{(t^{2k}+1)s^k - t^k(s^{2k}+1)}{(t-s)(ts-1)} = \frac{t^k - s^k}{t-s}\,\frac{(ts)^k - 1}{ts-1} \\ &= (t^{k-1} + t^{k-2}s + \cdots + s^{k-1})(1 + ts + \cdots + t^{k-1}s^{k-1})\,. \end{aligned}$$

For $k = 3$, the matrix with this generating polynomial is

$$B_+ = \begin{bmatrix} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}.$$

For a general symmetric $\mathbf{p} = (p_i)_{i=1}^7 \in \mathbb{F}^7$ and $\mathbf{q}$ as before, $\mathbf{q} = \mathbf{e}_4$, the split Bezoutian of $\mathbf{p}$ and $\mathbf{q}$ is given by

$$B_+ = \begin{bmatrix} & & p_1 & & \\ & p_1 & p_2 & p_1 & \\ p_1 & p_2 & p_1 + p_3 & p_2 & p_1 \\ & p_1 & p_2 & p_1 & \\ & & p_1 & & \end{bmatrix}.$$

Moreover, from this special case it is clear how the split Bezoutian of a general $\mathbf{p} \in \mathbb{F}_+^{2k+1}$ and $\mathbf{q} = \mathbf{e}_{k+1}$ looks like.

Recall that $P_\pm = \frac{1}{2}(I_n \pm J_n)$ are the projections from $\mathbb{F}^n$ onto $\mathbb{F}_\pm^n$ along $\mathbb{F}_\mp^n$. Our aim is to describe $BP_\pm = P_\pm BP_\pm$ for a symmetric T-Bezoutian B. As we know from Proposition 2.11, a symmetric T-Bezoutian B is the T-Bezoutian of a symmetric vector $\mathbf{u}_+ \in \mathbb{F}_+^{n+1}$ and a skewsymmetric vector $\mathbf{v}_- \in \mathbb{F}_-^{n+1}$. From these vectors we form the polynomials

$$\mathbf{p}_\pm(t) = (t \pm 1)\mathbf{u}_+(t) \quad \text{and} \quad \mathbf{q}_\pm(t) = (t \mp 1)\mathbf{v}_-(t)\,.$$

Clearly, $\mathbf{p}_+$ and $\mathbf{q}_+$ are symmetric, and $\mathbf{p}_-$ and $\mathbf{q}_-$ are skewsymmetric.

Proposition 2.15. *The symmetric T-Bezoutian* $B = \mathrm{Bez}_T(\mathbf{u}_+, \mathbf{v}_-)$ *can be represented as* $B = B_+ + B_-$, *where* $B_\pm = BP_\pm$ *and*

$$B_\pm = \mp\frac{1}{2}\,\mathrm{Bez}_{\mathrm{split}}(\mathbf{p}_\pm, \mathbf{q}_\pm)\,.$$

Proof. We compute the generating polynomial $B_+(t,s)$ of $B_+ = BP_+$. By definition we have

$$\begin{aligned} B_+(t,s) &= \frac{1}{2}\left(B(t,s) + B(t,s^{-1})s^{n-1}\right) \\ &= -\frac{1}{2}\left(\frac{\mathbf{u}_+(t)\mathbf{v}_-(s) + \mathbf{v}_-(t)\mathbf{u}_+(s)}{1-ts} + \frac{\mathbf{u}_+(t)\mathbf{v}_-(s) - \mathbf{v}_-(t)\mathbf{u}_+(s)}{t-s}\right) \\ &= -\frac{1}{2}\,\frac{(t+1)\mathbf{u}_+(t)(s-1)\mathbf{v}_-(s) - (t-1)\mathbf{v}_-(t)(s+1)\mathbf{u}_+(s)}{(t-s)(ts-1)}\,. \end{aligned}$$

This is just the generating polynomial of the matrix $-\frac{1}{2}\,\mathrm{Bez}_{\mathrm{split}}(\mathbf{p}_+,\mathbf{q}_+)$. The other case is proved analogously. ■

12. Relations between H- and T-Bezoutians. There is a simple relation between H- and T-Bezoutians, namely

$$\mathrm{Bez}_T(\mathbf{u},\mathbf{v}) = -\mathrm{Bez}_H(\mathbf{u},\mathbf{v})J_n\,.$$

More general relations can be described with the help of Frobenius-Fischer transformations. Analogously to Theorem 2.7 we obtain the following.

Theorem 2.16. *For any $\varphi \in \mathrm{GL}(\mathbb{F}^2)$, the transformation*

$$\Phi\;:\;B \mapsto K_n(\varphi)BK_n(J_2\varphi)^T$$

maps T-Bezoutians into H-Bezoutians. Moreover,

$$K_n(\varphi)\mathrm{Bez}_T(\mathbf{u},\mathbf{v})K_n(J_2\varphi)^T = \frac{1}{\det(\varphi J_2)}\,\mathrm{Bez}_H(\widetilde{\mathbf{u}},\widetilde{\mathbf{v}}) = \frac{1}{\det\varphi}\,\mathrm{Bez}_H(\widetilde{\mathbf{v}},\widetilde{\mathbf{u}})\,, \tag{2.18}$$

where $\widetilde{\mathbf{u}} = K_{n+1}(\varphi)\mathbf{u}$ and $\widetilde{\mathbf{v}} = K_{n+1}(\varphi)\mathbf{v}$.

In the case $\mathbb{F} = \mathbb{C}$ it is of particular interest to describe congruence transformations that transform Hermitian T-Bezoutians into real symmetric H-Bezoutians, in other words to describe a coordinate transformation that transforms Hermitian T-Bezoutian forms into real quadratic H-Bezoutian forms. The transformation Φ has this property if and only if $K_n(J_2\varphi)^T = K_n(\varphi)^*$ (up to multiples of I_2), which is equivalent to $J_2\varphi = \overline{\varphi}$. Suppose that $\varphi = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, then this is equivalent to $a = \overline{b}$ and $c = \overline{d}$. It can be easily checked that φ has this property if and only if the linear fractional function $\varphi(t) = \dfrac{at+b}{ct+d}$ maps the unit circle onto the real line (compare Section 1,5). Hence we have the following.

Corollary 2.17. *If $\varphi(t)$ maps the unit circle onto the real line, then the transformation $\Phi\,:\,B \mapsto K_n(\varphi)BK_n(\varphi)^*$ maps Hermitian T-Bezoutians into real symmetric H-Bezoutians. In particular, the signatures of B and $\Phi(B)$ coincide.*

3. Resultant Matrices and Matrix Representations of Bezoutians

In this section we show that Bezoutians are closely related to resultant matrices and that the relations between these two classes can be used to derive important matrix representations of Bezoutians. We present two kinds of relations between resultant matrices and Bezoutians. The first is due to Kravitsky and Russakovsky, the second an interpretation of Bezoutians as Schur complements in resultant matrices.

The resultant matrix $\mathrm{Res}(\mathbf{u},\mathbf{v})$ of two polynomials $\mathbf{u}(t) \in \mathbb{F}^{m+1}, \mathbf{v}(t) \in \mathbb{F}^{n+1}(t)$ was introduced in (2.4) as the $(m+n)\times(m+n)$ matrix

$$\mathrm{Res}\,(\mathbf{u},\mathbf{v}) = \begin{bmatrix} M_n(\mathbf{u})^T \\ M_m(\mathbf{v})^T \end{bmatrix}\,.$$

In this section we restrict ourselves to the case $m = n$, which is no restriction of generality when speaking about nonsingularity, rank and related quantities. Recall that $\mathrm{Res}\,(\mathbf{u},\mathbf{v})$ is nonsingular if and only if the polynomials $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are coprime and at least one of the leading coefficients of $\mathbf{u}(t)$ or $\mathbf{v}(t)$ is not zero.

1. Kravitsky-Russakovsky formulas. To begin with we generalize the resultant concept. Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be polynomials of degree n. The *p-resultant matrix* ($p = 0, 1, \dots$) of $\mathbf{u}(t)$ and $\mathbf{v}(t)$ is, by definition, the $(2n + 2p) \times (2n + p)$ matrix

$$\mathrm{Res}_p(\mathbf{u}, \mathbf{v}) = \begin{bmatrix} M_{n+p}(\mathbf{u})^T \\ M_{n+p}(\mathbf{v})^T \end{bmatrix} .$$

In the case $p = 0$ we have the resultant matrix in the former sense. For the sequel it is important to observe that

$$\mathrm{Res}_p(\mathbf{u}, \mathbf{v})\ell_{2n+p}(t) = \begin{bmatrix} \mathbf{u}(t)\ell_{n+p}(t) \\ \mathbf{v}(t)\ell_{n+p}(t) \end{bmatrix} , \tag{3.1}$$

where $\ell_m(t) = (t^{i-1})_{i=1}^m$.

Theorem 3.1. *Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be polynomials of degree n. Then*

1.
$$\mathrm{Res}_p(\mathbf{u}, \mathbf{v})^T \begin{bmatrix} O & J_{n+p} \\ -J_{n+p} & O \end{bmatrix} \mathrm{Res}_p(\mathbf{u}, \mathbf{v}) = \begin{bmatrix} O & O & -B_H \\ O & O & O \\ B_H & O & O \end{bmatrix} , \tag{3.2}$$
where $B_H = \mathrm{Bez}_H(\mathbf{u}, \mathbf{v})$, and

2.
$$\mathrm{Res}_p(\mathbf{u}, \mathbf{v})^T \begin{bmatrix} I_{n+p} & O \\ O & -I_{n+p} \end{bmatrix} \mathrm{Res}_p(\mathbf{v}^J, \mathbf{u}^J) = \begin{bmatrix} B_T & O & O \\ O & O & O \\ O & O & -B_T \end{bmatrix} , \tag{3.3}$$
where $B_T = \mathrm{Bez}_T(\mathbf{u}, \mathbf{v})$.

Proof. We compare the generating polynomials of the right-hand and of the left-hand sides. According to (3.1) we have

$$\begin{aligned} & \ell_{2n+p}(t)^T \mathrm{Res}_p(\mathbf{u}, \mathbf{v})^T \begin{bmatrix} O & J_{n+p} \\ -J_{n+p} & O \end{bmatrix} \mathrm{Res}_p(\mathbf{v}, \mathbf{u})\, \ell_{2n+p}(s) \\ = \ & (\mathbf{u}(t)\mathbf{v}(s) - \mathbf{v}(t)\mathbf{u}(s))\ell_{n+p}(t)^T J_{n+p}\, \ell_{n+p}(s) \\ = \ & (\mathbf{u}(t)\mathbf{v}(s) - \mathbf{v}(t)\mathbf{u}(s))\, \frac{t^{n+p} - s^{n+p}}{t - s} \\ = \ & (t^{n+p} - s^{n+p})\mathrm{Bez}_H(\mathbf{u}, \mathbf{v})(t, s) , \end{aligned}$$

which is the polynomial form of the first assertion.

To prove the second relation we observe that (3.1) implies

$$\begin{aligned} & \ell_{2n+p}(t)^T \mathrm{Res}_p(\mathbf{u}, \mathbf{v})^T \begin{bmatrix} I_{n+p} & O \\ O & -I_{n+p} \end{bmatrix} \mathrm{Res}_p(\mathbf{v}^J, \mathbf{u}^J)\ell_{2n+p}(s) \\ = \ & \frac{1 - (ts)^{n+p}}{1 - ts}\, (\mathbf{u}(t)\mathbf{v}^J(s) - \mathbf{v}(t)\mathbf{u}^J(s)) . \end{aligned}$$

This leads to the second assertion. ■

2. Matrix representations of Bezoutians. The Kravitsky-Russakovsky formulas (3.2) and (3.3) provide an elegant way to obtain matrix representations of Bezoutians in terms of triangular Toeplitz matrices. These formulas are very important in connection with inversion of Toeplitz and Hankel matrices. They represent so-called “inversion formulas”. Note that from computational point of view the formulas presented here are not the most efficient ones. Other, more efficient formulas for the cases $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$ can be found in [38], [40], [41], [42], [43].

We define, for $\mathbf{u} = (u_i)_{i=1}^{n+1}$, the lower triangular $n \times n$ Toeplitz matrix

$$T(\mathbf{u}) = \begin{bmatrix} u_1 & & \\ \vdots & \ddots & \\ u_n & \dots & u_1 \end{bmatrix} .$$

Note that $T(\mathbf{u})$ is the T-Bezoutian $B_-(\mathbf{u})$ of $\mathbf{u}(t)$ and t^n, which was introduced in (2.14). Note also that the matrix $T(\mathbf{u})$ is related to the H-Bezoutian $B(\mathbf{u})$ defined by (2.2) and the T-Bezoutian $B_+(\mathbf{u})$ defined by (2.14) via

$$B(\mathbf{u}) = J_n T(\mathbf{u}^J) , \quad B_+(\mathbf{u}) = -T(\mathbf{u}^J)^T .$$

Furthermore, let us mention that we have commutativity

$$T(\mathbf{u}_1)T(\mathbf{u}_2) = T(\mathbf{u}_2)T(\mathbf{u}_1)$$

and the relation $T(\mathbf{u})^T = T(\mathbf{u})^J$. The nonsingular matrices $T(\mathbf{u})$ form a commutative subgroup of $GL(\mathbb{F}^n)$. With this notation the resultant matrix $\mathrm{Res}(\mathbf{u}, \mathbf{v})$ for $\mathbf{u}, \mathbf{v} \in \mathbb{F}^{n+1}$ can be written in the form

$$\mathrm{Res}(\mathbf{u}, \mathbf{v}) = \begin{bmatrix} T(\mathbf{u})^T & T(\mathbf{u}^J) \\ T(\mathbf{v})^T & T(\mathbf{v}^J) \end{bmatrix} .$$

The application of Theorem 3.1 for $p = 0$ leads now to the following.

Theorem 3.2. *The H-Bezoutian of two polynomials* $\mathbf{u}(t), \mathbf{v}(t) \in \mathbb{F}^{n+1}$ *admits*

1. *the representations*

$$\mathrm{Bez}_H(\mathbf{u}, \mathbf{v}) = T(\mathbf{v}) J_n T(\mathbf{u}^J) - T(\mathbf{u}) J_n T(\mathbf{v}^J)$$

and

$$\mathrm{Bez}_H(\mathbf{u}, \mathbf{v}) = T(\mathbf{u}^J)^T J_n T(\mathbf{v})^T - T(\mathbf{v}^J)^T J_n T(\mathbf{u})^T .$$

2. *the representations*

$$\mathrm{Bez}_T(\mathbf{u}, \mathbf{v}) = T(\mathbf{u}) T(\mathbf{v}^J)^T - T(\mathbf{v}) T(\mathbf{u}^J)^T$$

and

$$\mathrm{Bez}_T(\mathbf{u}, \mathbf{v}) = T(\mathbf{v}^J) T(\mathbf{u})^T - T(\mathbf{u}^J) T(\mathbf{v})^T .$$

3. Bezoutians as Schur complements. We assume that the polynomial $\mathbf{u}(t)$ has degree n. Then the matrix $T(\mathbf{u}^J)$ is nonsingular. Now the second expression for $\mathrm{Bez}_H(\mathbf{u}, \mathbf{v})$ in Theorem 3.2,1 can be written in the form

$$C = T(\mathbf{u}^J)^{-1} J_n \mathrm{Bez}_H(\mathbf{u}, \mathbf{v}) = T(\mathbf{v})^T - T(\mathbf{v}^J) T(\mathbf{u}^J)^{-1} T(\mathbf{u})^T .$$

We see that C is the Schur complement of the left upper block in

$$\widetilde{R} = \mathrm{Res}\,(\mathbf{u}, \mathbf{v}) \begin{bmatrix} O & I_n \\ I_n & O \end{bmatrix} = \begin{bmatrix} T(\mathbf{u}^J) & T(\mathbf{u})^T \\ T(\mathbf{v}^J) & T(\mathbf{v})^T \end{bmatrix} .$$

Recall that the concept of Schur complement is defined in connection with the factorization of a block matrix

$$G = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_n & O \\ CA^{-1} & I_n \end{bmatrix} \begin{bmatrix} A & O \\ O & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I_n & A^{-1}B \\ O & I_n \end{bmatrix} ,$$

where A is assumed to be invertible. Here $D - CA^{-1}B$ is said to be the *Schur complement of A in G*. Applying this factorization to our case we obtain the following.

Proposition 3.3. *Let* $\mathbf{u}(t) \in \mathbb{F}^{n+1}(t)$ *be a polynomial of degree* n, $\mathbf{v}(t) \in \mathbb{F}^{n+1}(t)$. *Then the resultant of* $\mathbf{u}(t)$ *and* $\mathbf{v}(t)$ *can be represented in the form*

$$\operatorname{Res}(\mathbf{u},\mathbf{v}) = \begin{bmatrix} T(\mathbf{u}^J) & O \\ T(\mathbf{v}^J) & T(\mathbf{u}^J)^{-1}J_n \end{bmatrix} \begin{bmatrix} I_n & O \\ O & \operatorname{Bez}_H(\mathbf{u},\mathbf{v}) \end{bmatrix} \begin{bmatrix} T(\mathbf{u}^J)^{-1}T(\mathbf{u})^T & I_n \\ I_n & O \end{bmatrix}.$$

From this proposition we see that $\operatorname{Res}(\mathbf{u},\mathbf{v})$ is nonsingular if and only if $\operatorname{Bez}_H(\mathbf{u},\mathbf{v})$ has this property. Hence $\operatorname{Bez}_H(\mathbf{u},\mathbf{v})$ is nonsingular if and only if the polynomials $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are coprime. Taking (2.10) into account we conclude the following.

Corollary 3.4. *The nullity of* $\operatorname{Bez}_H(\mathbf{u},\mathbf{v})$ *is equal to the degree of the greatest common divisor of* $\mathbf{u}(t)$ *and* $\mathbf{v}(t)$.

Clearly, the same is also true for T-Bezoutians.

4. Inverses of Hankel and Toeplitz Matrices

The most striking property of H- and T-Bezoutians is that inverses of Hankel and Toeplitz matrices belong to these classes. In view of Theorem 3.2 a consequence of this fact is that inverses of Toeplitz and Hankel matrices can be represented as product sum of triangular Toeplitz matrices, which is important for fast matrix-vector multiplication. Later, in Section 7,6 and Section 8,7 we will see that, vice versa, inverses of H- and T-Bezoutians are Hankel or Toeplitz matrices, respectively. Let us start with the Hankel case.

1. Inverses of Hankel matrices. Let $H_n = [\, s_{i+j-1}\,]_{i,j=1}^n$ be a nonsingular Hankel matrix. Besides H_n we consider the $(n-1)\times(n+1)$ Hankel matrix ∂H_n which is obtained from H_n after deleting the last row and adding another column on the right so that the Hankel structure is preserved. That means

$$\partial H_n = \begin{bmatrix} s_1 & \dots & s_{n+1} \\ \vdots & .\cdot\,^{\cdot} & \vdots \\ s_{n-1} & \dots & s_{2n-1} \end{bmatrix}. \tag{4.1}$$

For H_n is nonsingular, ∂H_n has a two-dimensional nullspace. A basis $\{\,\mathbf{u},\mathbf{v}\,\}$ of the nullspace of ∂H_n will be called *fundamental system for* H_n. We consider for fixed $s \in \mathbb{F}$ the linear system of equations

$$H_n \mathbf{x}_s = \ell_n(s)\,, \tag{4.2}$$

where $\ell_n(s)$ is introduced in (1.1). It can be checked that

$$\partial H_n \begin{bmatrix} \mathbf{x}_s \\ 0 \end{bmatrix} = \ell_{n-1}(s) \quad \text{and} \quad \partial H_n \begin{bmatrix} 0 \\ \mathbf{x}_s \end{bmatrix} = s\ell_{n-1}(s)\,.$$

Hence $\begin{bmatrix} 0 \\ \mathbf{x}_s \end{bmatrix} - s \begin{bmatrix} \mathbf{x}_s \\ 0 \end{bmatrix}$ belongs to the kernel of ∂H_n. In polynomial language, this means that there are constants a_s and b_s such that

$$(t-s)\mathbf{x}_s(t) = a_s\mathbf{u}(t) - b_s\mathbf{v}(t)\,.$$

Now we consider s as a variable. From (4.2) it is clear that $\mathbf{x}_s(t) = \ell_n(t)^T H_n^{-1}\ell_n(s)$ is a polynomial in s of degree $n-1$. (It is just the generating polynomial of the matrix H_n^{-1}.) We conclude that $a_s = \mathbf{a}(s)$ and $b_s = \mathbf{b}(s) \in \mathbb{F}^{n+1}(s)$. Thus, H_n^{-1} is a quasi-H-Bezoutian. According to

Corollary 2.5, this implies that H_n^{-1} is an H-Bezoutian, which means that $\mathbf{a}(t) = \gamma\mathbf{v}(t)$ and $\mathbf{b}(t) = \gamma\mathbf{u}(t)$, and $H_n^{-1} = \gamma\,\mathrm{Bez}_H\,(\mathbf{u},\mathbf{v})$ for some nonzero constant γ. It remains to compute γ. For this we introduce the $2 \times (n+1)$ matrix

$$F = \begin{bmatrix} s_n & \dots & s_{2n-1} & s_{2n} \\ 0 & \dots & 0 & 1 \end{bmatrix} .$$

Here $s_{2n} \in \mathbb{F}$ is arbitrary. A fundamental system $\{\mathbf{u},\mathbf{v}\}$ will be called *canonical* if

$$F[\,\mathbf{u}\ \ \mathbf{v}\,] = I_2 \,.$$

Let $\{\mathbf{u},\mathbf{v}\}$ be canonical. Then, in particular, $u := \mathbf{e}_{n+1}^T\mathbf{u} = 0$ and $v = \mathbf{e}_{n+1}^T\mathbf{v} = 1$. Furthermore, if we consider $\mathbf{u}$ as a vector in $\mathbb{F}^n$, then it is just the last column of H_n^{-1}, i.e.

$$H_n\mathbf{u} = \mathbf{e}_n. \tag{4.3}$$

We compare $\mathbf{u}$ with the last column of $\mathrm{Bez}_H\,(\mathbf{u},\mathbf{v})$, which is equal to $\mathbf{v}u - \mathbf{u}v = -\mathbf{u}$ (cf. Theorem 3.2). Thus, $\gamma = -1$. Note that $\mathbf{v}$ is of the form $\mathbf{v} = \begin{bmatrix} -\mathbf{z} \\ 1 \end{bmatrix}$, where $\mathbf{z}$ is the solution of the system

$$H_n\mathbf{z} = \mathbf{g} \quad \text{with} \quad \mathbf{g} = (s_{n+i})_{i=1}^n \,. \tag{4.4}$$

Hereafter we need the following fact.

Proposition 4.1. *Let the equations* (4.3) *and* (4.4) *be solvable. Then H_n is nonsingular.*

Proof. Assume that H_n is singular, and let $\mathbf{v} = (v_j)_{j=1}^n$ be a nontrivial vector such that $H_n\mathbf{v} = \mathbf{0}$. Then applying $\mathbf{v}^T$ from the left side to the equations (4.3) and (4.4) leads to

$$\mathbf{v}^T H_n\mathbf{u} = \mathbf{v}^T\mathbf{e}_n = \mathbf{0} \text{ and } \mathbf{v}^T H_n\mathbf{z} = \mathbf{v}^T\mathbf{g} = \mathbf{0},$$

which means, in particular, that $v_n = 0$. Taking into account

$$H_nS_n - S_n^T H_n = \mathbf{e}_n\mathbf{g}^T - \mathbf{g}\mathbf{e}_n^T$$

we conclude $(S_n\mathbf{v})^T H_n = \mathbf{0}$. Repeating the above arguments for the $S_n\mathbf{v}$ instead of $\mathbf{v}$ shows that $v_{n-1} = 0$, and so on. Finally we have $\mathbf{v} = \mathbf{0}$ which is a contradiction. Thus, the nonsingularity of H_n is proved. ■

Now we consider a general fundamental system $\{\mathbf{u},\mathbf{v}\}$. The matrix $\varphi = F[\,\mathbf{u}\ \ \mathbf{v}\,]$ is nonsingular. In fact, suppose it is singular. Then there is a nontrivial linear combination $\mathbf{w}(t)$ of $\mathbf{u}(t)$ and $\mathbf{v}(t)$ such that $F\mathbf{w} = \mathbf{0}$. In particular the highest order coefficient vanishes, i.e. $\mathbf{w} \in \mathbb{F}^n$. Since $\mathbf{w} \in \ker \partial H_n$ we conclude that $H_n\mathbf{w} = \mathbf{0}$, which means that H_n is singular. The columns of $[\,\mathbf{u}\ \ \mathbf{v}\,]\varphi^{-1}$ form now a canonical fundamental system. It remains to apply Lemma 2.2 to obtain the following.

Theorem 4.2. *Let $\{\mathbf{u},\mathbf{v}\}$ be a fundamental system for H_n. Then*

$$H_n^{-1} = \frac{1}{\det\varphi}\,\mathrm{Bez}_H(\mathbf{v},\mathbf{u}) \,, \tag{4.5}$$

where $\varphi = F[\,\mathbf{u}\ \ \mathbf{v}\,]$.

Since $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ is nonsingular, the polynomials $\mathbf{u}(t)$ and $\mathbf{v}(t)$ must be coprime (cf. Corollary 3.4). Hence the following is true.

Corollary 4.3. *If $\{\mathbf{u},\mathbf{v}\}$ is a fundamental system for a nonsingular Hankel matrix, then the polynomials $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are coprime.*

2. Characterization of fundamental systems. There are several possibilities to characterize fundamental systems via solutions of special linear systems. We are mainly interested in characterizations by vectors that will be computed recursively using Levinson algorithms. In the Hankel case these vectors are the last columns $\mathbf{x}_k$ of H_k^{-1} or alternatively the monic solutions $\mathbf{u}_k$ of the Yule-Walker equations $H_k\mathbf{u}_k = \rho_k\mathbf{e}_k$, where ρ_k is so that $\mathbf{e}_k^T\mathbf{u}_k = 1$.

It is convenient to consider an $(n+1)\times(n+1)$ extension $H_{n+1} = [\, s_{i+j-1}\,]_{i,j=1}^{n+1}$. The matrix H_{n+1} is for almost all choices of s_{2n} and s_{2n+1} nonsingular. In fact, H_{n+1} is nonsingular if the Schur complement of the leading principal submatrix H_n in H_{n+1} is nonsingular. This Schur complement is equal to

$$s_{2n+1} - \mathbf{g}^T H_n^{-1}\mathbf{g}.$$

That means, for any s_{2n} there is only one value of s_{2n+1} for which H_{n+1} is singular. Now, since the vector $\begin{bmatrix} \mathbf{u}_n \\ 0 \end{bmatrix}$, which will be also denoted by $\mathbf{u}_n$, and the vector $\mathbf{u}_{n+1}$ are linearly independent and belong both to the kernel of ∂H_n they form a fundamental system for H_n. To compute the factor $\dfrac{1}{\det\varphi}$ in (4.5) we observe that

$$F\begin{bmatrix} \mathbf{u}_n & \mathbf{u}_{n+1} \end{bmatrix} = \begin{bmatrix} \rho_n & 0 \\ 0 & 1 \end{bmatrix}.$$

For the corresponding vectors $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$ we find that

$$F\begin{bmatrix} \mathbf{x}_n & \mathbf{x}_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \xi_{n+1} \end{bmatrix},$$

where ξ_{n+1} is the last component of $\mathbf{x}_{n+1}$.

Corollary 4.4. *The inverse of the Hankel matrix H_n is given by*

$$H_n^{-1} = \frac{1}{\rho_n}\,\mathrm{Bez}_H(\mathbf{u}_{n+1},\mathbf{u}_n) = \frac{1}{\xi_{n+1}}\,\mathrm{Bez}_H(\mathbf{x}_{n+1},\mathbf{x}_n)\,.$$

3. Christoffel-Darboux formula. We compare the first Bezoutian formula for Hankel matrix inversion of Corollary 4.4 with the UL-factorization of H_n^{-1} (see e.g. [33]) which can be written in polynomial language as

$$H_n^{-1}(t,s) = \sum_{k=1}^{n} \frac{1}{\rho_k}\mathbf{u}_k(t)\mathbf{u}_k(s)\,.$$

We conclude

$$\sum_{k=1}^{n} \frac{1}{\rho_k}\mathbf{u}_k(t)\mathbf{u}_k(s) = \frac{1}{\rho_n}\,\frac{\mathbf{u}_{n+1}(t)\mathbf{u}_n(s) - \mathbf{u}_n(t)\mathbf{u}_{n+1}(s)}{t-s}\,. \tag{4.6}$$

This relation is called *Christoffel-Darboux formula.* It is important in the theory of orthogonal polynomials.

4. Inverses of Toeplitz matrices. The proof of the fact that inverses of Toeplitz matrices are T-Bezoutians follows the same lines as that for Hankel matrices. We introduce the $(n-1)\times(n+1)$ Toeplitz matrix ∂T_n obtained from $T_n = [\, a_{i-j}\,]_{i,j=1}^{n}$ after deleting the first row and adding another column to the right by preserving the Toeplitz structure,

$$\partial T_n = \begin{bmatrix} a_1 & a_0 & \dots & a_{2-n} & a_{1-n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & \dots & a_0 & a_{-1} \end{bmatrix}. \tag{4.7}$$

If T_n is nonsingular, then ∂T_n has a two-dimensional nullspace. Each basis of this subspace is called *fundamental system for T_n*. The role of the matrix F is taken by

$$F = \begin{bmatrix} a_0 & \dots & a_{1-n} & a_{-n} \\ 0 & \dots & 0 & 1 \end{bmatrix},$$

where a_{-n} is arbitrary.

Theorem 4.5. *Let $\{\mathbf{u}, \mathbf{v}\}$ be a fundamental system for T_n. Then*

$$T_n^{-1} = \frac{1}{\det \varphi} \operatorname{Bez}_T(\mathbf{u}, \mathbf{v}),$$

where $\varphi = F[\,\mathbf{u}\ \ \mathbf{v}\,]$.

The Toeplitz analogue of Proposition 4.1 is now as follows.

Proposition 4.6. *Let the equations*

$$T_n \mathbf{y} = \mathbf{e}_1 \quad \text{and} \quad T_n \mathbf{z} = \mathbf{f}^J$$

with $\mathbf{f} = (a_{-i})_{i=1}^n$ be solvable. Then T_n is nonsingular.

Taking into account that

$$T_n S_n - S_n T_n = \mathbf{e}_1 \mathbf{f}^T - \mathbf{f}^J \mathbf{e}_n^T$$

the proof of this proposition is analogous to that one of Proposition 4.1.

5. Characterization of fundamental systems. In the Toeplitz case the Levinson algorithm computes recursively the first and last columns $\mathbf{x}_k^-$ and $\mathbf{x}_k^+$ of T_k^{-1} or alternatively the solutions $\mathbf{u}_k^\pm$ of the Yule-Walker equations

$$T_k \mathbf{u}_k^- = \rho_k^- \mathbf{e}_1, \qquad \text{and} \qquad T_k \mathbf{u}_k^+ = \rho_k^+ \mathbf{e}_k, \tag{4.8}$$

where $\rho_k^\pm \in \mathbb{F}$ are so that

$$\mathbf{e}_1^T \mathbf{u}_k^- = 1 \qquad \text{and} \qquad \mathbf{e}_k^T \mathbf{u}_k^+ = 1.$$

(In other words $\mathbf{u}_k^+(t)$ is assumed to be monic and $\mathbf{u}_k^-(t)$ *comonic*, which means that $(\mathbf{u}_k^-)^J(t)$ is monic.) So it is reasonable to describe the fundamental system with these vectors.

It can easily be seen that $\begin{bmatrix} \mathbf{x}_n^- \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ \mathbf{x}_n^+ \end{bmatrix}$ belong to the nullspace of ∂T_n and in the case where T_{n-1} is nonsingular they are linearly independent. Thus, they form a fundamental system. Likewise $\begin{bmatrix} \mathbf{u}_n^- \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ \mathbf{u}_n^+ \end{bmatrix}$ form a fundamental system. We find that

$$F \begin{bmatrix} \mathbf{u}_n^- & 0 \\ 0 & \mathbf{u}_n^+ \end{bmatrix} = \begin{bmatrix} \rho_n & * \\ 0 & 1 \end{bmatrix}, \quad F \begin{bmatrix} \mathbf{x}_n^- & 0 \\ 0 & \mathbf{x}_n^+ \end{bmatrix} = \begin{bmatrix} 1 & * \\ 0 & \xi_n \end{bmatrix},$$

where ξ_n is the first component of $\mathbf{x}_n^-$ which equals the last component of $\mathbf{x}_n^+$. Consequently, $\rho_n^+ = \rho_n^- = \rho_n$.

We will have a problem with these systems, if the submatrix T_{n-1} is singular. For example, in this case the solution $\mathbf{u}_n^+$ does not exist and $\xi_n = 0$. For this reason we also consider, like in the Hankel case, an $(n+1) \times (n+1)$ Toeplitz extension $T_{n+1} = [\,a_{i-j}\,]_{i,j=1}^{n+1}$. This extension is nonsingular for almost all choices of $a_{\pm n}$. The proof of this fact is, however, less trivial than in the Hankel case.

The Schur complement of T_n in T_{n+1} is given by

$$\sigma = a_0 - (\mathbf{g}_+ + a_n \mathbf{e}_1)^T T_n^{-1} (\mathbf{g}_- + a_{-n} \mathbf{e}_1),$$

where $\mathbf{g}_\pm = [\,0\; a_{\pm(n-1)}\; \dots\; a_{\pm 1}\,]^T$. Hence

$$-\sigma = \xi a_n a_{-n} + \eta_- a_n + \eta_+ a_{-n} + \zeta\,, \tag{4.9}$$

where $\xi = \mathbf{e}_1^T T_n^{-1}\mathbf{e}_1 = \mathbf{e}_1^T\mathbf{x}_n^-$, $\zeta = \mathbf{g}_+^T T_n^{-1}\mathbf{g}_- - a_0$, and

$$\eta_- = \mathbf{e}_1^T T_n^{-1}\mathbf{g}_- = \mathbf{g}_-^T\mathbf{x}_n^+, \quad \eta_+ = \mathbf{g}_+^T T_n^{-1}\mathbf{e}_1 = \mathbf{g}_+^T\mathbf{x}_n^-\,.$$

If $\xi \neq 0$, which is equivalent to the nonsingularity of T_{n-1}, the set of pairs (a_n, a_{-n}) for which T_{n+1} is singular is a quadratic curve in $\mathbb{F}^2$. (Choosing, for example, $a_n = a_{-n}$ there are at most 2 values of a_n for which T_{n+1} is singular.)

We show that if $\xi = 0$, then $\eta_\pm \neq 0$. In fact, in the case where $\xi = 0$ we have $T_n S_n^T \mathbf{x}_n^- = \eta_+\mathbf{e}_n$. Since T_n is assumed to be nonsingular, we have $\eta_+ \neq 0$. Analogously, $\eta_- \neq 0$. That means in the case $\xi = 0$ the pairs (a_n, a_{-n}) for which T_{n+1} is singular are on the graph of a polynomial of first degree. (Choose, for example, $a_{-n} = 0$, then T_{n+1} is nonsingular with the exception of one value of a_n.)

Let now T_{n+1} be a nonsingular Toeplitz extension of T_n, $\mathbf{x}_{n+1}^-$ the first and $\mathbf{x}_{n+1}^+$ the last column of T_{n+1}^{-1}. Furthermore, let $\mathbf{u}_{n+1}^\pm$ be the solutions of the corresponding Yule-Walker equations(4.8) for $k = n+1$. Then $\{\mathbf{x}_{n+1}^-, \mathbf{x}_{n+1}^+\}$ and $\{\mathbf{u}_{n+1}^-, \mathbf{u}_{n+1}^+\}$ are fundamental systems for T_n and

$$F\,[\;\mathbf{x}_{n+1}^- \quad \mathbf{x}_{n+1}^+\;] = \begin{bmatrix} 1 & 0 \\ * & \xi_{n+1} \end{bmatrix}, \quad F\,[\;\mathbf{u}_{n+1}^- \quad \mathbf{u}_{n+1}^+\;] = \begin{bmatrix} \rho_{n+1} & 0 \\ * & 1 \end{bmatrix}. \tag{4.10}$$

Corollary 4.7. *The inverse of the Toeplitz matrix T_n is given by*

$$T_n^{-1} = \frac{1}{\xi_{n+1}}\,\mathrm{Bez}_T(\mathbf{x}_{n+1}^-, \mathbf{x}_{n+1}^+) = \frac{1}{\rho_{n+1}}\,\mathrm{Bez}_T(\mathbf{u}_{n+1}^-, \mathbf{u}_{n+1}^+)\,.$$

6. Inverses of symmetric Toeplitz matrices. We discuss now the case of a symmetric Toeplitz matrix T_n. Let T_{n+1} be a symmetric Toeplitz extension of T_n. Since in this case $\mathbf{g}_+ = \mathbf{g}_-$ we have in (4.9) $\eta_+ = \eta_-$. From this we conclude that T_{n+1} is nonsingular with the exception of at most two values of a_n. Thus, we may assume that T_{n+1} is nonsingular.

Since we have $\mathbf{x}_{n+1} := \mathbf{x}_{n+1}^+ = (\mathbf{x}_{n+1}^-)^J$, the vectors $\mathbf{w}_{n+1}^+ = \mathbf{x}_{n+1} + \mathbf{x}_{n+1}^J$ and $\mathbf{w}_{n+1}^- = \mathbf{x}_{n+1} - \mathbf{x}_{n+1}^J$ form a fundamental system consisting of a symmetric and a skewsymmetric vector. The vectors $\mathbf{w}_{n+1}^\pm$ are the solutions of $T_{n+1}\mathbf{w}_{n+1}^\pm = \mathbf{e}_{n+1} \pm \mathbf{e}_1$ and

$$F\,[\;\mathbf{w}_{n+1}^- \quad \mathbf{w}_{n+1}^+\;] = \begin{bmatrix} -1 & 1 \\ -\mathbf{w}_{n+1}^-(0) & \mathbf{w}_{n+1}^+(0) \end{bmatrix}.$$

Corollary 4.8. *The inverse of a nonsingular symmetric Toeplitz matrix T_n is given by*

$$T_n^{-1} = \frac{1}{\gamma}\,\mathrm{Bez}_T(\mathbf{w}_{n+1}^-, \mathbf{w}_{n+1}^+)\,,$$

where $\gamma = \mathbf{w}_{n+1}^-(0) - \mathbf{w}_{n+1}^+(0)$.

One can show that for solving a system $T_n\mathbf{z} = \mathbf{b}$ it is sufficient to compute the vectors $\mathbf{w}_k^+$. So it is reasonable to ask whether it is possible to describe $\mathbf{w}_{n+1}^-$ in terms of $\mathbf{w}_k^+$. The following proposition gives an answer to this question. Let T_{n+2} be a nonsingular $(n+2)\times(n+2)$ symmetric Toeplitz extension of T_{n+1} and $\mathbf{w}_{n+2}^\pm$ the solutions of $T_{n+2}\mathbf{w}_{n+2}^\pm = \mathbf{e}_{n+2} \pm \mathbf{e}_1$.

Proposition 4.9. *The polynomials* $\mathbf{w}_{n+1}^{\pm}$ *are given by*

$$\mathbf{w}_{n+1}^{\pm}(t) = \frac{t\mathbf{w}_n^{+}(t) - c_{\pm}\mathbf{w}_{n+2}^{+}(t)}{1 \pm t}\,, \tag{4.11}$$

where $\mathbf{w}_{n+2}^{+}(1) \neq 0$ *and* $c_- = \mathbf{w}_n^{+}(1)/\mathbf{w}_{n+2}^{+}(1)$. *If* n *is odd, then* $\mathbf{w}_{n+2}^{+}(-1) \neq 0$ *and* $c_+ = -\mathbf{w}_n^{+}(-1)/\mathbf{w}_{n+2}^{+}(-1)$. *If* n *is even, then* $\mathbf{w}_{n+2}^{+}(-1) = 0$ *and* c_+ *is not determined by* $\mathbf{w}_n^{+}$ *and* $\mathbf{w}_{n+2}^{+}$ *alone.*

Proof. We have

$$T_{n+2}\begin{bmatrix} \mathbf{w}_{n+1}^{\pm} & 0 \\ 0 & \mathbf{w}_{n+1}^{\pm} \end{bmatrix} = \begin{bmatrix} \pm 1 & \pm a_{\pm} \\ 0 & \pm 1 \\ \mathbf{0} & \mathbf{0} \\ 1 & 0 \\ a_{\pm} & 1 \end{bmatrix}, \quad T_{n+2}\begin{bmatrix} 0 & \\ \mathbf{w}_n^{+} & \mathbf{w}_{n+2}^{+} \\ 0 & \end{bmatrix} = \begin{bmatrix} b & 1 \\ 1 & 0 \\ \mathbf{0} & \mathbf{0} \\ 1 & 0 \\ b & 1 \end{bmatrix}$$

for some $a_{\pm}, b \in \mathbb{F}$. Consequently,

$$\begin{bmatrix} \mathbf{w}_{n+1}^{\pm} \\ 0 \end{bmatrix} \pm \begin{bmatrix} 0 \\ \mathbf{w}_{n+1}^{\pm} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{w}_n^{+} \\ 0 \end{bmatrix} - c_{\pm} w_{n+2}^{+}$$

for some $c_{\pm} \in \mathbb{F}$. Writing this in polynomial language, we see that $\mathbf{w}_{n+1}^{\pm}(t) \pm t\mathbf{w}_{n+1}^{\pm}(t) = t\mathbf{w}_n^{+}(t) - c_{\pm}\mathbf{w}_{n+2}^{+}(t)$ and obtain (4.11).

To prove the rest of the proposition we recall that the polynomials $\mathbf{w}_{n+1}^{+}(t)$ and $\mathbf{w}_{n+1}^{-}(t)$ form a fundamental system. Therefore, they are coprime. Suppose that $\mathbf{w}_{n+2}^{+}(1) = 0$. Then (4.11) implies $\mathbf{w}_{n+1}^{+}(1) = 0$. But we have also $\mathbf{w}_{n+1}^{-}(1) = 0$, since $\mathbf{w}_{n+1}^{-}$ is skewsymmetric. This contradicts the coprimeness of $\mathbf{w}_{n+1}^{+}(t)$ and $\mathbf{w}_{n+1}^{-}(t)$. Consequently, $\mathbf{w}_{n+2}^{+}(1) \neq 0$. Analogously, if n is odd and $\mathbf{w}_{n+2}^{+}(-1) = 0$, then (4.11) implies $\mathbf{w}_{n+1}^{-}(-1) = 0$. But we have also $\mathbf{w}_{n+1}^{+}(-1) = 0$, since $\mathbf{w}_{n+1}^{+}$ is symmetric and has an even length. This contradiction shows that $\mathbf{w}_{n+2}^{+}(-1) \neq 0$. If n is even, then T_n is not completely determined by its restriction to symmetric vectors. That means $\mathbf{w}_{n+1}^{+}$ is not completely given by $\mathbf{w}_n^{+}$ and $\mathbf{w}_{n+2}^{+}$. ∎

If n is even, then the constant c_+ can be obtained by applying a test functional, which could be the multiplication by any row of T_{n+1}.

7. Inverses of skewsymmetric Toeplitz matrices. In the case of a nonsingular skewsymmetric Toeplitz matrix T_n, $n = 2m$, the Levinson-type algorithm can be used to compute vectors spanning the nullspace of T_{2k-1} for $k = 1, \dots, m$. So it is reasonable to ask for a fundamental system $\{\mathbf{u}, \mathbf{v}\}$ consisting of vectors of this kind.

Let $\mathbf{x}$ be any vector spanning the nullspace of T_{n-1}. From the relation $T_{n-1}^{J} = -T_{n-1}$ follows that also the vector $\mathbf{x}^J$ belongs to the nullspace of T_{n-1}. Thus $\mathbf{x}$ is either symmetric or skewsymmetric. We show that the latter is not possible.

Lemma 4.10. *The vector* $\mathbf{x}$ *is symmetric.*

Proof. Let $\mathbf{f}_j$ denote the jth row of T_{n-1}, $n = 2m$. State that the row $\mathbf{f}_m$ in the middle of T_{n-1} is skewsymmetric. We introduce vectors $\mathbf{f}_j^{\pm} = \mathbf{f}_j \mp \mathbf{f}_{n-j}$ for $j = 1, \dots, m-1$. Then the $\mathbf{f}_j^{+}$ are symmetric, the $\mathbf{f}_j^{-}$ are skewsymmetric, $\mathbf{f}_j^{\pm} \in \mathbb{F}_{\pm}^{n-1}$, and the system $T_{n-1}\mathbf{v} = O$ is equivalent to $\mathbf{f}_j^{\pm}\mathbf{v} = 0$ for $j = 1, \dots, m-1$ and $\mathbf{f}_m\mathbf{v} = 0$. Since $\dim \mathbb{F}_{+}^{n-1} = \frac{n}{2}$, there exists a symmetric vector $\mathbf{v} \neq O$ such that $\mathbf{f}_j^{+}\mathbf{v} = 0$ for $j = 1, \dots, m-1$. Since, obviously, $\mathbf{f}_j^{-}\mathbf{v} = O$ and $\mathbf{f}_m\mathbf{v} = 0$

we have $T_{n-1}\mathbf{v} = O$. Taking into account that $\dim\ker T_{n-1} = 1$ we conclude that $\mathbf{x} = c\mathbf{v}$ for some $c \in \mathbb{F}$. Thus, $\mathbf{x} \in \mathbb{F}_+^{n-1}$. ■

Now, by Lemma 4.10, $\mathbf{x}$ is symmetric and

$$\mathbf{u} = \begin{bmatrix} 0 \\ \mathbf{x} \\ 0 \end{bmatrix} \in \ker \partial T_n .$$

(Since we do not want to assume that T_{n-2} is nonsingular, we cannot assume that $\mathbf{x}$ is monic.) Furthermore, let T_{n+1} be any $(n+1)\times(n+1)$ skewsymmetric Toeplitz extension of T_n and $\mathbf{v}$ a (symmetric) vector spanning the nullspace of T_{n+1}. Since T_n is nonsingular, we may assume that $\mathbf{v}$ is monic. Now $\{\mathbf{u}, \mathbf{v}\}$ is a fundamental system, and

$$F\,[\ \mathbf{u} \quad \mathbf{v}\ ] = \begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix}, \quad \gamma = [\,a_1 \dots a_{n-1}\,]\mathbf{x} .$$

(Here $\gamma \neq 0$ since otherwise $\begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix}$ belongs to the kernel of T_n.) Thus we obtain the following.

Corollary 4.11. *The inverse of the nonsingular skewsymmetric Toeplitz matrix T_n is given by*

$$T_n^{-1} = \frac{1}{\gamma}\,\mathrm{Bez}_T(\mathbf{u}, \mathbf{v}) .$$

8. Inverses of Hermitian Toeplitz matrices. Finally we discuss the case of a nonsingular Hermitian Toeplitz matrix T_n. Besides T_n we consider an $(n+1)\times(n+1)$ Hermitian Toeplitz extension T_{n+1} of T_n. With similar arguments as above one can show that for almost all values of a_n the matrix T_{n+1} is nonsingular, so we may assume this. In the Hermitian case we have for the first and last columns $\mathbf{x}_{n+1}^-, \mathbf{x}_{n+1}^+$ of T_{n+1}^{-1} that

$$\mathbf{x}_{n+1} := \mathbf{x}_{n+1}^- = (\mathbf{x}_{n+1}^+)^\#$$

and for the solutions $\mathbf{u}_{n+1}^\pm$ of the Yule-Walker equations

$$\mathbf{u}_{n+1} := \mathbf{u}_{n+1}^- = (\mathbf{u}_{n+1}^+)^\# .$$

Taking Corollary 4.7 into account we obtain

$$T_n^{-1} = \frac{1}{\xi_{n+1}}\,\mathrm{Bez}_T(\mathbf{x}_{n+1}, \mathbf{x}_{n+1}^\#) = \frac{1}{\rho_{n+1}}\,\mathrm{Bez}_T(\mathbf{u}_{n+1}, \mathbf{u}_{n+1}^\#) , \tag{4.12}$$

where ξ_{n+1} is the first component of $\mathbf{x}_{n+1}$ and ρ_{n+1} is so that $\mathbf{u}_{n+1}(t)$ is comonic.

In the Levinson-type algorithm described in [53], [47] not the vectors $\mathbf{x}_k$ are computed but the solutions of the equations $T_k\mathbf{q}_k = \mathbf{e}$, where $\mathbf{e}$ is the vector all components of which are equal to 1. For an inversion formula we need the vectors $\mathbf{q}_n$ and $\mathbf{q}_{n+1}$. Since $\mathbf{q}_{n+1}$ and $\mathbf{q}_n$ are conjugate-symmetric, $\mathbf{q}_{n+1}(t) - t\mathbf{q}_n(t)$ is not identically equal to zero. Hence

$$\mathbf{x}_{n+1}(t) = b(\mathbf{q}_{n+1}(t) - t\mathbf{q}_n(t)) \tag{4.13}$$

for some nonzero $b \in \mathbb{C}$.

Besides $\mathbf{q}_n$ we consider the coefficient vector $\mathbf{w}$ of $\mathbf{w}(t) = \mathrm{i}\,(t-1)\mathbf{q}_n(t)$, which is obviously conjugate-symmetric.

Proposition 4.12. *The inverse of a nonsingular Hermitian Toeplitz matrix* T_n *is given by*

$$T_n^{-1} = \frac{\mathrm{i}}{c}\,\mathrm{Bez}_T(\mathbf{w}, \mathbf{q}_{n+1}) - \frac{1}{c}\,\mathbf{q}_n \overline{\mathbf{q}}_n^T\,, \tag{4.14}$$

where c is the real constant $\mathbf{q}_{n+1}(1) - \mathbf{q}_n(1)$.

Proof. We insert (4.13) into (4.12) and obtain, after an elementary calculation, formula (4.14) with $c = \dfrac{\xi_{n+1}}{|b|^2} \neq 0$. Taking into account that $\mathbf{q}_n(t) = (T_n^{-1}\mathbf{e})(t) = T_n^{-1}(t, 1)$ and that, due to (4.14), $T_n^{-1}(t,1) = \dfrac{1}{c}\,\mathbf{q}_n(t)(\overline{\mathbf{q}}_{n+1}(1) - \overline{\mathbf{q}}_n(1))$ we find that $c = \overline{c} = \mathbf{q}_{n+1}(1) - \mathbf{q}_n(1)$. ■

9. Solution of systems. The formulas for the inverses of Toeplitz and Hankel matrices presented in this section can be used in combination with the matrix representations of Bezoutians to solve Toeplitz and Hankel systems. This is in particular convenient if systems have to be solved with different right-hand sides and one and the same coefficient matrix. The advantage compared with factorization methods is that only $O(n)$ parameters have to be stored.

The application of the formulas requires 4 matrix-vector multiplications by triangular Toeplitz matrices. If these multiplications are carried out in the classical way, then $2\,n^2$ multiplications and $2\,n^2$ additions are needed, which is more than, for example, if back substitution in the LU-factorization is applied. However, due to the Toeplitz structure of the matrices there are faster methods, actually methods with a complexity less than $O(n^2)$, to do this. In the cases $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$ the Fast Fourier and related real trigonometric transformations with a computational complexity of $O(n \log n)$ can be applied.

5. Generalized Triangular Factorizations of Bezoutians

In this section we describe algorithms that lead to a generalized UL-factorization of Bezoutians. In the case of H-Bezoutians the algorithm is just the Euclidian algorithm.

1. Division with remainder. Suppose that $\mathbf{u} = (u_i)_{i=1}^{n+1} \in \mathbb{F}^{n+1}, \mathbf{v} = (v_i)_{i=1}^{m+1} \in \mathbb{F}^{m+1},\ m \leq n\,,$ and that the last components of $\mathbf{u}$ and $\mathbf{v}$ are not zero. Division with remainder means to find polynomials $\mathbf{q}(t) \in \mathbb{F}^{n-m+1}(t)$ and $\mathbf{r}(t) \in \mathbb{F}^m(t)$ such

$$\mathbf{u}(t) = \mathbf{q}(t)\mathbf{v}(t) + \mathbf{r}(t)\,. \tag{5.1}$$

In matrix language this means that we first solve the $(n-m+1) \times (n-m+1)$ triangular Toeplitz system

$$\begin{bmatrix} v_{m+1} & \dots & v_{n-2m+1} \\ & \ddots & \vdots \\ & & v_{m+1} \end{bmatrix} \mathbf{q} = \begin{bmatrix} u_{m+1} \\ \vdots \\ u_{n+1} \end{bmatrix},$$

where we put $v_i = 0$ for $i \notin \{1, \dots, m+1\}$. With the notation (2.3) we find $\mathbf{r}$ via

$$\begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix} = \mathbf{u} - M_{n-m+1}(\mathbf{v})\mathbf{q}\,.$$

2. Factorization step for H-Bezoutians. We clarify what means division with remainder in terms of the H-Bezoutian. From (5.1) we obtain for $B = \mathrm{Bez}_H\,(\mathbf{u}, \mathbf{v})$

$$B(t,s) = \mathbf{v}(t)\,\frac{\mathbf{q}(t) - \mathbf{q}(s)}{t-s}\,\mathbf{v}(s) + \frac{\mathbf{r}(t)\mathbf{v}(s) - \mathbf{v}(t)\mathbf{r}(s)}{t-s}\,,$$

which can be written in the form

$$\mathrm{Bez}_H(\mathbf{u},\mathbf{v}) = M_{n-m}(\mathbf{v})B(\mathbf{q})M_{n-m}(\mathbf{v})^T + \mathrm{Bez}_H(\mathbf{r},\mathbf{v})\,,$$

where $B(\mathbf{q})$ is defined in (2.2). This is equivalent to

$$\mathrm{Bez}_H(\mathbf{u},\mathbf{v}) = \begin{bmatrix} I_m & M_{n-m}(\mathbf{v}) \\ O & \end{bmatrix} \begin{bmatrix} \mathrm{Bez}_H(\mathbf{r},\mathbf{v}) & O \\ O & B(\mathbf{q}) \end{bmatrix} \begin{bmatrix} I_m & M_{n-m}(\mathbf{v}) \\ O & \end{bmatrix}^T . \tag{5.2}$$

Note that in this equation the left factor is upper triangular and the right factor is lower triangular.

3. Euclidian algorithm. The Euclidian algorithm is the successive application of division with remainder. We set $\mathbf{u}_0(t) = \mathbf{u}(t)$ and $\mathbf{u}_1(t) = \mathbf{v}(t)$. Then, for $i = 1, 2, \ldots$ we find $-\mathbf{u}_{i+1}(t)$ as remainder of the division of $\mathbf{u}_{i-1}(t)$ by $\mathbf{u}_i(t)$, i.e.

$$\mathbf{u}_{i+1}(t) = \mathbf{q}_i(t)\mathbf{u}_i(t) - \mathbf{u}_{i-1}(t)\,. \tag{5.3}$$

We take the minus sign with $\mathbf{u}_{i+1}(t)$ for convenience. With this definition one factorization step (5.2) reduces $\mathrm{Bez}_H(\mathbf{u}_{i-1},\mathbf{u}_i)$ to $\mathrm{Bez}_H(\mathbf{u}_i,\mathbf{u}_{i+1})$, without the minus to $\mathrm{Bez}_H(\mathbf{u}_{i+1},\mathbf{u}_i)$. If for some $i = l$ we have $\mathbf{u}_{l+1}(t) = 0$, then the algorithm is terminated and $\mathbf{u}_l(t)$ is the greatest common divisor of $\mathbf{u}(t)$ and $\mathbf{v}(t)$.

There are modifications of the Euclidian algorithm with different normalizations. One could assume, for example, that the polynomials $\mathbf{u}_i(t)$ are monic or that the polynomials $\mathbf{q}_i(t)$ are monic. For these two possibilities one has to admit a constant factor at $\mathbf{u}_{i-1}$.

4. Generalized UL-factorization of H-Bezoutians. Applying (5.2) successively to the polynomials $\mathbf{u}_i(t)$ appearing in the Euclidian algorithm (5.3) we arrive at the following.

Theorem 5.1. *Let $\mathbf{u}(t), \mathbf{v}(t) \in \mathbb{F}^{n+1}(t)$ where $\mathbf{u}(t)$ has a nonzero leading coefficient. Then $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ admits the factorization*

$$\mathrm{Bez}_H(\mathbf{u},\mathbf{v}) = UDU^T \tag{5.4}$$

where

$$U = \begin{bmatrix} I_d & M_{n_l}(\mathbf{u}_l) & \ldots & M_{n_1-n_2}(\mathbf{u}_2) & M_{n-n_1}(\mathbf{u}_1) \\ O & O & & O & \end{bmatrix},$$

$n_i = \deg \mathbf{u}_i(t)$, d is the degree of the greatest common divisor of $\mathbf{u}(t)$ and $\mathbf{v}(t)$, and

$$D = \mathrm{diag}\,(\,O_d,\ B(\mathbf{q}_l),\ \ldots\ ,\ B(\mathbf{q}_1)\,)\,. \tag{5.5}$$

Note that U is a nonsingular upper triangular matrix, $B(\mathbf{q}_i)$ are nonsingular upper triangular Hankel matrices defined by (2.2). From the theorem we can again conclude that the nullity of $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ is equal to d (cf. Corollary 3.4).

Let us discuss the case, in which $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are coprime and all polynomials $\mathbf{q}_i(t)$ have degree 1. This is called the *generic case*. Suppose that $\mathbf{q}_i(t) = \rho_i t + \delta_i$. Then (5.4) turns into an UL-factorization of the H-Bezoutian in which, U is the upper triangular matrix the kth row of which is equal to the transpose of $\begin{bmatrix} \mathbf{u}_{n+1-k} \\ \mathbf{0}_{k-1} \end{bmatrix}$ and $D = \mathrm{diag}\,(\rho_{n+1-i})_{i=1}^n$. The UL-factorization of the H-Bezoutian exists if and only if the matrix $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})^J$ is strongly nonsingular. That means in the generic case the matrix has this property. The converse is also true, since in the non-generic case the matrix $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})^J$ has singular leading principal submatrices.

5. Inertia computation. It is an important consequence of Theorem 5.1 that the signature of a real H-Bezoutian can be computed via running the Euclidian algorithm. In fact, in the case of real polynomials $\mathbf{u}(t)$ and $\mathbf{v}(t)$ the matrix $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ is congruent to the block diagonal matrix D given by (5.5). It remains to compute the signature of $B(\mathbf{q}_i)$.

Let ρ_i denote the leading coefficient of $\mathbf{q}_i(t)$. Then the signature of $B(\mathbf{q}_i)$ is equal to the signature of $\rho_i J_{m_i}$, $m_i = n_{i-1} - n_i$. This can be shown using a homotopy argument. Let $H(t) = t\rho_i J_{m_i} + (1-t)B(\mathbf{q}_i)$ for $0 \le t \le 1$. Then $H(0) = B(\mathbf{q}_i)$ and $H(1) = \rho_i J_{m_i}$. Furthermore, $H(t)$ is nonsingular for all t and depends continuously on t. Hence $\operatorname{sgn} H(t)$ is constant for $0 \le t \le 1$. The signature of $\rho_i J_{m_i}$ is obviously equal to zero if m_i is even and is equal to the sign of ρ_i if m_i is odd. Applying Sylvester's inertia law we conclude the following.

Corollary 5.2. *The signature of the real H-Bezoutian* $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ *is given by*

$$\operatorname{sgn} \mathrm{Bez}_H(\mathbf{u},\mathbf{v}) = \sum_{n_{i-1}-n_i \text{ odd}} \operatorname{sgn} \rho_i \,,$$

where ρ_i *are the antidiagonal entries of* $B(\mathbf{q}_i)$.

Since the Euclidian algorithm computes besides the signature s also the rank r of the H-Bezoutian, it gives a complete picture about the inertia of $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$,

$$\mathrm{In}\,\mathrm{Bez}_H(\mathbf{u},\mathbf{v}) = (s_+, s_-, d),$$

where $s_\pm = \frac{n-d\pm s}{2}$ and $d = n - r$.

6. Factorization step for T-Bezoutians in the generic case. We consider the problem of triangular factorization of a T-Bezoutian $B = \mathrm{Bez}_T(\mathbf{u},\mathbf{v})$, where $\mathbf{u},\mathbf{v} \in \mathbb{F}^{n+1}$. This problem is more complicated than for H-Bezoutians, unless the matrix is strongly nonsingular. We introduce the 2×2 matrix

$$\Gamma = \Gamma(\mathbf{u},\mathbf{v}) = \begin{bmatrix} \mathbf{e}_1^T\mathbf{u} & \mathbf{e}_1^T\mathbf{v} \\ \mathbf{e}_{n+1}^T\mathbf{u} & \mathbf{e}_{n+1}^T\mathbf{v} \end{bmatrix}.$$

The case of nonsingular Γ is referred to as *generic case*, the case of singular Γ as *non-generic case*. In this subsection we consider the generic case. Observe that $\gamma := B(0,0) = \det\Gamma$. That means that γ is the entry in the left upper corner of B. Thus, we have the generic case if B is strongly nonsingular. Note that γ is also the entry in the right lower corner of B, due to the persymmetry of B. In the generic case, $\begin{bmatrix} \widetilde{\mathbf{u}} & \widetilde{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}\Gamma^{-1}$ is of the form

$$\begin{bmatrix} \widetilde{\mathbf{u}} & \widetilde{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & 0 \\ 0 & \mathbf{v}_1 \end{bmatrix}, \quad \mathbf{u}_1, \mathbf{v}_1 \in \mathbb{F}^n.$$

According to Lemma 2.9 we have

$$\mathrm{Bez}_T(\widetilde{\mathbf{u}},\widetilde{\mathbf{v}}) = \frac{1}{\gamma}\,\mathrm{Bez}_T(\mathbf{u},\mathbf{v}).$$

Furthermore, for $\widetilde{B} = \mathrm{Bez}_T(\widetilde{\mathbf{u}},\widetilde{\mathbf{v}})$ we obtain

$$\begin{aligned} \widetilde{B}(t,s) &= \frac{\mathbf{u}_1(t)\mathbf{v}_1(s^{-1})s^{n-1} - ts\mathbf{v}_1(t)\mathbf{u}_1(s^{-1})s^{n-1}}{1-ts} \\ &= B_1(t,s) + \mathbf{v}_1(t)\mathbf{u}_1(s^{-1})s^{n-1}, \end{aligned}$$

where $B_1 = \mathrm{Bez}_T(\mathbf{u}_1,\mathbf{v}_1)$. We also have

$$\widetilde{B}(t,s) = ts\,B_1(t,s) + \mathbf{u}_1(t)\mathbf{v}_1(s^{-1})s^{n-1}.$$

In matrix language this can be written as

$$\mathrm{Bez}_T(\mathbf{u},\mathbf{v}) = \begin{bmatrix} I_{n-1} & \\ \mathbf{0}^T & \mathbf{v}_1 \end{bmatrix} \begin{bmatrix} \gamma\,\mathrm{Bez}_T(\mathbf{u}_1,\mathbf{v}_1) & \mathbf{0} \\ \mathbf{0}^T & \gamma \end{bmatrix} \begin{bmatrix} I_{n-1} & \\ \mathbf{0}^T & \mathbf{u}_1^J \end{bmatrix}^T$$

or

$$\mathrm{Bez}_T(\mathbf{u},\mathbf{v}) = \begin{bmatrix} \mathbf{u}_1 & \begin{matrix} I_{n-1} \\ \mathbf{0}^T \end{matrix} \end{bmatrix} \begin{bmatrix} \gamma & \mathbf{0}^T \\ \mathbf{0} & \gamma\,\mathrm{Bez}_T(\mathbf{u}_1,\mathbf{v}_1) \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^J & \begin{matrix} I_{n-1} \\ \mathbf{0}^T \end{matrix} \end{bmatrix}^T . \tag{5.6}$$

7. LU-factorization of T-Bezoutians. Let $B = \mathrm{Bez}_T(\mathbf{u},\mathbf{v})$ be strongly nonsingular, which is equivalent to the strongly nonsingularity of B^J, due to persymmetry. We can apply now the factorization step of the previous subsection, since the property of strongly nonsingularity is inherited after a factorization step. If we carry out the factorization step successively, then we obtain the following algorithm. We set $\mathbf{u}_1 = \mathbf{u}$ and $\mathbf{v}_1 = \mathbf{v}$ and find recursively polynomials $\mathbf{u}_k(t)$ and $\mathbf{v}_k(t)$ via

$$\begin{bmatrix} \mathbf{u}_{k+1}(t) & \mathbf{v}_{k+1}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{u}_k(t) & \mathbf{v}_k(t) \end{bmatrix} \Gamma_k^{-1} \begin{bmatrix} 1 & 0 \\ 0 & t^{-1} \end{bmatrix}, \tag{5.7}$$

where $\Gamma_k = \Gamma(\mathbf{u}_k,\mathbf{v}_k)$. This algorithm has the same structure as the Schur algorithm for Toeplitz matrices. We call it also *Schur algorithm.* Like for Toeplitz matrices, it can be slightly modified by replacing the matrix Γ_k^{-1} by a matrix of the form $\begin{bmatrix} 1 & * \\ * & 1 \end{bmatrix}$. This will reduce the number of operations.

To simplify the notation we agree upon the following. For a sequence $(\mathbf{w}_j)_{j=1}^n$ with $\mathbf{w}_j \in \mathbb{F}^{n+1-j}$, by $L(\mathbf{w}_j)_{j=1}^n$ will be denoted the lower triangular matrix the kth column of which is equal to

$$L(\mathbf{w}_j)_{j=1}^n \mathbf{e}_k = \begin{bmatrix} \mathbf{0}_{k-1} \\ \mathbf{w}_k \end{bmatrix}.$$

Now we conclude the following from (5.6).

Theorem 5.3. *Let $B = \mathrm{Bez}_T(\mathbf{u},\mathbf{v})$ be strongly nonsingular, and let $\mathbf{u}_k(t)$ and $\mathbf{v}_k(t)$ be the polynomials obtained by the Schur algorithm* (5.7)*. Then B admits an LU-factorization*

$$B = LDU\,,$$

where

$$L = L(\mathbf{u}_i)_{i=1}^n\,, \quad U = (L(\mathbf{v}_i)_{i=1}^n)^T$$

and

$$D = \mathrm{diag}\,(\widetilde{\gamma}_i^{-1})_{i=1}^n\,, \quad \widetilde{\gamma}_i = \prod_{j=1}^i \gamma_j\,, \quad \gamma_j = \det \Gamma_j\,.$$

8. Non-generic case for T-Bezoutians. Now we consider the case where the matrix $\Gamma = \Gamma(\mathbf{u},\mathbf{v})$ is singular. If Γ has a zero row, then $\mathbf{u}(t)$ and $\mathbf{v}(t)$ or $\mathbf{u}^J(t)$ and $\mathbf{v}^J(t)$ have a common factor t. Suppose that $\mathbf{u}(t) = t^{\mu_-}\mathbf{u}_0(t)$, $\mathbf{v}(t) = t^{\mu_-}\mathbf{v}_0(t)$, $\mathbf{u}^J(t) = t^{\mu_+}\mathbf{u}_0^J(t)$, and $\mathbf{v}^J(t) = t^{\mu_+}\mathbf{v}_0^J(t)$ such that $\Gamma(\mathbf{u}_0,\mathbf{v}_0)$ has no zero row. Then $B(t,s) = t^{\mu_-}s^{\mu_+}B_0(t,s)$, where $B_0 = \mathrm{Bez}_T(\mathbf{u}_0,\mathbf{v}_0)$ or, in matrix language

$$B = \begin{bmatrix} O & O & O \\ O & B_0 & O \\ O & O & O \end{bmatrix},$$

where the zero matrix in the left upper corner is $\mu_- \times \mu_+$.

Now we assume that Γ is singular but has no zero row. Then there is a 2×2 matrix Φ with $\det \Phi = 1$ such that the last column of $\Gamma\Phi$ is zero, but the first column consists of nonzero elements. We set

$$\begin{bmatrix} \widetilde{\mathbf{u}}(t) & \widetilde{\mathbf{v}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{u}(t) & \mathbf{v}(t) \end{bmatrix} \Phi.$$

Then, according to Lemma 2.9, we have $B = \mathrm{Bez}_T(\widetilde{\mathbf{u}}, \widetilde{\mathbf{v}}) = \mathrm{Bez}_T(\mathbf{u}, \mathbf{v})$. Furthermore, let us write $\widetilde{\mathbf{v}}$ in the form

$$\widetilde{\mathbf{v}} = \begin{bmatrix} \mathbf{0}_{\nu_-} \\ \mathbf{w} \\ \mathbf{0}_{\nu_+} \end{bmatrix}$$

with some vector $\mathbf{w} = (w_i)_{i=1}^{m+1} \in \mathbb{F}^{m+1}$, $m+\nu_+ + \nu_- = n$, with nonzero first and last components. We apply now a two-sided division with remainder to find polynomials $\mathbf{q}_-(t) \in \mathbb{F}^{\nu_-}(t)$, $\mathbf{q}_+(t) \in \mathbb{F}^{\nu_+ +1}(t)$, and $\mathbf{r}(t) \in \mathbb{F}^m(t)$ such that

$$\widetilde{\mathbf{u}}(t) = (t^{\nu_-}\mathbf{q}_+(t) + \mathbf{q}_-(t))\mathbf{w}(t) + t^{\nu_-}\mathbf{r}(t)\,.$$

The vectors $\mathbf{q}_\pm$ can be found by solving the triangular Toeplitz systems

$$\begin{bmatrix} w_1 & & \\ \vdots & \ddots & \\ w_{\nu_-} & \dots & w_1 \end{bmatrix} \mathbf{q}_- = \begin{bmatrix} \widetilde{u}_1 \\ \vdots \\ \widetilde{u}_{\nu_-} \end{bmatrix},$$

$$\begin{bmatrix} w_{m+1} & \dots & w_{m-\nu_+ +1} \\ & \ddots & \vdots \\ & & w_{m+1} \end{bmatrix} \mathbf{q}_+ = \begin{bmatrix} \widetilde{u}_{n-\nu_+ +1} \\ \vdots \\ \widetilde{u}_{n+1} \end{bmatrix}.$$

Then we have

$$\mathbf{u}^J(t) = (t^{\nu_+ +1}\mathbf{q}_-^J(t) + \mathbf{q}_+^J(t))\mathbf{w}^J(t) + t^{\nu_+ +1}\mathbf{r}^J(t)$$

and

$$\begin{aligned} B(t,s) &= \mathbf{w}(t)\left(\frac{\mathbf{q}_-(t) - t^{\nu_-}\mathbf{q}_-^J(s)s}{1-ts}\, s^{\nu_+} + t^{\nu_-}\,\frac{\mathbf{q}_+(t)s^{\nu_+} - \mathbf{q}_+^J(s)}{1-ts}\right)\mathbf{w}^J(s) \\ &\quad + t^{\nu_-}\,\frac{\mathbf{r}(t)\mathbf{w}^J(s) - \mathbf{w}(t)\mathbf{r}^J(s)s}{1-ts}\, s^{\nu_+} \\ &= \mathbf{w}(t)\left(\mathrm{Bez}_T(\mathbf{q}_-, \mathbf{e}_{\nu_- +1})(t,s)s^{\nu_+} + t^{\nu_-}\mathrm{Bez}_T(\mathbf{q}_+, \mathbf{e}_1)(t,s)\right)\mathbf{w}^J(s) \\ &\quad + t^{\nu_-}\mathrm{Bez}_T(\mathbf{r}, \mathbf{w})(t,s)s^{\nu_+}\,. \end{aligned}$$

In matrix form this can be written as

$$B = M_{\nu_+ + \nu_-}(\mathbf{w})\begin{bmatrix} O & B_-(\mathbf{q}_-) \\ B_+(\mathbf{q}_+) & O \end{bmatrix} M_{\nu_+ + \nu_-}(\mathbf{w}^J)^T + \begin{bmatrix} O & O & O \\ O & B_1 & O \\ O & O & O \end{bmatrix},$$

where $B_1 = \mathrm{Bez}_T(\mathbf{r}, \mathbf{w})$ is of order m, $B_+(\mathbf{q}_+) = \mathrm{Bez}_T(\mathbf{q}_+, \mathbf{e}_1)$ and $B_-(\mathbf{q}_-) = \mathrm{Bez}_T(\mathbf{q}, \mathbf{e}_{\nu_- +1})$ are of order $\nu_\pm$ (cf. (2.14)), and the zero matrix in the left upper corner of the last term has size $\nu_- \times \nu_+$.

9. Hermitian T-Bezoutians. We discuss now the specifics of the case of an Hermitian T-Bezoutian B. Our main attention is dedicated to the question how to compute the signature, because this is the most important application of the procedure. First we remember that there are two possibilities to represent Hermitian T-Bezoutian. The first is $B = \mathrm{Bez}_T(\mathbf{u}, \mathbf{u}^\#)$ for a general vector $\mathbf{u} \in \mathbb{C}^{n+1}$, the second is $B = \mathrm{i}\,\mathrm{Bez}_T(\mathbf{u}_+, \mathbf{u}_-)$ for two conjugate-symmetric vectors $\mathbf{u}_\pm$ (see Section 2,10).

In the generic case we use the first representation. In the representation (5.6) we have $\mathbf{v}_1^J = \overline{\mathbf{u}}_1$, and γ is real. Thus, for a strongly nonsingular B, Theorem 5.3 provides a factorization

$B = LDL^*$. Consequently,

$$\operatorname{sgn} B = \sum_{i=1}^{n} \operatorname{sgn} \widetilde{\gamma}_i \,.$$

That means that the signature of B can be computed via the Schur algorithm in $O(n^2)$ operations.

In the non-generic case, i.e. in the case where Γ is singular, we switch from the first representation of B to the second one. This is done as follows. Suppose B is given as $B = \operatorname{Bez}_T(\mathbf{u}, \mathbf{u}^{\#})$. Then Γ is centro-Hermitian. Hence the homogeneous equation

$$\Gamma \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has a nontrivial conjugate-symmetric solution, which is a solution with $\beta = \overline{\alpha}$. We can assume that $|\alpha| = 1$. We set

$$\Phi = \begin{bmatrix} \alpha \mathrm{i} & \alpha \\ -\overline{\alpha} \mathrm{i} & \overline{\alpha} \end{bmatrix}$$

and

$$\begin{bmatrix} \mathbf{u}_+(t) & \mathbf{u}_-(t) \end{bmatrix} = \begin{bmatrix} \mathbf{u}(t) & \mathbf{u}^{\#}(t)) \end{bmatrix} \Phi \,.$$

Then $\mathbf{u}_\pm$ are conjugate-symmetric and according to Lemma 2.9 we obtain

$$B = \frac{1}{2\mathrm{i}} \operatorname{Bez}_T(\mathbf{u}_+, \mathbf{u}_-) \,.$$

We can now apply the reduction step described in Section 5,8 for $\widetilde{\mathbf{u}} = \mathbf{u}_+$ and $\widetilde{\mathbf{v}} = \mathbf{u}_-$. Due to the Hermitian symmetry we have $\nu_- = \nu_+ =: \nu$. The vector $\mathbf{q}_-^{\#}$ is just the vector $\mathbf{q}_+$ after cancelling its first component. The vectors $\mathbf{w}$ and $\mathbf{r}$, both considered as elements of $\mathbb{F}^{m+1}$, are conjugate-symmetric. This leads to

$$2\mathrm{i}\, B = M_{2\nu}(\mathbf{w}) \begin{bmatrix} O & B_-(\mathbf{q}_-) \\ (B_-(\mathbf{q}_-))^* & O \end{bmatrix} M_{2\nu}(\mathbf{w})^* + \begin{bmatrix} O & O & O \\ O & B_1 & O \\ O & O & O \end{bmatrix} ,$$

where $B_1 = \operatorname{Bez}_T(\mathbf{r}, \mathbf{w})$ is $m \times m$. Taking into account that $B_-(\mathbf{q})$ and the zero matrices in the corners of the last term are $\nu \times \nu$ matrices, the sum of the ranks of the two terms on the right hand side is equal to the rank of B. This is also true for the signature. The signature of the first term is equal to zero. Hence

$$\operatorname{sgn} B = \operatorname{sgn} B_1 \,.$$

If $\operatorname{Bez}_T(\mathbf{r}, \mathbf{w})$ is singular, then we carry out another non-generic step. If $\operatorname{Bez}_T(\mathbf{r}, \mathbf{w})$ is nonsingular we go over to the first Bezoutian representation by introducing $\mathbf{v} = \frac{1}{2}(\mathbf{r} - \mathrm{i}\mathbf{w})$ and obtain $B_1 = \operatorname{Bez}_T(\mathbf{v}, \mathbf{v}^{\#})$. Now we can apply a generic step. Summing up, we have described a procedure that computes the signature of an arbitrary Hermitian T-Bezoutian in $O(n^2)$ operations.

6. Bezoutians and Companion Matrices

In this section we show that Bezoutians are related to functions of companion matrices.

1. Factorization of the companion. The companion matrix of the monic polynomial $\mathbf{u}(t) = \sum_{k=1}^{n+1} u_k t^{k-1} \in \mathbb{F}^{n+1}$ is, by definition, the $n \times n$ matrix

$$C(\mathbf{u}) = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -u_1 & -u_2 & \dots & -u_n \end{bmatrix}$$

It is easy to show that the characteristic polynomial of $C(\mathbf{u})$, $\det(tI_n - C(\mathbf{u}))$, is equal to $\mathbf{u}(t)$. This is also a consequence of the following useful relation.

Lemma 6.1. *We have*

$$tI_n - C(\mathbf{u}) = \begin{bmatrix} 0 & -1 & & \\ & & \ddots & \\ & & & -1 \\ 1 & \mathbf{u}_1(t) & \dots & \mathbf{u}_{n-1}(t) \end{bmatrix} \begin{bmatrix} \mathbf{u}(t) & \\ & I_{n-1} \end{bmatrix} \begin{bmatrix} 1 & & & \\ -t & 1 & & \\ & \ddots & \ddots & \\ & & -t & 1 \end{bmatrix}, \tag{6.1}$$

where $\mathbf{u}_k(t) = u_{k+1} + u_{k+2}t + \dots + u_{n+1}t^{n-k}$.

Proof. It is immediately checked that

$$(tI_n - C(\mathbf{u})) \begin{bmatrix} 1 & & & & \\ t & 1 & & & \\ t^2 & t & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ t^{n-1} & t^{n-2} & \dots & t & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & & \\ & & \ddots & \\ & & & -1 \\ \mathbf{u}(t) & \mathbf{u}_1(t) & \dots & \mathbf{u}_{n-1}(t) \end{bmatrix}.$$

This equality can be rearranged to (6.1). ∎

The polynomials $\mathbf{u}_k(t)$ appearing in the first factor of the right side of (6.1) are the *Horner polynomials* of $\mathbf{u}^J(t)$. They satisfy the recursion

$$\mathbf{u}_k(t) = t\mathbf{u}_{k+1}(t) + u_{k+1}$$

and can be represented as

$$[\, \mathbf{u}_1(t) \, \dots \, \mathbf{u}_n(t) \,] = \ell_n(t)^T B(\mathbf{u}) \, .$$

where $B(\mathbf{u})$ is introduced in (2.2). For $t_0 \in \mathbb{F}$, the matrix $t_0 I_n - C(\mathbf{u})$ is a special case of a resultant matrix (cf. (2.4)). In fact,

$$t_0 I_n - C(\mathbf{u}) = \operatorname{Res}(t_0 - t, \mathbf{u}(t) + t^{n-1}(t_0 - t)) \, .$$

Since the resultant matrix is nonsingular if and only if the polynomials are coprime, we conclude again that $t_0 I_n - C(\mathbf{u})$ is singular if and only if $t - t_0$ is a divisor of $\mathbf{u}(t)$, i.e. $\mathbf{u}(t_0) = 0$.

2. Functional calculus. Before we continue with companions and Bezoutians we recall some general definitions and facts concerning functions of a matrix. Let A be an $n \times n$ matrix and $\mathbf{u}(t) = \sum\limits_{k=1}^{m} u_k t^{k-1}$ a polynomial. Then $\mathbf{u}(A)$ denotes the matrix

$$\mathbf{u}(A) = \sum_{k=1}^{m} u_k A^{k-1}$$

in which we set $A^0 = I_n$. The matrices of the form $\mathbf{u}(A)$ form a commutative matrix algebra and the transformation $\mathbf{u}(t) \mapsto \mathbf{u}(A)$ is a linear operator and a ring homomorphism. In particular, if $\mathbf{u}(t) = \mathbf{u}_1(t)\mathbf{u}_2(t)$, then $\mathbf{u}(A) = \mathbf{u}_1(A)\mathbf{u}_2(A)$. If $\mathbf{u}(t)$ is the characteristic polynomial of A, then, according to the Cayley-Hamilton theorem, $\mathbf{u}(A) = O$. Let a polynomial $\mathbf{v}(t)$ and the characteristic polynomial $\mathbf{u}(t)$ of A be coprime. Then the Bezout equation

$$\mathbf{v}(t)\mathbf{x}(t) + \mathbf{u}(t)\mathbf{y}(t) = 1 \tag{6.2}$$

has a solution $(\mathbf{x}(t), \mathbf{y}(t))$. Replacing t by A we obtain that $\mathbf{v}(A)\mathbf{x}(A) = I_n$. That means $\mathbf{v}(A)$ is nonsingular and $\mathbf{x}(A)$ is its inverse.

3. Barnett's formula. The following remarkable formula is due to *S. Barnett*.

Theorem 6.2. *Let $\mathbf{u}(t), \mathbf{v}(t) \in \mathbb{F}^{n+1}(t)$ and $\mathbf{u}(t)$ be monic. Then*

$$\mathrm{Bez}_H(\mathbf{u}, \mathbf{v}) = B(\mathbf{u})\mathbf{v}(C(\mathbf{u}))\,, \tag{6.3}$$

where $B(\mathbf{u})$ is introduced in (2.2).

Proof. Due to linearity, it is sufficient to prove the formula for $\mathbf{v}(t) = \mathbf{e}_k(t) = t^{k-1}$. We set $B_k = \mathrm{Bez}_H(\mathbf{u}, \mathbf{e}_k)$. Since $B_k = B(\mathbf{u})C(\mathbf{u})^{k-1}$ is true for $k = 1$ we still have to show that $B_{k+1} = B_k C(\mathbf{u})$. Taking into account that

$$C(\mathbf{u})\ell_n(s) = s\ell_n(s) - \mathbf{u}(s)\mathbf{e}_n\,,$$

we obtain

$$\ell_n(t)^T B_k C(\mathbf{u})\ell_n(s) = sB_k(t,s) - \mathbf{u}(s)\ell_n(t)^T B_k \mathbf{e}_n = sB_k(t,s) - t^{k-1}\mathbf{u}(s)\,. \tag{6.4}$$

On the other hand,

$$B_{k+1}(t,s) = \frac{(\mathbf{u}(t)s^{k-1} - t^{k-1}\mathbf{u}(s))s}{t-s} - t^{k-1}\mathbf{u}(s) = sB_k(t,s) - t^{k-1}\mathbf{u}(s)\,. \tag{6.5}$$

Comparing (6.4) and (6.5) we obtain the recursion $B_{k+1} = B_k C(\mathbf{u})$. ■

From this theorem we can conclude again (cf. Corollary 3.4) that the H-Bezoutian of $\mathbf{u}(t)$ and $\mathbf{v}(t)$ is nonsingular if $\mathbf{u}(t)$ and $\mathbf{v}(t)$ ($\mathbf{u}(t) \in \mathbb{F}^{n+1}(t)$ monic) are coprime and that its inverse is given by

$$\mathrm{Bez}_H(\mathbf{u}, \mathbf{v})^{-1} = \mathbf{x}(C(\mathbf{u}))B(\mathbf{u})^{-1}\,, \tag{6.6}$$

where $\mathbf{x}(t)$ is from the solution of the Bezout equation (6.2). In the next subsection we will show that $\mathbf{x}(C(\mathbf{u}))$ is actually a Hankel matrix.

4. Barnett's formula for T-Bezoutians. Now we consider T-Bezoutians. Let $\mathbf{u}(t)$ be a comonic polynomial of degree $\leq n$ and $B_k = \mathrm{Bez}_T(\mathbf{u}, \mathbf{e}_k)$. Then

$$B_k(t,s) = B_{k+1}(t,s)s - t^{k-1}\mathbf{u}^J(s)\,.$$

This can be written as

$$B_k = B_{k+1}C(\mathbf{u}^J)\,.$$

With the notation of (2.14) we obtain the *Toeplitz analogue of Barnett's formula.*

Theorem 6.3. *Let $\mathbf{u}(t), \mathbf{v}(t) \in \mathbb{F}^{n+1}(t)$, where $\mathbf{u}(t)$ is comonic. Then*

$$\mathrm{Bez}_T(\mathbf{u}, \mathbf{v}) = B_-(\mathbf{u})\mathbf{v}^J(C(\mathbf{u}^J))\,.$$

In particular, for $\mathbf{v}(t) = 1$ we obtain the equality

$$B_+(\mathbf{u}) = B_-(\mathbf{u})C(\mathbf{u}^J)^n\,, \tag{6.7}$$

which yields an LU-factorization of $C(\mathbf{u}^J)^n$ and will be applied below to prove an inversion formula for T-Bezoutions.

7. Hankel Matrices Generated by Rational Functions

In this section we consider Hankel matrices generated by rational functions and show that they are closely related to H-Bezoutians. We understand "rational functions" in an abstract sense as elements of the quotient field of the ring of polynomials. But occasionally, in particular if we restrict ourselves to the case $\mathbb{F} = \mathbb{C}$, we interpret them in the analytic sense as functions defined in $\mathbb{F}$.

By a *proper* rational function we mean a rational function for which the degree of the numerator polynomial is not greater than the degree of the denominator polynomial. We say that the representation $\mathbf{f}(t) = \frac{\mathbf{p}(t)}{\mathbf{u}(t)}$ is in *reduced form* if $\mathbf{u}(t)$ and $\mathbf{p}(t)$ are coprime and $\mathbf{u}(t)$ is monic. This representation is unique. The *degree of a proper rational function* is the degree of the denominator polynomial in the reduced representation.

1. Generating functions of Hankel matrices. A proper rational function $\mathbf{f}(t)$ can be represented as

$$\mathbf{f}(t) = h_0 + h_1 t^{-1} + h_2 t^{-2} + \dots . \tag{7.1}$$

If $\mathbb{F} = \mathbb{C}$, then (7.1) can be interpreted as the Laurent series expansion of $\mathbf{f}(t)$ at infinity converging outside a disk with center 0. For a general field $\mathbb{F}$ (7.1) has a meaning as quotient of two formal power series. The coefficients h_i can be obtained recursively by an obvious formula. For $\mathbf{f}(t)$ having a Laurent expansion (7.1), we set $\mathbf{f}(\infty) = h_0$ and write $\mathbf{f}(t) = O(t^{-m})$ if $h_0 = \dots = h_{m-1} = 0$.

Note that if $\mathbf{f}(t)$ is given by (7.1), then we have

$$\frac{\mathbf{f}(t) - \mathbf{f}(s)}{t - s} = \sum_{k=1}^{\infty} h_k \, \frac{t^{-k} - s^{-k}}{t - s} = - \sum_{i,j=1}^{\infty} h_{i+j-1} \, t^{-i} s^{-j} \, . \tag{7.2}$$

This relation suggests the following definition. For $n = 1, 2, \dots$, the $n \times n$ *Hankel matrix generated by* $\mathbf{f}(t)$ is, by definition, the matrix

$$H_n(\mathbf{f}) = [\, h_{i+j-1} \,]_{i,j=1}^{n} .$$

Let us point out that the entry h_0 does not enter the definition of $H_n(\mathbf{f})$. If for some $n \times n$ Hankel matrix H_n there is a function $\mathbf{f}$ so that $H_n = H_n(\mathbf{f})$, then $\mathbf{f}(t)$ will be called *generating function* of H_n.

Example 7.1. *As an example, let us compute the Hankel matrices generated by a partial fraction* $\dfrac{1}{(t-c)^m}$ *(*$c \in \mathbb{F}$, $m = 1, 2, \dots, 2n-1$*). We denote*

$$L_n(c, m) = \frac{1}{(m-1)!} \, H_n \left(\frac{1}{(t-c)^m} \right)$$

and write $L_n(c)$ *instead of* $L_n(c, 1)$. *In view of*

$$\frac{1}{t - c} = t^{-1} + c \, t^{-2} + c^2 \, t^{-3} + \dots \tag{7.3}$$

we have

$$L_n(c) = [\, c^{i+j-2} \,]_{i,j=1}^{n} \, . \tag{7.4}$$

Differentiating the equality (7.3) *we obtain*

$$L_n(c, 2) = [\, (i + j - 2) c^{i+j-3} \,]_{i,j=1}^{n}$$

and in general, for $m = 1, \dots, 2n-1$,

$$L_n(c,m) = \left[\begin{pmatrix} i+j-2 \\ m-1 \end{pmatrix} c^{i+j-1-m} \right]_{i,j=1}^{n} .$$

It is obvious that the rank of $L_n(c,m)$ *is equal to* m.

The matrices $L_n(c,m)$ are called *elementary Hankel matrices.*

Example 7.2. *For our second example we assume that* $\mathbb{F}$ *is algebraically closed. Let* $\mathbf{u}(t)$ *be a polynomial of degree* n *and let* $t_1, \dots, t_n$ *be the zeros of* $\mathbf{u}(t)$*. The* Newton sums s_i *(*$i = 1, 2, \dots$*) are given by*

$$s_i = \sum_{k=1}^{n} t_k^{i-1} .$$

We form the Hankel matrix $H_n = [\, s_{i+j-1} \,]_{i,j=1}^{n}$*. Then we have*

$$H_n = H_n \left(\frac{\mathbf{u}'(t)}{\mathbf{u}(t)} \right) .$$

This follows from the obvious relation

$$\frac{\mathbf{u}'(t)}{\mathbf{u}(t)} = \sum_{k=1}^{n} \frac{1}{t - t_k} .$$

The transformation

$$\mathcal{H} : \mathbf{f}(t) \longrightarrow H_n(\mathbf{f})$$

is clearly a linear operator from the vector space of all proper rational functions to the space of $n \times n$ Hankel matrices. The kernel of this operator consists of all proper rational function $\mathbf{f}(t)$ for which $\mathbf{f}(t) - \mathbf{f}(\infty) = O(t^{-2n})$. We show that this transformation is onto. That means any $k \times k$ Hankel matrix can be regarded as generated by a proper rational function.

Proposition 7.3. *Let* $\mathbf{u}(t)$ *be a fixed monic polynomial of degree* $2n-1$*. Then any* $n \times n$ *Hankel matrix* H_n *can be represented uniquely in the form*

$$H_n = H_n \left(\frac{\mathbf{p}}{\mathbf{u}} \right) \tag{7.5}$$

for some $\mathbf{p} \in \mathbb{F}^{2n-1}$.

Proof. Clearly, a matrix of the form (7.5) with $\mathbf{p} \in \mathbb{F}^{2n-1}$ does not belong to the kernel of the transformation $\mathcal{H}$. That means that the mapping of the vector $\mathbf{p} \in \mathbb{F}^{2n-1}$ to the Hankel matrix $H_n \left(\frac{\mathbf{p}}{\mathbf{u}} \right)$ is one-to-one. Hence the dimension of its range equals $2n-1$. This is just the dimension of the space of $n \times n$ Hankel matrices. Thus the mapping is onto. ∎

Since in an algebraically closed field $\mathbb{F}$ any proper rational function has a partial fraction decomposition we conclude that in this case any Hankel matrix can be represented as a linear combination of elementary Hankel matrices. The reader may observe that the problem to find the generating function of a Hankel matrix is closely related to the Padé approximation problem at infinity and the partial realization problem.

In connection with these and other problems the question about a generating function of minimal degree arises. We will see later that the degree of the generating function is at least equal to the rank of H_n. But it can be bigger. For example, the rank-one Hankel matrix $H_n = \mathbf{e}_n \mathbf{e}_n^T$ has no generating function of degree less than $2n-1$. Here we restrict ourselves to the nonsingular case. For a nonsingular $n \times n$ Hankel matrix, a generating function of degree n always exists, as the next proposition shows. Let us use the notation of Section 4,1.

Theorem 7.4. *Let $H_n = [\, s_{i+j-1} \,]_{i,j=1}^n$ be a nonsingular Hankel matrix, $\{\mathbf{u}(t), \mathbf{v}(t)\}$ be a fundamental system of H_n, where $\mathbf{u}(t)$ is monic and $\deg \mathbf{v}(t) < n$. Then, for any $\alpha \in \mathbb{F}$, there is a vector $\mathbf{p} \in \mathbb{F}^n$ such that*

$$H_n = H_n \left(\frac{\mathbf{p}(t)}{\mathbf{u}(t) - \alpha \mathbf{v}(t)} \right) .$$

Proof. We consider the $(n-1) \times (n+1)$ matrix ∂H_n, which was introduced in (4.1). The vector $\mathbf{w} = \mathbf{u} - \alpha \mathbf{v}$ is a monic vector belonging to the nullspace of ∂H_n. We set

$$\mathbf{p} = \begin{bmatrix} 0 & s_1 & \dots & s_n \\ & \ddots & \ddots & \vdots \\ & & 0 & s_1 \end{bmatrix} \mathbf{w} .$$

From this definition and from $\mathbf{w} \in \ker \partial H_n$ we can see that in the expansion

$$\frac{\mathbf{p}(t)}{\mathbf{w}(t)} = h_1 t^{-1} + h_2 t^{-2} + \dots$$

we have $h_i = s_i$, $i = 1, \dots, 2n-1$. Hence, $H_n = H_n \left(\frac{\mathbf{p}(t)}{\mathbf{w}(t)} \right)$. ■

Example 7.5. *Let us find generating functions of degree n of the matrix $H_n = J_n$. For this matrix $\{\mathbf{e}_{n+1}, \mathbf{e}_1\}$ is a fundamental system. Furthermore, $\mathbf{p} = \mathbf{e}_1$. Thus, for any $\alpha \in \mathbb{F}$,*

$$J_n = H_n \left(\frac{1}{t^n - \alpha} \right) .$$

Let us present a special case of Theorem 7.4 involving the solutions of the equations

$$H_k \mathbf{u}_k = \rho_k \mathbf{e}_k \,, \quad \mathbf{e}_k^T \mathbf{u}_k = 1 \tag{7.6}$$

for $k = n, n+1$. Here H_{n+1} is a nonsingular extension of H_n. As we already know from Section 4,2 these monic solutions form a fundamental system for H_n. Thus, the following is immediately clear.

Corollary 7.6. *Let H_n and H_{n+1} be as above and $\mathbf{u}_n, \mathbf{u}_{n+1}$ be the solutions of (7.6) for $k = n, n+1$. Then, for an $\alpha \in \mathbb{F}$, there are vectors $\mathbf{p}_{n+1} \in \mathbb{F}^n$ and $\mathbf{p}_n \in \mathbb{F}^{n-1}$ such that*

$$H_n = H_n \left(\frac{\mathbf{p}_{n+1}(t) - \alpha \mathbf{p}_n(t)}{\mathbf{u}_{n+1}(t) - \alpha \mathbf{u}_n(t)} \right) .$$

2. Vandermonde factorization of Hankel matrices. In this subsection we assume that $\mathbb{F} = \mathbb{C}$. Let H_n be an $n \times n$ nonsingular Hankel matrix. Then, by Theorem 7.4, it has a generating function of degree n. We can assume that the denominator polynomial of this function has only simple roots, which follows from the following lemma. Recall from Corollary 4.3 that the polynomials forming a fundamental system of a nonsingular Hankel matrix are coprime.

Lemma 7.7. *Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be coprime. Then for all $\alpha \in \mathbb{C}$, with the exception of a finite number of points, the polynomial $\mathbf{w}(t) = \mathbf{u}(t) - \alpha \mathbf{v}(t)$ has only simple roots.*

Proof. Suppose that τ_0 is a multiple root of $\mathbf{w}(t)$. Then $\mathbf{u}(\tau_0) = \alpha\mathbf{v}(\tau_0)$ and $\mathbf{u}'(\tau_0) = \alpha\mathbf{v}'(\tau_0)$. Since $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are coprime, $\mathbf{v}(\tau_0) \neq 0$. Hence τ_0 is a root of the nonzero polynomial $\mathbf{z}(t) = \mathbf{u}(t)\mathbf{v}'(t) - \mathbf{v}(t)\mathbf{u}'(t)$. Now we choose α such that $\mathbf{u}(\tau) \neq \alpha\mathbf{v}(\tau)$ for all roots τ of $\mathbf{z}(t)$. Then none of the τ is a root of $\mathbf{w}(t)$, so $\mathbf{w}(t)$ has no multiple roots. ■

Let $\mathbf{f}(t) = \frac{\mathbf{p}(t)}{\mathbf{w}(t)}$ with $(\infty) = \mathbf{0}$ be a proper rational function in reduced form such that $\mathbf{w}(t)$ has simple roots $t_1, \dots, t_n$. Then $\mathbf{f}(t)$ has a partial fraction decomposition

$$\mathbf{f}(t) = \sum_{i=1}^{n} \frac{\delta_i}{t - t_i},$$

where

$$\delta_i = \frac{\mathbf{p}(t_i)}{\mathbf{w}'(t_i)} = \left(\left(\frac{1}{\mathbf{f}}\right)'(t_i)\right)^{-1}. \tag{7.7}$$

Hence $H_k(\mathbf{f})$ can be represented as a linear combination of elementary Hankel matrices $L_k(t_i)$ defined by (7.4)

$$H_k(\mathbf{f}) = \sum_{i=1}^{n} \delta_i L_k(t_i). \tag{7.8}$$

This relation can be stated as a matrix factorization. In fact, observe that $L_k(t_i)$ is equal to $\ell_k(t_i)\ell_k(t_i)^T$, where $\ell_k(t_i) = (t_i^{j-1})_{j=1}^{k}$. We form the $n \times k$ Vandermonde matrix $V_k(\mathbf{t})$, $\mathbf{t} = (t_i)_{i=1}^n$, with the rows $\ell_k(t_i)^T$ $(i = 1, \dots, n,)$

$$V_k(\mathbf{t}) = [\, t_i^{j-1} \,]_{i=1, j=1}^{n \quad k}.$$

Now (7.8) is equivalent to the following.

Proposition 7.8. *Let $\mathbf{f}(t)$ and $\mathbf{t}$ be as above. Then for $k \geq n$, the Hankel matrix $H_k = H_k(\mathbf{f})$ admits a representation*

$$H_k = V_k(\mathbf{t})^T D V_k(\mathbf{t}), \tag{7.9}$$

where $D = \operatorname{diag}(\delta_i)_{i=1}^n$ and the δ_i are given by (7.7).

Example 7.9. *For the Hankel matrix of Example 7.2 we obtain the following factorization,*

$$H_n\left(\frac{\mathbf{u}'}{\mathbf{u}}\right) = V_n(\widetilde{\mathbf{t}})^T \operatorname{diag}(\nu_i)_{i=1}^r V_n(\widetilde{\mathbf{t}}), \tag{7.10}$$

where $\widetilde{\mathbf{t}}$ is the vector of different zeros $\widetilde{t}_i$ $(i = 1, \dots, r)$ of $\mathbf{u}(t)$, and ν_i are their multiplicities.

A consequence of this Proposition 7.8 is that $\operatorname{rank} H_k(\mathbf{f}) = n$ for all $k \geq n$. In Section 7,5 we will show that this is true for a general field $\mathbb{F}$. Combining Proposition 7.8 with Proposition 7.4 we obtain the following.

Corollary 7.10. *Let H_n be a nonsingular $n \times n$ Hankel matrix, $\{\mathbf{u}, \mathbf{v}\}$ a fundamental system of H_n, and $\alpha \in \mathbb{C}$ such that $\mathbf{w}(t) = \mathbf{u}(t) - \alpha\mathbf{v}(t)$ has simple roots $t_1, \dots, t_n$. Then H_n admits a representation*

$$H_n = V_n(\mathbf{t})^T D V_n(\mathbf{t})$$

with a diagonal matrix D.

Let us find the Vandermonde factorization for Example 7.5, i.e. for $H_n = J_n$. The polynomial $t^n - \alpha$ has simple roots for all $\alpha \neq 0$, and these roots t_i are the nth complex roots of α. The diagonal matrix is given by

$$D = \frac{1}{n} \operatorname{diag}(t_1^{1-n}, \ldots, t_n^{1-n}).$$

3. Real Hankel matrices. We consider the special case of a real, nonsingular Hankel matrix H_n. In this case the fundamental system of H_n is also real. We choose $\alpha \in \mathbb{R}$, since then the non-real roots of $\mathbf{w}(t) = \mathbf{u}(t) - \alpha \mathbf{v}(t)$ appear in conjugate complex pairs. Let $t_1, \ldots, t_r$ be the real roots and $t_{r+1}, t_{r+2} = \bar{t}_{r+1}, \ldots, t_{n-1}, t_n = \bar{t}_{n-1}$ be the non-real roots of $\mathbf{w}(t)$. Then

$$V_n(\mathbf{t})^T = V_n(\mathbf{t})^* \operatorname{diag}(I_r, \underbrace{J_2, \ldots, J_2}_{l}),$$

where $r + 2l = n$. Thus, we obtain from Proposition 7.8 the following.

Corollary 7.11. *If the Hankel matrix H_k $(k \geq n)$ in Proposition 7.8 is real, then it admits a representation*

$$H_k = V_k(\mathbf{t})^* D_1 V_k(\mathbf{t}),$$

where

$$D_1 = \operatorname{diag}\left(\delta_1, \ldots, \delta_r, \begin{bmatrix} 0 & \overline{\delta}_{r+1} \\ \delta_{r+1} & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \overline{\delta}_{n-1} \\ \delta_{n-1} & 0 \end{bmatrix}\right).$$

In particular, the matrices H_n and D_1 are congruent.

Combining this with Sylvester's inertia law, with (7.9), and with the fact that the signature of a matrix $\begin{bmatrix} 0 & \overline{\delta} \\ \delta & 0 \end{bmatrix}$ equals 0, we conclude the following.

Corollary 7.12. *Let $\mathbf{f}(t)$ be a real rational function of degree n with simple poles $t_1, \ldots, t_n$, where $t_1, \ldots, t_r$ are real and the other poles non-real. Then for $k \geq n$ the signature of the Hankel matrix $H_k = H_k(\mathbf{f})$ is given by*

$$\operatorname{sgn} H_k = \sum_{i=1}^{r} \operatorname{sgn} \delta_i,$$

where δ_i is defined by (7.7). In particular, H_n is positive definite if and only if all roots of $\mathbf{w}(t)$ are real and $\delta_i > 0$ for all i.

Let us specify the criterion of positive definiteness further.

4. The Cauchy index. Let $\mathcal{C}$ be an oriented closed curve in the extended complex plane $\mathbb{C} \cup \{\infty\}$ and $\mathbf{f}(t)$ a rational function with real values on $\mathcal{C}$ with the exception of poles. A pole c of $\mathbf{f}(t)$ on $\mathcal{C}$ is said to be of *positive type* if

$$\lim_{\substack{t \to c- \\ t \in \mathcal{C}}} \mathbf{f}(t) = -\infty \quad \text{and} \quad \lim_{\substack{t \to c+ \\ t \in \mathcal{C}}} \mathbf{f}(t) = \infty.$$

It is said to be of *negative type* if c is of positive type for $-\mathbf{f}(t)$. If a pole is not of positive or negative type, then it is called *neutral.* The *Cauchy index of* $\mathbf{f}(t)$ *along* $\mathcal{C}$ is, by definition, the integer

$$\operatorname{ind}_{\mathcal{C}} \mathbf{f}(t) = p_+ - p_-$$

where p_+ is the number of poles of positive and p_- the number of poles of negative type. The pole c is of positive (negative) type if and only if the function $\frac{1}{\mathbf{f}(t)}$ is increasing (decreasing) in a neighborhood of c.

It is clear that if c is a pole of positive or negative type on $\mathcal{C}$, then a small perturbation of the coefficients of $\mathbf{f}(t)$ leads only to a small change of the pole on $\mathcal{C}$ by preserving its type (which is not true for neutral poles). Now we are in the position to relate the signature of Hankel matrices generated by a rational function with the Cauchy index of this function along $\mathbb{R}$.

Proposition 7.13. *Let $\mathbf{f}(t)$ be a real proper rational function of degree n. Then*

$$\operatorname{sgn} H_n(\mathbf{f}) = \operatorname{ind}_{\mathbb{R}} \mathbf{f}(t)\,.$$

Proof. Suppose that $\mathbf{f}(t) = \frac{\mathbf{p}(t)}{\mathbf{u}(t)}$ is the reduced representation as quotient of polynomials. Let us first assume that $\mathbf{u}(t)$ has simple roots $t_1, \dots, t_n$, i.e. $\mathbf{f}(t)$ has simple poles. Simple poles cannot be neutral. If t_i is a simple pole of positive type, then $\left(\frac{1}{\mathbf{f}}\right)'(t_i) > 0$. Comparing this with (7.7) we conclude that $\delta_i > 0$. Analogously, we have $\delta_i < 0$ for a pole t_i of negative type. Now it remains to apply Corollary 7.12.

Now let $\mathbf{u}(t)$ have multiple roots. Neutral poles of $\mathbf{f}(t)$ correspond to roots of $\mathbf{u}(t)$ of even order and do not contribute to the Cauchy index of $\mathbf{f}(t)$. It is easy to see that we can disturb $\mathbf{u}(t)$ additively with an $\alpha \in \mathbb{R}$ as small as we want such that the disturbed roots of even order disappear or become simple roots of $\mathbf{u}(t) + \alpha$, so that the respective pairs of poles do not contribute to the Cauchy index of $\mathbf{f}_\alpha(t) = \frac{\mathbf{p}(t)}{\mathbf{u}(t)+\alpha}$. The other poles remain simple and of the same type. So the first part of the proof applies to $\mathbf{f}_\alpha(t)$. Due to Proposition 7.17 below $H_n(\mathbf{f})$ is nonsingular. Taking into account that the signature of a nonsingular Hankel matrix is invariant with respect to small perturbations the assertion follows. ■

Some readers might be unsatisfied with the analytic argument in the proof of the algebraic Proposition 7.13. For those readers we note that a purely algebraic proof of this proposition is possible if more general Vandermonde representations of Hankel matrices are considered.

Let us discuss the question how positive definiteness of $H_n(\mathbf{f})$ can be characterized in terms of $\mathbf{f}(t)$. According to Proposition 7.13, $H_n(\mathbf{f})$ is positive definite if and only if the Cauchy index of $\mathbf{f}(t)$ along $\mathbb{R}$ is equal to n. That means that $\mathbf{f}(t)$ must have n poles of positive type. Between two poles of positive type there must be a zero of $\mathbf{f}(t)$, i.e. a root of the numerator polynomial. In other words, the poles and zeros of $\mathbf{f}(t)$ must interlace.

We say that the *real roots of two polynomials* $\mathbf{u}(t)$ *and* $\mathbf{p}(t)$ *interlace* if between two roots of $\mathbf{u}(t)$ there is exactly one root of $\mathbf{p}(t)$. Polynomials with roots that interlace are coprime. Let us summarize our discussion.

Corollary 7.14. *Let $\mathbf{f}(t)$ be a real rational function of degree n, and let $\mathbf{f}(t) = \frac{\mathbf{p}(t)}{\mathbf{u}(t)}$ be its reduced representation. Then $H_n(\mathbf{f})$ is positive definite if and only if the polynomials $\mathbf{u}(t)$ and $\mathbf{p}(t)$ have only real simple roots that interlace.*

5. Congruence to H-Bezoutians. In Section 4 we showed that inverses of Hankel matrices are H-Bezoutians. Now we are going to explain another relation between Hankel matrices and H-Bezoutians. In the real case this relation just means that Hankel matrices and H-Bezoutians are congruent.

Throughout this subsection, let $\mathbf{u}(t)$ be a polynomial of degree n, $\mathbf{v}(t) \in \mathbb{F}^{n+1}(t)$, and $\mathbf{f}(t) = \frac{\mathbf{v}(t)}{\mathbf{u}(t)}$. Then $B = \operatorname{Bez}_H(\mathbf{u}, \mathbf{v})$ is an $n \times n$ matrix. For $k > n$, we identify this matrix with the $k \times k$ matrix obtained from B by adding $k - n$ zero rows and zero columns at the bottom and on the right. The same we do for $B(\mathbf{u}) = \operatorname{Bez}_H(\mathbf{u}, \mathbf{e}_1)$ introduced in (2.2).

Proposition 7.15. *For $k \geq n$, the $k \times k$ Hankel matrix generated by $\mathbf{f}(t) = \frac{\mathbf{v}(t)}{\mathbf{u}(t)}$ is related to the H-Bezoutian of $\mathbf{u}(t)$ and $\mathbf{v}(t)$ via*

$$\mathrm{Bez}_H(\mathbf{u},\mathbf{v}) = B(\mathbf{u})\, H_k\,(\mathbf{f})\, B(\mathbf{u})\,. \tag{7.11}$$

Proof. For $B = \mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ and $\mathbf{u} = (u_i)_{i=1}^{n+1}$ we have in view of (7.2)

$$\begin{aligned} B(t,s) &= -\mathbf{u}(t)\,\frac{\mathbf{f}(t)-\mathbf{f}(s)}{t-s}\,\mathbf{u}(s) \\ &= \sum_{i,j=1}^{n+1}\sum_{p,q=1}^{\infty} u_i\, h_{p+q-1}\, u_j\, t^{i-p-1}s^{j-q-1} \\ &= \sum_{m,l=1}^{n}\sum_{p,q=1}^{\infty} u_{m+p}\, h_{p+q-1}\, u_{q+l}\, t^{m-1}s^{l-1}\,, \end{aligned}$$

where we set $u_j = 0$ for $j > n+1$. The coefficient matrix of this polynomial can be written as a product of three matrices, as it is claimed. ■

Recall from Corollary 3.4 that the nullity of $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ is equal to the degree of the greatest common divisor of $\mathbf{u}(t)$ and $\mathbf{v}(t)$. This implies that the rank of $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ is equal to the degree of the rational function $\mathbf{f}(t)$.

Corollary 7.16. *In the real case the matrices $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ and $H_n\left(\frac{\mathbf{u}}{\mathbf{v}}\right)$ are congruent.*

Proposition 7.17. *Let $\mathbf{f}(t)$ be a rational function of degree n. Then, for $k \geq n$, the rank of $H_k(\mathbf{f})$ is equal to n. In particular, $H_n(\mathbf{f})$ is nonsingular.*

Proof. Let $\mathbf{f}(t) = \frac{\mathbf{v}(t)}{\mathbf{u}(t)}$ be in reduced form and $\deg \mathbf{u}(t) = n$. Since the matrix $B(\mathbf{u})$ in (7.11) as an $k \times k$ matrix is singular for $k > n$, the assertion cannot be concluded directly from Proposition 7.15. Thus, for $k > n$, define $\mathbf{u}_k$ and $\mathbf{v}_k$ by $\mathbf{u}_k(t) = t^k\mathbf{u}(t)$ and $\mathbf{v}_k(t) = t^k\mathbf{v}(t)$, respectively. The $\mathbf{f}(t) = \frac{\mathbf{v}_k(t)}{\mathbf{u}_k(t)}$ and, due to Proposition 7.15,

$$\mathrm{Bez}_H(\mathbf{u}_k,\mathbf{v}_k) = B(\mathbf{u}_k)\, H_k(\mathbf{f})\, B(\mathbf{u}_k)\,, \tag{7.12}$$

where $B(\mathbf{u}_k)$ is nonsingular. Since the nullity of $\mathrm{Bez}_H(\mathbf{u}_k,\mathbf{v}_k)$ equals k, we have that

$$\mathrm{rank}\,\mathrm{Bez}_H(\mathbf{u}_k,\mathbf{v}_k) = n = \mathrm{rank}\, H_k(\mathbf{f})\,.$$

■

Note that under the assumptions of Proposition 7.8 the assertion of Proposition 7.17 follows already from the Vandermonde factorization of $H_k(\mathbf{f})$ given in (7.9).

In the real case, (7.12) is a congruence relation. Applying Sylvester's inertia law we can conclude the following for the case $\mathbb{F} = \mathbb{R}$.

Proposition 7.18. *If $\mathbf{u}(t) \in \mathbb{R}^{n+1}(t)$ with $\deg \mathbf{u}(t) = n$ and $\mathbf{v}(t) \in \mathbb{R}^{n+1}(t)$, then the matrices $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ (considered as $k \times k$ matrix) and $H_k\left(\frac{\mathbf{v}}{\mathbf{u}}\right)$ have the same inertia.*

Proof. Due to formula (7.12), which is also true if $\frac{\mathbf{v}(t)}{\mathbf{u}(t)}$ is not in reduced form, and since $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ and $\mathrm{Bez}_H(\mathbf{u}_k,\mathbf{v}_k)$ have the same rank it remains to show that these two Bezoutians have the same signature. But, this follows from

$$\mathrm{Bez}_H(\mathbf{u}_k,\mathbf{v}_k) = M_n(\mathbf{e}_{k+1})\mathrm{Bez}_H(\mathbf{u},\mathbf{v})M_n(\mathbf{e}_{k+1})^T$$

and from Corollary 1.2. ■

6. Inverses of H-Bezoutians. Comparing (7.11) with Barnett's formula (6.3) we obtain that the Hankel matrix $H_n\left(\frac{\mathbf{p}}{\mathbf{u}}\right)$ admits a representation

$$H_n\left(\frac{\mathbf{p}}{\mathbf{u}}\right) = \mathbf{p}(C(\mathbf{u}))B(\mathbf{u})^{-1}\,. \tag{7.13}$$

Together with (6.6) this immediately leads to the following.

Theorem 7.19. *Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be coprime and $(\mathbf{q}(t),\mathbf{p}(t))$, be the (unique) solution of the Bezout equation*

$$\mathbf{u}(t)\mathbf{q}(t) + \mathbf{v}(t)\mathbf{p}(t) = 1 \tag{7.14}$$

with $\mathbf{q}(t),\mathbf{p}(t) \in \mathbb{F}^n(t)$. Then

$$\mathrm{Bez}_H(\mathbf{u},\mathbf{v})^{-1} = H_n\left(\frac{\mathbf{p}}{\mathbf{u}}\right)\,.$$

An immediate consequence of this theorem is the converse of Theorem 4.2.

Corollary 7.20. *The inverse of a nonsingular H-Bezoutian is a Hankel matrix.*

Furthermore, Theorem 7.19 tells us that the inverse of a Bezoutian or the inverse of a Hankel matrix generated by a rational function can be computed with an algorithm which solves the Bezout equation (7.14). We show next that this equation can be solved with the help of the Euclidian algorithm.

7. Solving the Bezout equation. Let $\mathbf{u}_i(t)$ be the polynomials computed by the Euclidian algorithm, as described in Section 5,3, then with the help of the data of the Euclidian algorithm we can recursively solve the Bezout equations

$$\mathbf{u}(t)\mathbf{x}_i(t) + \mathbf{v}(t)\mathbf{y}_i(t) = \mathbf{u}_i(t) \quad (i = 0, 1, \dots), \tag{7.15}$$

where, for initialization, we have

$$\begin{array}{rcl} \mathbf{x}_0(t) &=& 1 \\ \mathbf{y}_0(t) &=& 0 \end{array}, \qquad \begin{array}{rcl} \mathbf{x}_1(t) &=& 0 \\ \mathbf{y}_1(t) &=& 1 \end{array}.$$

The recursion is given by

$$\begin{array}{rcl} \mathbf{x}_{i+1}(t) &=& \mathbf{x}_{i-1}(t) - \mathbf{q}_i(t)\mathbf{x}_i(t) \\ \mathbf{y}_{i+1}(t) &=& \mathbf{y}_{i-1}(t) - \mathbf{q}_i(t)\mathbf{y}_i(t)\,. \end{array} \tag{7.16}$$

In fact,

$$\begin{array}{rcl} \mathbf{u}(t)\mathbf{x}_{i+1}(t) + \mathbf{v}(t)\mathbf{y}_{i+1}(t) &=& \mathbf{u}(t)(\mathbf{x}_{i-1} - \mathbf{q}_i(t)\mathbf{x}_i(t)) + \mathbf{v}(t)(\mathbf{y}_{i-1}(t) - \mathbf{q}_i(t)\mathbf{y}_i(t)) \\ &=& \mathbf{u}_{i-1}(t) - \mathbf{q}_i(t)\mathbf{u}_i(t) = \mathbf{u}_{i+1}(t)\,. \end{array}$$

Introducing the 2×2 matrix polynomials

$$X_i(t) = \begin{bmatrix} \mathbf{x}_{i-1}(t) & \mathbf{x}_i(t) \\ \mathbf{y}_{i-1}(t) & \mathbf{y}_i(t) \end{bmatrix} \quad \text{and} \quad \Phi_i(t) = \begin{bmatrix} 0 & 1 \\ 1 & -\mathbf{q}_i(t) \end{bmatrix}$$

we can write the recursion (7.16) as

$$X_{i+1}(t) = X_i(t)\Phi_i(t)\,, \qquad X_1(t) = I_2\,.$$

If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are coprime, then we have for some $i = l$ that $\mathbf{u}_l(t) = c = \text{const}$. Thus, the solution of (7.14) is obtained from $(\mathbf{x}_l(t), \mathbf{y}_l(t))$ by dividing by c.

As mentioned in Section 5,3 the Euclidian algorithm for finding the greatest common divisor of two polynomials is closely related to a Schur-type algorithm for Hankel matrices. The Euclidian algorithm for solving the Bezout equation is related to a mixed Levinson-Schur-type algorithm for Hankel matrices. It is also possible to design a pure Levinson-type algorithm for the solution of the Bezout equation.

8. Toeplitz Matrices Generated by Rational Functions

This section is the Toeplitz counterpart of the previous one. We define and study Toeplitz matrices generated by rational functions and show that they are closely related to T-Bezoutians. Some features are completely analogous to the Hankel case, but there are also some significant differences.

1. Generating functions of Toeplitz matrices. Let $\mathbf{f}(t) = \frac{\mathbf{p}(t)}{\mathbf{u}(t)}$ be a proper rational function with $\mathbf{u}(0) \neq 0$. Then $\mathbf{f}(t)$ admits series expansions in powers of t and as well as in powers of t^{-1},

$$\mathbf{f}(t) = a_0^+ + a_1 t + a_2 t^2 + \dots , \quad \mathbf{f}(t) = -a_0^- - a_{-1}t^{-1} - a_{-2}t^{-2} - \dots . \tag{8.1}$$

If $\mathbb{F} = \mathbb{C}$, then (8.1) can be understood as the Laurent series expansion at $t = 0$ and at $t = \infty$, respectively. For a general $\mathbb{F}$, (8.1) makes sense as the quotient of two formal Laurent series. The coefficients can be obtained recursively by obvious relations. Note that, in a formal sense,

$$\frac{\mathbf{f}(t) - \mathbf{f}(s^{-1})}{1 - ts} = \sum_{i,j=0}^{\infty} a_{i-j} t^i s^j , \tag{8.2}$$

where $a_0 = a_0^+ + a_0^-$. The latter follows from the obvious relation

$$(1 - ts)T(t, s) = a_0 + \sum_{i=1}^{\infty} a_i t^i + \sum_{j=1}^{\infty} a_{-j} s^j ,$$

where $T = [\, a_{i-j} \,]_{i,j=1}^{\infty}$. This relation makes the following definition natural.

For $n = 1, 2, \dots$ the $n\times n$ *Toeplitz matrix generated by* $\mathbf{f}(t)$ with the expansions (8.1) is, by definition, the matrix $T_n(\mathbf{f}) = [\, a_{i-j} \,]_{i,j=1}^{n}$, where $a_0 = a_0^+ + a_0^-$. If $T_n = T_n(\mathbf{f})$, then the function $\mathbf{f}(t)$ is called *generating function* of T_n. Obviously, $T_n(\mathbf{f})$ is the zero matrix if $\mathbf{f}$ is a constant function. Note that finding the generating function of a given Toeplitz matrix is a two-point Padé approximation problem for the points 0 and ∞.

Example 8.1. *Since for $c \neq 0$*

$$\frac{1}{1 - ct} = \sum_{k=0}^{\infty} c^k t^k \quad \text{and} \quad \frac{1}{1 - ct} = -\sum_{k=1}^{\infty} c^{-k} t^{-k} ,$$

we have

$$T_n\left(\frac{1}{1 - ct}\right) = [\, c^{i-j} \,]_{i,j=1}^{n} = \ell_n(c)\ell_n\left(c^{-1}\right)^T , \tag{8.3}$$

where $\ell_n(c)$ is defined in (1.1)*.*

Example 8.2. *Our second example is the Toeplitz analogue of Example* 7.2*. Let $\mathbb{F}$ be algebraically closed and $\mathbf{u}(t)$ a polynomial of degree n with the roots $t_1, \dots, t_n$ and $\mathbf{u}(0) \neq 0$. We define*

$$c_i = \sum_{k=1}^{n} t_k^i \quad (i = 0, \pm 1, \pm 2, \dots)$$

and form the Toeplitz matrix $T_n = [\,c_{i-j}\,]_{i,j=1}^n$. *Then* $T_n = T_n(\mathbf{f})$ *for*

$$\mathbf{f}(t) = \sum_{k=1}^n \frac{1}{1 - t_k t} \,.$$

Taking $\mathbf{u}^J(t) = \prod_{k=1}^n (1 - t_k t)$ *into account we find that*

$$\mathbf{f}(t) = \frac{(\mathbf{u}')^J(t)}{\mathbf{u}^J(t)} \,.$$

Here $\mathbf{u}'$ *has to be considered as a vector in* $\mathbb{F}^n$, *that means* $(\mathbf{u}')^J(t) = t^{n-1}\mathbf{u}'(t^{-1})$.

Like for Hankel matrices, it can be shown that, for any given polynomial $\mathbf{u}(t)$ of degree $2n-2$ with $\mathbf{u}(0) \neq 0$, any $n \times n$ Toeplitz matrix has a generating function with denominator polynomial $\mathbf{u}(t)$. More important is the following Toeplitz analogue of Theorem 7.4 about generating functions of nonsingular Hankel matrices. Note that its proof is somehow different to the Hankel case.

Theorem 8.3. *Let* $T_n = [\,a_{i-j}\,]_{i,j=1}^n$ *be a nonsingular Toeplitz matrix,* $\{\mathbf{u}(t), \mathbf{v}(t)\}$ *be a fundamental system of* T_n. *Furthermore, let* $\alpha, \beta \in \mathbb{F}$ *be such that* $\mathbf{w}(t) = \alpha\mathbf{u}(t) + \beta\mathbf{v}(t)$ *is of degree* n *and* $\mathbf{w}(0) \neq 0$. *Then there is a* $\mathbf{p} \in \mathbb{F}^{n+1}$ *such that*

$$T_n = T_n\left(\frac{\mathbf{p}}{\mathbf{w}}\right) \,.$$

Proof. Let ∂T_n be the matrix defined in (4.7). Then $\mathbf{w}$ belongs to the nullspace of ∂T_n. We find a_n and a_{-n} via the equations

$$\left[\begin{array}{cccc} a_n & a_{n-1} & \dots & a_0 \end{array}\right] \mathbf{w} = \left[\begin{array}{cccc} a_0 & \dots & a_{1-n} & a_{-n} \end{array}\right] \mathbf{w} = 0$$

and form the $(n+1)\times(n+1)$ Toeplitz matrix $T_{n+1} = [\,a_{i-j}\,]_{i,j=1}^{n+1}$. Then we have $T_{n+1}\mathbf{w} = \mathbf{0}$. Now we represent T_{n+1} as $T_{n+1} = T_{n+1}^+ + T_{n+1}^-$, where T_{n+1}^+ is a lower triangular and T_{n+1}^- is an upper triangular Toeplitz matrix and define

$$\mathbf{p} = T_{n+1}^+ \mathbf{w} \,. \tag{8.4}$$

We have also $\mathbf{p} = -T_{n+1}^- \mathbf{w}$. A comparison of coefficients reveals that

$$\frac{\mathbf{p}(t)}{\mathbf{w}(t)} = a_0^+ + a_1 t + \dots + a_n t^n + \dots \quad \text{and} \quad \frac{\mathbf{p}(t)}{\mathbf{w}(t)} = -a_0^- - a_{-1}t^{-1} - \dots - a_{-n}t^{-n} - \dots \,,$$

where $a_0^\pm$ are the diagonal entries of $T_{n+1}^\pm$, respectively. Thus $T_n = T_n(\mathbf{f})$ for $\mathbf{f}(t) = \frac{\mathbf{p}(t)}{\mathbf{w}(t)}$. ∎

Example 8.4. *Let us compute generating functions for the identity matrix* I_n. *We observe first that* $\{\mathbf{e}_1, \mathbf{e}_{n+1}\}$ *is a fundamental system. To meet the conditions of the theorem we choose* $\alpha \neq 0, \beta \neq 0$. *Then we find that* $a_n = -\frac{\beta}{\alpha}$ *and* $a_{-n} = -\frac{\alpha}{\beta}$. *We can choose now*

$$T_{n+1}^+ = \gamma I_{n+1} - \frac{\beta}{\alpha}\,\mathbf{e}_{n+1}\mathbf{e}_1^T \,,$$

where γ *is arbitrary. For* $\gamma = 0$ *this leads to* $\mathbf{p} = -\beta\mathbf{e}_{n+1}$. *Thus generating functions for* I_n *are given by*

$$\mathbf{f}(t) = \frac{-\beta t^n}{\alpha + \beta t^n} \,.$$

If we choose a different γ, *then the resulting function differs from that function only by a constant.*

2. Matrices with symmetry properties. It is a little bit surprising that if $\mathbf{f}(t)$ is symmetric in the sense that

$$\mathbf{f}(t^{-1}) = \mathbf{f}(t)\,, \tag{8.5}$$

then the matrix $T_n(\mathbf{f})$ becomes skewsymmetric. Symmetric matrices $T_n(\mathbf{f})$ are obtained if $\mathbf{f}(t)$ satisfies

$$\mathbf{f}(t^{-1}) = -\mathbf{f}(t)\,. \tag{8.6}$$

In the case $\mathbb{F} = \mathbb{C}$ the matrices $T_n(\mathbf{f})$ are Hermitian if

$$\mathbf{f}(\,\bar{t}^{\,-1}) = -\overline{\mathbf{f}(t)}\,. \tag{8.7}$$

This is equivalent to saying that $\mathbf{f}(t)$ takes purely imaginary values on the unit circle. We show that the converse is, in a sense, also true.

Proposition 8.5. *If T_n is a nonsingular symmetric, skewsymmetric or Hermitian Toeplitz matrix, then there exists a generating function $\mathbf{f}(t)$ for T_n of degree n that satisfies the conditions (8.6), (8.5), (8.7), respectively.*

Proof. If T_n is symmetric, then the fundamental system consists of a symmetric vector $\mathbf{u}$ and a skewsymmetric vector $\mathbf{v}$. If the last component of $\mathbf{u}$ does not vanish, then we can choose $\mathbf{w} = \mathbf{u}$ to satisfy the conditions of Theorem 8.3. Further, we obtain $a_{-n} = a_n$, thus T_{n+1} is symmetric, and the choice $T^-_{n+1} = (T^+_{n+1})^T$ is possible. Hence we have

$$\mathbf{p} = T^+_{n+1}\mathbf{u} = -(T^+_{n+1})^T\mathbf{u} = -T^+_{n+1}\mathbf{u}^J = -\mathbf{p}^J\,.$$

That means that $\mathbf{p}$ is skewsymmetric, which implies that $\mathbf{f}(t) = \frac{\mathbf{p}(t)}{\mathbf{u}(t)}$ satisfies (8.6).

If the last component of $\mathbf{u}$ vanishes, then the last component of $\mathbf{v}$ must be nonzero and we can choose $\mathbf{w} = \mathbf{v}$. Again we obtain $a_{-n} = a_n$, so that T_{n+1} is symmetric. With the choice $T^-_{n+1} = (T^+_{n+1})^T$ we have

$$\mathbf{p} = T^+_{n+1}\mathbf{v} = -(T^+_{n+1})^T\mathbf{v} = -T^+_{n+1}\mathbf{v}^J = \mathbf{p}^J\,.$$

Thus, $\mathbf{p}$ is symmetric and $\mathbf{f}(t) = \frac{\mathbf{p}(t)}{\mathbf{v}(t)}$ satisfies (8.6). The proof of the other cases is analogous. We have to take into account that a fundamental system of a nonsingular skewsymmetric Toeplitz matrix consists of two symmetric vectors and the fundamental system of a nonsingular Hermitian Toeplitz matrix of two conjugate-symmetric vectors. ■

To discuss this proposition we consider the Examples 8.1, 8.4. The Toeplitz matrix $T_n = [\,c^{i-j}\,]_{i,j=1}^n$ in Example 8.1 is Hermitian if c is on the unit circle, but its generating function $\frac{1}{1-ct}$ does not satisfy (8.7). However, since $\frac{1}{1-ct} = \frac{1}{2}\left(\frac{1+ct}{1-ct} + 1\right)$ we have also

$$T_n = T_n\left(\frac{1}{2}\,\frac{1+ct}{1-ct}\right)\,,$$

and this generating function satisfies (8.7). Generating functions for the identity matrix I_n satisfying (8.6) and so reflecting its symmetry are

$$\mathbf{f}(t) = \frac{1}{2}\,\frac{1-t^n}{1+t^n} \quad \text{and} \quad \mathbf{f}(t) = \frac{1}{2}\,\frac{1+t^n}{1-t^n}\,.$$

3. Vandermonde factorization of nonsingular Toeplitz matrices. Let T_n be a nonsingular $n \times n$ Toeplitz matrix with complex entries and $\mathbf{f}(t) = \frac{\mathbf{p}(t)}{\mathbf{w}(t)}$ be a generating function of degree n with $\mathbf{f}(\infty) = 0$. According to Theorem 8.3 and Lemma 7.7 such a function exists and, due to the freedom in the choice of $\mathbf{w}(t)$ we can assume that $\mathbf{w}(t)$ has only simple roots $t_1, \dots, t_n$. Using the partial fraction decomposition of $\mathbf{f}(t)$ in the form

$$\mathbf{f}(t) = \sum_{i=1}^{n} -\frac{1}{t_i}\frac{\gamma_i}{1 - t_i^{-1}t}$$

as well as (8.3) we can conclude, in analogy to the Hankel case, the following.

Proposition 8.6. *Let T_n be a nonsingular $n \times n$ Toeplitz matrix, $\{\mathbf{u}, \mathbf{v}\}$ a fundamental system of T_n and $\alpha, \beta \in \mathbb{C}$ such that $\mathbf{w}(t) = \alpha\mathbf{u}(t) + \beta\mathbf{v}(t)$ is of degree n, satisfies $\mathbf{w}(0) \neq 0$, and has simple roots $t_1, \dots, t_n$. Then T_n admits a representation*

$$T_n = V_n(\mathbf{t}^{-1})^T D V_n(\mathbf{t}) \tag{8.8}$$

where $\mathbf{t} = (t_i)_{i=1}^n$ and $\mathbf{t}^{-1} = (t_i^{-1})_{i=1}^n$, and D is diagonal, $D = \operatorname{diag}\left(-t_k^{-1}\gamma_k\right)_{k=1}^n$.

The diagonal matrix can be expressed in terms of the generating function. If we set $D = \operatorname{diag}(\delta_i)_{i=1}^n$, then

$$\delta_i = -\frac{1}{t_i}\frac{\mathbf{p}(t_i)}{\mathbf{w}'(t_i)} = -\frac{1}{t_i}\left(\left(\frac{1}{\mathbf{f}}\right)'(t_i)\right)^{-1} . \tag{8.9}$$

Note that, like for Hankel matrices, the Vandermonde factorization for a nonsingular Toeplitz matrix $T_n(\mathbf{f})$ extends to all Toeplitz matrices $T_k(\mathbf{f})$ with $k \geq n$ as

$$T_k(\mathbf{f}) = V_k(\mathbf{t}^{-1})^T D V_k(\mathbf{t}) . \tag{8.10}$$

4. Hermitian Toeplitz matrices. Let the assumptions of Proposition 8.6 be satisfied. We consider the special case of an Hermitian Toeplitz matrix T_n. In this case T_n has a fundamental system consisting of two conjugate-symmetric vectors. If we choose α and β as reals, then the vector $\mathbf{w}$ in Proposition 8.6 is also conjugate-symmetric. For a polynomial with a conjugate-symmetric coefficient vector the roots are symmetric with respect to the unit circle $\mathbb{T}$. That means if t_0 is a root, then $\overline{t_0}^{-1}$ is also a root. In particular, the roots not on $\mathbb{T}$ appear in pairs that are symmetric with respect to $\mathbb{T}$.

Let $t_1, \dots, t_r$ be the roots of $\mathbf{w}(t)$ on $\mathbb{T}$ and $t_{r+1}, t_{r+2} = \overline{t}_{r+1}^{-1}, \dots, t_{n-1}, t_n = \overline{t}_{n-1}^{-1}$ be the roots of $\mathbf{w}(t)$ which are not on $\mathbb{T}$. Note that for the coefficients γ_i in the partial fraction decomposition of $\mathbf{f}(t)$, which are the residuals at t_i, we have $\delta_{r+2} = \overline{\delta}_{r+1}, \dots, \delta_n = \overline{\delta}_{n-1}$. Furthermore,

$$V_n(\mathbf{t}^{-1})^T = V_n(\mathbf{t})^* \operatorname{diag}(I_r, \underbrace{J_2, \dots, J_2}_{l}) ,$$

where $r + 2l = n$. Now Proposition 8.6 leads to the following.

Corollary 8.7. *If the Toeplitz matrix in Proposition* 8.6 *is Hermitian, then it admits a representation*

$$T_n = V_n(\mathbf{t})^* D_1 V_n(\mathbf{t}) ,$$

where

$$D_1 = \operatorname{diag}\left(\delta_1, \dots, \delta_r, \begin{bmatrix} 0 & \overline{\delta}_{r+1} \\ \delta_{r+1} & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \overline{\delta}_{n-1} \\ \delta_{n-1} & 0 \end{bmatrix}\right)$$

In particular, the matrices T_n and D_1 are congruent.

5. Signature and Cauchy index. Corollary 8.7 allows us to express the signature of T_n in terms of the signs of the diagonal elements δ_i $(i = 1, \dots, r)$ of D_1 (cf. Corollary 7.12). Our aim is now to express it in terms of the Cauchy index.

Let $\mathbf{f}(t)$ be a rational function of degree n satisfying (8.7). Then the function $\frac{1}{\mathrm{i}}\,\mathbf{f}(t)$ takes real values on the unit circle, thus the Cauchy index (see Section 7,4) $\mathrm{ind}_{\mathbb{T}}\frac{1}{\mathrm{i}}\,\mathbf{f}(t)$ is well defined. It is the difference of the number of poles on $\mathbb{T}$ of positive type and the number of poles of negative type. Let $t_j = e^{\mathrm{i}\theta_j}$ be a pole on $\mathbb{T}$. Then t_j is of positive (negative) type if and only if the (real-valued) function

$$\varphi(\theta) = \frac{\mathrm{i}}{\mathbf{f}(e^{\mathrm{i}\theta})}$$

is increasing (decreasing) in a neighborhood of θ_j. For a simple pole this is equivalent to $\varphi'(\theta_j) > 0$ $(\varphi'(\theta_j) < 0)$. We have

$$\varphi'(\theta_j) = -t_j \left(\frac{1}{\mathbf{f}}\right)'(t_j)\,.$$

Comparing this with (8.9) we conclude that $\varphi(\theta_j) = \delta_j^{-1}$. We arrive at the following statement for the case of simple poles. It can be generalized to multiple poles by using the continuity arguments from the proof of Proposition 7.13.

Proposition 8.8. *Let $\mathbf{f}(t)$ be a proper rational function with degree n that takes imaginary values on the unit circle. Then*

$$\mathrm{sgn}\, T_n(\mathbf{f}) = \mathrm{ind}_{\mathbb{T}}\,\frac{1}{\mathrm{i}}\mathbf{f}(t)\,.$$

We also obtain a criterion of positive definiteness.

Corollary 8.9. *Let $\mathbf{f}(t)$ be a proper rational function of degree n, and let*

$$\mathbf{f}(t) = \frac{\mathrm{i}\mathbf{p}(t)}{\mathbf{u}(t)}$$

be its reduced representation in which $\mathbf{u}$ and $\mathbf{p}$ are conjugate-symmetric. Then $T_n(\mathbf{f})$ is positive definite if and only if the polynomials $\mathbf{u}(t)$ and $\mathbf{p}(t)$ have only roots on the unit circle, these roots are simple and interlaced.

6. Congruence to T-Bezoutians. The Toeplitz analogue of Proposition 7.15 is as follows. (Concerning the order of $\mathrm{Bez}_T(\mathbf{u},\mathbf{v})$ and of $B_\pm(\mathbf{u})$ compare the remarks before Proposition 7.15.)

Proposition 8.10. *Let $\mathbf{u},\mathbf{v} \in \mathbb{F}^{n+1}$, where $\mathbf{u}$ has nonvanishing first and last components. Then, for $k \geq n$, the T-Bezoutian of $\mathbf{u}(t)$ and $\mathbf{v}(t)$ is related to the $k\times k$ Toeplitz matrix $T_k(\mathbf{f})$ generated by $\mathbf{f}(t) = \frac{\mathbf{v}(t)}{\mathbf{u}(t)}$ via*

$$\mathrm{Bez}_T\,(\mathbf{u},\mathbf{v}) = B_-(\mathbf{u})\,T_k(\mathbf{f})\,B_+(\mathbf{u})\;,$$

where $B_\pm(\mathbf{u})$ are introduced in (2.14).

Proof. For $B = \mathrm{Bez}_T(\mathbf{u},\mathbf{v})$ we have

$$B(t,s) = \mathbf{u}(t)\,\frac{\mathbf{f}(t) - \mathbf{f}(s^{-1})}{1-ts}\,(-\mathbf{u}^J(s))\,.$$

Applying (8.2) we obtain

$$B = \begin{bmatrix} u_1 & & & \\ \vdots & \ddots & & O \\ u_n & \dots & u_1 & \end{bmatrix} T \begin{bmatrix} -u_{n+1} & \dots & -u_2 \\ & \ddots & \vdots \\ & & -u_{n+1} \\ & O & \end{bmatrix},$$

where $\mathbf{u} = (u_i)_{i=1}^{n+1}$ and $T = [\, a_{i-j}\,]_{i,j=1}^{\infty}$. Hence $B = B_-(\mathbf{u})T_k(\mathbf{f})B_+(\mathbf{u})$. ■

Corollary 8.11. *If $T_n\left(\frac{\mathbf{v}}{\mathbf{u}}\right)$ is Hermitian and if* $\mathbf{u}$ *is a conjugate-symmetric (or conjugate-skew-symmetric) polynomial of degree $n+1$ then* $\mathrm{Bez}_T(\mathbf{v},\mathbf{u})$ *(or* $\mathrm{Bez}_T(\mathbf{u},\mathbf{v})$*) and* $T_n(\mathbf{f})$ *are congruent.*

Corollary 8.12. *For $k \ge n$ the rank of $T_k(\mathbf{f})$ is equal to the degree of $\mathbf{f}(t)$. In particular, if $\mathbf{f}(t)$ has degree n, then $T_n(\mathbf{f})$ is nonsingular.*

If we combine Proposition 8.10 with Barnett's formula in Theorem 6.3, then we obtain the representation

$$T_n\left(\frac{\mathbf{v}}{\mathbf{u}}\right) = \mathbf{v}^J(C(\mathbf{u}^J))B_+(\mathbf{u})^{-1}\,, \tag{8.11}$$

where we assume that $\mathbf{u}(t)$ is comonic.

7. Inverses of T-Bezoutians. Now we show that relation (8.11) leads to an inversion formula for T-Bezoutians.

Theorem 8.13. *Let $\mathbf{u},\mathbf{v} \in \mathbb{F}^{n+1}$ be such that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are coprime and the first and last components of $\mathbf{u}$ do not vanish. If $(\mathbf{q}(t),\mathbf{p}(t))$, $\mathbf{q},\mathbf{p} \in \mathbb{F}^{n+1}$, is the solution of the Diophantine equation*

$$\mathbf{u}(t)\mathbf{q}(t) + \mathbf{v}(t)\mathbf{p}(t) = t^n\,, \tag{8.12}$$

then $\mathrm{Bez}_T(\mathbf{u},\mathbf{v})$ *is invertible and the inverse is given by*

$$\mathrm{Bez}_T(\mathbf{u},\mathbf{v})^{-1} = T_n\left(\frac{\mathbf{p}}{\mathbf{u}}\right)\,.$$

Proof. First we note that (8.12) is equivalent to

$$\mathbf{u}^J(t)\mathbf{q}^J(t) + \mathbf{v}^J(t)\mathbf{p}^J(t) = t^n\,.$$

Thus, $\mathbf{v}^J(t)\mathbf{p}^J(t) \equiv t^n$ modulo $\mathbf{u}^J(t)$. From Theorem 6.3, the representation (8.11), and the Cayley-Hamilton theorem, now we obtain

$$\begin{aligned} \mathrm{Bez}_T(\mathbf{u},\mathbf{v})T_n\left(\frac{\mathbf{p}}{\mathbf{u}}\right) &= B_-(\mathbf{u})\mathbf{v}^J(C(\mathbf{u}^J))p^J(C(\mathbf{u}^J))B_+(\mathbf{u})^{-1} \\ &= B_-(\mathbf{u})C(\mathbf{u}^J)^n B_+(\mathbf{u})^{-1}\,. \end{aligned}$$

Taking (6.7) into account this leads to

$$\mathrm{Bez}_T\,(\mathbf{u},\mathbf{v})T_n\left(\frac{\mathbf{p}}{\mathbf{u}}\right) = B_+(\mathbf{u})B_+(\mathbf{u})^{-1} = I_n\,,$$

and the theorem is proved. ■

8. Relations between Toeplitz and Hankel matrices. We know that if T_k is a $k \times k$ Toeplitz matrix, then J_kT_k is Hankel, and vice versa. We show how the generating functions are related.

Proposition 8.14. *Let* $\mathbf{u}, \mathbf{q} \in \mathbb{F}^{n+1}$, *where* $\mathbf{u}$ *has nonvanishing first and last components, and let* $\frac{\mathbf{q}(t)}{\mathbf{u}(t)}$ *be a generating function of* T_k. *For* $k \geq n$, *let* $\mathbf{p} \in \mathbb{F}^n$ *be such that* $-\mathbf{p}(t) \in \mathbb{F}^n(t)$ *is the remainder polynomial of* $t^k \mathbf{q}^J(t)$ *divided by* $\mathbf{u}^J(t)$. *Then*

$$J_k T_k \left(\frac{\mathbf{q}}{\mathbf{u}}\right) = H_k \left(\frac{\mathbf{p}}{\mathbf{u}}\right) .$$

Proof. According to the definition of $\mathbf{p}$ we have

$$t^k \mathbf{q}^J(t) = -\mathbf{p}^J(t) + \mathbf{r}(t)\mathbf{u}^J(t)$$

for some $\mathbf{r}(t) \in \mathbb{F}^{k+1}(t)$. This is equivalent to $\mathbf{q}(t) = -t^k \mathbf{p}(t) + t^k \mathbf{r}(t^{-1})\mathbf{u}(t)$, and we obtain

$$\frac{\mathbf{q}(t)}{\mathbf{u}(t)} = -t^k \frac{\mathbf{p}(t)}{\mathbf{u}(t)} + t^k \mathbf{r}(t^{-1}).$$

Thus

$$t^k \mathbf{r}(t^{-1}) = a_0^+ + a_1 t + \cdots + a_k t^k .$$

On the other hand,

$$\frac{\mathbf{p}(t)}{\mathbf{u}(t)} = \mathbf{r}(t^{-1}) - t^{-k} \frac{\mathbf{q}(t)}{\mathbf{u}(t)} .$$

Consequently,

$$\frac{\mathbf{p}(t)}{\mathbf{u}(t)} = a_k + a_{k-1} t^{-1} + \cdots + a_0 t^{-k} a_{-1} t^{-k-1} + \dots ,$$

where $a_0 = a_0^+ + a_0^-$. This means that

$$H_k = H_k \left(\frac{\mathbf{p}}{\mathbf{u}}\right) = \begin{bmatrix} a_{k-1} & a_{k-2} & \cdots & a_0 \\ a_{k-2} & & \iddots & \\ \vdots & \iddots & & \vdots \\ a_0 & \cdots & & a_{1-k} \end{bmatrix} .$$

Thus $H_k = J_k T_k \left(\frac{\mathbf{q}}{\mathbf{u}}\right)$. ■

9. Vandermonde Reduction of Bezoutians

In this section the underlying field is the field of complex numbers, $\mathbb{F} = \mathbb{C}$. Some of the results can be extended to general algebraically closed fields.

In Sections 7 and 8 we showed that nonsingular Hankel and Toeplitz matrices can be represented as a product of the transpose of a Vandermonde matrix V_n^T, a diagonal matrix, and the Vandermonde matrix V_n, and we called this Vandermonde factorization. Since inverses of Hankel and Toeplitz matrices are H- and T-Bezoutians, respectively, this is equivalent to the fact that nonsingular Bezoutians can be reduced to diagonal form by multiplying them by V_n from the left and by V_n^T from the right. We call this kind of factorization *Vandermonde reduction of Bezoutians.* In this section we give a direct derivation of Vandermonde reduction of Bezoutians and generalize it to general, not necessarily nonsingular Bezoutians.

Hereafter, the notations *confluent* and *non-confluent* are used in connection with confluent and non-confluent Vandermonde matrices, respectively.

1. Non-confluent Hankel case. To begin with, let us recall that, for $\mathbf{t} = (t_i)_{i=1}^n$, the (non-confluent) Vandermonde matrix $V_m(\mathbf{t})$ is defined by

$$V_m(\mathbf{t}) = \begin{bmatrix} 1 & t_1 & \dots & t_1^{m-1} \\ 1 & t_2 & \dots & t_2^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & t_n & \dots & t_n^{m-1} \end{bmatrix} .$$

If $m = n$ and the t_i are distinct, then $V_n(\mathbf{t})$ is nonsingular.

Obviously, for $\mathbf{x} \in \mathbb{C}^m$, $V_m(\mathbf{t})\mathbf{x} = (\mathbf{x}(t_i))_{i=1}^n$. Furthermore, for an $n \times n$ matrix B and $\mathbf{s} = (s_j)_{j=1}^n$,

$$V_n(\mathbf{t}) B V_n(\mathbf{s})^T = [\, B(t_i, s_j)\,]_{i,j=1}^n \,. \tag{9.1}$$

We specify this for $\mathbf{t} = \mathbf{s}$ with pairwise distinct components and an H-Bezoutian $B = \mathrm{Bez}_H(\mathbf{u}, \mathbf{v})$, where $\mathbf{u}(t), \mathbf{v}(t) \in \mathbb{C}^{n+1}(t)$ and $\mathbf{u}(t)$ has degree n. In this case the off diagonal entries c_{ij} of $C = V_n(\mathbf{t}) B V_n(\mathbf{t})^T$ are given by

$$c_{ij} = \frac{\mathbf{u}(t_i)\mathbf{v}(t_j) - \mathbf{v}(t_i)\mathbf{u}(t_j)}{t_i - t_j} \,. \tag{9.2}$$

Our aim is to find $\mathbf{t}$ such that C is diagonal. One possibility is to choose the zeros of $\mathbf{u}(t)$. This is possible if $\mathbf{u}(t)$ has only simple zeros. A more general case is presented next.

Proposition 9.1. *Let $\alpha \in \mathbb{C}$ be such that $\mathbf{w}(t) = \mathbf{u}(t) - \alpha \mathbf{v}(t)$ has simple roots $t_1, \dots, t_n$, and let $\mathbf{t} = (t_i)_{i=1}^n$. Then*

$$V_n(\mathbf{t}) \mathrm{Bez}_H(\mathbf{u}, \mathbf{v}) V_n(\mathbf{t})^T = \mathrm{diag}(\gamma_i)_{i=1}^n \,, \tag{9.3}$$

where

$$\gamma_i = \mathbf{u}'(t_i)\mathbf{v}(t_i) - \mathbf{u}(t_i)\mathbf{v}'(t_i) \,. \tag{9.4}$$

Proof. According to Lemma 2.2 we have $\mathrm{Bez}_H(\mathbf{u}, \mathbf{v}) = \mathrm{Bez}_H(\mathbf{w}, \mathbf{v})$. From (9.2) we see that the off diagonal elements of the matrix $V_n(\mathbf{t}) \mathrm{Bez}_H(\mathbf{u}, \mathbf{v}) V_n(\mathbf{t})^T$ vanish. The expression (9.4) follows from (9.2), $\gamma_i = \lim\limits_{t \to t_i} \frac{\mathbf{u}(t)\mathbf{v}(t_i) - \mathbf{v}(t)\mathbf{u}(t_i)}{t - t_i}$. ■

Remark 9.2. If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are coprime, which is the same as saying that $\mathrm{Bez}_H(\mathbf{u}, \mathbf{v})$ is nonsingular, then according to Lemma 7.7 for almost all values of α the polynomial $\mathbf{w}(t)$ has simple roots.

Remarkably the special case $\mathbf{v}(t) = 1$ of (9.3) leads to a conclusion concerning the inverse of a Vandermonde matrix.

Corollary 9.3. *Let $\mathbf{t} = (t_i)_{i=1}^n$ have pairwise distinct components, and let $\mathbf{u}(t)$ be defined by $\mathbf{u}(t) = \prod_{i=1}^n (t - t_i)$. Then the inverse of $V_n(\mathbf{t})$ is given by*

$$V_n(\mathbf{t})^{-1} = B(\mathbf{u}) V_n(\mathbf{t})^T \mathrm{diag} \left(\frac{1}{\mathbf{u}'(t_i)} \right)_{i=1}^n , \tag{9.5}$$

where $B(\mathbf{u})$ is the upper triangular Hankel matrix introduced in (2.2).

Now we consider the H-Bezoutian of real polynomials $\mathbf{u}(t)$ and $\mathbf{v}(t)$, which is an Hermitian matrix. Similarly to Corollary 7.11, Proposition 9.1 leads to a matrix congruence.

Corollary 9.4. *Let in Proposition* 9.1 *the polynomials* $\mathbf{u}(t)$ *and* $\mathbf{v}(t)$ *be real and* $\alpha \in \mathbb{R}$. *Furthermore, let* $t_1, \ldots, t_r$ *be the (simple) real and* $(t_{r+1}, \overline{t}_{r+1}), \ldots, (t_{n-1}, \overline{t}_{n-1})$ *the (simple) non-real roots of* $\mathbf{w}(t)$. *Then*

$$V_n(\mathbf{t})\mathrm{Bez}_H(\mathbf{u},\mathbf{v})V_n(\mathbf{t})^* = \mathrm{diag}\left(\gamma_1, \ldots, \gamma_r, \begin{bmatrix} 0 & \overline{\gamma}_{r+1} \\ \gamma_{r+1} & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \overline{\gamma}_{n-1} \\ \gamma_{n-1} & 0 \end{bmatrix}\right).$$

In particular,

$$\mathrm{sgn}\,\mathrm{Bez}_H(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{r} \mathrm{sgn}\,\gamma_i\,.$$

Note that it follows from the Hermitian symmetry of the matrix that the numbers $\gamma_1, \ldots, \gamma_r$ are real.

2. Non-confluent Toeplitz case. Let $\mathbf{t} = (t_i)_{i=1}^n$ have distinct nonzero components, $\mathbf{t}^{-1} = \left(\frac{1}{t_i}\right)_{i=1}^n$. If we specify (9.1) for a T-Bezoutian $\mathrm{Bez}_T(\mathbf{u},\mathbf{v})$, then we obtain for the off diagonal entries c_{ij} of $C = V_n(\mathbf{t})\mathrm{Bez}_T(\mathbf{u},\mathbf{v})V_n(\mathbf{t}^{-1})^T$ the relation

$$c_{ij} = -\frac{\mathbf{u}(t_i)\mathbf{v}(t_j) - \mathbf{v}(t_i)\mathbf{u}(t_j)}{t_i - t_j}\, t_j^{1-n}\,.$$

For the diagonal entries we obtain using l'Hospital's rule

$$c_{ii} = (\mathbf{v}'(t_i)\mathbf{u}(t_i) - \mathbf{u}'(t_i)\mathbf{v}(t_i))t_i^{1-n}\,.$$

In the same way as in Proposition 9.1 we derive the following.

Proposition 9.5. *Let* $\alpha \in \mathbb{C}$ *be such that* $\mathbf{w}(t) = \mathbf{u}(t) - \alpha\mathbf{v}(t)$ *has simple nonzero roots* $t_1, \ldots, t_n$, *and let* $\mathbf{t} = (t_i)_{i=1}^n$. *Then*

$$V_n(\mathbf{t})\mathrm{Bez}_T(\mathbf{u},\mathbf{v})V_n(\mathbf{t}^{-1})^T = \mathrm{diag}(\gamma_i)_{i=1}^n\,,$$

where

$$\gamma_i = (\mathbf{v}'(t_i)\mathbf{u}(t_i) - \mathbf{u}'(t_i)\mathbf{v}(t_i))t_i^{1-n}\,.$$

Let $\mathbf{u}$ be conjugate-symmetric, $\mathbf{v}$ conjugate-skewsymmetric. Then the matrix $\mathrm{Bez}_T(\mathbf{u},\mathbf{v})$ is Hermitian. For purely imaginary α, the roots of $\mathbf{w}(t) = \mathbf{u}(t) - \alpha\mathbf{v}(t)$ are located symmetrically with respect to the unit circle $\mathbb{T}$.

Corollary 9.6. *Let in Proposition* 9.5 *the polynomial* $\mathbf{u}$ *be conjugate-symmetric and* $\mathbf{v}(t)$ *be conjugate-skewsymmetric, and let* $\alpha \in \mathrm{i}\mathbb{R}$. *Furthermore, let* $t_1, \ldots, t_r$ *be the (simple) roots of* $\mathbf{w}(t)$ *on* $\mathbb{T}$ *and* $t_{r+1}, t_{r+2} = \overline{t}_{r+1}^{-1}, \ldots, t_{n-1}, t_n = \overline{t}_{n-1}^{-1}$ *the (simple) roots of* $\mathbf{w}(t)$ *outside* $\mathbb{T}$. *Then*

$$V_n(\mathbf{t})\mathrm{Bez}_T(\mathbf{u},\mathbf{v})V_n(\mathbf{t})^* = \mathrm{diag}\left(\gamma_1, \ldots, \gamma_r, \begin{bmatrix} 0 & \overline{\gamma}_{r+1} \\ \gamma_{r+1} & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \overline{\gamma}_{n-1} \\ \gamma_{n-1} & 0 \end{bmatrix}\right).$$

In particular,

$$\mathrm{sgn}\,\mathrm{Bez}_T(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{r} \mathrm{sgn}\,\gamma_i\,.$$

3. Confluent case. Here we need the following generalization of a Vandermonde matrix. Let $\mathbf{t} = (t_i)_{i=1}^m$ and $\mathbf{r} = (r_i)_{i=1}^m \in \mathbb{N}^m$. We denote by $V_n(t_i, r_i)$ the $r_i \times n$ matrix

$$V_n(t_i, r_i) = \begin{bmatrix} 1 & t_i & t_i^2 & & \dots & t_i^{n-1} \\ 0 & 1 & 2t_i & & \dots & (n-1)t_i^{n-2} \\ \vdots & & \ddots & & & \vdots \\ 0 & \dots & 0 & 1 & \dots & \binom{n-1}{r_i-1} t_i^{n-r_i} \end{bmatrix}$$

and introduce the matrix

$$V_n(\mathbf{t}, \mathbf{r}) = \begin{bmatrix} V_n(t_1, r_1) \\ \vdots \\ V_n(t_m, r_m) \end{bmatrix},$$

which is called *confluent Vandermonde matrix.*

Now we show that in case that $\mathbf{u}(t)$ has multiple roots Bezoutians can be reduced to block diagonal form with the help of confluent Vandermonde matrices $V_n(\mathbf{t}, \mathbf{r})$. We restrict ourselves to the case of H-Bezoutians. The case of T-Bezoutians is analogous.

First we consider the special single node case $\mathbf{t} = 0$, $\mathbf{r} = r$. Suppose that $\mathbf{u} = (u_i)_{i=1}^{n+1}$ and $\mathbf{u}(t)$ has the root $t = 0$ with multiplicity r, i.e. $u_1 = u_2 = \cdots = u_r = 0$. Obviously, $V_n(0, r) = \begin{bmatrix} I_r & O \end{bmatrix}$. Hence $V_n(0, r) B V_n(0, r)^T$ is the $r \times r$ leading principal submatrix of $B = \mathrm{Bez}_H(\mathbf{u}, \mathbf{v})$. We denote this matrix by $\Gamma(0)$ and observe, taking Theorem 3.2 into account, that

$$\Gamma(0) = \begin{bmatrix} v_1 & & \\ \vdots & \ddots & \\ v_r & \dots & v_1 \end{bmatrix} \begin{bmatrix} & & u_{r+1} \\ & \cdot^{\cdot^{\cdot}} & \vdots \\ u_{r+1} & \dots & u_{2r+1} \end{bmatrix} = \begin{bmatrix} & & w_1 \\ & \cdot^{\cdot^{\cdot}} & \vdots \\ w_1 & \dots & w_r \end{bmatrix},$$

where

$$\sum_{i=1}^{2n-r+1} w_i t^{i-1} = \mathbf{u}(t)\mathbf{v}(t)t^{-r}.$$

Taylor expansion gives us

$$u_i = \frac{1}{(i-1)!}\, \mathbf{u}^{(i-1)}(0)$$

and an analogous expression for v_i.

Suppose now that t_0 is a root of $\mathbf{u}(t)$ with multiplicity r. We consider the polynomials $\widetilde{\mathbf{u}}(t) = \mathbf{u}(t - t_0)$ and $\widetilde{\mathbf{v}}(t) = \mathbf{v}(t - t_0)$ and $\widetilde{B} = [\,\widetilde{b}_{ij}\,]_{i,j=1}^n = \mathrm{Bez}_H(\widetilde{\mathbf{u}}, \widetilde{\mathbf{v}})$. Then, for $k = 0, 1, \dots,$

$$\widetilde{\mathbf{u}}^{(k)}(0) = \mathbf{u}^{(k)}(t_0), \quad \widetilde{\mathbf{v}}^{(k)}(0) = \mathbf{v}^{(k)}(t_0)$$

and

$$\begin{aligned} \widetilde{b}_{ij} &= \frac{1}{(i-1)!(j-1)!}\, \frac{\partial^{i+j-2}}{\partial t^{i-1} \partial s^{j-1}}\, \widetilde{B}(t, s)\,|_{(0,0)} \\ &= \frac{1}{(i-1)!(j-1)!}\, \frac{\partial^{i+j-2}}{\partial t^{i-1} \partial s^{j-1}}\, B(t, s)\,|_{(t_0,t_0)}. \end{aligned}$$

Hence

$$\Gamma(t_0) = \begin{bmatrix} \widetilde{v}_1 & & \\ \vdots & \ddots & \\ \widetilde{v}_r & \dots & \widetilde{v}_1 \end{bmatrix} \begin{bmatrix} & & \widetilde{u}_{r+1} \\ & \cdot^{\cdot^{\cdot}} & \vdots \\ \widetilde{u}_{r+1} & \dots & \widetilde{u}_{2r+1} \end{bmatrix} = \begin{bmatrix} & & w_1 \\ & \cdot^{\cdot^{\cdot}} & \vdots \\ w_1 & \dots & w_r \end{bmatrix},$$

where $\widetilde{u}_i = \dfrac{1}{(i-1)!}\,\mathbf{u}^{(i-1)}(t_0)$, analogously for $\widetilde{v}_i$, and $\sum_{i=1}^{2n-r+1} w_i(t-t_0)^{i-1} = \mathbf{u}(t)\mathbf{v}(t)(t-t_0)^{-r}$. We arrive at the following.

Proposition 9.7. *Let $t_1,\dots,t_m$ be the (different) roots of $\mathbf{u}(t)$ and $r_1,\dots,r_m$ the corresponding multiplicities, $\mathbf{t}=(t_i)_{i=1}^m$ and $\mathbf{r}=(r_i)_{i=1}^m$. Then*

$$V_n(\mathbf{t},\mathbf{r})\mathrm{Bez}_H(\mathbf{u},\mathbf{v})V_n(\mathbf{t},\mathbf{r})^T = \mathrm{diag}(\Gamma_i)_{i=1}^m\,,$$

where

$$\Gamma_i = \begin{bmatrix} & & w_{i1} \\ & \iddots & \vdots \\ w_{i1} & \dots & w_{ir_i} \end{bmatrix} \quad \textit{and} \quad \sum_{j=1}^{2n-r_i+1} w_{ij}(t-t_i)^{j-1} = \mathbf{u}(t)\mathbf{v}(t)(t-t_i)^{-r_i}\,.$$

The case $\mathbf{v}(t)=1$ provides a formula for the inverse of confluent Vandermonde matrices.

Corollary 9.8. *Let $\mathbf{t}=(t_1,\dots,t_q)$ have distinct components, and let*

$$\mathbf{u}(t) = \prod_{i=1}^q (t-t_i)^{r_i},$$

where $n = r_1+\dots+r_q$. Then the inverse of the confluent Vandermonde matrix $V_n(\mathbf{t},\mathbf{r})$, $\mathbf{r} = (r,\dots,r_q)$, is given by

$$V_n(\mathbf{t},\mathbf{r})^{-1} = B(\mathbf{u})V_n(\mathbf{t},\mathbf{r})^T\mathrm{diag}\,(\Gamma_i^{-1})_{i=1}^m\,,$$

where $\Gamma_i = \begin{bmatrix} & & w_{i1} \\ & \iddots & \vdots \\ w_{i1} & \dots & w_{ir_i} \end{bmatrix}$ *with* $w_{ij} = \frac{\mathbf{u}^{(j+r_i)}(t_i)}{(j+r_i-1)!}$.

We leave it to the reader to state the analogous properties of T-Bezoutians.

10. Root Localization Problems

In this section we show the importance of Bezoutians, Hankel and Toeplitz matrices for root localization problems. Throughout the section, let $\mathbb{F}=\mathbb{C}$.

1. Inertia of polynomials. Let $\mathcal{C}$ be a simple oriented closed curve in the extended complex plane $\mathbb{C}^\infty = \mathbb{C}\cup\{\infty\}$ dividing it into an "interior" part Ω_+ and an "exterior" part Ω_-. We assume that the domain Ω_+ is situated left from $\mathcal{C}$ if a point moves along $\mathcal{C}$ in positive direction. The *inertia of the polynomial* $\mathbf{u}(t)\in\mathbb{C}^{n+1}(t)$ *with respect to* $\mathcal{C}$ is, by definition, a triple of nonnegative integers

$$\mathrm{in}_{\mathcal{C}}(\mathbf{u}) = (\pi_+(\mathbf{u}),\pi_-(\mathbf{u}),\pi_0(\mathbf{u}))\,,$$

where $\pi_\pm(\mathbf{u})$ is the number of zeros of $\mathbf{u}(t)$ in $\Omega_\pm$, respectively, and $\pi_0(\mathbf{u})$ is the number of zeros on $\mathcal{C}$. In all cases multiplicities are counted. We say that $\mathbf{u}(t)\in\mathbb{C}^{n+1}(t)$ has a *zero at* ∞ *with multiplicity* r if the r leading coefficients of $\mathbf{u}(t)$ are zero. By a *root localization problem* we mean the problem to find the inertia of a given polynomial with respect to a curve $\mathcal{C}$.

In the sequel we deal only with the cases that $\mathcal{C}$ is the real line $\mathbb{R}$, the imaginary line $\mathrm{i}\,\mathbb{R}$, or the unit circle $\mathbb{T}$. We relate the inertia of polynomials to inertias of Hermitian matrices, namely to Bezoutians. Recall that the inertia of an Hermitian matrix A is the triple

$$\mathrm{In}\,A = (p_+(A),p_-(A),p_0(A))\,,$$

where $p_+(A)$ is the number of positive, $p_-(A)$ the number of negative eigenvalues (counting multiplicities), and $p_0(A)$ the nullity of A. Clearly, the inertia of A is completely determined by the rank and the signature of A. The importance of the relation consists in the fact that the inertia of Bezoutians can be computed via recursive algorithms for triangular factorization, which were described in Section 5.

2. Inertia with respect to the real line. Let $\mathbf{u}(t)$ be a given monic polynomial of degree n, $\mathbf{q}(t)$ its real and $\mathbf{p}(t)$ its imaginary part, i.e. $\mathbf{u}(t) = \mathbf{q}(t) + \mathrm{i}\,\mathbf{p}(t)$. We consider the matrix

$$B = \frac{1}{2\mathrm{i}} \operatorname{Bez}_H(\mathbf{u}, \overline{\mathbf{u}}). \tag{10.1}$$

For example, if $\mathbf{u}(t) = t - c$, then $B = \frac{1}{2\mathrm{i}}(c - \overline{c}) = \operatorname{Im} c$. Hence c is in the upper half plane if and only if $B > 0$. In general, we have

$$\begin{aligned}(t-s)B(t,s) &= \left[\frac{1}{2\mathrm{i}}(\mathbf{q}(t) + \mathrm{i}\mathbf{p}(t))(\mathbf{q}(s) - \mathrm{i}\mathbf{p}(s)) - (\mathbf{q}(t) - \mathrm{i}\mathbf{p}(t))(\mathbf{q}(s) + \mathrm{i}\mathbf{p}(s))\right] \\ &= \mathbf{p}(t)\mathbf{q}(s) - \mathbf{q}(t)\mathbf{p}(s)\,.\end{aligned}$$

Hence,

$$B = \operatorname{Bez}_H(\mathbf{p}, \mathbf{q})\,. \tag{10.2}$$

In particular, we see that B is a real symmetric matrix. The following is usually referred to as *Hermite's theorem.*

Theorem 10.1. *Let $\mathbf{u}(t)$ be a monic polynomial of degree n,*

$$\operatorname{in}_{\mathbb{R}}(\mathbf{u}) = (\pi_+(\mathbf{u}), \pi_-(\mathbf{u}), \pi_0(\mathbf{u})),$$

and B be defined by (10.1) or (10.2). Then the signature of B is given by

$$\operatorname{sgn} B = \pi_+(\mathbf{u}) - \pi_-(\mathbf{u})\,.$$

In particular, B is positive definite if and only if $\mathbf{u}(t)$ has all its roots in the upper half-plane. Furthermore, if $\mathbf{u}(t)$ and $\overline{\mathbf{u}}(t)$ are coprime, then $\operatorname{In} B = \operatorname{in}_{\mathbb{R}}(\mathbf{u})$.

Proof. Let $\mathbf{d}(t)$ be the greatest common divisor of $\mathbf{u}(t)$ and $\overline{\mathbf{u}}(t)$ and δ its degree. Then $\mathbf{d}(t)$ is also the greatest common divisor of $\mathbf{p}(t)$ and $\mathbf{q}(t)$, and let $\mathbf{u}(t) = \mathbf{d}(t)\mathbf{u}_0(t)$. Then $\overline{\mathbf{u}}(t) = \mathbf{d}(t)\overline{\mathbf{u}}_0(t)$, since $\mathbf{d}(t)$ is real. According to (2.5) we have

$$B = \operatorname{Res}(\mathbf{d}, \mathbf{u}_0)^* \begin{bmatrix} B_0 & O \\ O & O \end{bmatrix} \operatorname{Res}(\mathbf{d}, \mathbf{u}_0)\,,$$

where $B_0 = \frac{1}{2\mathrm{i}} \operatorname{Bez}_H(\mathbf{u}_0, \overline{\mathbf{u}}_0)$. Since $\mathbf{d}(t)$ and $\mathbf{u}_0(t)$ are coprime, $\operatorname{Res}(\mathbf{d}, \mathbf{u}_0)$ is nonsingular. By Sylvester's inertia law we have $\operatorname{sgn} B = \operatorname{sgn} B_0$. We find now $\operatorname{sgn} B_0$.

Let z_1 be a root of $\mathbf{u}_0(t)$ and $\mathbf{u}_0(t) = (t - z_1)\mathbf{u}_1(t)$. Then $\overline{\mathbf{u}}_0(t) = (t - \overline{z}_1)\overline{\mathbf{u}}_1(t)$. Taking into account that

$$\frac{1}{2\mathrm{i}} \frac{(t - z_1)(s - \overline{z}_1) - (t - \overline{z}_1)(s - z_1)}{t - s} = \operatorname{Im} z_1$$

we obtain using (2.5)

$$B_0 = \operatorname{Res}(t - \overline{z}_1, \mathbf{u}_1)^* \begin{bmatrix} B_1 & \mathbf{0} \\ \mathbf{0}^T & \operatorname{Im} z_1 \end{bmatrix} \operatorname{Res}(t - \overline{z}_1, \mathbf{u}_1)\,,$$

where $B_1 = \frac{1}{2\mathrm{i}} \operatorname{Bez}_H(\mathbf{u}_1, \overline{\mathbf{u}}_1)$. Repeating these arguments for the other roots z_k of $\mathbf{u}_0(t)$ ($k = 2, \dots, n - \delta$) we conclude that there is a matrix R such that

$$B_0 = R^* \operatorname{diag}(\operatorname{Im} z_1, \dots, \operatorname{Im} z_{n-\delta})\, R\,.$$

Thus, B_0 is congruent to the diagonal matrix of the $\operatorname{Im} z_i$. Applying Sylvester's inertia law we obtain

$$\operatorname{sgn} B = \operatorname{sgn} B_0 = \sum_{i=1}^{n-\delta} \operatorname{sgn}(\operatorname{Im} z_i) = \pi_+(\mathbf{u}) - \pi_-(\mathbf{u}),$$

which proves the main part of the theorem.

If $\mathbf{u}(t)$ and $\overline{\mathbf{u}}(t)$ are coprime, then B is nonsingular, thus $p_0(B) = 0$, and $\pi_0(\mathbf{u}) = 0$. Hence $\pi_+(\mathbf{u}) + \pi_-(\mathbf{u}) = n$. Consequently $\pi_\pm(\mathbf{u}) = p_\pm(B)$, and the theorem is proved. ■

If $\mathbf{u}(t)$ is a real polynomial, then the Bezoutian B is zero, so all information about the polynomial is lost. It is remarkable that in the other cases information about the polynomial is still contained in B.

Example 10.2. *Let $\mathbf{u}(t) = (t - z_0)\mathbf{d}(t)$, where $\mathbf{d}(t)$ is real and z_0 is in the upper half-plane. Then $B = \operatorname{Im} z_0\, \mathbf{d}\mathbf{d}^*$. This matrix has signature 1 saying that $\pi_+(\mathbf{u}) - \pi_-(\mathbf{u}) = 1$ but not specifying the location of the roots of $\mathbf{d}(t)$. Nevertheless, the polynomial $\mathbf{d}(t)$ and so information about $\mathbf{u}(t)$ can be recovered from B.*

Recall that according to the results in Section 5 the Euclidian algorithm applied to the polynomials $\mathbf{p}(t)$ and $\mathbf{q}(t)$ provides a method to compute the signature of B in $O(n^2)$ operations.

We know from Proposition 7.18 that the Bezoutian $B = \operatorname{Bez}_H(\mathbf{p}, \mathbf{q})$ and the Hankel matrix $H_n = H_n\left(-\frac{\mathbf{p}}{\mathbf{q}}\right)$ have the same inertia. Hence we can conclude the following.

Corollary 10.3. *Let $\mathbf{p}(t)\,,\ \mathbf{q}(t) \in \mathbb{R}^{n+1}(t)$ be two coprime polynomials, where $\mathbf{q}(t)$ is monic with degree n and $\mathbf{u}(t) = \mathbf{q}(t) + \mathrm{i}\mathbf{p}(t)$. Then*

$$\operatorname{in}_{\mathbb{R}}(\mathbf{u}) = \operatorname{In} H_n\left(-\frac{\mathbf{p}}{\mathbf{q}}\right).$$

Theorem 10.1, gives a complete answer to the problem to find the inertia of a polynomial only if $\mathbf{u}(t)$ has no real and conjugate complex roots. In the other cases we have only partial information. More precisely, if δ denotes the number of real roots of $\mathbf{u}(t)$, then

$$\pi_\pm(\mathbf{u}) = p_\pm(B) + \frac{1}{2}(p_0(B) - \delta).$$

Note that $\frac{1}{2}(p_0(B) - \delta)$ is the number of conjugate complex pairs of roots of $\mathbf{u}(t)$. The number δ is also the number of real roots of the greatest common divisor $\mathbf{d}(t)$ of $\mathbf{u}(t)$ and $\overline{\mathbf{u}}(t)$. Clearly, $\mathbf{d}(t)$ is a real polynomial. Thus we are led to the problem to count the number of real roots of a real polynomial. This problem will be discussed next.

3. Real roots of real polynomials. Let $\mathbf{p}(t)$ be a real polynomial of degree n. We consider the rational function $\frac{\mathbf{p}'(t)}{\mathbf{p}(t)}$ and the Hankel matrix $H_n\left(\frac{\mathbf{p}'(t)}{\mathbf{p}(t)}\right)$. This is just our Example 7.2. According to Proposition 7.17 the rank of this matrix is equal to the number of *different* roots of $\mathbf{p}(t)$ and, due to Corollaries 7.11 and 1.2, the signature is the number of *different real* roots of $\mathbf{p}(t)$. Let $\pi_0'(\mathbf{p})$ denote the number of different real roots of $\mathbf{p}(t)$. Taking also into account Proposition 7.18, we have now the following *Theorem of Jacobi-Borchardt.*

Theorem 10.4. *The number of different real roots $\pi_0'(\mathbf{p})$ of the real polynomial $\mathbf{p}$ is given by*

$$\pi_0'(\mathbf{p}) = \operatorname{sgn} \operatorname{Bez}_H(\mathbf{p}, \mathbf{p}') = \operatorname{sgn} H_n\left(\frac{\mathbf{p}'(t)}{\mathbf{p}(t)}\right).$$

Example 10.5. *Let* $\mathbf{p}(t) = t^4 - 1$. *Then*

$$H_4\left(\frac{\mathbf{p}'(t)}{\mathbf{p}(t)}\right) = 4\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathrm{Bez}_H(\mathbf{p}, \mathbf{p}') = 4\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Both matrices have signature 2, which confirms Theorem 10.4.

Theorem 10.4 completely solves the root localization problem for real polynomials in case of simple roots. The multiple roots are just the roots of the greatest common divisor $\mathbf{p}_1(t)$ of $\mathbf{p}(t)$ and $\mathbf{p}'(t)$. If we find the numbers of different real roots of $\mathbf{p}_1(t)$ by Theorem 10.4, then we can obtain the number of real roots with multiplicity of at least two. If we continue we obtain the number of different real roots with multiplicity of at least 3, and so on.

From the algorithmic view point we have to apply the Euclidian algorithm again and again. If it terminates at a certain polynomial $\mathbf{d}(t)$, then we continue with $\mathbf{d}(t)$ and $\mathbf{d}'(t)$ until the remainder is a constant. It is remarkable that all together there are not more than n steps.

This also concerns the general root localization problem for a complex polynomial $\mathbf{u}(t)$ which was discussed above in Section 10,2. First we apply the Euclidian algorithm to the real and imaginary parts of $\mathbf{u}(t)$, and when it terminates at a non-constant $\mathbf{d}(t)$, then we continue with the Euclidian algorithm for $\mathbf{d}(t)$ and $\mathbf{d}'(t)$. Again we have at most n steps if the degree of the original polynomial is n.

4. Inertia with respect to the imaginary axis. To find the inertia of a *real* polynomial with respect to the imaginary axis $\mathrm{i}\mathbb{R}$ is a very important task in many applications, because it is related to the question of stability of systems. In order to avoid confusion, let us point out that if $\mathrm{in}_{\mathrm{i}\mathbb{R}}(\mathbf{u}) = (\pi_+(\mathbf{u}), \pi_-(\mathbf{u}), \pi_0(\mathbf{u}))$, then according to the definition above $\pi_+(\mathbf{u})$ is the number of roots of $\mathbf{u}(t)$ with *negative* real part and $\pi_-(\mathbf{u})$ those with *positive* real part.

Clearly, the imaginary axis case can easily be transformed into the real line case by a transformation of the variable. It remains to study the specifics that arises from the fact that the polynomial under investigation is real. Suppose that $\mathbf{p}(t)$ is a real polynomial of degree n. We set $\mathbf{u}(t) = \mathbf{p}(\mathrm{i}t)$. Then

$$\mathrm{in}_{\,\mathrm{i}\mathbb{R}}(\mathbf{p}) = \mathrm{in}_{\,\mathbb{R}}(\mathbf{u}).$$

Furthermore, if $\mathbf{p}(t) = \sum_{j=1}^{n+1} p_j t^{j-1}$, then $\mathbf{u}(t)$ admits a representation

$$\mathbf{u}(t) = \mathbf{a}(t^2) + \mathrm{i}\, t\, \mathbf{b}(t^2),$$

where, for odd $n = 2m+1$,

$$\mathbf{a}(t) = p_1 - p_3 t + \cdots + (-1)^m p_{2m+1} t^m, \quad \mathbf{b}(t) = p_2 - p_4 t + \cdots + (-1)^m p_{2m+2} t^m, \tag{10.3}$$

and for even $n = 2m$,

$$\mathbf{a}(t) = p_1 - p_3 t + \cdots + (-1)^m p_{2m+1} t^m, \quad \mathbf{b}(t) = p_2 - p_4 t + \cdots + (-1)^{m-1} p_{2m} t^{m-1}. \tag{10.4}$$

From (10.2) and Proposition 2.8 we conclude that the matrix $B = \frac{1}{2\mathrm{i}} \mathrm{Bez}_H(\mathbf{u}, \overline{\mathbf{u}})$ is congruent to the direct sum of the matrices

$$B_0 = \mathrm{Bez}_H(t\mathbf{b}, \mathbf{a}) \quad \text{and} \quad B_1 = \mathrm{Bez}_H(\mathbf{b}, \mathbf{a}). \tag{10.5}$$

Using Theorem 10.1 we arrive at the following.

Theorem 10.6. *Let* $\mathbf{p}(t)$ *be a real polynomial of degree* n *and let* $\mathbf{a}(t)$ *and* $\mathbf{b}(t)$ *be defined by* (10.3) *or* (10.4)*, depending on whether* n *is odd or even, and let* B_0 *and* B_1 *be given by* (10.5)*. Then*

$$\operatorname{sgn} B_0 + \operatorname{sgn} B_1 = \pi_+(\mathbf{p}) - \pi_-(\mathbf{p}),$$

where $\pi_+(\mathbf{p})$ *denotes the number of roots of* $\mathbf{p}(t)$ *in the left and* $\pi_-(\mathbf{p})$ *the number of roots in the right half plane. In particular, the roots of* $\mathbf{p}(t)$ *lie entirely in the left half-plane if and only both matrices* B_0 *and* B_1 *are positive definite. Furthermore, if* $\mathbf{a}(t)$ *and* $\mathbf{b}(t)$ *are coprime, then*

$$\operatorname{in}_{\mathrm{i}\mathbb{R}}(\mathbf{p}) = \operatorname{In} B_0 + \operatorname{In} B_1 .$$

5. Roots on the imaginary axis and positive real roots of real polynomials. In order to get a full picture about the location of the roots of the real polynomial $\mathbf{p}(t)$ with respect to the imaginary axis we have to find the number π_0 of all roots on the imaginary axis, counting multiplicities. As a first step we find the number π_0' of different roots on $\mathrm{i}\mathbb{R}$. This number is equal to the number of different real roots of the greatest common divisor of $\mathbf{a}(t^2)$ and $t\mathbf{b}(t^2)$, where $\mathbf{a}(t)$ and $\mathbf{b}(t)$ are defined by (10.3) or (10.4). Let $\mathbf{p}(0) \neq 0$ and $\mathbf{d}(t)$ be the greatest common divisor of $\mathbf{a}(t)$ and $\mathbf{b}(t)$. Then the greatest common divisor of $\mathbf{a}(t^2)$ and $t\mathbf{b}(t^2)$ equals $\mathbf{d}(t^2)$. The number of real roots of the polynomial $\mathbf{d}(t^2)$ equals twice the number of positive real roots of $\mathbf{d}(t)$. Thus, we are led to the problem to count the roots of a real polynomial on the positive real half axis.

Let $\mathbf{p}(t)$ be a real polynomial of degree n and $\mathbf{p}(0) \neq 0$. We consider the function $\mathbf{f}_1(t) = \frac{t\mathbf{p}'(t)}{\mathbf{p}(t)}$. This function has a partial fraction decomposition

$$\mathbf{f}_1(t) = \sum_{i=1}^{n} \frac{t}{t - t_i} = n + \sum_{i=1}^{n} \frac{t_i}{t - t_i} ,$$

where $t_1, \dots, t_n$ are the roots of $\mathbf{p}(t)$. This leads to a Vandermonde factorization similar to the factorization (7.10) in Example 7.9. From this factorization we conclude that

$$\operatorname{sgn} H_n(\mathbf{f}_1) = \delta_+ - \delta_- ,$$

where δ_+ denotes the number of different positive and δ_- the number of different negative real roots of $\mathbf{p}(t)$. Using Proposition 7.18 and the result of Theorem 10.4, which is $\operatorname{sgn} \operatorname{Bez}_H(\mathbf{p}, \mathbf{p}') = \delta_+ + \delta_-$, we conclude the following.

Theorem 10.7. *Let* $\mathbf{p}(t)$ *be as in Theorem* 10.6 *with* $\mathbf{p}(0) \neq 0$. *The number of positive real roots* δ_+ *of* $\mathbf{p}(t)$ *is given by*

$$\delta_+ = \frac{1}{2}(\operatorname{sgn} \operatorname{Bez}_H(\mathbf{p}, \mathbf{p}') + \operatorname{sgn} \operatorname{Bez}_H(\mathbf{p}, t\mathbf{p}')) .$$

Furthermore, all roots of $\mathbf{p}(t)$ *are real and positive if and only if* $\operatorname{Bez}_H(\mathbf{p}, t\mathbf{p}')$ *is positive definite.*

Example 10.8. *For* $\mathbf{p}(t) = t^4 - 1$ *we obtain*

$$\operatorname{Bez}_H(\mathbf{p}, t\mathbf{p}') = 4\, J_4 .$$

This matrix has a signature equal to zero. Thus (cf. Example 10.5*),*

$$\frac{1}{2}(\operatorname{sgn} \operatorname{Bez}_H(\mathbf{p}, \mathbf{p}') + \operatorname{sgn} \operatorname{Bez}_H(\mathbf{p}, t\mathbf{p}')) = 1$$

which confirms Theorem 10.7.

6. Inertia with respect to the unit circle. Now we discuss the problem how to find the inertia $\mathrm{in}_{\mathbb{T}}(\mathbf{u}) = (\pi_+(\mathbf{u}), \pi_-(\mathbf{u}), \pi_0(\mathbf{u}))$ of a complex monic polynomial $\mathbf{u}(t)$ of degree n with respect to the unit circle $\mathbb{T}$. According to the definition in Section 10,1, $\pi_+(\mathbf{u})$ is the number of roots inside the unit circle, $\pi_-(\mathbf{u})$ is the number of roots outside the unit circle, and $\pi_0(\mathbf{u})$ the number of roots on the unit circle. We consider the matrix

$$B = \mathrm{Bez}_T(\mathbf{u}^\#, \mathbf{u}) . \tag{10.6}$$

For example, if $n = 1$ and $\mathbf{u}(t) = t - c$, then $B = 1 - |c|^2$. Thus c belongs to the open unit disk if and only if $B > 0$. A general $\mathbf{u}(t)$ can be represented as $\mathbf{u}(t) = \mathbf{u}_+(t) + \mathrm{i}\mathbf{u}_-(t)$, where $\mathbf{u}_\pm$ are conjugate symmetric. We have now that the polynomial $(1 - ts)B(t, s)$ is equal to

$$(\mathbf{u}_+(t) - \mathrm{i}\mathbf{u}_-(t))(\overline{\mathbf{u}}_+(s) + \mathrm{i}\overline{\mathbf{u}}_-(s)) - ((\mathbf{u}_+(t) + \mathrm{i}\mathbf{u}_-(t))(\overline{\mathbf{u}}_+(s) - \mathrm{i}\overline{\mathbf{u}}_-(s))).$$

Thus $(1 - ts)B(t, s) = 2\mathrm{i}(\mathbf{u}_+(t)\overline{\mathbf{u}}_-(s) - \mathbf{u}_-(t)\overline{\mathbf{u}}_+(s))$, which means

$$B = 2\mathrm{i}\,\mathrm{Bez}_T(\mathbf{u}_+, \mathbf{u}_-) . \tag{10.7}$$

The following is usually referred to as the *Schur-Cohn theorem*. It can be proved with the same arguments as Hermite's theorem (Theorem 10.1).

Theorem 10.9. *Let $\mathbf{u}(t)$ be a monic polynomial of degree n and B be defined by (10.6) or (10.7). Then the signature of B is given by*

$$\mathrm{sgn}\, B = \pi_+(\mathbf{u}) - \pi_-(\mathbf{u}) .$$

In particular, B is positive definite if and only if $\mathbf{u}(t)$ has all its roots in the open unit disk. Furthermore, if $\mathbf{u}(t)$ and $\mathbf{u}^\#(t)$ are coprime, then $\mathrm{In}\, B = \mathrm{in}_{\mathbb{T}}(\mathbf{u})$.

Theorem 10.9 provides full information about the inertia of $\mathbf{u}(t)$ only if $\mathbf{u}(t)$ has no roots on the unit circle and symmetric with respect to the unit circle or, what is the same if $\mathbf{u}(t)$ and $\mathbf{u}^\#(t)$ are coprime. To complete the picture we still have to find the inertia of the greatest common divisor of $\mathbf{u}(t)$ and $\mathbf{u}^\#(t)$, which is a conjugate-symmetric polynomial.

7. Roots of conjugate-symmetric polynomials. Let $\mathbf{w}(t)$ be a monic conjugate-symmetric polynomial of degree n and $t_1, \dots, t_n$ its roots. Then

$$\mathbf{w}(t) = \prod_{k=1}^{n}(t - t_k) = \mathbf{w}^\#(t) = \prod_{k=1}^{n}(1 - \bar{t}_k t) ,$$

which implies $\bar{t}_k^{-1} = t_k$, $k = 1, \dots, n$. We consider the function $\mathbf{f}(t) = \frac{(\mathbf{w}')^\#(t)}{\mathbf{w}^\#(t)}$, where $(\mathbf{w}')^\# = J_n\overline{\mathbf{w}'}$ This function has a partial fraction decomposition (cf. Example 8.2)

$$\mathbf{f}(t) = \sum_{k=1}^{n} \frac{1}{1 - \bar{t}_k t} .$$

From this representation we can see that the Toeplitz matrix $T_n(\mathbf{f})$ is Hermitian and its signature is equal to the number of different roots of $\mathbf{w}(t)$ on $\mathbb{T}$ (cf. Corollary 8.7, (8.10), and Corollary 1.2).

Theorem 10.10. *For a conjugate-symmetric polynomial $\mathbf{w}(t)$, $\mathrm{Bez}_T((\mathbf{w}')^\#, \mathbf{w}^\#)$ is Hermitian, and its signature is equal to the number of different roots of $\mathbf{w}(t)$ on the unit circle.*

11. Toeplitz-plus-Hankel Bezoutians

Some important results for the Toeplitz and Hankel case can be generalized to matrices which are the sum of such structured matrices. In particular, we will show that the inverse of a (nonsingular) matrix which is the sum of a Toeplitz plus a Hankel matrix possesses again a (generalized) Bezoutian structure. To be more precise we define the following.

1. Definition. An $n \times n$ matrix B is called *Toeplitz-plus-Hankel Bezoutian* (*T+H-Bezoutian*) if there are eight polynomials $\mathbf{g}_i(t)$, $\mathbf{f}_i(t)$ $(i = 1, 2, 3, 4)$ of $\mathbb{F}^{n+2}(t)$ such that

$$B(t,s) = \frac{\sum\limits_{i=1}^{4} \mathbf{g}_i(t)\mathbf{f}_i(s)}{(t-s)(1-ts)} \,. \tag{11.1}$$

In analogy to the Hankel or Toeplitz case we use here the notation

$$B = \mathrm{Bez}_{T+H}((\mathbf{g}_i, \mathbf{f}_i)_1^4).$$

H-Bezoutians or T-Bezoutians are also T+H-Bezoutians. For example, the flip matrix J_n introduced in (1.2) is an H-Bezoutian, $J_n(t,s)$ can be written as

$$J_n(t,s) = \frac{t^n - s^n}{t-s} = \frac{t^n - s^n - t^{n+1}s + ts^{n+1}}{(t-s)(1-ts)} \,,$$

which shows that J_n is the T+H-Bezoutian (11.1), where

$$\mathbf{g}_1 = -\mathbf{f}_2 = \mathbf{e}_{n+1},\ \mathbf{g}_2 = \mathbf{f}_1 = 1,\ \mathbf{g}_3 = \mathbf{f}_4 = \mathbf{e}_{n+2},\ \mathbf{g}_4 = -\mathbf{f}_3 = \mathbf{e}_2 \,.$$

The shift matrix (1.8) is a T-Bezoutian and a T+H-Bezoutian,

$$S_n(t,s) = \frac{t - t^n s^{n-1}}{1-ts} = \frac{t^2 - t^{n+1}s^{n-1} - ts + t^n s^n}{(t-s)(1-ts)} \,.$$

For these examples the sum $S_n + J_n$ is also a T+H-Bezoutian,

$$(S_n + J_n)(t,s) = \frac{(t^n + t^2) - t^{n+1}(s + s^{n-1}) + (t^n - 1)s^n + t(s^{n+1} - s)}{(t-s)(1-ts)} \,.$$

But for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{g}, \mathbf{h} \in \mathbb{F}^{n+1}, n > 3$, the rank of the matrix with the generating polynomial

$$(1-ts)(\mathbf{u}(t)\mathbf{v}(s) - \mathbf{v}(t)\mathbf{u}(s)) + (t-s)(\mathbf{g}(t)\mathbf{h}^J(s) - \mathbf{h}(t)\mathbf{g}^J(s))$$

is not expected to be less or equal to 4. This means that the sum of a T- and an H-Bezoutian $\mathrm{Bez}_H(\mathbf{u}, \mathbf{v}) + \mathrm{Bez}_T(\mathbf{g}, \mathbf{f})$ is, in general, not a T+H-Bezoutian.

2. The transformation $\boldsymbol{\nabla_{T+H}}$. The T+H analogue of the transformations ∇_H or ∇_T (introduced in Section 2,2 and in Section 2,8) is the transformation ∇_{T+H} mapping a matrix $A = [\, a_{ij}\,]_{i,j=1}^n$ of order n onto a matrix of order $n+2$ according to

$$A = [\, a_{i-1,j} - a_{i,j-1} + a_{i-1,j-2} - a_{i-2,j-1}\,]_{i,j=1}^{n+2} \,.$$

Here we put $a_{ij} = 0$ if $i \notin \{1, 2, \ldots, n\}$ or $j \notin \{1, 2, \ldots, n\}$. Denoting $W_n = S_n + S_n^T$ we have

$$\nabla_{T+H} A = \begin{bmatrix} 0 & -\mathbf{e}_1^T A & 0 \\ A\mathbf{e}_1 & AW_n - W_n A & A\mathbf{e}_n \\ 0 & -\mathbf{e}_n^T A & 0 \end{bmatrix} . \tag{11.2}$$

The generating polynomial of $\nabla_{T+H} A$ is

$$(\nabla_{T+H} A)(t,s) = (t-s)(1-ts)A(t,s) \,. \tag{11.3}$$

Hence a matrix B is a T+H-Bezoutian if and only if

$$\mathrm{rank} \nabla_{T+H} B \leq 4 \,.$$

Recall that the $n \times n$ matrix in the center of (11.2) is the matrix $\nabla(A)$ introduced in (1.11). In other words, the transformation ∇ is a restriction of $\nabla_{\mathrm{T+H}}$, and it is clear that T+H-Bezoutians are quasi-T+H matrices, but not vice versa.

3. Uniqueness. Different vector systems $\{\mathbf{g}_i, \mathbf{f}_i\}_{i=1}^4$, $\{\widetilde{\mathbf{g}}_i, \widetilde{\mathbf{f}}_i\}_{i=1}^4$ may produce the same T+H-Bezoutian.

Note that $B = \mathrm{Bez}_{T+H}((\mathbf{g}_i, \mathbf{f}_i)_1^4)$ is equal to $\widetilde{B} = \mathrm{Bez}_{T+H}((\widetilde{\mathbf{g}}_i, \widetilde{\mathbf{f}}_i)_1^4)$ if and only if $\nabla_{T+H} B = \nabla_{T+H} \widetilde{B}$. To answer the questions under which conditions this happens we use the following lemma.

Lemma 11.1. *Let G_j, F_j $(j = 1,2)$ be full rank matrices of order $m \times r, n \times r$, respectively, $r = \mathrm{rank}\, G_j = \mathrm{rank}\, F_j$. Then*

$$G_1 F_1^T = G_2 F_2^T \tag{11.4}$$

if and only if there is a nonsingular $r \times r$ matrix φ such that

$$G_2 = G_1 \varphi \,, \;\; F_1 = F_2 \varphi^T \,. \tag{11.5}$$

Proof. Assume there is a nonsingular φ so that $G_2 = G_1\varphi$ and $F_2^T = \varphi^{-1} F_1^T$, then $G_1 F_1^T = G_2 F_2^T$. Now let (11.4) be satisfied and $A = G_1 F_1^T$. The image of A is spanned by the columns of G_1 as well as of G_2. Thus there exists a nonsingular matrix φ so that $G_2 = G_1\varphi$. With the same arguments for A^T we obtain that there is a nonsingular matrix ψ so that $F_2 = F_1\psi$. Hence

$$G_1 F_1^T = G_2 F_2^T = G_1 \varphi \psi^T F_1^T \,. \tag{11.6}$$

Since G_1, F_1 are of full rank they are one-sided invertible, and we conclude from (11.6) that $\varphi \cdot \psi^T = I_r$. ■

Let $B, \widetilde{B}$ be $n \times n$ T+H-Bezoutians and $G, \widetilde{G}, F, \widetilde{F}$ be full rank matrices with

$$r = \mathrm{rank}\, G = \mathrm{rank}\, F \leq 4 \,, \; \widetilde{r} = \mathrm{rank}\, \widetilde{G} = \mathrm{rank}\, \widetilde{F} \leq 4$$

such that the matrices $\nabla_{\mathrm{T+H}} B$ and $\nabla_{\mathrm{T+H}} \widetilde{B}$ allow the following rank decompositions

$$\nabla_{\mathrm{T+H}} B = G F^T \,, \;\; \nabla_{\mathrm{T+H}} \widetilde{B} = \widetilde{G} \widetilde{F}^T \,.$$

Proposition 11.2. *The T+H-Bezoutians B and $\widetilde{B}$ coincide if and only if $r = \widetilde{r}$, and there is a nonsingular $r \times r$ matrix φ so that*

$$\widetilde{G} = G\varphi \,, \; F = \widetilde{F} \varphi^T \,.$$

To specify this for nonsingular Bezoutians we make the following observation.

Proposition 11.3. *Let B be an $n \times n$ matrix ($n \geq 2$) with rank $\nabla_{\mathrm{T+H}} B < 4$. Then B is a singular matrix.*

Proof. Let us prove this by contradiction. Assume B is nonsingular and $\nabla_{\mathrm{T+H}} B < 4$. Taking (11.2) into account elementary considerations show that $\nabla_{\mathrm{T+H}} B$ allows the following decomposition

$$\nabla_{\mathrm{T+H}} B = \begin{bmatrix} 0 \\ B\mathbf{e}_1 \\ 0 \end{bmatrix} [1 \; * \; 0\,] + \begin{bmatrix} 0 \\ B\mathbf{e}_n \\ 0 \end{bmatrix} [\,0 \; * \; 1] - \begin{bmatrix} 1 \\ * \\ 0 \end{bmatrix} [\,0 \;\, \mathbf{e}_1^T B \;\, 0\,] - \begin{bmatrix} 0 \\ * \\ 1 \end{bmatrix} [\,0 \;\, \mathbf{e}_n^T B \;\, 0\,], \tag{11.7}$$

where $*$ stands for some vector of $\mathbb{F}^n$. Due to the nonsingularity of B its first and last rows as well as its first and last columns are linearly independent. Thus,

$$\operatorname{rank}\begin{bmatrix} 0 & 0 & 1 & 0 \\ B\mathbf{e}_1 & B\mathbf{e}_n & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & * & 0 \\ 0 & * & 1 \\ 0 & \mathbf{e}_1^T B & 0 \\ 0 & \mathbf{e}_n^T B & 0 \end{bmatrix} = 4\,,$$

which contradicts rank $\nabla_{\mathrm{T+H}} B < 4$. ■

Corollary 11.4. *If rank* $\nabla_{\mathrm{T+H}} B < 4$ *then the first and the last rows (or the first and the last columns) of* B *are linearly dependent.*

For T-(or H-)Bezoutians B, the condition rank $\nabla_T B < 2$ (or rank $\nabla_H B < 2$) leads to $B \equiv 0$. But in the T+H case nontrivial T+H-Bezoutians B with rank $\nabla_{\mathrm{T+H}} B < 4$ exist. Examples are $B = I_n + J_n$ $(n \geq 2)$ and split Bezoutians introduced in Section 2,11. In these cases rank $\nabla B \leq 2$. Now we present the result for the nonsingular case.

Proposition 11.5. *The nonsingular T+H-Bezoutians*

$$B = \mathrm{Bez}_{\mathrm{T+H}}((\mathbf{g}_i, \mathbf{f}_i)_1^4) \text{ and } \widetilde{B} = \mathrm{Bez}_{\mathrm{T+H}}((\widetilde{\mathbf{g}}_i, \widetilde{\mathbf{f}}_i)_1^4)$$

coincide if and only if there is a nonsingular 4×4 *matrix* φ *such that*

$$[\,\mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3\ \mathbf{g}_4\,]\,\varphi = [\,\widetilde{\mathbf{g}}_1\ \widetilde{\mathbf{g}}_2\ \widetilde{\mathbf{g}}_3\ \widetilde{\mathbf{g}}_4\,]$$

and

$$[\,\widetilde{\mathbf{f}}_1\ \widetilde{\mathbf{f}}_2\ \widetilde{\mathbf{f}}_3\ \widetilde{\mathbf{f}}_4\,]\,\varphi^T = [\,\mathbf{f}_1\ \mathbf{f}_2\ \mathbf{f}_3\ \mathbf{f}_4\,]\,.$$

4. Inverses of T+H-Bezoutians. Recall that in the Hankel and Toeplitz cases we proved that a nonsingular matrix is an H- or a T-Bezoutian if and only if it is the inverse of a Hankel or of a Toeplitz matrix, respectively (see Sections 4,1, 4,4, 7,6, 8,7). Such an assertion is also true in the T+H case. We start with proving the following part of it.

Theorem 11.6. *Let* B *be a nonsingular T+H-Bezoutian. Then* B^{-1} *is a T+H matrix.*

Proof. Taking Proposition 11.3 into account we have rank $\nabla_{T+H} B = 4$, and a rank decomposition of $\nabla_{T+H} B$ is of the form (11.7). In particular, this means that there are vectors $\mathbf{z}_i \in \mathbb{F}^n, i = 1, 2, 3, 4$, such that

$$BW_n - W_n B = B\mathbf{e}_1\,\mathbf{z}_1^T + B\mathbf{e}_n\,\mathbf{z}_2^T + \mathbf{z}_3\,\mathbf{e}_1^T B + \mathbf{z}_4\,\mathbf{e}_n^T B\,.$$

Applying B^{-1} from both sides this equality leads to

$$B^{-1}W_n - W_n B^{-1} = -(\mathbf{e}_1\,\mathbf{z}_1^T B^{-1} + \mathbf{e}_n\,\mathbf{z}_2^T B^{-1} + B^{-1}\mathbf{z}_3\,\mathbf{e}_1^T + B^{-1}\mathbf{z}_4\,\mathbf{e}_n^T)\,.$$

Thus, the matrix of order $n-2$ in the center of $\nabla(B^{-1})$ is the zero matrix. By Proposition 1.4 this proves that B^{-1} is a T+H matrix. ■

In the next section we will show that the converse is also true, i.e., the inverse of a (nonsingular) T+H matrix is a T+H-Bezoutian.

12. Inverses of T+H-matrices

We consider now $n \times n$ matrices R_n which are the sum of a Toeplitz matrix T_n and a Hankel matrix H_n. For our purposes it is convenient to use a representation (1.5) for $m = n$,

$$T_n = T_n(\mathbf{a})\,,\ \mathbf{a} = (a_i)_{i=1-n}^{n-1}\,, \quad H_n = T_n(\mathbf{b})J_n\,,\ \mathbf{b} = (b_i)_{i=1-n}^{n-1}\,,$$

$$R_n = T_n(\mathbf{a}) + T_n(\mathbf{b})J_n = \begin{bmatrix} a_0 & \dots & a_{1-n} \\ \vdots & \ddots & \vdots \\ a_{n-1} & \dots & a_0 \end{bmatrix} + \begin{bmatrix} b_{1-n} & \dots & b_0 \\ \vdots & ⋰ & \vdots \\ b_0 & \dots & b_{n-1} \end{bmatrix}. \tag{12.1}$$

We want to prove that the inverse of a T+H matrix R_n is a T+H-Bezoutian and even more, we want to present inversion formulas

$$R_n^{-1} = \mathrm{Bez}_{T+H}((\mathbf{g}_i, \mathbf{f}_i)_1^4)\,.$$

Thus, we have to answer the question how to obtain the vectors $\mathbf{g}_i, \mathbf{f}_i\,,\ i = 1,2,3,4$. Note that representations of inverses of T+H matrices as T+H-Bezoutians allow fast matrix-vector multiplication by these matrices (in case $\mathbb{F} = \mathbb{C}$ see [38], in case $\mathbb{F} = \mathbb{R}$ [40], [42]).

1. Fundamental systems. Besides the nonsingular T+H matrix R_n of (12.1) we consider the $(n-2)\times(n+2)$ T+H matrices $\partial R_n, \partial R_n^T$ obtained from R_n, R_n^T after deleting the first and last rows and adding one column to the right and to the left by preserving the T+H structure,

$$\partial R_n = \begin{bmatrix} a_2 & a_1 & \dots & a_{2-n} & a_{1-n} \\ a_3 & a_2 & \dots & a_{3-n} & a_{2-n} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} & a_{n-2} & \dots & a_{-1} & a_{-2} \end{bmatrix} + \begin{bmatrix} b_{1-n} & b_{2-n} & \dots & b_1 & b_2 \\ b_{2-n} & b_{3-n} & \dots & b_2 & b_3 \\ \vdots & \vdots & & \vdots & \vdots \\ b_{-2} & b_{-1} & \dots & b_{n-2} & b_{n-1} \end{bmatrix}, \tag{12.2}$$

$$\partial R_n^T = \begin{bmatrix} a_{-2} & a_{-1} & \dots & a_{n-2} & a_{n-1} \\ a_{-3} & a_{-2} & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{1-n} & a_{2-n} & \dots & a_1 & a_2 \end{bmatrix} + \begin{bmatrix} b_{1-n} & b_{2-n} & \dots & b_1 & b_2 \\ b_{2-n} & b_{3-n} & \dots & b_2 & b_3 \\ \vdots & \vdots & & \vdots & \vdots \\ b_{-2} & b_{-1} & \dots & b_{n-2} & b_{n-1} \end{bmatrix}. \tag{12.3}$$

Since R_n is nonsingular both matrices ∂R_n and ∂R_n^T are of full rank, which means

$$\dim\ker\partial R_n = \dim\ker\partial R_n^T = 4\,.$$

Any system of eight vectors $\{\mathbf{u}_i\}_{i=1}^4, \{\mathbf{v}_i\}_{i=1}^4$, where $\{\mathbf{u}_i\}_{i=1}^4$ is a basis of $\ker\partial R_n$ and $\{\mathbf{v}_i\}_{i=1}^4$ is a basis of $\ker\partial R_n^T$, is called *fundamental system for* R_n. The reason for this notation is that these vectors completely determine the inverse R_n^{-1}. In order to understand this we consider first a special fundamental system.

Hereafter we use the following notation. For a given vector $\mathbf{a} = (a_j)_{j=1-n}^{n-1}$ we define

$$\mathbf{a}_\pm = (a_{\pm j})_{j=1}^n\,, \tag{12.4}$$

where $a_{\pm n}$ can be arbitrarily chosen. The matrix $\nabla(R_n) = R_nW_n - W_nR_n$ allows a rank decomposition of the form,

$$\nabla(R_n) = -(\mathbf{a}_+ + \mathbf{b}_-^J)\mathbf{e}_1^T - (\mathbf{a}_-^J + \mathbf{b}_+)\mathbf{e}_n^T + \mathbf{e}_1(\mathbf{a}_- + \mathbf{b}_-^J)^T + \mathbf{e}_n(\mathbf{a}_+^J + \mathbf{b}_+)^T\,. \tag{12.5}$$

Multiplying (12.5) from both sides by R_n^{-1} we obtain a rank decomposition of $\nabla(R_n^{-1})$.

Proposition 12.1. *We have*

$$\nabla\left(R_n^{-1}\right) = \mathbf{x}_1\mathbf{y}_1^T + \mathbf{x}_2\mathbf{y}_2^T - \mathbf{x}_3\mathbf{y}_3^T - \mathbf{x}_4\mathbf{y}_4^T\,, \tag{12.6}$$

where $\mathbf{x}_i$ $(i=1,2,3,4)$ *are the solutions of*

$$R_n\mathbf{x}_1 = \mathbf{a}_+ + \mathbf{b}_-^J\,,\ R_n\mathbf{x}_2 = \mathbf{a}_-^J + \mathbf{b}_+\,,\ R_n\mathbf{x}_3 = \mathbf{e}_1\,,\ R_n\mathbf{x}_4 = \mathbf{e}_n\,, \tag{12.7}$$

and $\mathbf{y}_i$ $(i=1,2,3,4)$ *are the solutions of*

$$R_n^T\mathbf{y}_1 = \mathbf{e}_1\,,\ R_n^T\mathbf{y}_2 = \mathbf{e}_n\,,\ R_n^T\mathbf{y}_3 = \mathbf{a}_- + \mathbf{b}_-^J\,,\ R_n^T\mathbf{y}_4 = \mathbf{a}_+^J + \mathbf{b}_+\,. \tag{12.8}$$

According to (12.2), (12.3) we obtain the following fundamental system for R_n.

Proposition 12.2. *Let* $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{F}^n$ *be defined by (*12.7*), (*12.8*). The vector system*

$$\left\{\mathbf{u}_1 = \begin{bmatrix} 1 \\ -\mathbf{x}_1 \\ 0 \end{bmatrix},\ \mathbf{u}_2 = \begin{bmatrix} 0 \\ -\mathbf{x}_2 \\ 1 \end{bmatrix},\ \mathbf{u}_3 = \begin{bmatrix} 0 \\ \mathbf{x}_3 \\ 0 \end{bmatrix},\ \mathbf{u}_4 = \begin{bmatrix} 0 \\ \mathbf{x}_4 \\ 0 \end{bmatrix}\right\} \tag{12.9}$$

is a basis of $\ker \partial R_n$*, the vector system*

$$\left\{\mathbf{v}_1 = \begin{bmatrix} 0 \\ \mathbf{y}_1 \\ 0 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 0 \\ \mathbf{y}_2 \\ 0 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} 1 \\ -\mathbf{y}_3 \\ 0 \end{bmatrix},\ \mathbf{v}_4 = \begin{bmatrix} 0 \\ -\mathbf{y}_4 \\ 1 \end{bmatrix}\right\} \tag{12.10}$$

is a basis of $\ker \partial R_n^T$.

2. Inversion of T+H matrices. The special fundamental system of Proposition 12.2 deliver the parameters needed in a Bezoutian formula for R_n^{-1}. This is the initial point for our further considerations.

Theorem 12.3. *Let* R_n *be the nonsingular T+H matrix (*12.1*) and* $\{\mathbf{u}_i\}_{i=1}^4$, $\{\mathbf{v}_i\}_{i=1}^4$ *be the fundamental system for* R_n *given by (*12.9*), (*12.7*), (*12.10*), (*12.8*). Then* R_n^{-1} *is the T+H-Bezoutian defined by its generating polynomial as follows,*

$$R_n^{-1}(t,s) = \frac{\mathbf{u}_3(t)\mathbf{v}_3(s) + \mathbf{u}_4(t)\mathbf{v}_4(s) - \mathbf{u}_1(t)\mathbf{v}_1(s) - \mathbf{u}_2(t)\mathbf{v}_2(s)}{(t-s)(1-ts)}\,. \tag{12.11}$$

Proof. Since $\mathbf{x}_3$ is the first, $\mathbf{x}_4$ the last column, $\mathbf{y}_1^T$ is the first, $\mathbf{y}_2^T$ the last row of R_n^{-1} we conclude from (11.2)

$$\nabla_{T+H} R_n^{-1} = \begin{bmatrix} 0 & -\mathbf{y}_1^T & 0 \\ \mathbf{x}_3 & \nabla(R_n^{-1}) & \mathbf{x}_4 \\ 0 & -\mathbf{y}_2^T & 0 \end{bmatrix}.$$

Taking (12.6) into account this leads to

$$\nabla_{T+H} R_n^{-1} = [\,-\mathbf{u}_1\ \ -\mathbf{u}_2\ \ \mathbf{u}_3\ \ \mathbf{u}_4\,]\,[\,\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\,]^T\,,$$

where the vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ are defined in (12.9), (12.10). The inversion formula follows now from (11.3). ■

In particular, this theorem shows that if we want to use the vectors of any fundamental system in a Bezoutian formula for R_n^{-1} a “normalization” of them is necessary. For this purpose we introduce the following $(n+2)\times 4$ matrices

$$F = [\,\mathbf{e}_1\ \mathbf{e}_{n+2}\ \mathbf{f}_1\ \mathbf{f}_2\,]\,,\quad G = [\,\mathbf{g}_1\ \mathbf{g}_2\ \mathbf{e}_1\ \mathbf{e}_{n+2}\,]\,,$$

where

$$\mathbf{f}_1 = (a_{1-i} + b_{i-n})_{i=0}^{n+1} \ , \ \mathbf{f}_2 = (a_{n-i} + b_{i-1})_{i=0}^{n+1} \ ,$$
$$\mathbf{g}_1 = (a_{i-1} + b_{i-n})_{i=0}^{n+1} \ , \ \mathbf{g}_2 = (a_{i-n} + b_{i-1})_{i=0}^{n+1} \ ,$$

whith $a_{\pm n}, b_{\pm n}$ arbitrarily chosen. We call a fundamental system $\{\mathbf{u}_i\}_{i=1}^4, \{\mathbf{v}_i\}_{i=1}^4$ for R_n *canonical* if

$$F^T [\, \mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\ \mathbf{u}_4 \,] = G^T [\, \mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4 \,] = I_4 \,. \tag{12.12}$$

Proposition 12.4. *A fundamental system $\{\mathbf{u}_i\}_{i=1}^4, \{\mathbf{v}_i\}_{i=1}^4$ for R_n is canonical if and only if $\mathbf{u}_i$ is of the form* (12.9)*,* (12.7) *and $\mathbf{v}_i$ is of the form* (12.10)*,* (12.8) *for $i = 1, 2, 3, 4$.*

Proof. If $\{\mathbf{u}_i\}_{i=1}^4$ and $\{\mathbf{v}_i\}_{i=1}^4$ are canonical then (12.12) means, in particular, that the first component of $\mathbf{u}_1$ and $\mathbf{v}_3$ as well as the last components of $\mathbf{u}_2$ and $\mathbf{v}_4$ are one. The first and last components of the other vectors are zero. Hence there are vectors $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{F}^n$ such that $\mathbf{u}_i, \mathbf{v}_i$ are of the form (12.9), (12.10). Now by (12.12) we have

$$[\, I_{+-}\mathbf{f}_1\ I_{+-}\mathbf{f}_2 \,]^T [\, \mathbf{x}_3\ \mathbf{x}_4 \,] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{12.13}$$

Here, for a given vector $\mathbf{h} = (h_i)_{i=0}^{n+1} \in \mathbb{F}^{n+2}$ the vector $I_{+-}\mathbf{h} \in \mathbb{F}^n$ is defined by

$$I_{+-}\mathbf{h} = (h_i)_{i=1}^n . \tag{12.14}$$

Since

$$(I_{+-}\mathbf{f}_1)^T = e_1^T R_n \,, \ (I_{+-}\mathbf{f}_2)^T = e_n^T R_n$$

and since $\begin{bmatrix} 0 \\ \mathbf{x}_3 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ \mathbf{x}_4 \\ 0 \end{bmatrix}$ are in $\ker \partial R_n$ equality (12.13) leads to

$$R_n \mathbf{x}_3 = e_1 \,, \ R_n \mathbf{x}_4 = e_n \,.$$

Moreover, $\begin{bmatrix} 1 \\ -\mathbf{x}_1 \\ 0 \end{bmatrix} \in \ker \partial R_n$ means that $R_n \mathbf{x}_1 = \mathbf{a}_+ + \mathbf{b}_-^J$ and $\begin{bmatrix} 0 \\ -\mathbf{x}_2 \\ 1 \end{bmatrix} \in \ker \partial R_n$ means that $R_n \mathbf{x}_2 = \mathbf{a}_-^J + \mathbf{b}_+$. Similar arguments show that $\mathbf{y}_i$, $i = 1, 2, 3, 4$, are the solutions of (12.8), and the necessity part of the proof is complete.

If $\{\mathbf{u}_i\}_{i=1}^4, \{\mathbf{v}_i\}_{i=1}^4$ are of the form (12.9), (12.7), and (12.10), (12.8) then, obviously, (12.12) is satisfied. ■

Given an arbitrary fundamental system $\{\widetilde{\mathbf{u}}_i\}_{i=1}^4, \{\widetilde{\mathbf{v}}_i\}_{i=1}^4$ we define two 4×4 nonsingular matrices Γ_F, Γ_G,

$$F^T [\, \widetilde{\mathbf{u}}_1\ \widetilde{\mathbf{u}}_2\ \widetilde{\mathbf{u}}_3\ \widetilde{\mathbf{u}}_4 \,] = \Gamma_F \,, \quad G^T [\, \widetilde{\mathbf{v}}_1\ \widetilde{\mathbf{v}}_2\ \widetilde{\mathbf{v}}_3\ \widetilde{\mathbf{v}}_4 \,] = \Gamma_G \,.$$

We conclude that by

$$[\, \mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\ \mathbf{u}_4 \,] = [\, \widetilde{\mathbf{u}}_1\ \widetilde{\mathbf{u}}_2\ \widetilde{\mathbf{u}}_3\ \widetilde{\mathbf{u}}_4 \,] \, \Gamma_F^{-1} \tag{12.15}$$

and

$$[\, \mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4 \,] = [\, \widetilde{\mathbf{v}}_1\ \widetilde{\mathbf{v}}_2\ \widetilde{\mathbf{v}}_3\ \widetilde{\mathbf{v}}_4 \,] \, \Gamma_G^{-1} \tag{12.16}$$

a canonical fundamental system $\{\mathbf{u}_i\}_{i=1}^4, \{\mathbf{v}_i\}_{i=1}^4$ is given. Note that for fixed $a_{\pm n}, b_{\pm n}$ the canonical fundamental system is unique. The following becomes clear.

Theorem 12.5. *Let R_n be the nonsingular T+H matrix* (12.1) *and $\{\widetilde{\mathbf{u}}_i\}_{i=1}^4, \{\widetilde{\mathbf{v}}_i\}_{i=1}^4$ be a fundamental system of R_n. Then the inverse R_n^{-1} is the T+H-Bezoutian* (12.11)*, where $\{\mathbf{u}_i\}_{i=1}^4$ and $\{\mathbf{v}_i\}_{i=1}^4$ are given by* (12.15) *and* (12.16)*, respectively.*

Let R_n be given by (12.1). Hereafter we use also a representation of R_n which involves the projections $P_\pm = \frac{1}{2}(I_n \pm J_n)$ onto $\mathbb{F}^n_\pm$ introduced in (1.3) and the vectors

$$\mathbf{c} = (c_j)_{j=1-n}^{n-1} = \mathbf{a} + \mathbf{b}\,,\ \mathbf{d} = (d_j)_{j=1-n}^{n-1} = \mathbf{a} - \mathbf{b}\,,$$

namely

$$R_n = T_n(\mathbf{c})P_+ + T_n(\mathbf{d})P_-\,. \tag{12.17}$$

Instead of the solutions $\mathbf{x}_i$ of (12.7) and the solutions $\mathbf{y}_i$ of (12.8) we consider now the solutions of the following equations the right hand sides of which depend on $\mathbf{c}, \mathbf{d}$ and $\widetilde{\mathbf{c}} = \mathbf{a}^J + \mathbf{b}$, $\widetilde{\mathbf{d}} = \mathbf{a}^J - \mathbf{b}$,

$$R_n\mathbf{w}_1 = \frac{1}{2}(\mathbf{c}_+ + \mathbf{c}_-^J)\,,\ R_n\mathbf{w}_2 = \frac{1}{2}(\mathbf{d}_+ - \mathbf{d}_-^J)\,,\ R_n\mathbf{w}_3 = P_+\mathbf{e}_1\,,\ R_n\mathbf{w}_4 = P_-\mathbf{e}_1 \tag{12.18}$$

and

$$R_n^T\mathbf{z}_1 = P_+\mathbf{e}_1\,,\ R_n^T\mathbf{z}_2 = P_-\mathbf{e}_1\,,\ R_n^T\mathbf{z}_3 = \frac{1}{2}(\widetilde{\mathbf{c}}_+ + \widetilde{\mathbf{c}}_-^J)\,,\ R_n^T\mathbf{z}_4 = \frac{1}{2}(\widetilde{\mathbf{d}}_+ - \widetilde{\mathbf{d}}_-^J)\,. \tag{12.19}$$

Here we use the notation (12.4). We introduce the vectors

$$\begin{aligned}
&\breve{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ -2\mathbf{w}_1 \\ 1 \end{bmatrix},\quad \breve{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ -2\mathbf{w}_2 \\ -1 \end{bmatrix},\quad \breve{\mathbf{u}}_3 = \begin{bmatrix} 0 \\ \mathbf{w}_3 \\ 0 \end{bmatrix},\quad \breve{\mathbf{u}}_4 = \begin{bmatrix} 0 \\ \mathbf{w}_4 \\ 0 \end{bmatrix}, \\
&\breve{\mathbf{v}}_1 = \begin{bmatrix} 0 \\ \mathbf{z}_1 \\ 0 \end{bmatrix},\quad \breve{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ \mathbf{z}_2 \\ 0 \end{bmatrix},\quad \breve{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ -2\mathbf{z}_3 \\ 1 \end{bmatrix},\quad \breve{\mathbf{v}}_4 = \begin{bmatrix} 1 \\ -2\mathbf{z}_4 \\ -1 \end{bmatrix}.
\end{aligned} \tag{12.20}$$

Now an inversion formula which involves these vectors follows from formula (12.11).

Proposition 12.6. *Let R_n be the nonsingular T+H matrix (*12.17*). Then the inverse R_n^{-1} is given by*

$$R_n^{-1}(t,s) = \frac{\breve{\mathbf{u}}_3(t)\breve{\mathbf{v}}_3(s) + \breve{\mathbf{u}}_4(t)\breve{\mathbf{v}}_4(s) - \breve{\mathbf{u}}_1(t)\breve{\mathbf{v}}_1(s) - \breve{\mathbf{u}}_2(t)\breve{\mathbf{v}}_2(s)}{(t-s)(1-ts)}\,, \tag{12.21}$$

*where $\{\breve{\mathbf{u}}_i\}_{i=1}^4$, $\{\breve{\mathbf{v}}_i\}_{i=1}^4$ are defined in (*12.20*).*

Proof. Since

$$[\,\breve{\mathbf{u}}_1\ \breve{\mathbf{u}}_2\ \breve{\mathbf{u}}_3\ \breve{\mathbf{u}}_4\,] = [\,\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\ \mathbf{u}_4\,]\,\varphi$$

and

$$[\,\breve{\mathbf{v}}_1\ \breve{\mathbf{v}}_2\ \breve{\mathbf{v}}_3\ \breve{\mathbf{v}}_4\,] = [\,\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\,]\,\varphi^{-1}\,,$$

where φ is the block diagonal matrix

$$\varphi = \operatorname{diag}\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right),$$

the proposition follows from Proposition 11.5 and (12.11). ■

3. Inversion of symmetric T+H matrices. It is easy to see that a T+H matrix is symmetric if and only if the Toeplitz part has this property. Let R_n be a nonsingular, symmetric T+H matrix (12.1). Then the solutions of (12.7) and (12.8) coincide,

$$\mathbf{y}_1 = \mathbf{x}_3\,,\ \mathbf{y}_2 = \mathbf{x}_4\,,\ \mathbf{y}_3 = \mathbf{x}_1\,,\ \mathbf{y}_4 = \mathbf{x}_2\,.$$

Using the inversion formula (12.11) R_n^{-1} is given by the vectors $\{\mathbf{u}_i\}_{i=1}^4$ of (12.9),

$$R_n^{-1}(t,s) = \frac{\mathbf{u}_3(t)\mathbf{u}_1(s) - \mathbf{u}_1(t)\mathbf{u}_3(s) + \mathbf{u}_4(t)\mathbf{u}_2(s) - \mathbf{u}_2(t)\mathbf{u}_4(s)}{(t-s)(1-ts)}\,. \tag{12.22}$$

Since $\mathbf{a} = \mathbf{a}^J$ we have $\mathbf{c} = \widetilde{\mathbf{c}}$, $\mathbf{d} = \widetilde{\mathbf{d}}$, and the inversion formula (12.21) can be simplified as well,

$$R_n^{-1}(t,s) = \frac{\breve{\mathbf{u}}_3(t)\breve{\mathbf{u}}_1(s) - \breve{\mathbf{u}}_1(t)\breve{\mathbf{u}}_3(s) + \breve{\mathbf{u}}_4(t)\breve{\mathbf{u}}_2(s) - \breve{\mathbf{u}}_2(t)\breve{\mathbf{u}}_4(s)}{(t-s)(1-ts)} . \tag{12.23}$$

If we have any basis $\{\widetilde{\mathbf{u}}_i\}_{i-1}^4$ of $\ker \partial R_n$, it remains to compute Γ_F, and $\{\mathbf{u}_i\}_{i=1}^4$ is given by (12.15).

We will not consider the skewsymmetric case, since a skewsymmetric T+H matrix is always a pure Toeplitz matrix. (For the skewsymmetric Toeplitz case see Section 4,7.)

4. Inversion of centrosymmetric T+H matrices. If R_n from (12.1) is centrosymmetric, i.e. $R_n^J = R_n$, then in view of $J_n T_n(\mathbf{a}) J_n = T_n(\mathbf{a}^J)$,

$$R_n = \frac{1}{2}(R_n + R_n^J) = T_n\left(\frac{1}{2}(\mathbf{a}+\mathbf{a}^J)\right) + T_n\left(\frac{1}{2}(\mathbf{b}+\mathbf{b}^J)\right) J_n .$$

Together with Exercises 15, 16 we conclude the following.

Proposition 12.7. *Let R_n be an $n \times n$ T+H matrix. Then the following assertions are equivalent.*

1. *R_n is centrosymmetric.*
2. *In the representation* (12.1) *(resp.* (12.17)*) the Toeplitz matrices $T_n(\mathbf{a})$ and $T_n(\mathbf{b})$ (resp. $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$) are symmetric.*
3. *In the representation* (12.1) *(resp.* 12.17)*) $\mathbf{a}$ and $\mathbf{b}$ (resp. $\mathbf{c}$ and $\mathbf{d}$) are symmetric vectors.*

Corollary 12.8. *A centrosymmetric T+H matrix R_n is also symmetric.*

Moreover, in the centrosymmetric case the representation (12.17) can be written in the form

$$R_n = P_+ T_n(\mathbf{c}) P_+ + P_- T_n(\mathbf{d}) P_- . \tag{12.24}$$

Now we specify the results for general T+H matrices to centrosymmetric T+H matrices R_n. Since R_n is symmetric we can use the simplifications of the previous subsection. To begin with we observe that the right hand sides of the first and the third equations of (12.18) are symmetric and of the second and the fourth equations are skewsymmetric if we choose

$$c_n = c_{-n},\ d_n = d_{-n} .$$

Since centrosymmetric matrices map symmetric (skewsymmetric) vectors into symmetric (skewsymmetric) vectors, we conclude that the solutions $\mathbf{w}_1, \mathbf{w}_3$ of (12.18) as well as their extensions $\breve{\mathbf{u}}_1, \breve{\mathbf{u}}_3$ of (12.20) are symmetric, whereas $\mathbf{w}_2, \mathbf{w}_4$ and $\breve{\mathbf{u}}_2, \breve{\mathbf{u}}_4$ are skewsymmetric vectors. This leads to further simplications of the inversion formula (12.23). But before presenting this formula let us introduce a more unified notation, where the subscript + designates symmetric, − skewsymmetric vectors in the fundamental system,

$$\mathbf{u}_+ = \begin{bmatrix} 0 \\ \mathbf{w}_3 \\ 0 \end{bmatrix}, \ \mathbf{u}_- = \begin{bmatrix} 0 \\ \mathbf{w}_4 \\ 0 \end{bmatrix}, \ \mathbf{v}_+ = \begin{bmatrix} 1 \\ -2\mathbf{w}_1 \\ 1 \end{bmatrix}, \ \mathbf{v}_- = \begin{bmatrix} 1 \\ -2\mathbf{w}_2 \\ -1 \end{bmatrix} . \tag{12.25}$$

Here $\mathbf{w}_i$ are the solutions of (12.18) which turn obviously into pure Toeplitz equations,

$$T_n(\mathbf{c})\mathbf{w}_1 = P_+\mathbf{c}_+ ,\ T_n(\mathbf{d})\mathbf{w}_2 = P_-\mathbf{d}_+ ,\ T_n(\mathbf{c})\mathbf{w}_3 = P_+\mathbf{e}_1 ,\ T_n(\mathbf{d})\mathbf{w}_4 = P_-\mathbf{e}_1 . \tag{12.26}$$

Note that these equations have unique symmetric or skewsymmetric solutions. Thus, the inversion formula (12.23) can be rewritten as a sum of a split Bezoutian of (+)-type and a split Bezoutian of (−)-type. These special Bezoutians were introduced in Section 2,11. Let us use the notations adopted there.

Theorem 12.9. *Let R_n be a nonsingular, centrosymmetric T+H matrix given by (*12.17*) and $\mathbf{u}_\pm, \mathbf{v}_\pm$ be the vectors of $\mathbb{F}_\pm^{n+2}$ defined in (*12.25*), where the $\mathbf{w}_i$ are the unique symmetric or skewsymmetric solutions of(*12.26*). Then*

$$R_n^{-1} = B_+ + B_- \, ,$$

where $B_\pm$ are the split Bezoutians of $(\pm)$*-type*

$$B_\pm = \mathrm{Bez_{split}}(\mathbf{v}_\pm, \mathbf{u}_\pm) \, .$$

Similar ideas as those of Section 4,5 lead to a slight modification of the last theorem. We extend the nonsingular centrosymmetric T+H matrix R_n given by (12.17) to a nonsingular centrosymmetric T+H matrix R_{n+2}, such that R_n is its central submatrix of order n.

$$R_{n+2} = T_{n+2}(\mathbf{c})P_+ + T_{n+2}(\mathbf{d})P_- \, . \tag{12.27}$$

Here $\mathbf{c}$ and $\mathbf{d}$ are extensions of the original vectors $\mathbf{c}$ and $\mathbf{d}$ by corresponding components $c_{-n} = c_n \, , \, d_{-n} = d_n \, , \, c_{-n-1} = c_{n+1} \, , \, d_{-n-1} = d_{n+1}$. Let $\mathbf{x}_{n+2}^\pm, \mathbf{x}_n^\pm$ be the unique symmetric or skewsymmetric solutions of

$$\begin{array}{ll} T_{n+2}(\mathbf{c})\mathbf{x}_{n+2}^+ = P_+\mathbf{e}_1 \, , & T_n(\mathbf{c})\mathbf{x}_n^+ = P_+\mathbf{e}_1 \, , \\ T_{n+2}(\mathbf{d})\mathbf{x}_{n+2}^- = P_-\mathbf{e}_1 \, , & T_n(\mathbf{d})\mathbf{x}_n^- = P_-\mathbf{e}_1 \, . \end{array} \tag{12.28}$$

(Note that $\mathbf{x}_n^+ = \mathbf{w}_3 \, , \mathbf{x}_n^- = \mathbf{w}_4$. The solutions $\mathbf{x}_{n+2}^\pm$ are up to a constant factor equal to the vectors $\mathbf{v}_\pm$.)

Corollary 12.10. *Let R_{n+2} be a nonsingular, centrosymmetric extension (*12.27*) of R_n. Then the equations (*12.28*) have unique symmetric or skewsymmetric solutions and*

$$R_n^{-1} = \frac{1}{r_+}\mathrm{Bez_{split}}(\mathbf{x}_{n+2}^+, \mathbf{u}_+) + \frac{1}{r_-}\mathrm{Bez_{split}}(\mathbf{x}_{n+2}^-, \mathbf{u}_-) \, ,$$

where $r_\pm$ are the first components of $\mathbf{x}_{n+2}^\pm$ and $\mathbf{u}_\pm = \begin{bmatrix} 0 \\ \mathbf{x}_n^\pm \\ 0 \end{bmatrix}$.

If $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$ are nonsingular then R_n is nonsingular. Indeed, taking (12.24) into account $R_n\mathbf{u} = 0$ leads to

$$P_+T_n(\mathbf{c})P_+\mathbf{u} = -P_-T_n(\mathbf{d})P_-\mathbf{u} \, .$$

Hence $P_+\mathbf{u} = \mathbf{0}$ and $P_-\mathbf{u} = \mathbf{0}$ which means $\mathbf{u} = \mathbf{0}$. The converse is not true. Take, for example, $\mathbf{c} = (1, 1, 1)$ and $\mathbf{d} = (-1, 1, -1)$, then $T_2(\mathbf{c})$ and $T_2(\mathbf{d})$ are singular, whereas $R_2 = 2I_2$ is nonsingular. One might conjecture that for a nonsingular R_n there is always a representation (12.17) with nonsingular $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$. For $n = 2$ this is true. But this fails to be true for $n = 3$. Consider, for example,

$$\mathbf{c} = (1, 0, 1, 0, 1) \text{ and } \mathbf{d} = (0, 0, 1, 0, 0) \, .$$

Then

$$R_3 = T_3(\mathbf{c})P_+ + T_3(\mathbf{d})P_- = \frac{1}{2}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

is nonsingular. But $T_3(\mathbf{c})$ is a chess-board matrix (1.4) with $c = 1, b = 0$ which is singular and uniquely determined in the representation of R_3 (cf. Exercise 16).

Let us consider besides $R_n = T_n(\mathbf{a}) + T_n(\mathbf{b})J_n$ the matrix $R_n^- = T(\mathbf{a}) - T(\mathbf{b})J_n$. If R_n is represented in the form (12.24) then the corresponding representation of R_n^- is

$$R_n^- = P_+ T_n(\mathbf{d})P_+ + P_- T_n(\mathbf{c})P_- \,,$$

which means that the roles of $\mathbf{c}$ and $\mathbf{d}$ are interchanged. We conclude the following

Proposition 12.11. *The (symmetric) Toeplitz matrices $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$ are nonsingular if and only if both R_n and R_n^- are nonsingular.*

Proof. We have already shown that the nonsingularity of $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$ implies the nonsingularity of R_n. The nonsingularity of R_n^- follows with the same arguments. It remains to show that the singularity of $T_n(\mathbf{c})$ (or $T_n(\mathbf{d})$) leads to the singularity of R_n or R_n^-. Let $\mathbf{u}$ be a nontrivial vector such that $T_n(\mathbf{c})\mathbf{u} = 0$. We split $\mathbf{u}$ into its symmetric and skewsymmetric parts

$$\mathbf{u} = \mathbf{u}_+ + \mathbf{u}_- \quad (\mathbf{u}_\pm \in \mathbb{F}^n_\pm).$$

Clearly, at least one of the vectors $\mathbf{u}_+$ or $\mathbf{u}_-$ is nonzero, and $T_n(\mathbf{c})\mathbf{u}_+ = T_n(\mathbf{c})\mathbf{u}_- = 0$. Since

$$R_n\mathbf{u}_+ = T_n(\mathbf{c})\mathbf{u}_+ \,, \quad R_n^-\mathbf{u}_- = T_n(\mathbf{c})\mathbf{u}_-$$

we obtain that R_n or R_n^- is singular. This is also obtained if we assume that $T_n(\mathbf{d})$ is singular. ■

5. Inversion of centro-skewsymmetric T+H matrices. In this subsection let us consider T+H matrices R_n which are centro-skewsymmetric, $R_n = -R_n^J$. Since for an $n \times n$ centro-skewsymmetric matrix A, $\det A = (-1)^n \det A$, all centro-skewsymmetric matrices of odd order are singular. Hence we consider here mainly matrices of even order. The centro-skewsymmetric counterpart of Proposition 12.7 is as follows.

Proposition 12.12. *Let R_n be an $n \times n$ T+H matrix. Then the following assertions are equivalent.*

1. *R_n is centro-skewsymmetric.*
2. *There is a representation (*12.1*) (resp. (*12.17*)) such that the Toeplitz matrices $T_n(\mathbf{a})$ and $T_n(\mathbf{b})$ (resp. $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$) are skewsymmetric.*
3. *There is a representation (*12.1*) (resp. (*12.17*)) such that* $\mathbf{a}$ *and* $\mathbf{b}$ *(resp.* $\mathbf{c}$ *and* $\mathbf{d}$*) are skewsymmetric vectors.*

In the remaining part of this subsection we only use such representations. In this case (12.17) can be rewritten as

$$R_n = P_- T_n(\mathbf{c})P_+ + P_+ T_n(\mathbf{d})P_- \,.$$

Its transposed matrix is given by

$$R_n^T = -(P_- T_n(\mathbf{d})P_+ + P_+ T_n(\mathbf{c})P_-) \,.$$

In the equations (12.19) we have $\widetilde{\mathbf{c}} = -\mathbf{d}$ and $\widetilde{\mathbf{d}} = -\mathbf{c}$.

In general, R_n is neither symmetric nor skewsymmetric, thus a connection between the solutions of (12.18) and (12.19) is not obvious. If we choose $c_n = -c_{-n}$ and $d_n = -d_{-n}$ than $\mathbf{c}_- = -\mathbf{c}_+$, $\mathbf{d}_- = -\mathbf{d}_+$. Hence the right-hand sides of the equations (12.18), (12.19) are either symmetric or skewsymmetric. Since R_n as a centro-skewsymmetric matrix maps $\mathbb{F}^n_\pm$ to $\mathbb{F}^n_\mp$, we obtain that the solutions are also either symmetric or skewsymmetric. Let us indicate these symmetry properties again by denoting

$$\mathbf{w}_+ = \mathbf{w}_1 \,, \mathbf{w}_- = \mathbf{w}_2 \,, \mathbf{x}_- = \mathbf{w}_3 \,, \mathbf{x}_+ = \mathbf{w}_4,$$
$$\widetilde{\mathbf{x}}_- = \mathbf{z}_1 \,, \ \widetilde{\mathbf{x}}_+ = \mathbf{z}_2 \,, \ \widetilde{\mathbf{w}}_+ = \mathbf{z}_3 \,, \ \widetilde{\mathbf{w}}_- = \mathbf{z}_4.$$

Since these symmetries pass to the augmented vectors $\breve{\mathbf{u}}_j$, $\breve{\mathbf{v}}_j$ of (12.20) we set

$$\begin{array}{l}\mathbf{v}_+ = \breve{\mathbf{u}}_1\,,\ \mathbf{v}_- = \breve{\mathbf{u}}_2\,,\ \mathbf{u}_- = \breve{\mathbf{u}}_3\,,\ \mathbf{u}_+ = \breve{\mathbf{u}}_4\,,\\ \widetilde{\mathbf{v}}_+ = \breve{\mathbf{v}}_3\,,\ \widetilde{\mathbf{v}}_- = \breve{\mathbf{v}}_4\,,\ \widetilde{\mathbf{u}}_- = \breve{\mathbf{v}}_1\,,\ \widetilde{\mathbf{u}}_+ = \breve{\mathbf{v}}_2\,.\end{array} \tag{12.29}$$

The equations (12.18), (12.19) turn into Toeplitz equations,

$$T_n(\mathbf{c})\mathbf{x}_+ = P_-\mathbf{e}_1\,,\ T_n(\mathbf{c})\mathbf{w}_+ = P_-\mathbf{c}_+\,,\ T_n(\mathbf{d})\mathbf{x}_- = P_+\mathbf{e}_1\,,\ T_n(\mathbf{d})\mathbf{w}_- = P_+\mathbf{d}_+ \tag{12.30}$$

and

$$T_n(\mathbf{c})\widetilde{\mathbf{x}}_- = -P_+\mathbf{e}_1\,,\ T_n(\mathbf{c})\widetilde{\mathbf{w}}_- = P_+\mathbf{c}_+\,,\ T_n(\mathbf{d})\widetilde{\mathbf{x}}_+ = -P_-\mathbf{e}_1\,,\ T_n(\mathbf{d})\widetilde{\mathbf{w}}_+ = P_-\mathbf{d}_+\,. \tag{12.31}$$

According to Proposition 12.6 and (11.3) R_n^{-1} given by the augmented vectors (12.29) of these solutions via

$$\nabla_{T+H} R_n^{-1} = \mathbf{u}_-\widetilde{\mathbf{v}}_+^T - \mathbf{v}_+\widetilde{\mathbf{u}}_-^T - \mathbf{v}_-\widetilde{\mathbf{u}}_+^T + \mathbf{u}_+\widetilde{\mathbf{v}}_-^T\,. \tag{12.32}$$

Now we show how the solutions of (12.30) and (12.31) are related. First we compare the equations $T_n(\mathbf{c})\widetilde{\mathbf{x}}_- = P_+\mathbf{e}_1$ and $T_n(\mathbf{c})\mathbf{x}_+ = P_-\mathbf{e}_1$ for any $\mathbf{c} \in \mathbb{F}_-^{2n-1}$. The following lemma shows that there is an essential difference between the centrosymmetric and centro-skewsymmetric cases.

Lemma 12.13. *Let $T_n(\mathbf{c})$ be skewsymmetric. If the equation $T_n(\mathbf{c})\widetilde{\mathbf{x}}_- = -P_+\mathbf{e}_1$ is solvable, then equation $T_n(\mathbf{c})\mathbf{x}_+ = P_-\mathbf{e}_1$ is also solvable, and if n is even, then the converse is also true. If $\widetilde{\mathbf{x}}_-$ is a skewsymmetric solution of the first equation, then a solution of the second equation is given by*

$$\mathbf{x}_+(t) = \frac{1+t}{1-t}\,\widetilde{\mathbf{x}}_-(t) \tag{12.33}$$

Proof. If $T_n(\mathbf{c})\widetilde{\mathbf{x}}_- = -P_+\mathbf{e}_1$ is solvable, then there exists a skewsymmetric solution $\widetilde{\mathbf{x}}_-$. Since $\widetilde{\mathbf{x}}_-$ is skewsymmetric, we have $\widetilde{\mathbf{x}}_-(1) = 0$. Hence (12.33) defines a polynomial $\mathbf{x}_+(t)$. Moreover, the coefficient vector $\mathbf{x}_+$ is symmetric.

Let $\mathbf{z} \in \mathbb{F}^n$ be defined by $\mathbf{z}(t) = \frac{1}{t-1}\,\widetilde{\mathbf{x}}_-(t)$ and $\mathbf{z}^1(t) = t\mathbf{z}(t)$. If now $T_n(\mathbf{c})\mathbf{z} = (r_k)_{k=1}^n$, then $T_n(\mathbf{c})\mathbf{z}^1 = (r_{k-1})_{k=1}^n$, where r_0 is some number. In view of $T_n(\mathbf{c})(\mathbf{z}^1 - \mathbf{z}) = -P_+\mathbf{e}_1$, we have

$$r_0 - r_1 = -\frac{1}{2},\ r_1 = r_2 = \cdots = r_{n-1},\ r_{n-1} - r_n = -\frac{1}{2}.$$

Since the $(n-1)\times(n-1)$ principal submatrix T_{n-1} of $T_n(\mathbf{c})$ is skewsymmetric and the vector $\mathbf{z}' \in \mathbb{F}^{n-1}$ obtained from $\mathbf{z}$ by deleting the last (zero) component is symmetric, the vector $T_{n-1}\mathbf{z}' = (r_k)_{k=1}^{n-1}$ is skewsymmetric. Hence

$$r_0 = -\frac{1}{2},\ r_1 = r_2 = \cdots = r_{n-1} = 0,\ r_n = \frac{1}{2}\,.$$

We conclude that $T_n(\mathbf{c})(\mathbf{z} + \mathbf{z}^1) = -P_-\mathbf{e}_1$. This means that $\mathbf{x}_+ = \mathbf{z} + \mathbf{z}^1$.

The proof of the converse direction follows the same lines. One has to take into account that if n is even and $\mathbf{x}_+$ is symmetric, then $\mathbf{x}_+(-1) = 0$. Hence $\mathbf{z}(t) = \frac{1}{t+1}\mathbf{x}_+(t)$ is a polynomial. ■

Note that the converse direction of Lemma 12.13 is not true if n is odd. If, for example,

$$T_3(\mathbf{c}) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

then $T_3(\mathbf{c})\mathbf{x}_+ = P_-\mathbf{e}_1$ is solvable but $T_n(\mathbf{c})\widetilde{\mathbf{x}}_- = P_+\mathbf{e}_1$ is not.

The relation between the solutions $\mathbf{x}_+$ and $\widetilde{\mathbf{x}}_-$ extends to the augmented vectors $\mathbf{u}_+$ and $\widetilde{\mathbf{u}}_-$. We have

$$\mathbf{u}_+(t) = \frac{1+t}{1-t}\,\widetilde{\mathbf{u}}_-(t).$$

Replacing $\mathbf{c}$ by $\mathbf{d}$ we obtain

$$\widetilde{\mathbf{u}}_+(t) = \frac{1+t}{1-t}\,\mathbf{u}_-(t).$$

Now we compare the equations $T_n(\mathbf{c})\mathbf{w}_+ = P_-\mathbf{c}_+$ and $T_n(\mathbf{c})\widetilde{\mathbf{w}}_- = P_+\mathbf{c}_+$. More precisely, we compare the augmented vectors $\mathbf{v}_+$ and $\widetilde{\mathbf{v}}_-$.

Lemma 12.14. *Let $T_n(\mathbf{c})$ be skewsymmetric. If the equation $T_n(\mathbf{c})\widetilde{\mathbf{w}}_- = P_+\mathbf{c}_+$ is solvable, then the equation $T_n(\mathbf{c})\mathbf{w}_+ = P_-\mathbf{c}_+$ is also solvable, and the augmented vectors of these solutions are related via*

$$\mathbf{v}_+(t) = \frac{1+t}{1-t}\,\widetilde{\mathbf{v}}_-(t). \tag{12.34}$$

If n is even, then the solvability of $T_n(\mathbf{c})\mathbf{w}_+ = P_-\mathbf{c}_+$ implies the solvability of $T_n(\mathbf{c})\widetilde{\mathbf{w}}_- = P_+\mathbf{c}_+$.

Proof. Let $\widetilde{T}$ denote the $n\times(n+2)$ matrix $\widetilde{T} = [c_{i-j+1}]_{i=0}^{n-1}\,{}_{j=0}^{n+1}$ with $c_{-n} = c_n$. If $T_n(\mathbf{c})\widetilde{\mathbf{w}}_- = P_+\mathbf{c}_+$, then $\widetilde{T}\widetilde{\mathbf{v}}_- = \mathbf{0}$. Furthermore, if $\widetilde{\mathbf{w}}_-$ is skewsymmetric, then $\widetilde{\mathbf{v}}_-$ is skewsymmetric. Hence $\widetilde{\mathbf{v}}_-(1) = 0$ and $\mathbf{z}(t) = \frac{1}{1-t}\,\widetilde{\mathbf{v}}_-(t)$ is a polynomial. We consider the coefficient vector $\mathbf{z}$ of $\mathbf{z}(t)$ as a vector in $\mathbb{F}^{n+2}$ and denote the coefficient vector of $t\mathbf{z}(t)$ by $\mathbf{z}^1$.

Suppose that $\widetilde{T}\mathbf{z} = (r_k)_1^n$, then $\widetilde{T}\mathbf{z}^1 = (r_{k-1})_1^n$, where r_0 is some number. Since $\mathbf{z}-\mathbf{z}^1 = \widetilde{\mathbf{v}}_-$ and $\widetilde{T}\widetilde{\mathbf{v}}_- = 0$, we conclude that $r_0 = \cdots = r_n$.

Let $T_{n+1}(\mathbf{c})$ denote the $(n+1)\times(n+1)$ matrix $[c_{i-j}]_{i,j=0}^n$ and $\mathbf{z}' \in \mathbb{F}^{n+1}$ the vector obtained from $\mathbf{z}$ deleting the last (zero) component. Then $T_{n+1}(c)\mathbf{z}' = (r_k)_{k=0}^n$. Here $T_{n+1}(\mathbf{c})$ is skewsymmetric and $\mathbf{z}'$ is symmetric, thus the vector $(r_k)_{k=0}^n$ is skewsymmetric. Since all components are equal, it must be the zero vector. We obtain $\widetilde{T}(\mathbf{z}+\mathbf{z}^1) = \mathbf{0}$. Observe that $\mathbf{z}+\mathbf{z}^1$ is symmetric and that its first component is equal to 1. Therefore, $\mathbf{z}+\mathbf{z}^1 = \mathbf{v}_+ = [\,1\ -2\mathbf{w}_+^T\ 1\,]^T$ for some symmetric vector $\mathbf{w}_+ \in \mathbb{F}^n$. This vector is now a solution of the equation $T_n(\mathbf{c})\mathbf{w}_+ = P_-\mathbf{c}_+$.

The converse direction is proved analogously taking into account that if n is even, then the length of $\mathbf{v}_+$, which is $n+2$, is even. Hence $\mathbf{v}_+(-1) = 0$ and $\mathbf{z}(t) = \frac{1}{1+t}\mathbf{v}_+(t)$ is well defined. ∎

Replacing $\mathbf{c}$ by $\mathbf{d}$ we obtain

$$\widetilde{\mathbf{v}}_+(t) = \frac{1+t}{1-t}\,\mathbf{v}_-(t).$$

Taking (12.32), Lemma 12.13 and Lemma 12.14 together we arrive at

$$\begin{aligned}\nabla_{T+H}(R_n^{-1})(t,s) \;=\; & \mathbf{u}_-(t)\frac{1+s}{1-s}\mathbf{v}_-(s) - \mathbf{v}_-(t)\frac{1+s}{1-s}\mathbf{u}_-(s) \\ & -\mathbf{v}_+(t)\frac{1-s}{1+s}\mathbf{u}_+(s) + \mathbf{u}_+(t)\frac{1-s}{1+s}\mathbf{v}_+(s)\,,\end{aligned} \tag{12.35}$$

which finally leads to the following theorem taking (11.3) into account.

Theorem 12.15. *Let the centro-skewsymmetric T+H matrix R_n be nonsingular and given by (12.17). Then the equations (12.30) are solvable and the generating function of the inverse matrix is given by the augmented vectors of the solutions of these equations via*

$$R_n^{-1}(t,s) = B_+(t,s)\,\frac{s-1}{s+1} + B_-(t,s)\,\frac{s+1}{s-1}$$

and

$$B_\pm = \mathrm{Bez}_{\mathrm{split}}(\mathbf{u}_\pm, \mathbf{v}_\pm)\,.$$

Note that for a nonsingular matrix R_n all equations (12.30) and (12.31) are uniquely solvable. Moreover, we observe that $\mathbf{x} = \mathbf{x}_+ - \widetilde{\mathbf{x}}_-$ is a solution of $T_n(\mathbf{c})\mathbf{x} = \mathbf{e}_1$ and $\mathbf{w} = \mathbf{w}_+ - \widetilde{\mathbf{w}}_-$ is a solution of $T_n(\mathbf{c})\mathbf{w} = \mathbf{c}_-^J$. Taking Proposition 4.6 into account we obtain the nonsingularity of $T_n(\mathbf{c})$. Analogously, $T_n(\mathbf{d})$ is nonsingular. This leads to the following surprising conclusion.

Corollary 12.16. *For a centro-skewsymmetric T+H matrix*

$$R_n = T(\mathbf{a}) + T(\mathbf{b})J_n = T(\mathbf{c})P_+ + T(\mathbf{d})P_-$$

with skewsymmetric vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, *the following assertions are equivalent:*

1. R_n *is nonsingular.*
2. $R_n^- = T(\mathbf{a}) - T(\mathbf{b})J_n$ *is nonsingular.*
3. $T(\mathbf{c})$ *and* $T(\mathbf{d})$ *are nonsingular.*

To present the counterpart of Corollary 12.10 let us extend the nonsingular centro-skewsymmetric T+H matrix R_n given by (12.17) to a nonsingular centro-skewsymmetric T+H matrix R_{n+2}, such that R_n is its central submatrix of order n.

$$R_{n+2} = T_{n+2}(\mathbf{c})P_+ + T_{n+2}(\mathbf{d})P_-\,, \tag{12.36}$$

where $\mathbf{c}$ and $\mathbf{d}$ are extensions of the original vectors $\mathbf{c}$ and $\mathbf{d}$ by corresponding components $c_{-n} = -c_n$, $d_{-n} = -d_n$, $c_{-n-1} = -c_{n+1}$, $d_{-n-1} = -d_{n+1}$. Let $\mathbf{x}_{n+2}^\pm, \mathbf{x}_n^\pm$ be the unique symmetric or skewsymmetric solutions of

$$\begin{array}{ll} T_{n+2}(\mathbf{c})\mathbf{x}_{n+2}^- = P_+\mathbf{e}_1\,, & T_n(\mathbf{c})\mathbf{x}_n^- = P_+\mathbf{e}_1\,, \\ T_{n+2}(\mathbf{d})\mathbf{x}_{n+2}^+ = P_-\mathbf{e}_1\,, & T_n(\mathbf{d})\mathbf{x}_n^+ = P_-\mathbf{e}_1\,. \end{array} \tag{12.37}$$

Note that $\mathbf{x}_n^\pm = -\widetilde{\mathbf{x}}_\pm$, (solutions of (12.31)), thus $-\mathbf{u}_\pm$ are the augmented vectors defined by $\mathbf{u}_\pm(t) = t\mathbf{x}_n^\pm(t)$. The solutions $\mathbf{x}_{n+2}^\pm$ are up to a constant factor equal to the vectors $\mathbf{v}_\pm$.

Corollary 12.17. *Let R_{n+2} be a nonsingular and centro-skewsymmetric extension (12.36) of R_n. Then the equations (12.37) have unique symmetric or skewsymmetric solutions and*

$$R_n^{-1} = \frac{1}{r_+}\mathrm{Bez}_{\mathrm{split}}(\mathbf{x}_{n+2}^+, \mathbf{u}_+)\frac{s-1}{s+1} + \frac{1}{r_-}\mathrm{Bez}_{\mathrm{split}}(\mathbf{x}_{n+2}^-, \mathbf{u}_-)\frac{s+1}{s-1}\,,$$

where $r_\pm$ are the first components of $\mathbf{x}_{n+2}^\pm$.

13. Exercises

1. An $n \times n$ matrix B is called *quasi-T-Bezoutian* if $\nabla_T B$ introduced in (2.17) has rank 2 at most.
   (a) Show that B is a quasi-T-Bezoutian if and only if BJ_n is a quasi-H-Bezoutian (introduced in Section 2,4).
   (b) State and prove a proposition about the representation of a quasi-T-Bezoutian that is analogous to Proposition 2.4.
2. The special Toeplitz matrix

$$Z_n^\alpha(\mathbf{a}) = \begin{bmatrix} a_0 & \alpha a_{n-1} & \dots & \alpha a_1 \\ a_1 & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha a_{n-1} \\ a_{n-1} & \dots & a_1 & a_0 \end{bmatrix} \qquad (\alpha \in \mathbb{F})$$

   is called α-*circulant.*
   (a) Show that the T-Bezoutian of a polynomial $\mathbf{u}(t) \in \mathbb{F}^{n+1}$ and $t^n - \alpha$ is an α-circulant and each α-circulant is of this form.
   (b) Show that a T-Bezoutian is a Toeplitz matrix if and only if it is an α-circulant for some α or an upper triangular Toeplitz matrix.
3. Let $\mathbf{u}(t)$ be a polynomial of degree n and $\mathbf{v}(t)$ a polynomial of degree $\leq n$. Describe the nullspace of the transpose of $\mathrm{Res}_p(\mathbf{u}, \mathbf{v})$ (introduced in Section 3,1) in terms of the greatest common divisor of $\mathbf{u}(t)$ and $\mathbf{v}(t)$. Use this to show that the nullity of $\mathrm{Res}_p(\mathbf{u}, \mathbf{v})$ is, independently of p, equal to the degree of the greatest common divisor of $\mathbf{u}(t)$ and $\mathbf{v}(t)$.
4. (a) Show that an $n \times n$ matrix A is Toeplitz if and only if $I_{+-} \nabla_T(A) I_{+-}^T$ is the zero matrix, where I_{+-} is introduced in (12.14).
   (b) Show that the product of two nonzero Toeplitz matrices is Toeplitz again if and only if both factors are α-circulants for the same α or both are upper (or lower) triangular Toeplitz matrices.
5. Prove that the nonsingularity of a Toeplitz matrix $T_n = [\, a_{i-j} \,]_{i,j=1}^n$ follows from the solvability of the equations

$$T_n \mathbf{y} = \mathbf{e}_1 \quad \text{and} \quad T_n \mathbf{z} = (a_{-n+j-1})_{j=1}^n \,,$$

   where a_{-n} is arbitrarily chosen. Construct a fundamental system from these solutions.
6. Design a Levinson-type algorithm for the solution of the Bezout equation (7.14), i.e. an algorithm that does not rely on successive polynomial division. Compare the complexity of this algorithm with the complexity of the algorithm described in Section 7,7.
   Hint. Consider first the “regular” case in which the degrees of all quotients $\mathbf{q}_i(t)$ are equal to 1.
7. Let $\mathbf{p}(t) = p_1 + p_2 t + p_3 t^2 + t^3$ be a monic real polynomial. Show the following *theorem of Vyshnegradsky.* The polynomial $\mathbf{p}(t)$ has all its roots in the left half plane if and only if all coefficients are positive and $p_2 p_3 > p_1$.
8. The factorizations presented in this paper can be used to derive formulas for the determinants of Bezoutians, Hankel, Toeplitz and resultant matrices. To solve the following problems one can use Vandermonde factorization or reduction and take into account that

$$\det V_n(\mathbf{t}) = \prod_{i>j} (t_i - t_j) \,,$$

where $\mathbf{t} = (t_1, \ldots, t_n)$ or apply Barnett's formula and

$$\det \mathbf{p}(C(\mathbf{u})) = \prod_{i=1}^{n} \mathbf{p}(t_i)$$

if $\mathbf{u}(t) = \prod_{i=1}^{n}(t - t_i)$. Suppose that $\mathbf{v}(t) = \prod_{i=1}^{m}(t - s_i)$ and $\mathbf{u}(t) = \prod_{i=1}^{n}(t - t_i)$ are complex polynomials and $m \leq n$.

(a) Show that

$$\det H_n\left(\frac{\mathbf{v}}{\mathbf{u}}\right) = (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^{n}\prod_{j=1}^{m}(t_i - s_j)\,,$$

and find an analogous formula for $\det T_n\left(\frac{\mathbf{v}}{\mathbf{u}}\right)$.

(b) Derive from (a) formulas for the determinants of $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$, $\mathrm{Bez}_T(\mathbf{u},\mathbf{v})$, $\mathrm{Res}\,(\mathbf{u},\mathbf{v})$.

9. Find the Toeplitz matrices

$$T_n\left(\frac{1}{(1-ct)^m}\right) \quad \text{and} \quad T_n\left(\left(\frac{1+ct}{1-ct}\right)^m\right)$$

10. Let $\mathbf{u}(t)$ and $\mathbf{v}(t) = \mathbf{v}_1(t)\mathbf{v}_2(t)$ be polynomials of degree n.

(a) Show that

$$\mathrm{Bez}_H(\mathbf{u},\mathbf{v}) = \mathrm{Bez}_H(\mathbf{u},\mathbf{v}_1)B(\mathbf{u})^{-1}\mathrm{Bez}_H(\mathbf{u},\mathbf{v}_2)$$

(b) If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are monic, show that

$$\mathrm{Bez}_H(\mathbf{u},\mathbf{v}) = B(\mathbf{u})J_nB(\mathbf{v})(C(\mathbf{u})^n - C(\mathbf{v})^n)\,.$$

11. Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be complex polynomials of degree n and m, respectively, where $m \leq n$, and let $t_1, \ldots, t_r$ be the different roots of the greatest common divisor of $\mathbf{u}(t)$ and $\mathbf{v}(t)$ and $\nu_1, \ldots, \nu_r$ their multiplicities. Let vectors $\ell_k(c,\nu)$ be defined by

$$\ell_k(c,\nu) = \left(\binom{i-1}{\nu-1}c^{i-k}\right)_{i=1}^{k}\,.$$

Show that the vectors $\ell_n(t_i,k)$, where $k = 1, \ldots, \nu_i$, $i = 1, \ldots, r$ form a basis of the nullspace of $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$ and the corresponding vectors $\ell_{m+n+p}(t_i,k)$ a basis of the nullspace of $\mathrm{Res}_p(\mathbf{u},\mathbf{v})$ introduced in Section 3,1.

12. Let $\mathbf{u}(t)$ and $\mathbf{p}(t)$ be coprime polynomials and $\deg \mathbf{p}(t) \leq \deg \mathbf{u}(t) = n$. Show that for $k > n$ the coefficient vectors of $t^j\mathbf{u}(t)$, $j = 1, \ldots, k-n$ form a basis of the nullspace of $H_k\left(\frac{\mathbf{p}}{\mathbf{u}}\right)$.

13. Let $\mathbf{u}(t)$ be a polynomial with real coefficients of degree n.

(a) Describe the number of different positive real roots in terms of the signatures of the matrices $H_n\left(\frac{t\mathbf{u}'(t)}{\mathbf{u}(t)}\right)$ and $H_n\left(\frac{\mathbf{u}'}{\mathbf{u}}\right)$.

(b) Let a and b be real numbers, $a < b$. Describe the number of different real roots of $\mathbf{u}(t)$ in the interval $[\,a,b\,]$ in terms of the signatures of the matrices $H_n(\mathbf{g})$ and $H_n\left(\frac{\mathbf{u}'}{\mathbf{u}}\right)$, where

$$\mathbf{g}(t) = \frac{(t-a)(b-t)\mathbf{u}'(t) + nt^2\mathbf{u}(t)}{\mathbf{u}(t)}\,.$$

(c) Prove a representation of $\mathrm{Res}(\mathbf{u},\mathbf{v})$ which is analoguos to that one of Proposition 3.3 but involves $\mathrm{Bez}_T(\mathbf{u},\mathbf{v})$ instead of $\mathrm{Bez}_H(\mathbf{u},\mathbf{v})$.

14. Prove that a matrix A is a T+H matrix if and only if the matrix $I_{+-}\nabla(A)I_{+-}^T$ is the zero matrix, where $\nabla(A)$ is introduced in (1.11) and I_{+-} in (12.14).

15. Let $\mathbf{e}$ and $\mathbf{e}_\sigma$ denote the vectors of $\mathbb{F}^{2n-1}$

$$\mathbf{e} = (\,1,\ 1\ldots,\ 1\,) \quad \text{and} \quad \mathbf{e}_\sigma = (\,(-1)^i\,)_{i=1}^{2n-1}\,.$$

Show that a T+H matrix $R_n = T_n(\mathbf{a}) + T_n(\mathbf{b})J_n$ is equal to $R'_n = T_n(\mathbf{a}') + T_n(\mathbf{b}')J_n$ if and only if, for some $\alpha, \beta \in \mathbb{F}$, $\mathbf{a}' = \mathbf{a} + \alpha\mathbf{e} + \beta\mathbf{e}_\sigma$ and $\mathbf{b}' = \mathbf{b} - \alpha\mathbf{e} - \beta(-1)^{n-1}\mathbf{e}_\sigma$.

16. Let R_n be an $n\times n$ T+H matrix given by (12.17) and by $R_n = T_n(\mathbf{c}')P_+ + T_n(\mathbf{d}')P_-$. Show that
    (a) If n is odd, then $\mathbf{c}' = \mathbf{c}$, i.e. $\mathbf{c}$ is unique, and $\mathbf{d}'$ is of the form $\mathbf{d}' = \mathbf{d} + \alpha\mathbf{e} + \beta\mathbf{e}_\sigma$ for $\alpha, \beta \in \mathbb{F}$.
    (b) If n is even, then $\mathbf{c}'$ is of the form $\mathbf{c}' = \mathbf{c} + \alpha\mathbf{e}_\sigma$ and $\mathbf{d}'$ of the form $\mathbf{d}' = \mathbf{d} + \beta\mathbf{e}$ for $\alpha, \beta \in \mathbb{F}$.

    Here $\mathbf{e}$ and $\mathbf{e}_\sigma$ are as above.

17. Let R_n be an $n\times n$ nonsingular, centro-skewsymmetric T+H matrix given by (12.17). Show that

$$R_n^{-1} = T_n(\mathbf{c})^{-1}P_- + T_n(\mathbf{d})^{-1}P_+\,.$$

14. Notes

1. The Bezoutian and the resultant matrix have a long history, which goes back to the 18th century. Both concepts grew up from the work of Euler [5] in connection with the elimination of variables for the solution of systems of nonlinear algebraic equations. In 1764, Bezout generalized a result of Euler [1]. In his solution the determinant of a matrix occured which was only in 1857 shown by Cayley [3] to be the same as that being today called the (Hankel) Bezoutian. For more detailed information see [66].
2. The classical studies of Jacobi [51] and Sylvester [65] utilized Bezoutians in the theory of separation of polynomial roots. Hermite [49] studied the problem of counting the roots of a polynomial in the upper half-plane. Clearly, this is equivalent to finding the number of roots in the left half-plane, which is important for stability considerations. A nice review of classical results concerning root localization problems is given in the survey paper [52]; see also [50], [8], [63], [61], [60].
3. The importance of Bezoutians for the inversion of Hankel and Toeplitz matrices became clear much later. Only in 1974, Lander [55] established the fundamental result that the inverses of a (nonsingular) Hankel matrix can be represented as a Bezoutian of two polynomials and that, conversely, any nonsingular Bezoutian is the inverse of a Hankel matrix. Similar results are true for Toeplitz matrices. In [55] also a Vandermonde factorization of Bezoutians was presented.
4. There is a huge number of papers and books dedicated to Bezoutians, resultant matrices and connected problems. Let me recommend some books and survey papers (see also the references therein) to light the younger history and recent developments, to pursue and to accentuate the topic in different directions. (This list is far away from being complete!)

   Books: Gohberg, Lancaster, and Rodman [10], Heinig and Rost [33], Lancaster and Tismenetsky [54], Bini and Pan [2], Fuhrmann [6], Pan [62], Lascoux [56].

   Papers: Gohberg, Kaashoek, Lerer, and Rodman [9], Lerer and Tismenetsky [58], Lerer and Rodman [57], Fuhrmann and Datta [7], Gohberg and Shalom [11], Emiris and Mourrain [4], Mourrain and Pan [59].

References

[1] E. Bezout. Recherches sur le degre des equations resultants de l'evanouissement des inconnues, et sur les moyens qu'il convenient d'employer pour trouver ces equations. *Mem Acad. Roy. Sci. Paris*, pages 288–338, 1764.

[2] D. Bini and V. Y. Pan. *Polynomial and matrix computations. Vol. 1.* Progress in Theoretical Computer Science. Birkhäuser Boston Inc., Boston, MA, 1994. Fundamental algorithms.

[3] A. Cayley. Note sur la methode d'elimination de Bezout. *J. Reine Angew. Math.*, 53:366–367, 1857.

[4] I. Z. Emiris and B. Mourrain. Matrices in elimination theory. *J. Symbolic Comput.*, 28(1-2):3–44, 1999. Polynomial elimination—algorithms and applications.

[5] L. Euler. *Introductio in analysin infinitorum. Tomus primus.* Sociedad Andaluza de Educación Matemática "Thales", Seville, 2000. Reprint of the 1748 original.

[6] P. A. Fuhrmann. *A polynomial approach to linear algebra.* Universitext. Springer-Verlag, New York, 1996.

[7] P. A. Fuhrmann and B. N. Datta. On Bezoutians, Vandermonde matrices, and the Liénard-Chipart stability criterion. In *Proceedings of the Fourth Haifa Matrix Theory Conference (Haifa, 1988)*, volume 120, pages 23–37, 1989.

[8] F. R. Gantmacher. *Matrizentheorie.* Hochschulbücher für Mathematik [University Books for Mathematics], 86. VEB Deutscher Verlag der Wissenschaften, Berlin, 1986. With a foreword by D. P. Želobenko, Translated from the Russian by Helmut Boseck, Dietmar Soyka and Klaus Stengert.

[9] I. Gohberg, M. A. Kaashoek, L. Lerer, and L. Rodman. Common multiples and common divisors of matrix polynomials. II. Vandermonde and resultant matrices. *Linear and Multilinear Algebra*, 12(3):159–203, 1982/83.

[10] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix polynomials.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1982. Computer Science and Applied Mathematics.

[11] I. Gohberg and T. Shalom. On Bezoutians of nonsquare matrix polynomials and inversion of matrices with nonsquare blocks. *Linear Algebra Appl.*, 137/138:249–323, 1990.

[12] I. C. Gohberg and G. Heinig. The inversion of finite Toeplitz matrices. *Mat. Issled.*, 8(3(29)):151–156, 183, 1973.

[13] I. C. Gohberg and G. Heinig. Inversion of finite Toeplitz matrices consisting of elements of a noncommutative algebra. *Rev. Roumaine Math. Pures Appl.*, 19:623–663, 1974.

[14] I. C. Gohberg and G. Heinig. The resultant matrix and its generalizations. I. The resultant operator for matrix polynomials. *Acta Sci. Math. (Szeged)*, 37:41–61, 1975.

[15] I. C. Gohberg and G. Heinig. The resultant matrix and its generalizations. II. The continual analogue of the resultant operator. *Acta Math. Acad. Sci. Hungar.*, 28(3-4):189–209, 1976.

[16] G. Heinig. The notion of Bezoutian and of resultant for operator pencils. *Funkcional. Anal. i Priložen.*, 11(3):94–95, 1977.

[17] G. Heinig. Über Block-Hankelmatrizen und den Begriff der Resultante für Matrixpolynome. *Wiss. Z. Techn. Hochsch. Karl-Marx-Stadt*, 19(4):513–519, 1977.

[18] G. Heinig. Verallgemeinerte Resultantenbegriffe bei beliebigen Matrixbüscheln. I. Einseitiger Resultantenoperator. *Wiss. Z. Tech. Hochsch. Karl-Marx-Stadt*, 20(6):693–700, 1978.

[19] G. Heinig. Verallgemeinerte Resultantenbegriffe bei beliebigen Matrixbüscheln. II. Gemischter Resultantenoperator. *Wiss. Z. Tech. Hochsch. Karl-Marx-Stadt*, 20(6):701–703, 1978.

[20] G. Heinig. Bezoutiante, Resultante und Spektralverteilungsprobleme für Operatorpolynome. *Math. Nachr.*, 91:23–43, 1979.

[21] G. Heinig. Generalized resultant operators and classification of linear operator pencils up to strong equivalence. In *Functions, series, operators, Vol. I, II (Budapest, 1980)*, volume 35 of *Colloq. Math. Soc. János Bolyai*, pages 611–620. North-Holland, Amsterdam, 1983.

[22] G. Heinig. Chebyshev-Hankel matrices and the splitting approach for centrosymmetric Toeplitz-plus-Hankel matrices. *Linear Algebra Appl.*, 327(1-3):181–196, 2001.

[23] G. Heinig and F. Hellinger. On the Bezoutian structure of the Moore-Penrose inverses of Hankel matrices. *SIAM J. Matrix Anal. Appl.*, 14(3):629–645, 1993.

[24] G. Heinig and F. Hellinger. Displacement structure of generalized inverse matrices. *Linear Algebra Appl.*, 211:67–83, 1994. Generalized inverses (1993).

[25] G. Heinig and F. Hellinger. Moore-Penrose inversion of square Toeplitz matrices. *SIAM J. Matrix Anal. Appl.*, 15(2):418–450, 1994.

[26] G. Heinig and U. Jungnickel. Zur Lösung von Matrixgleichungen der Form $AX - XB = C$. *Wiss. Z. Tech. Hochsch. Karl-Marx-Stadt*, 23(4):387–393, 1981.

[27] G. Heinig and U. Jungnickel. On the Routh-Hurwitz and Schur-Cohn problems for matrix polynomials and generalized Bezoutians. *Math. Nachr.*, 116:185–196, 1984.

[28] G. Heinig and U. Jungnickel. On the Bezoutian and root localization for polynomials. *Wiss. Z. Tech. Hochsch. Karl-Marx-Stadt*, 27(1):62–65, 1985.

[29] G. Heinig and U. Jungnickel. Hankel matrices generated by Markov parameters, Hankel matrix extension, partial realization, and Padé-approximation. In *Operator theory and systems (Amsterdam, 1985)*, volume 19 of *Oper. Theory Adv. Appl.*, pages 231–253. Birkhäuser, Basel, 1986.

[30] G. Heinig and U. Jungnickel. Hankel matrices generated by the Markov parameters of rational functions. *Linear Algebra Appl.*, 76:121–135, 1986.

[31] G. Heinig and U. Jungnickel. Lyapunov equations for companion matrices. *Linear Algebra Appl.*, 76:137–147, 1986.

[32] G. Heinig and K. Rost. *Algebraic methods for Toeplitz-like matrices and operators*, volume 19 of *Mathematical Research*. Akademie-Verlag, Berlin, 1984.

[33] G. Heinig and K. Rost. *Algebraic methods for Toeplitz-like matrices and operators*, volume 13 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1984.

[34] G. Heinig and K. Rost. On the inverses of Toeplitz-plus-Hankel matrices. *Linear Algebra Appl.*, 106:39–52, 1988.

[35] G. Heinig and K. Rost. Inversion of matrices with displacement structure. *Integral Equations Operator Theory*, 12(6):813–834, 1989.

[36] G. Heinig and K. Rost. Matrices with displacement structure, generalized Bezoutians, and Moebius transformations. In *The Gohberg anniversary collection, Vol. I (Calgary, AB, 1988)*, volume 40 of *Oper. Theory Adv. Appl.*, pages 203–230. Birkhäuser, Basel, 1989.

[37] G. Heinig and K. Rost. Matrix representations of Toeplitz-plus-Hankel matrix inverses. *Linear Algebra Appl.*, 113:65–78, 1989.

[38] G. Heinig and K. Rost. DFT representations of Toeplitz-plus-Hankel Bezoutians with application to fast matrix-vector multiplication. *Linear Algebra Appl.*, 284(1-3):157–175, 1998. ILAS Symposium on Fast Algorithms for Control, Signals and Image Processing (Winnipeg, MB, 1997).

[39] G. Heinig and K. Rost. Representations of Toeplitz-plus-Hankel matrices using trigonometric transformations with application to fast matrix-vector multiplication. In *Proceedings of the Sixth Conference of the International Linear Algebra Society (Chemnitz, 1996)*, volume 275/276, pages 225–248, 1998.

[40] G. Heinig and K. Rost. Hartley transform representations of inverses of real Toeplitz-plus-Hankel matrices. In *Proceedings of the International Conference on Fourier Analysis and Applications (Kuwait, 1998)*, volume 21, pages 175–189, 2000.

[41] G. Heinig and K. Rost. Hartley transform representations of symmetric Toeplitz matrix inverses with application to fast matrix-vector multiplication. *SIAM J. Matrix Anal. Appl.*, 22(1):86–105 (electronic), 2000.

[42] G. Heinig and K. Rost. Representations of inverses of real Toeplitz-plus-Hankel matrices using trigonometric transformations. In *Large-scale scientific computations of engineering and environmental problems, II (Sozopol, 1999)*, volume 73 of *Notes Numer. Fluid Mech.*, pages 80–86. Vieweg, Braunschweig, 2000.

[43] G. Heinig and K. Rost. Efficient inversion formulas for Toeplitz-plus-Hankel matrices using trigonometric transformations. In *Structured matrices in mathematics, computer science, and engineering, II (Boulder, CO, 1999)*, volume 281 of *Contemp. Math.*, pages 247–264. Amer. Math. Soc., Providence, RI, 2001.

[44] G. Heinig and K. Rost. Centro-symmetric and centro-skewsymmetric Toeplitz matrices and Bezoutians. *Linear Algebra Appl.*, 343/344:195–209, 2002. Special issue on structured and infinite systems of linear equations.

[45] G. Heinig and K. Rost. Centrosymmetric and centro-skewsymmetric Toeplitz-plus-Hankel matrices and Bezoutians. *Linear Algebra Appl.*, 366:257–281, 2003. Special issue on structured matrices: analysis, algorithms and applications (Cortona, 2000).

[46] G. Heinig and K. Rost. Fast algorithms for centro-symmetric and centro-skewsymmetric Toeplitz-plus-Hankel matrices. *Numer. Algorithms*, 33(1-4):305–317, 2003. International Conference on Numerical Algorithms, Vol. I (Marrakesh, 2001).

[47] G. Heinig and K. Rost. Split algorithms for Hermitian Toeplitz matrices with arbitrary rank profile. *Linear Algebra Appl.*, 392:235–253, 2004.

[48] G. Heinig and K. Rost. Split algorithms for centrosymmetric Toeplitz-plus-Hankel matrices with arbitrary rank profile. In *The extended field of operator theory*, volume 171 of *Oper. Theory Adv. Appl.*, pages 129–146. Birkhäuser, Basel, 2007.

[49] C. Hermite. Extrait d'une lettre de Mr. Ch. Hermite de Paris a Mr. Borchard de Berlin, sur le nombre des racines d'une equation algebrique comprises entre des limits donees. *J. Reine Angew. Math.*, 52:39–51, 1856.

[50] A. S. Householder. Bezoutiants, elimination and localization. *SIAM Rev.*, 12:73–78, 1970.

[51] C. Jacobi. De eliminatione variablis e duabus aequatione algebraicis. *J. Reine Angew. Math.*, 15:101–124, 1836.

[52] M. G. Krein and M. A. Naimark. The method of symmetric and Hermitian forms in the theory of the separation of the roots of algebraic equations. *Linear and Multilinear Algebra*, 10(4):265–308, 1981. Translated from the Russian by O. Boshko and J. L. Howland.

[53] B. Krishna and H. Krishna. Computationally efficient reduced polynomial based algorithms for Hermitian Toeplitz matrices. *SIAM J. Appl. Math.*, 49(4):1275–1282, 1989.

[54] P. Lancaster and M. Tismenetsky. *The theory of matrices*. Computer Science and Applied Mathematics. Academic Press Inc., Orlando, FL, second edition, 1985.

[55] F. I. Lander. The Bezoutian and the inversion of Hankel and Toeplitz matrices (in russian). *Mat. Issled.*, 9(2 (32)):69–87, 249–250, 1974.

[56] A. Lascoux. *Symmetric functions and combinatorial operators on polynomials*, volume 99 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2003.

[57] L. Lerer and L. Rodman. Bezoutians of rational matrix functions. *J. Funct. Anal.*, 141(1):1–36, 1996.

[58] L. Lerer and M. Tismenetsky. The Bezoutian and the eigenvalue-separation problem for matrix polynomials. *Integral Equations Operator Theory*, 5(3):386–445, 1982.

[59] B. Mourrain and V. Y. Pan. Multivariate polynomials, duality, and structured matrices. *J. Complexity*, 16(1):110–180, 2000. Real computation and complexity (Schloss Dagstuhl, 1998).

[60] A. Olshevsky and V. Olshevsky. Kharitonov's theorem and Bezoutians. *Linear Algebra Appl.*, 399:285–297, 2005.

[61] V. Olshevsky and L. Sakhnovich. An operator identities approach to bezoutians. A general scheme and examples. In *Proc.of the MTNS'04 Conference*. 2004.

[62] V. Y. Pan. *Structured matrices and polynomials*. Birkhäuser Boston Inc., Boston, MA, 2001. Unified superfast algorithms.

[63] M. M. Postnikov. *Ustoichivye mnogochleny (Stable polynomials)*. “Nauka”, Moscow, 1981.

[64] K. Rost. Toeplitz-plus-Hankel Bezoutians and inverses of Toeplitz and Toeplitz-plus-Hankel matrices. *Oper. Matrices*, in print.

[65] I. Sylvester. On a theory of syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's functions, and that of the greatest algebraical common measure. *Phylos. Trans. Roy. Soc. London*, 143:407–548, 1853.

[66] H. K. Wimmer. On the history of the Bezoutian and the resultant matrix. *Linear Algebra Appl.*, 128:27–34, 1990.

Georg Heinig and Karla Rost
Dept. of Mathematics
Chemnitz University of Technology
Reichenhainer Straße 39
D-09126 Chemnitz
Germany
e-mail: `karla.rost@mathematik.tu-chemnitz.de`