Banach Algebras of Structured Matrix Sequences

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Abstract

We discuss the asymptotic behavior of matrix sequences belonging to a special class of non-commutative Banach algebras and study, in particular, the stability, and more general the Fredholm property of such sequences. The abstract results are applied to finite sections of band-dominated operators, especially in the case $l^p(\mathbb{Z})$, $1 \leq p \leq \infty$.

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Introduction

The study of operator sequences is an important task in numerical analysis and asymptotic spectral theory. There is a bulk of papers dealing with these problems. The now classical publications [6] and [10] were devoted to a deep study of the stability problem for projection methods aimed at the approximate solution of convolution-type equations. The remarkable paper [10] contains the first appearence of the use of localization techniques in the study of stability. The idea is to introduce a suitable Banach algebra which contains the sequences under consideration and to take advantage of the fact that the stability of a sequence (A_n) is equivalent to the invertibility of (A_n) modulo the ideal of sequences tending to zero in the norm. Later on, it became clear that Banach algebra techniques provide the right tools to study the behavior of singular values, pseudospectra and numerical ranges of the matrices which constitute the sequences under consideration. The most complete results are available for C^*-algebras (see for instance the books [3], [7], [8], [14]). These C^*-algebras typically arise as follows.

Let $\mathbf{H}$ be an infinite dimensional Hilbert space and $(\mathbf{H}_n)$ be a sequence of closed subspaces of $\mathbf{H}$ such that the orthogonal projections P_n from $\mathbf{H}$ onto $\mathbf{H}_n$ converge strongly to the identity operator I on $\mathbf{H}$. We denote by $\mathcal{F}$ the set of all bounded sequences $\{A_n\}$ of operators $A_n \in \mathcal{L}(\mathbf{H}_n)$ and equip $\mathcal{F}$ with a C^*-algebra structure in the usual way, that is by introducing the algebraic operations componentwise. Let $\mathcal{G}$ denote the closed ideal of $\mathcal{F}$ consisting of all sequences $\{G_n\} \in \mathcal{F}$ with $\|G_n\| \to 0$ as $n \to \infty$.

Let further T be a (possibly infinite) index set. Suppose that for each $t \in T$ we are given an infinite dimensional Hilbert space $\mathbf{H}^t$ with identity operator I^t as well as a sequence (E_n^t) of partial isometries $E_n^t : \mathbf{H}^t \to \mathbf{H}$ such that

- the initial projections P_n^t of E_n^t converge strongly to I^t as $n \to \infty$,
- the range projection of E_n^t is P_n and
- the separation condition

$$(E_n^s)^* E_n^t \rightharpoonup 0 \text{ weakly as } n \to \infty$$

holds true for every $s, t \in T$ with $s \neq t$.

For brevity, we write E_n^{-t} instead of $(E_n^t)^*$. Let $\mathcal{F}^T$ stand for the set of all sequences $\{A_n\} \in \mathcal{F}$ for which the strong limits

$$\operatorname*{s-lim}_{n\to\infty} E_n^{-t} A_n E_n^t \quad \text{and} \quad \operatorname*{s-lim}_{n\to\infty} (E_n^{-t} A_n E_n^t)^*$$

exist for every $t \in T$, and define mappings $W^t : \mathcal{F}^T \to \mathcal{L}(\mathbf{H}^t)$ by

$$W^t\{A_n\} := \operatorname*{s-lim}_{n\to\infty} E_n^{-t} A_n E_n^t.$$

The set $\mathcal{F}^T$ forms a C^*-subalgebra of $\mathcal{F}$ and the separation condition ensures that, for every $t \in T$ and every compact operator $K^t \in \mathcal{K}(\mathbf{H}^t)$, the sequence $\{E_n^t K^t E_n^{-t}\}$ belongs to $\mathcal{F}^T$, and that for all $s \in T$

$$W^s\{E_n^t K^t E_n^{-t}\} = \begin{cases} K^t & \text{if } s = t \\ 0 & \text{if } s \neq t. \end{cases}$$

A sequence $\{A_n\} \in \mathcal{F}^T$ is said to be stable if there is an n_0 such that the operators $A_n \in \mathcal{L}(\mathbf{H}_n)$ are invertible for $n \geq n_0$ and

$$\sup_{n \geq n_0} \|A_n^{-1} P_n\| < \infty.$$

There is a variety of concrete operator sequences which belong to algebras of the type $\mathcal{F}^T$ (see for instance [3], [7], [8], [14]). The importance of these algebras is given by the following observation. Let $\mathcal{J}^T \subset \mathcal{F}^T$ be the smallest closed ideal of $\mathcal{F}^T$ which contains all sequences $\{E_n^t K^t E_n^{-t}\}$ with $t \in T$ and $K^t \in \mathcal{K}(\mathbf{H}^t)$ as well as all sequences $\{G_n\} \in \mathcal{G}$.

Theorem 1. *1. A sequence $\{A_n\} \in \mathcal{F}^T$ is stable if and only if the operators $W^t\{A_n\}$ are invertible in $\mathcal{L}(\boldsymbol{H}^t)$ for every $t \in T$ and if the coset $\{A_n\} + \mathcal{J}^T$ is invertible in the quotient algebra $\mathcal{F}^T/\mathcal{J}^T$.*

2. If $\{A_n\} \in \mathcal{F}^T$ is a sequence with invertible coset $\{A_n\} + \mathcal{J}^T$ then all operators $W^t\{A_n\}$ are Fredholm on $\boldsymbol{H}^t$ and the number of the non-invertible operators among them is finite.

Notice that as a rule $\mathcal{F}^T/\mathcal{J}^T$ is non-commutative and one often has to apply local principles in order to prove invertibility.

If $\{A_n\} \in \mathcal{F}^T$ is subject to condition 2 in Theorem 1 then the sequence $\{A_n\}$ is called Fredholm sequence. The reason for this notion is that in case $\mathcal{F}$ is an algebra formed by matrix sequences, then $\{A_n\}$ is invertible (in $\mathcal{F}$) modulo the ideal of so-called centrally compact sequences (see [8], Chapter 6), and such sequences are called Fredholm. It is worth noticing that $\{A_n\} \in \mathcal{F}^{T_1}$ can be non-Fredholm, but $\{A_n\} \in \mathcal{F}^{T_2}$ with $T_1 \subset T_2$ can be Fredholm. This means that the ideal $\mathcal{J}^{T_1}$ is somewhat too small.

Now, let the spaces $\mathbf{H}_n$ be finite dimensional for every $n \in \mathbb{N}$, that is, $\mathcal{F}$ consists of matrix sequences. For $A \in \mathcal{L}(\mathbf{H}_n)$ we put $m := \dim \mathbf{H}_n$ and let $\sigma_k(A)$ (for $k = 0, 1, \ldots, m$) denote the singular values of A, i.e. the eigenvalues of $(A^*A)^{\frac{1}{2}}$ in such a way, that $\sigma_1(A) \leq \sigma_2(A) \leq \ldots \leq \sigma_m(A)$. The following theorem was essentially proved by S. Roch and one of the authors.

Theorem 2. *1. Let $\{A_n\} \in \mathcal{F}^T$ be a Fredholm sequence. Then*

$$\liminf_{n\to\infty} \sigma_{k+1}(A_n) > 0 \quad \textit{and} \quad \lim_{n\to\infty} \sigma_k(A_n) = 0,$$

where $k = k\{A_n\} := \sum_{t \in T} \dim \ker W^t\{A_n\}$.

2. $\sum_{t \in T} \operatorname{ind} W^t\{A_n\} = 0$ for any Fredholm sequence $\{A_n\} \in \mathcal{F}^T$.

3. If at least one of the operators $W^t\{A_n\}$ for a sequence $\{A_n\}$ is not Fredholm, then $\lim_{n\to\infty} \sigma_k(A_n) = 0$ for every $k = 0, 1, \ldots$.

Later on, S. Roch proved that for the C^*-algebra $\mathcal{F}$ the following assertions are equivalent (see [8]):

1. $\{A_n\} \in \mathcal{F}$ is invertible modulo the ideal of centrally compact sequences (that is $\{A_n\}$ is Fredholm).

2. There is a number $k \in \mathbb{N} \cup \{0\}$ such that

$$\liminf_{n\to\infty} \sigma_{k+1}(A_n) > 0.$$

In general, it is a difficult task to state that a given sequence is Fredholm.

If the spaces under consideration are Banach spaces and $\mathcal{F}$ is merely a Banach algebra, considerably less is known. In particular, one needs appropriate substitutes for the singular values, which can be given by the so-called approximation numbers. For an m-dimensional normed space $\mathbf{X}$ and an operator $A \in \mathcal{L}(\mathbf{X})$ the k-th approximation number $s_k(A)$ of A is defined as

$$s_k(A) := \operatorname{dist}(A, \mathcal{F}_{m-k}(\mathbf{X})) := \inf\{\|A - F\|_{\mathcal{L}(\mathbf{X})} : F \in \mathcal{F}_{m-k}(\mathbf{X})\}, \tag{1}$$

$(k = 0, 1, ..., m)$, where $\mathcal{F}_{m-k}(\mathbf{X})$ denotes the collection of all operators from $\mathcal{L}(\mathbf{X})$ having the image of the dimension at most $m - k$. It is clear that

$$0 = s_0(A) \le s_1(A) \le s_2(A) \le ... \le s_m(A) = \|A\|_{\mathcal{L}(\mathbf{X})}.$$

Moreover one can show that (see [2], Proposition 9.2)

$$s_1(A) = \begin{cases} 1/\|A^{-1}\|_{\mathcal{L}(\mathbf{X})} & \text{if } A \text{ is invertible} \\ 0 & \text{if } A \text{ is not invertible.} \end{cases} \tag{2}$$

Note also that in case $\mathbf{X}$ is a Hilbert space the approximation numbers coincide with the singular values of A.

A. Böttcher studied the asymptotic behavior of the approximation numbers of the finite sections of Toeplitz operators with scalar-valued generating functions in spaces $l^p(\mathbb{Z}_+)$, $1 < p < \infty$ in [2], and obtained results like Theorem 2. His approach has the advantage that the speed of convergence of $s_k(A_n)$ to zero can be estimated. On the other hand, it seems that his approach is limited to scalar valued generation functions.

Algebras of the type $\mathcal{F}^T$ can also be introduced in the Banach space context. This is well known, see for instance [4] or [7]. However one has to suppose that not only (P_n) tends strongly to the identity operator, but also (P_n^*). Theorem 2 holds also true in this situation (with approximation numbers instead of singular values) as was shown by A. Rogozhin and one of the authors in the recent paper [23]. The related proofs are quite different from the previous ones.

As a matter of fact all these results are not applicable say to finite section sequences of operators acting on $l^\infty(\mathbb{Z}^K)$ spaces. Therefore one may ask if there is a general framework which also includes this case. The present paper presents such a general framework with applications to the finite section method for band-dominated operators acting on spaces $l^p(\mathbb{Z}^K)$, $1 \le p \le \infty$. For the spaces $l^p(\mathbb{Z})$, $1 \le p \le \infty$ the results are most complete and extend results of S. Roch for $p = 2$ [21] and of the authors for $1 < p < \infty$ [24].

The paper is organized as follows. In Section 1 we present the general framework for the analysis of the asymptotic behavior of operator sequences in the Banach space situation. For this we introduce a more suitable concept of convergence and "almost invertibility", the so-called $\mathcal{P}$-strong convergence and $\mathcal{P}$-Fredholmness, which has already been considered in [14] and [17], for example. Then we establish the Fredholm property for sequences in Banach algebras of the type $\mathcal{F}^T$ and prove a stability result similar to Theorem 1. The main result on the asymptotic behavior of the approximation numbers of sequences $\{A_n\} \in \mathcal{F}^T$ is stated in Section 1.2.5 and generalizes Theorem 2.

Section 2 is devoted to the finite sections of band-dominated operators. An important tool in that business and actually the link between the finite section sequences and our general theory is given by the notion of limit operators. The book [17] provides a comprehensive picture of the limit operator method, and [12] is also a recent introduction to this topic. The crucial step for the study of sequences arising from a certain class of band-dominated operators acting on spaces $l^p(\mathbb{Z}^K)$ is to determine whether the Fredholm property of the sequence $\{A_n\}$ is completely described by the Fredholm properties of the operators $W^t\{A_n\}$. This is done in 2.3.4

In the case $K = 1$ it is even possible to consider general band-dominated operators by studying appropriate subsequences. Moreover, we present an idea how to deal with "almost stable" sequences.

1 General theory

1.1 $\mathcal{P}$-compact and $\mathcal{P}$-Fredholm operators, $\mathcal{P}$-strong convergence

Let $\mathbf{X}$ be a Banach space. We denote by $\mathcal{L}(\mathbf{X})$ the Banach algebra of all bounded and linear operators on $\mathbf{X}$ and by $\mathcal{K}(\mathbf{X})$ the closed ideal of all compact operators in $\mathcal{L}(\mathbf{X})$. For $A \in \mathcal{L}(\mathbf{X})$ we put

$$\begin{aligned} \ker A &:= \{x \in \mathbf{X} : Ax = 0\}, \\ \operatorname{im} A &:= A(\mathbf{X}) = \{Ax : x \in \mathbf{X}\}, \\ \operatorname{coker} A &:= \mathbf{X}/\operatorname{im} A. \end{aligned}$$

Fredholm operators An operator $A \in \mathcal{L}(\mathbf{X})$ is called Fredholm operator if $\dim\ker A < \infty$ and $\dim\operatorname{coker} A < \infty$. If A is Fredholm then the integer $\operatorname{ind} A := \dim\ker A - \dim\operatorname{coker} A$ is referred to as the index of A. The collection of all Fredholm operators will be denoted by $\Phi(\mathbf{X})$. The following theorem records some important and essentially well known properties of such operators.

Theorem 3. *Let $\boldsymbol{X}$ be a Banach space and $A \in \mathcal{L}(\boldsymbol{X})$. Then the following are equivalent:*

1. *A is Fredholm.*
2. $\dim\ker A < \infty$, $\dim\ker A^* < \infty$ *and* $\operatorname{im} A$ *is closed.*
3. *The coset $A + \mathcal{K}(\boldsymbol{X})$ is invertible in the Calkin algebra $\mathcal{L}(\boldsymbol{X})/\mathcal{K}(\boldsymbol{X})$.*
4. *There exist projections $P, P' \in \mathcal{K}(\boldsymbol{X})$ s.t.* $\operatorname{im} P = \ker A$ *and* $\ker P' = \operatorname{im} A$.

5. *The following is NOT true: For each $l \in \mathbb{N}$ and each $\epsilon > 0$ there exists a projection $Q \in \mathcal{K}(\boldsymbol{X})$ with $\operatorname{rank} Q \geq l$ such that $\|AQ\| < \epsilon$ or $\|QA\| < \epsilon$.*

Moreover, we have:

- *$\Phi(\boldsymbol{X})$ is an open subset of $\mathcal{L}(\boldsymbol{X})$ and for $A \in \Phi(\boldsymbol{X})$ we have*

$$\dim \ker(A + C) \leq \dim \ker A, \ \dim \operatorname{coker}(A + C) \leq \dim \operatorname{coker} A$$

  whenever C has sufficiently small norm. The mapping $\operatorname{ind} : \Phi(\boldsymbol{X}) \to \mathbb{Z}$ is constant on the connected components of $\Phi(\boldsymbol{X})$.

- *Let $A \in \Phi(\boldsymbol{X})$ and $K \in \mathcal{K}(\boldsymbol{X})$. Then $A + K \in \Phi(\boldsymbol{X})$ and $\operatorname{ind} A + K = \operatorname{ind} A$.*

- *(Atkinson's theorem). Let $A, B \in \Phi(\boldsymbol{X})$. Then we have $AB \in \Phi(\boldsymbol{X})$ and $\operatorname{ind} AB = \operatorname{ind} A + \operatorname{ind} B$.*

Proof. The stated properties and the equivalence of the conditions 1 till 4 are well known. Here we only consider condition 5.

Let the conditions 1 to 4 be fulfilled and let $P, P' \in \mathcal{K}(\mathbf{X})$ be projections such that $\operatorname{im} P = \ker A$ and $\ker P' = \operatorname{im} A$. Then $\dim \operatorname{im} P$ and $\dim \operatorname{im} P'$ are finite. Moreover, as a consequence of the Banach inverse mapping theorem, the operator $A|_{\ker P} : \ker P \to \ker P'$ is invertible. Let $A^{(-1)}$ be its inverse and $B := (I - P)A^{(-1)}(I - P')$. Then $B \in \mathcal{L}(\mathbf{X})$ and $AB = I - P'$, $BA = I - P$. Now we assume that there is a compact projection Q with $\operatorname{rank} Q > \operatorname{rank} P$ such that $\|AQ\| < (2\|B\|)^{-1}$. Then there is an $x \in \operatorname{im} Q \cap \operatorname{im}(I - P)$ with $x \neq 0$ hence $\|(I - P)Qx\| = \|x\|$, but on the other hand $\|(I - P)Q\| = \|BAQ\| < 1/2$, a contradiction. An analogous argument for $\|QA\|$ shows that 5 is true.

Now we prove that 5 implies 2. Assume $\dim \ker A = \infty$. Then for every given $l \in \mathbb{N}$ we can choose an l-dimensional subspace of the kernel and a bounded linear projection Q onto this subspace and we get $AQ = 0$, which contradicts 5.
Assume $\dim \ker A^* = \infty$. For a given $l \in \mathbb{N}$ we choose l linearly independent bounded functionals from $\ker A^*$ and denote by $\mathbf{X}'$ the intersection of their kernels. Obviously, $\mathbf{X}'$ is a Banach space with an l-dimensional complement in $\mathbf{X}$ and $\operatorname{im} A \subset \mathbf{X}'$. Now we can choose a bounded projection Q parallel to $\mathbf{X}'$ onto one complement of $\mathbf{X}'$ and get $QA = 0$, again a contradiction.
Finally, assume that $\operatorname{im} A$ is not closed. We want to show that for every $l \in \mathbb{N}$ and for every $\delta > 0$ there exists an l-dimensional subspace $\mathbf{X}_l$ of $\mathbf{X}$ such that $\|A|_{\mathbf{X}_l}\| < \delta$. Fix $\delta > 0$. Then there exists $x_1 \in \mathbf{X}$, $\|x_1\| = 1$ such that $\|Ax_1\| < \delta$, otherwise A would be an invertible operator and hence its range would be closed. Assume that the assertion is true for $l - 1$. One can show that there exist a complement $\mathbf{Y}$ of $\mathbf{X}_{l-1}$ and a projection Q onto $\mathbf{X}_{l-1}$ with $\ker Q = \mathbf{Y}$ and $\|Q\| \leq l$. $A(\mathbf{Y})$ is not closed (otherwise $A(\mathbf{X}) = A(\mathbf{X}_{l-1}) + A(\mathbf{Y})$ would be closed). Now we can choose $x_l \in \mathbf{Y}$ with $\|x_l\| = 1$ and $\|Ax_l\| < \delta$ and for arbitrary scalars α, β and $x \in \mathbf{X}_{l-1}$ we get

$$\begin{aligned}
\|A(\alpha x + \beta x_l)\| &\leq \|A(\alpha x)\| + \|A(\beta x_l)\| < \delta(\|\alpha x\| + \|\beta x_l\|) \\
\|\alpha x\| &= \|Q(\alpha x)\| = \|Q(\alpha x + \beta x_l)\| \leq \|Q\| \|\alpha x + \beta x_l\| \\
\|\beta x_l\| &= \|(I - Q)(\beta x_l)\| = \|(I - Q)(\alpha x + \beta x_l)\| \\
&\leq (1 + \|Q\|) \|\alpha x + \beta x_l\|,
\end{aligned}$$

i.e. $\|A(\alpha x + \beta x_l)\| < \delta(1+2l)\|\alpha x + \beta x_l\|$. Since $\delta > 0$ is arbitrary the assertion follows by induction. Consequently, for every $l \in \mathbb{N}$ and $\epsilon > 0$ we can choose $0 < \delta < \epsilon/l$, an l-dimensional subspace of $\mathbf{X}$ and an appropriate projection Q onto this subspace with $\|Q\| \leq l$, such that $\|AQ\| \leq \delta\|Q\| < \epsilon$. □

Convergence A sequence (A_n) of operators $A_n \in \mathcal{L}(\mathbf{X})$ is said to converge to an operator $A \in \mathcal{L}(\mathbf{X})$

- *strongly* if $\|A_n x - Ax\| \to 0$ for each $x \in \mathbf{X}$ (we write $A = \operatorname{s-lim}_{n\to\infty} A_n$),
- *uniformly* if $\|A_n - A\|_{\mathcal{L}(\mathbf{X})} \to 0$.

Proposition 1. *Let $\boldsymbol{X}$ be a Banach space and $A, A_n \in \mathcal{L}(\boldsymbol{X})$. Then*

- *$A_n \to A$ strongly $\Leftrightarrow$ $A_n K \to AK$ uniformly for every $K \in \mathcal{K}(\boldsymbol{X})$.*
- *$A_n^* \to A^*$ strongly $\Rightarrow$ $KA_n \to KA$ uniformly for every $K \in \mathcal{K}(\boldsymbol{X})$.*

The aim of the subsequent steps is to generalize the concepts of compactness, Fredholmness and strong convergence.

Definition 1. Let $\mathbf{X}$ be a Banach space and let $\mathcal{P} = (P_n)_{n\in\mathbb{N}}$ be a bounded sequence of operators in $\mathcal{L}(\mathbf{X})$ with the following properties:

- For every $m \in \mathbb{N}$ there is an $N \in \mathbb{N}$ s.t. $P_n P_m = P_m P_n = P_m$ if $n \geq N$,
- $P_n \neq 0$ and $P_n \neq I$ for all $n \in \mathbb{N}$.

Then $\mathcal{P}$ is said to be an approximate projection. $\mathcal{P}$ is called approximate identity if in addition $\sup_n \|P_n x\| \geq \|x\|$ holds for each $x \in \mathbf{X}$.

Given an approximate projection $\mathcal{P}$, we set $S_1 := P_1$ and $S_n := P_n - P_{n-1}$ for $n > 1$. Further, for every bounded $U \subset \mathbb{R}$, we define $P_U := \sum_{k\in\mathbb{N}\cap U} S_k$. $\mathcal{P}$ is said to be uniform if $C_{\mathcal{P}} := \sup \|P_U\| < \infty$, the supremum over all bounded $U \subset \mathbb{R}$.

$\mathcal{P}$-compactness Let $\mathcal{P}$ be an approximate projection. A bounded linear operator K is called $\mathcal{P}$-compact if $\|KP_n - K\|, \|P_n K - K\| \to 0$ as $n \to \infty$. By $\mathcal{K}(\mathbf{X}, \mathcal{P})$ we denote the set of all $\mathcal{P}$-compact operators on $\mathbf{X}$ and by $\mathcal{L}(\mathbf{X}, \mathcal{P})$ the set of all operators $A \in \mathcal{L}(\mathbf{X})$ for which AK and KA are $\mathcal{P}$-compact whenever K is $\mathcal{P}$-compact.

Theorem 4. *(see [17], Proposition 1.1.8 and Theorem 1.1.9)*
Let $\mathcal{P}$ be an approximate projection. $\mathcal{L}(\boldsymbol{X}, \mathcal{P})$ is a closed subalgebra of $\mathcal{L}(\boldsymbol{X})$, it contains the identity operator, and $\mathcal{K}(\boldsymbol{X}, \mathcal{P})$ is a closed ideal of $\mathcal{L}(\boldsymbol{X}, \mathcal{P})$. An operator $A \in \mathcal{L}(\boldsymbol{X})$ belongs to $\mathcal{L}(\boldsymbol{X}, \mathcal{P})$ if and only if, for every $k \in \mathbb{N}$,

$$\|P_k A(I - P_n)\| \to 0 \text{ and } \|(I - P_n)AP_k\| \to 0 \text{ as } n \to \infty.$$

If $\mathcal{P}$ is uniform, then $\mathcal{L}(\boldsymbol{X}, \mathcal{P})$ is inverse closed in $\mathcal{L}(\boldsymbol{X})$.

Theorem 3 and Proposition 1 show that Fredholmness and strong convergence are closely related to the ideal of compact operators. This suggests the definition of generalizations of Fredholmness and strong convergence applying $\mathcal{P}$-compact operators.

$\mathcal{P}$-strong convergence Let $\mathcal{P}$ be an approximate projection and for each $n \in \mathbb{N}$ let $A_n \in \mathcal{L}(\mathbf{X}, \mathcal{P})$. The sequence (A_n) converges $\mathcal{P}$-strongly to $A \in \mathcal{L}(\mathbf{X})$ if, for all $K \in \mathcal{K}(\mathbf{X}, \mathcal{P})$, both $\|(A_n - A)K\|$ and $\|K(A_n - A)\|$ tend to 0 as $n \to \infty$. In this case we write $A_n \to A$ $\mathcal{P}$-strongly or $A = \mathcal{P}\text{-}\lim_{n\to\infty} A_n$.

Proposition 2. *(see [17], Proposition 1.1.14)*
If (A_n) is a bounded sequence in $\mathcal{L}(\boldsymbol{X}, \mathcal{P})$ then (A_n) converges $\mathcal{P}$-strongly to $A \in \mathcal{L}(\boldsymbol{X})$ if and only if

$$\|(A_n - A)P_m\| \to 0 \text{ and } \|P_m(A_n - A)\| \to 0 \text{ for every fixed } P_m \in \mathcal{P}.$$

Theorem 5. *(see [17], corollary 1.1.16f)*
Let $\mathcal{P}$ be an approximate identity and let $(A_n), (B_n) \subset \mathcal{L}(\boldsymbol{X}, \mathcal{P})$ be sequences converging $\mathcal{P}$-strongly to $A, B \in \mathcal{L}(\boldsymbol{X})$, respectively. Then

- *A is uniquely determined and $A \in \mathcal{L}(\boldsymbol{X}, \mathcal{P})$. (A_n) is uniformly bounded, i.e. $\|(A_n)\| := \sup_n \|A_n\| < \infty$, and $\|A\| \leq \|(P_n)\| \liminf_n \|A_n\|$.*
- *$A_n + B_n \to A + B$ and $A_n B_n \to AB$ $\mathcal{P}$-strongly.*

$\mathcal{P}$-Fredholm operators Let $\mathcal{P}$ be an approximate projection. An operator $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ is said to be $\mathcal{P}$-Fredholm if the coset $A + \mathcal{K}(\mathbf{X}, \mathcal{P})$ is invertible in the quotient algebra $\mathcal{L}(\mathbf{X}, \mathcal{P})/\mathcal{K}(\mathbf{X}, \mathcal{P})$.

Theorem 3 shows that the usual Fredholm property of linear bounded operators can be described in terms of compact projections. In what follows, we want to give a description of $\mathcal{P}$-Fredholmness in terms of $\mathcal{P}$-compact projections as well. For this, let $\mathcal{P} = (P_n)$ be a uniform approximate projection with

$$\operatorname{rank} P_n < \infty \text{ for all } n.$$

Then it is not hard to prove, that

$$\mathcal{K}(\mathbf{X}^*, \mathcal{P}^*) = \{K^* : K \in \mathcal{K}(\mathbf{X}, \mathcal{P})\} \text{ and } \mathcal{L}(\mathbf{X}^*, \mathcal{P}^*) \supset \{A^* : A \in \mathcal{L}(\mathbf{X}, \mathcal{P})\}.$$

A sequence $(A_n) \subset \mathcal{L}(\mathbf{X}, \mathcal{P})$ converges $\mathcal{P}$-strongly to $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ if and only if

$$\|(A_n - A)K\| \to 0 \text{ and } \|K(A_n - A)\| \to 0 \text{ for all } K \in \mathcal{K}(\mathbf{X}, \mathcal{P}).$$

Obviously, this is equivalent to

$$\|K^*(A_n^* - A^*)\| \to 0 \text{ and } \|(A_n^* - A^*)K^*\| \to 0 \text{ for all } K^* \in \mathcal{K}(\mathbf{X}^*, \mathcal{P}^*),$$

and thus to the $\mathcal{P}^*$-strong convergence of the sequence (A_n^*) to A^*.

Definition 2. An operator $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ is called proper $\mathcal{P}$-Fredholm, if there exist projections $P, P' \in \mathcal{K}(\mathbf{X}, \mathcal{P})$ s.t. $\operatorname{im} P = \ker A$ and $\ker P' = \operatorname{im} A$ and it is called proper deficient if, for each $l \in \mathbb{N}$ and each $\epsilon > 0$, there exists a projection $Q \in \mathcal{K}(\mathbf{X}, \mathcal{P})$ with $\operatorname{rank} Q \geq l$ such that $\|QA\| < \epsilon$ or $\|AQ\| < \epsilon$. Further one says that A enjoys the $\mathcal{P}$-dichotomy, if A is proper $\mathcal{P}$-Fredholm or proper deficient. The set of all operators $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ having the $\mathcal{P}$-dichotomy will be denoted by $\mathcal{D}(\mathbf{X}, \mathcal{P})$.

Obviously, all invertible operators are proper $\mathcal{P}$-Fredholm.

Proposition 3. *Let $\mathcal{P} = (P_n)_{n\in\mathbb{N}}$ be a uniform approximate projection with* $\operatorname{rank} P_n < \infty$ *for all $n \in \mathbb{N}$. For $A \in \mathcal{D}(\boldsymbol{X}, \mathcal{P})$ the following are equivalent:*

1. *A is proper $\mathcal{P}$-Fredholm.*
2. *A is $\mathcal{P}$-Fredholm.*
3. *A ist Fredholm.*
4. *A is not proper deficient.*
5. *There exists a $B \in \mathcal{L}(\boldsymbol{X}, \mathcal{P})$ s.t. $P := I - BA, P' := I - AB \in \mathcal{K}(\boldsymbol{X}, \mathcal{P})$ are projections,* $\operatorname{im} P = \ker A$ *and* $\ker P' = \operatorname{im} A$.

Moreover, if $A \in \mathcal{D}(\boldsymbol{X}, \mathcal{P})$ is proper $\mathcal{P}$-Fredholm or proper deficient then the operator $A^* \in \mathcal{D}(\boldsymbol{X}^*, \mathcal{P}^*)$ is of the same kind.

The proof of this proposition is given in the appendix.

Corollary 1. *Let $\mathcal{P}$ be a uniform approximate projection,* $\operatorname{rank} P_n < \infty$ *for all $n \in \mathbb{N}$, and assume that $P_n \to I$, $P_n^* \to I$ strongly. Then $\mathcal{K}(\boldsymbol{X}, \mathcal{P}) = \mathcal{K}(\boldsymbol{X})$ and $\mathcal{D}(\boldsymbol{X}, \mathcal{P}) = \mathcal{L}(\boldsymbol{X}, \mathcal{P}) = \mathcal{L}(\boldsymbol{X})$. A sequence $(A_n) \subset \mathcal{L}(\boldsymbol{X})$ is $\mathcal{P}$-strongly convergent if and only if (A_n) as well as (A_n^*) are strongly convergent.*

Proof. For every $K \in \mathcal{K}(\mathbf{X}, \mathcal{P})$ we have $\|P_n K P_n - K\| \to 0$, that is, K can be approximated by finite-rank operators. Hence K is compact. Conversely, every compact operator is $\mathcal{P}$-compact, due to Proposition 1. Now we obviously have $\mathcal{L}(\mathbf{X}) = \mathcal{L}(\mathbf{X}, \mathcal{P})$ and from Theorem 3 we obtain $\mathcal{D}(\mathbf{X}, \mathcal{P}) = \mathcal{L}(\mathbf{X})$.

A $\mathcal{P}$-strongly or strongly convergent sequence (A_n) is uniformly bounded (see Theorem 5 or the Banach-Steinhaus theorem). Hence Proposition 1, Proposition 2 an the relations

$$\begin{aligned}
\|(A_n - A)x\| &\le \|(A_n - A)P_m\|\|x\| + \|A_n - A\|\|(I - P_m)x\| \\
\|(A_n^* - A^*)f\| &\le \|(A_n^* - A^*)P_m^*\|\|f\| + \|A_n^* - A^*\|\|(I^* - P_m^*)f\| \\
&= \|P_m(A_n - A)\|\|f\| + \|A_n - A\|\|(I^* - P_m^*)f\|
\end{aligned}$$

easily complete the proof. □

Remark 1. It is still an open question under which conditions $\mathcal{D}(\mathbf{X}, \mathcal{P})$ is really a proper subset of $\mathcal{L}(\mathbf{X}, \mathcal{P})$. In Section 2 we will show that the class of band-dominated operators is contained in $\mathcal{D}(\mathbf{X}, \mathcal{P})$.

1.2 Sequence algebras, Fredholm sequences and approximation numbers

1.2.1 Sequence algebras

Let $(\mathbf{E}_n)$ be a sequence of finite dimensional Banach spaces and let (L_n) denote the sequence of the identities on $\mathbf{E}_n$, respectively. We denote by $\mathcal{F}$ the set of all bounded sequences $\{A_n\}$ of bounded linear operators $A_n \in \mathcal{L}(\mathbf{E}_n)$. Provided with the operations

$$\alpha\{A_n\} + \beta\{B_n\} := \{\alpha A_n + \beta B_n\}, \{A_n\}\{B_n\} := \{A_n B_n\},$$

and the supremum norm $\|\{A_n\}\|_{\mathcal{F}} := \sup_n \|A_n\|_{\mathcal{L}(\mathbf{E}_n)} < \infty$, $\mathcal{F}$ becomes a Banach algebra with identity $\{L_n\}$. The set

$$\mathcal{G} := \{\{G_n\} \in \mathcal{F} : \|G_n\|_{\mathcal{L}(\mathbf{E}_n)} \to 0\}$$

forms a closed ideal in $\mathcal{F}$.

Further, let T be a (possibly infinite) index set and suppose that, for every $t \in T$, there is a Banach space $\mathbf{E}^t$ with the identity I^t, and a bounded sequence (L_n^t) of projections L_n^t on $\mathbf{E}^t$ forming an approximate identity $\mathcal{P}^t := (L_n^t)$. Then

$$c^t := \|(L_n^t)\| = \sup_n \|L_n^t\|_{\mathcal{L}(\mathbf{E}^t)} < \infty \text{ for every } t \in T.$$

Further suppose that, for every $t \in T$, there is a sequence (E_n^t) of invertible homomorphisms

$$E_n^t : \mathcal{L}(\operatorname{im} L_n^t) \to \mathcal{L}(\mathbf{E}_n),$$

such that (for brevity, we write E_n^{-t} instead of $(E_n^t)^{-1}$)

$$M^t := \sup_n \{\|E_n^t\|, \|E_n^{-t}\|\} < \infty. \tag{I}$$

We denote by $\mathcal{F}^T$ the collection of all sequences $\{A_n\} \in \mathcal{F}$, for which there exist operators $W^t\{A_n\} \in \mathcal{L}(\mathbf{E}^t, \mathcal{P}^t)$ for all $t \in T$, such that all

$$A_n^{(t)} := E_n^{-t}(A_n)L_n^t \to W^t\{A_n\} \ \mathcal{P}^t\text{-strongly.}$$

These operators are uniquely determined (see Theorem 5). It is easy to show that $\mathcal{F}^T$ is a closed subalgebra of $\mathcal{F}$ which contains the identity and the ideal $\mathcal{G}$. Both, the mappings E_n^t and $W^t : \mathcal{F}^T \to \mathcal{L}(\mathbf{E}^t, \mathcal{P}^t), \{A_n\} \mapsto W^t\{A_n\}$ are unital homomorphisms.

Roughly spoken, the mappings E_n^t allow us to transform a given sequence $\{A_n\} \in \mathcal{F}^T$ and to generate snapshots $W^t\{A_n\}$ of $\{A_n\}$ from different angles. In what follows, we will examine the connections between the properties of $\{A_n\}$ and the properties of its snapshots, e.g. stability, Fredholmness, etc.

1.2.2 Fredholm sequences

Here we will introduce a class of Fredholm sequences. A sequence will be called Fredholm, if it is "invertible modulo an ideal of compact sequences", where these "compact sequences" will be generated by lifting $\mathcal{P}^t$-compact operators. For this, it turns out to be reasonable to suppose that the perspectives from which a sequence can be looked at, and which are given by the homomorphisms W^t, are separated in a sense.

Therefore, we suppose that the separation condition

$$W^\tau\{E_n^t(L_n^t K^t L_n^t)\} = \begin{cases} K^t & \text{if } t = \tau \\ 0 & \text{if } t \neq \tau \end{cases} \tag{II}$$

holds for all $\tau, t \in T$ and every $K^t \in \mathcal{K}^t := \mathcal{K}(\mathbf{E}^t, \mathcal{P}^t)$.

Definition 3. We put

$$\mathcal{J}^t := \{\{E_n^t(L_n^t K^t L_n^t)\} + \{G_n\} : K^t \in \mathcal{K}^t, \|G_n\| \to 0\} \quad (\forall\, t \in T),$$
$$\mathcal{J}^T := \mathrm{clos}_{\mathcal{F}^T} \left\{ \sum_{i=1}^{m} \{J_n^{t_i}\} : m \in \mathbb{N},\ t_i \in T,\ \{J_n^{t_i}\} \in \mathcal{J}^{t_i} \right\}.$$

Proposition 4. *$\mathcal{J}^T$ and all $\mathcal{J}^t$, $t \in T$ are closed ideals in $\mathcal{F}^T$.*

Proof. For $t \in T$, $K \in \mathcal{K}^t$ and $\{A_n\} \in \mathcal{F}^T$ we have

$$\begin{aligned}
A_n E_n^t(L_n^t K L_n^t) &= E_n^t(A_n^{(t)} K L_n^t) \\
&= E_n^t(L_n^t W^t\{A_n\} K L_n^t) + E_n^t(L_n^t(A_n^{(t)} - W^t\{A_n\}) K L_n^t), \\
E_n^t(L_n^t K L_n^t) A_n &= E_n^t(L_n^t K A_n^{(t)}) \\
&= E_n^t(L_n^t K W^t\{A_n\} L_n^t) + E_n^t(L_n^t K(A_n^{(t)} - W^t\{A_n\}) L_n^t),
\end{aligned}$$

where $W^t\{A_n\} \in \mathcal{L}(\mathbf{E}^t, \mathcal{P}^t)$ and hence $W^t\{A_n\}K, KW^t\{A_n\} \in \mathcal{K}^t$. Due to the condition I and since $A_n^{(t)} \to W^t\{A_n\}$ $\mathcal{P}^t$-strongly, the last summands tend to zero in the norm as $n \to \infty$ in both cases. Consequently, all $\mathcal{J}^t$ as well as $\mathcal{J}^T$ are (two-sided) ideals in $\mathcal{F}^T$ and $\mathcal{J}^T$ is closed by the definition.

Let $(\{J_n^k\})_{k\in\mathbb{N}} = (\{E_n^t(L_n^t K_k^t L_n^t)\} + \{G_n^k\})_{k\in\mathbb{N}} \subset \mathcal{J}^t$ be a Cauchy sequence. From Theorem 5 we obtain

$$\begin{aligned}
\|K_k^t - K_l^t\| &= \|W^t\left(\{E_n^t(L_n^t(K_k^t - K_l^t)L_n^t)\} + \{G_n^k\} - \{G_n^l\}\right)\| \\
&\le c^t M^t \|(\{E_n^t(L_n^t K_k^t L_n^t)\} + \{G_n^k\}) - (\{E_n^t(L_n^t K_l^t L_n^t)\} + \{G_n^l\})\|,
\end{aligned}$$

therefore the sequence $(K_k^t)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{K}^t$ and since $\mathcal{K}^t$ is closed, it possesses a limit $K^t \in \mathcal{K}^t$. Analogously, the estimate

$$\begin{aligned}
\|\{G_n^k\} - \{G_n^l\}\| &\le \|(\{E_n^t(L_n^t K_k^t L_n^t)\} + \{G_n^k\}) - (\{E_n^t(L_n^t K_l^t L_n^t)\} + \{G_n^l\})\| \\
&\quad + \|\{E_n^t(L_n^t(K_l^t - K_k^t)L_n^t)\}\| \\
&\le \|\{J_n^k\} - \{J_n^l\}\| + M^t(c^t)^2\|(K_l^t - K_k^t)\|
\end{aligned}$$

shows that $\{G_n^k\}$ converges to a certain $\{G_n\} \in \mathcal{G}$. Now it's easy to see that $\{E_n^t(L_n^t K^t L_n^t)\} + \{G_n\} \in \mathcal{J}^t$ is the limit of $(\{J_n^k\})_k$ and the closedness of $\mathcal{J}^t$ is proven. □

Definition 4. We introduce a class of Fredholm sequences by calling a sequence $\{A_n\} \in \mathcal{F}^T$ Fredholm if the coset $\{A_n\} + \mathcal{J}^T$ is invertible in the quotient algebra $\mathcal{F}^T/\mathcal{J}^T$.

The following basic properties of Fredholm sequences are obvious:

- The set of Fredholm sequences is open in $\mathcal{F}^T$.
- The sum of a Fredholm sequence and a sequence from the ideal $\mathcal{J}^T$ is Fredholm.
- The product of two Fredholm sequences is Fredholm.

In the subsequent propositions we will show that there are close relations between the Fredholm property of a sequence $\{A_n\} \in \mathcal{F}^T$ and the $\mathcal{P}^t$-Fredholm property of the corresponding operators $W^t\{A_n\}$, which justify the notion "Fredholm sequence". Since rank $L_n^t < \infty$ for all n, we have $\mathcal{K}^t \subset \mathcal{K}(\mathbf{E}^t)$ and every $\mathcal{P}^t$-Fredholm operator $W^t\{A_n\}$ is Fredholm, i.e. possesses finite dimensional kernel and cokernel and a finite index. We will introduce analogues to the kernel dimension, the cokernel dimension and the index for Fredholm sequences, having very similar properties.

We start with a result on the regularisation of Fredholm sequences.

Proposition 5. *Let $\{A_n\} \in \mathcal{F}^T$ be a Fredholm sequence. Then there exist a $\delta > 0$ and finite subsets $\{t_1, ..., t_m\}, \{\tau_1, ..., \tau_l\}$ of T such that the following holds:*
For each $\{\tilde{A}_n\} \in \mathcal{F}^T$ with $\|\{\tilde{A}_n\} - \{A_n\}\| < \delta$ there are $\{B_n\}, \{C_n\} \in \mathcal{F}^T$ and $\{G_n\}, \{\hat{G}_n\} \in \mathcal{G}$ as well as operators $K^{t_i} \in \mathcal{K}^{t_i}, K^{\tau_i} \in \mathcal{K}^{\tau_i}$ such that

$$\{B_n\}\{\tilde{A}_n\} = \{L_n\} + \sum_{i=1}^{m} \{E_n^{t_i}(L_n^{t_i} K^{t_i} L_n^{t_i})\} + \{G_n\}, \tag{3}$$

$$\{\tilde{A}_n\}\{C_n\} = \{L_n\} + \sum_{i=1}^{l} \{E_n^{\tau_i}(L_n^{\tau_i} K^{\tau_i} L_n^{\tau_i})\} + \{\hat{G}_n\}. \tag{4}$$

In particular, all operators $W^t\{\tilde{A}_n\}$ with $t \in T \backslash (\{t_1, ..., t_m\} \cup \{\tau_1, ..., \tau_l\})$ are invertible.

Proof. Let $\{A_n\}$ be Fredholm. Then there are sequences $\{\hat{A}_n\} \in \mathcal{F}^T$ and $\{J_n\}, \{K_n\} \in \mathcal{J}^T$ such that

$$\{\hat{A}_n\}\{A_n\} = \{L_n\} + \{J_n\} \text{ and } \{A_n\}\{\hat{A}_n\} = \{L_n\} + \{K_n\}.$$

By the definition of the ideal $\mathcal{J}^T$, there exist finite subsets $\{t_1, ..., t_m\}$ and $\{\tau_1, ..., \tau_l\}$ of T and sequences $\{\hat{J}_n^{t_i}\} \in \mathcal{J}^{t_i}, \{\hat{K}_n^{\tau_i}\} \in \mathcal{J}^{\tau_i}$ as well as $\{\hat{J}_n\}, \{\hat{K}_n\} \in \mathcal{J}^T$ with $\|\{\hat{J}_n\}\|, \|\{\hat{K}_n\}\| < 1/4$ such that

$$\{J_n\} = \sum_{i=1}^{m} \{\hat{J}_n^{t_i}\} + \{\hat{J}_n\} \text{ and } \{K_n\} = \sum_{i=1}^{l} \{\hat{K}_n^{\tau_i}\} + \{\hat{K}_n\}.$$

We put $\delta := 1/(4\|\{\hat{A}_n\}\|)$ and for $\{\tilde{A}_n\} \in \mathcal{F}^T$ with $\|\{\tilde{A}_n\} - \{A_n\}\| < \delta$ let

$$\{D_n\} := \{\tilde{A}_n\} - \{A_n\}.$$

Then we get

$$\{\hat{A}_n\}\{\tilde{A}_n\} = \{L_n\} + \sum_{i=1}^{m} \{\hat{J}_n^{t_i}\} + \{\hat{J}_n\} + \{\hat{A}_n\}\{D_n\}$$

$$\{\tilde{A}_n\}\{\hat{A}_n\} = \{L_n\} + \sum_{i=1}^{l} \{\hat{K}_n^{\tau_i}\} + \{\hat{K}_n\} + \{D_n\}\{\hat{A}_n\}$$

where $\|\{\hat{J}_n\} + \{\hat{A}_n\}\{D_n\}\|, \|\{\hat{K}_n\} + \{D_n\}\{\hat{A}_n\}\| < 1/2$. Since the sequences $\{L_n\} + \{\hat{J}_n\} + \{\hat{A}_n\}\{D_n\}$ and $\{L_n\} + \{\hat{K}_n\} + \{D_n\}\{\hat{A}_n\}$ are invertible in the Banach algebra $\mathcal{F}^T$, we can define

$$
\begin{aligned}
\{B_n\} &:= (\{L_n\} + \{\hat{J}_n\} + \{\hat{A}_n\}\{D_n\})^{-1}\{\hat{A}_n\} \in \mathcal{F}^T, \\
\{C_n\} &:= \{\hat{A}_n\}(\{L_n\} + \{\hat{K}_n\} + \{D_n\}\{\hat{A}_n\})^{-1} \in \mathcal{F}^T, \\
\{J_n^{t_i}\} &:= (\{L_n\} + \{\hat{J}_n\} + \{\hat{A}_n\}\{D_n\})^{-1}\{\hat{J}_n^{t_i}\} \in \mathcal{J}^{t_i}, i = 1, ..., m, \\
\{K_n^{\tau_i}\} &:= \{\hat{K}_n^{\tau_i}\}(\{L_n\} + \{\hat{K}_n\} + \{D_n\}\{\hat{A}_n\})^{-1} \in \mathcal{J}^{\tau_i}, i = 1, ..., l.
\end{aligned}
$$

Due to the definition of the ideals $\mathcal{J}^t$, there are sequences $\{G_n\}, \{\hat{G_n}\} \in \mathcal{G}$ and operators $K^{t_i} \in \mathcal{K}^{t_i}$, $K^{\tau_i} \in \mathcal{K}^{\tau_i}$ such that

$$
\begin{aligned}
\{B_n\}\{\tilde{A}_n\} &= \{L_n\} + \sum_{i=1}^{m} \{J_n^{t_i}\} = \{L_n\} + \sum_{i=1}^{m} \{E_n^{t_i}(L_n^{t_i} K^{t_i} L_n^{t_i})\} + \{G_n\}, \\
\{\tilde{A}_n\}\{C_n\} &= \{L_n\} + \sum_{i=1}^{l} \{K_n^{\tau_i}\} = \{L_n\} + \sum_{i=1}^{l} \{E_n^{\tau_i}(L_n^{\tau_i} K^{\tau_i} L_n^{\tau_i})\} + \{\hat{G}_n\}.
\end{aligned}
$$

Finally, applying $W^t, t \in T$ to these equations, and with the help of the separation condition, we arrive at

$$
\begin{aligned}
W^t\{B_n\}W^t\{\tilde{A}_n\} &= \begin{cases} I^t & \text{if } t \notin \{t_1, ..., t_m\} \\ I^{t_i} + K^{t_i} & \text{if } t = t_i \in \{t_1, ..., t_m\}, \end{cases} \\
W^t\{\tilde{A}_n\}W^t\{C_n\} &= \begin{cases} I^t & \text{if } t \notin \{\tau_1, ..., \tau_l\} \\ I^{\tau_i} + K^{\tau_i} & \text{if } t = \tau_i \in \{\tau_1, ..., \tau_l\}. \end{cases}
\end{aligned}
\tag{5}
$$

Thus, for each $t \in T \backslash (\{t_1, ..., t_m\} \cup \{\tau_1, ..., \tau_l\})$, the operator $W^t\{\tilde{A}_n\}$ is invertible. □

By the equations (5) the following theorem is proven as well:

Theorem 6. *If a sequence $\{A_n\} \in \mathcal{F}^T$ is Fredholm, then all corresponding operators $W^t\{A_n\}$ are $\mathcal{P}^t$-Fredholm (as well as Fredholm) on $\boldsymbol{E}^t$ and the number of the non-invertible operators among them is finite.*

Definition 5. This allows us to introduce three finite numbers for a Fredholm sequence $\{A_n\} \in \mathcal{F}^T$, its nullity $\alpha(\{A_n\})$, deficiency $\beta(\{A_n\})$ and index $\operatorname{ind}(\{A_n\})$, by

$$
\begin{aligned}
\alpha(\{A_n\}) &:= \sum_{t \in T} \dim \ker W^t\{A_n\}, \\
\beta(\{A_n\}) &:= \sum_{t \in T} \dim \operatorname{coker} W^t\{A_n\} \text{ and} \\
\operatorname{ind}(\{A_n\}) &:= \alpha(\{A_n\}) - \beta(\{A_n\}).
\end{aligned}
$$

Applying the well known properties of Fredholm operators (see Theorem 3) and Proposition 5, it is not hard to prove the following proposition.

Proposition 6. *Let $\{A_n\} \in \mathcal{F}^T$ be Fredholm and $\{B_n\} \in \mathcal{F}^T$. Then we have:*

- *If $\|\{B_n\}\|$ is sufficiently small, then $\alpha(\{A_n\} + \{B_n\}) \leq \alpha(\{A_n\})$, $\beta(\{A_n\} + \{B_n\}) \leq \beta(\{A_n\})$ and $\operatorname{ind}(\{A_n\} + \{B_n\}) = \operatorname{ind}(\{A_n\})$.*
- *If $\{B_n\} \in \mathcal{J}^T$, then $\operatorname{ind}(\{A_n\} + \{B_n\}) = \operatorname{ind}(\{A_n\})$.*
- *If $\{B_n\} \in \mathcal{F}^T$ is Fredholm, then $\operatorname{ind}(\{A_n\}\{B_n\}) = \operatorname{ind}(\{A_n\}) + \operatorname{ind}(\{B_n\})$.*

1.2.3 Stability of a sequence $\{A_n\} \in \mathcal{F}^T$

In what follows, let $\mathcal{P}^t$ be uniform approximate identities for all $t \in T$.

Definition 6. A sequence $\{A_n\} \in \mathcal{F}$ is called stable, if there is an index n_0 such that all operators A_n, $n \geq n_0$ are invertible and $\sup_{n \geq n_0} \|A_n^{-1}\| < \infty$.

A sequence $\{A_n\} \in \mathcal{F}^T$ is said to enjoy the $\mathcal{P}$-dichotomy, if all operators $W^t\{A_n\}$ have the $\mathcal{P}^t$-dichotomy, that is $W^t\{A_n\} \in \mathcal{D}(\mathbf{E}^t, \mathcal{P}^t)$ for every $t \in T$.

It is well known, that a sequence $\{A_n\} \in \mathcal{F}$ is stable if and only if the coset $\{A_n\} + \mathcal{G}$ is invertible in $\mathcal{F}/\mathcal{G}$. Utilizing the higher structure of the given setting, namely the existence of $\mathcal{P}^t$-strong limits $W^t\{A_n\}$, we can prove a stronger result.

Theorem 7. *A sequence $\{A_n\} \in \mathcal{F}^T$ is stable and has the $\mathcal{P}$-dichotomy if and only if $\{A_n\}$ is a Fredholm sequence and all $W^t\{A_n\}$ $(t \in T)$ are invertible. In particular, $\mathcal{F}^T/\mathcal{G}$ is inverse closed in $\mathcal{F}/\mathcal{G}$.*

Proof. Let $\{A_n\} \in \mathcal{F}^T$ be stable and have the $\mathcal{P}$-dichotomy. Then for large n, each $t \in T$ and every $K \in \mathcal{K}^t$, we have

$$\|E_n^{-t}(A_n)L_n^t K\| = \frac{\|E_n^{-t}(A_n^{-1})L_n^t\|}{\|E_n^{-t}(A_n^{-1})L_n^t\|}\|E_n^{-t}(A_n)L_n^t K\| \geq \frac{1}{\|E_n^{-t}(A_n^{-1})L_n^t\|}\|L_n^t K\|.$$

For $n \to \infty$, we obtain

$$\|W^t\{A_n\}K\| \geq C^t\|K\| \text{ and analogously } \|KW^t\{A_n\}\| \geq C^t\|K\|$$

for each $t \in T$ and every $K \in \mathcal{K}^t$, where $C^t \geq 1/(M^t c^t \sup \|A_n^{-1}\|) > 0$ is constant. Thus, $W^t\{A_n\}$ must be proper $\mathcal{P}^t$-Fredholm (see Proposition 3). Suppose that the kernel of $W^t\{A_n\}$ is not trivial. Then there is a projection $P \in \mathcal{K}^t, P \neq 0$ such that $0 = \|W^t\{A_n\}P\| \geq C^t\|P\| \geq C^t$, a contradiction. Thus $W^t\{A_n\}$ is injective. Analogously one shows that $W^t\{A_n\}$ is surjective and hence invertible, due to the Banach inverse mapping theorem. Since $\{A_n\}$ is stable, we get for large n and every $K \in \mathcal{K}^t$ that

$$\begin{aligned}&\left[E_n^{-t}(A_n^{-1})L_n^t - (W^t\{A_n\})^{-1}\right] K \\ &= E_n^{-t}(A_n^{-1})\left[L_n^t - E_n^{-t}(A_n)L_n^t(W^t\{A_n\})^{-1}\right] K + \left[L_n^t - I^t\right](W^t\{A_n\})^{-1}K,\end{aligned}$$

where the right-hand side obviously converges to zero in the norm as $n \to \infty$. In the same way we get $K\left[E_n^{-t}(A_n^{-1})L_n^t - (W^t\{A_n\})^{-1}\right] \to 0$ in the norm. With the notation $B_n := A_n^{-1}$ if A_n is invertible and $B_n := L_n$ otherwise, we obtain from this the $\mathcal{P}^t$-strong convergence of $E_n^{-t}(B_n)L_n^t$ to $(W^t\{A_n\})^{-1}$ for every $t \in T$ and therefore $\{B_n\} \in \mathcal{F}^T$. Moreover $\{B_n\} + \mathcal{J}^T$ is the inverse of $\{A_n\} + \mathcal{J}^T$. Thus, $\{A_n\}$ is a Fredholm sequence.

Conversely, let $\{A_n\}$ be Fredholm and all $W^t\{A_n\}$ be invertible. Then $\{A_n\}$ has the $\mathcal{P}$-dichotomy since every invertible operator is proper $\mathcal{P}^t$-Fredholm, and there are a sequence $\{B_n\} \in \mathcal{F}^T$, operators $K^i \in \mathcal{K}^{t_i}$ $(t_1, ..., t_m \in T)$ as well as a sequence $\{G_n\} \in \mathcal{G}$, such that (see Proposition 5)

$$B_n A_n = L_n + \underbrace{\sum_{i=1}^m E_n^{t_i}(L_n^{t_i} K^i L_n^{t_i})}_{\in \mathcal{J}^T} + G_n\,.$$

Defining $B_n' := B_n - \sum_{i=1}^m E_n^{t_i}(L_n^{t_i} K^i (W^{t_i}\{A_n\})^{-1} L_n^{t_i})$, we get $\{B_n'\} \in \mathcal{F}$ and

$$B_n' A_n = L_n + G_n + \sum_{i=1}^m E_n^{t_i}(L_n^{t_i} \underbrace{K^i(L_n^{t_i} - (W^{t_i}\{A_n\})^{-1} A_n^{(t_i)})}_{\to 0 \text{ in the norm}} L_n^{t_i}) = L_n + \tilde{G}_n,$$

with $\{\tilde{G}_n\} \in \mathcal{G}$. Thus, A_n is invertible from the left for sufficiently large n and hence invertible from both sides since $\mathbf{E}_n$ is finite dimensional. Together with $\{B_n'\} \in \mathcal{F}$ this shows the stability of $\{A_n\}$. □

This theorem discloses, that in case of Fredholm sequences, the invertibility of all limit operators $W^t\{A_n\}$ ensures that also the sequence itself and the operators A_n are "sufficiently well invertible".

In the subsequent sections we will treat the question, what consequences we have to expect on the "almost-invertibility" of a sequence, if its limit operators are not longer invertible, but, i.e. still Fredholm. For this we need a kind of measure for the "almost-invertibility" of the operators A_n and we will see that the approximation numbers (see (1)) provide a suitable tool.

1.2.4 Systems of projections

For fixed $t \in T$ let $\mathbf{X}^t$ be an m-dimensional subspace of $\mathbf{E}^t$ and $P^t \in \mathcal{K}^t$ a $\mathcal{P}^t$-compact projection with $\operatorname{im} P^t = \mathbf{X}^t$. We set $\hat{P}^t := I^t - P^t$ and obtain

$$I^t - (I^t - L_n^t)P^t = L_n^t P^t + \hat{P}^t.$$

Since $P^t \in \mathcal{K}^t$, there is an $n_t \in \mathbb{N}$, such that $\|(I^t - L_n^t)P^t\|, \|P^t(I^t - L_n^t)\| < 1$ for all $n \geq n_t$. Thus, for $n \geq n_t$, we obtain the invertibility of $I^t - (I^t - L_n^t)P^t$ and moreover the inverses are

$$(I^t - (I^t - L_n^t)P^t)^{-1} = \sum_{k=0}^\infty ((I^t - L_n^t)P^t)^k \in \mathcal{L}(\mathbf{E}^t, \mathcal{P}^t).$$

This justifies the following definition for $n \geq n_t$:

$$I^t = \underbrace{L_n^t P^t \sum_{k=0}^\infty ((I^t - L_n^t)P^t)^k}_{=:P_n^t} + \underbrace{\hat{P}^t \sum_{k=0}^\infty ((I^t - L_n^t)P^t)^k}_{=:\hat{P}_n^t}.$$

For these operators $P_n^t, \hat{P}_n^t$ we have

$$\begin{aligned}
\hat{P}^t \hat{P}_n^t &= \hat{P}^t \hat{P}^t \sum_{k=0}^\infty ((I^t - L_n^t)P^t)^k = \hat{P}^t \sum_{k=0}^\infty ((I^t - L_n^t)P^t)^k = \hat{P}_n^t, \\
\hat{P}_n^t \hat{P}^t &= \hat{P}^t \sum_{k=0}^\infty ((I^t - L_n^t)P^t)^k \hat{P}^t = \hat{P}^t \left(I^t \hat{P}^t + 0 \right) = \hat{P}^t \text{ and} \\
\hat{P}_n^t \hat{P}_n^t &= \hat{P}_n^t(\hat{P}^t \hat{P}_n^t) = (\hat{P}_n^t \hat{P}^t)\hat{P}_n^t = \hat{P}^t \hat{P}_n^t = \hat{P}_n^t.
\end{aligned}$$

Due to the last equation, all $\hat{P}_n^t$ are projections, and from the first and second equation we get $\operatorname{im}\hat{P}_n^t = \operatorname{im}\hat{P}^t$. Thus, the operators $P_n^t \in \mathcal{K}^t$ are projections and $\dim\operatorname{im}P_n^t = m$. Moreover, we have $\operatorname{im}P_n^t \subset \operatorname{im}L_n^t$ and finally

$$\begin{aligned}\|P^t - P_n^t\| &= \|P^t - I^t + I^t - P_n^t\| = \|\hat{P}_n^t - \hat{P}^t\| \\ &= \|\hat{P}^t \sum_{k=1}^{\infty}((I^t - L_n^t)P^t)^k\| \leq \|\hat{P}^t\| \left(\frac{\|(I^t - L_n^t)P^t\|}{1 - \|(I^t - L_n^t)P^t\|} \right) \to 0.\end{aligned}$$

What have we shown?

For a given m-dimensional subspace $\mathbf{X}^t \subset \mathbf{E}^t$ and a corresponding $\mathcal{P}^t$-compact projection $P^t \in \mathcal{K}^t$ we can construct a uniformly bounded sequence (P_n^t) of projections, whose ranges are of the dimension m, are contained in $\operatorname{im}L_n^t$, respectively, and approximate $\mathbf{X}^t$ in a sense (namely $P_n^t \to P^t$ uniformly).

Now, applying the homomorphisms E_n^t, we can lift the compressions of these P_n^t to $\operatorname{im}L_n^t$ to a sequence in $\mathcal{F}^T$. For this, we define for every n

$$R_n^t := E_n^t(P_n^t L_n^t)$$

and obtain projections $R_n^t \in \mathcal{L}(\mathbf{E}_n)$, respectively. It is easy to show, that $\dim\operatorname{im}R_n^t = \dim\operatorname{im}P_n^t = m$, that $\{R_n^t\}$ is uniformly bounded, since

$$\|R_n^t\| = \|E_n^t(P_n^t L_n^t)\|_{\mathcal{L}(\mathbf{E}_n)} \leq \|E_n^t\| \|P_n^t L_n^t\|_{\mathcal{L}(\operatorname{im}L_n^t)} \leq M^t \|P_n^t\|$$

and that $\{R_n^t\} \in \mathcal{J}^t$:

$$\|R_n^t - E_n^t(L_n^t P^t L_n^t)\| = \|E_n^t(L_n^t(P_n^t - P^t)L_n^t)\| \leq M^t c^t \|P_n^t - P^t\| \to 0.$$

Definition 7. For $t \in T$ let $\mathbf{X}^t$ be an m-dimensional subspace of $\mathbf{E}^t$ and $P^t \in \mathcal{K}^t$ a projection with $\operatorname{im}P^t = \mathbf{X}^t$. The system $(P^t, \hat{P}^t, P_n^t, \hat{P}_n^t, R_n^t)$ of operators (with $n \geq n_t$), which we can construct as above, is called an $\mathbf{X}^t$-corresponding system of projections.

The following proposition shows that, applying such $\mathbf{X}^t$-corresponding systems of projections, certain properties of boundedness of the operators $W^t\{A_n\}$ can be devolved to the sequence $\{A_n\}$.

Proposition 7. *Let $\{A_n\} \in \mathcal{F}^T$ and $\boldsymbol{X}^t$ be an m-dimensional subspace of $\boldsymbol{E}^t$ which possesses a corresponding system of projections $(P^t, \hat{P}^t, P_n^t, \hat{P}_n^t, R_n^t)$. Then*

$$\begin{aligned}\limsup_n \|A_n R_n^t\| &\leq M^t \|W^t\{A_n\}P^t\|, \\ \limsup_n \|R_n^t A_n\| &\leq M^t c^t \|P^t W^t\{A_n\}\|.\end{aligned}$$

Proof. From the above considerations we get

$$\begin{aligned}\|A_n R_n^t\| &= \|A_n E_n^t(P_n^t L_n^t)\|_{\mathcal{L}(\mathbf{E}_n)} = \|E_n^t(A_n^{(t)} P_n^t L_n^t)\|_{\mathcal{L}(\mathbf{E}_n)} \\ &\leq M^t \|A_n^{(t)} P_n^t\|_{\mathcal{L}(\operatorname{im}L_n^t)} \leq M^t \|A_n^{(t)} P^t \sum_{k=0}^{\infty}((I^t - L_n^t)P^t)^k\| \\ &\leq \frac{M^t}{1 - \|(I^t - L_n^t)P^t\|} \|A_n^{(t)} P^t\| \to M^t \|W^t\{A_n\}P^t\|,\end{aligned}$$

since $\|A_n^{(t)}P^t\| \leq \|W^t\{A_n\}P^t\| + \|(A_n^{(t)} - W^t\{A_n\})P^t\|$, where the last term tends to 0 as $n \to \infty$. Analogously

$$\begin{aligned}
\|R_n^t A_n\| &\leq M^t \|P_n^t A_n^{(t)}\|_{\mathcal{L}(\operatorname{im} L_n^t)} \leq M^t c^t \|P^t \sum_{k=0}^{\infty} ((I^t - L_n^t)P^t)^k A_n^{(t)}\| \\
&= M^t c^t \| \left(\sum_{k=0}^{\infty} (P^t(I^t - L_n^t))^k \right) P^t A_n^{(t)}\| \\
&\leq \frac{M^t c^t}{1 - \|P^t(I^t - L_n^t)\|} \|P^t A_n^{(t)}\| \to M^t c^t \|P^t W^t\{A_n\}\|.
\end{aligned}$$

□

1.2.5 The approximation numbers of $\{A_n\}$ and the operators $W^t\{A_n\}$

Theorem 8. *Let $\{A_n\} \in \mathcal{F}^T$ and $t \in T$ such that $W^t\{A_n\}$ is a proper $\mathcal{P}^t$-Fredholm operator. Then $s_k(A_n) \to 0$ for all $k \in \mathbb{N}$ with $k \leq \dim\ker W^t\{A_n\}$ or $k \leq \dim\operatorname{coker} W^t\{A_n\}$ as $n \to \infty$.*
If, for one $t \in T$, the operator $W^t\{A_n\}$ is proper deficient then $s_k(A_n) \to 0$ for every $k \in \mathbb{N}$.

Proof. Let $\mathbf{X}^t \subset \mathbf{E}^t$ be an m-dimensional subspace and $(P^t, \hat{P}^t, P_n^t, \hat{P}_n^t, R_n^t)$ an $\mathbf{X}^t$-corresponding system of projections. Due to Proposition 7 we obtain for the m-th approximation numbers of $\{A_n\}$ (for $n \geq n_t$) that

$$\begin{aligned}
s_m(A_n) &= \inf\{\|A_n + F\|_{\mathcal{L}(\mathbf{E}_n)} : F \in \mathcal{F}_{\dim \mathbf{E}_n - m}(\mathbf{E}_n)\} \\
&\leq \|A_n - A_n(L_n - R_n^t)\|_{\mathcal{L}(\mathbf{E}_n)} = \|A_n R_n^t\| \\
&\leq \frac{M^t}{1 - \|(I^t - L_n^t)P^t\|} \|A_n^{(t)} P^t\| \to M^t \|W^t\{A_n\}P^t\|, \text{ and} \\
s_m(A_n) &= \inf\{\|A_n + F\|_{\mathcal{L}(\mathbf{E}_n)} : F \in \mathcal{F}_{\dim \mathbf{E}_n - m}(\mathbf{E}_n)\} \\
&\leq \|A_n - (L_n - R_n^t)A_n\|_{\mathcal{L}(\mathbf{E}_n)} = \|R_n^t A_n\| \\
&\leq \frac{M^t c^t}{1 - \|P^t(I^t - L_n^t)\|} \|P^t A_n^{(t)}\| \to M^t c^t \|P^t W^t\{A_n\}\|.
\end{aligned} \tag{6}$$

If $W^t\{A_n\}$ is proper $\mathcal{P}^t$-Fredholm, then there are projections $P, P' \in \mathcal{K}^t$ such that $\operatorname{im} P = \ker W^t\{A_n\}$ and $\ker P' = \operatorname{im} W^t\{A_n\}$. From this we obtain systems of projections which correspond to $\ker W^t\{A_n\}$ or to a complement of $\operatorname{im} W^t\{A_n\}$, respectively. In view of (6), this shows the first assertion.

Now, suppose that $W^t\{A_n\}$ is proper deficient. Then for each $k \in \mathbb{N}$ and each $\epsilon > 0$ there is a projection $Q \in \mathcal{K}^t$, $\operatorname{rank} Q \geq k$ such that $\|W^t\{A_n\}Q\| < \epsilon$ or $\|QW^t\{A_n\}\| < \epsilon$. With $\mathbf{X}^t := \operatorname{im} Q$, $P^t := Q$ and due to the inequalities (6) we arrive at $\limsup_n s_k(A_n) = 0$, since ϵ can be chosen arbitrarily. □

Note that the inequalities (6) give us an estimate for the convergence speed of the approximation numbers.

Main theorem From Theorem 6, Theorem 8 and the subsequent two sections (see Propositions 9 and 10) we obtain an important generalization of our stability result.

Theorem 9. *Let $\{A_n\} \in \mathcal{F}^T$ have the $\mathcal{P}$-dichotomy.*
If $\{A_n\}$ is a Fredholm sequence then all operators $W^t\{A_n\}$ are Fredholm operators, the number of the non-invertible operators among them is finite and the approximation numbers of $\{A_n\}$ have the k-splitting-property with $k = \alpha(\{A_n\})$, i.e.

$$\lim_{n\to\infty} s_{\alpha(\{A_n\})}(A_n) = 0 \text{ and } \liminf_{n\to\infty} s_{\alpha(\{A_n\})+1}(A_n) > 0.$$

However, if one operator $W^t\{A_n\}$ is not Fredholm, then $\lim_{n\to\infty} s_k(A_n) = 0$ for every $k \in \mathbb{N}$.

1.2.6 The $\alpha(\{A_n\})$-th approximation numbers of Fredholm sequences $\{A_n\}$

Let $\{A_n\} \in \mathcal{F}^T$ be a Fredholm sequence having the $\mathcal{P}$-dichotomy. Due to equation (3)

$$\{B_n\}\{A_n\} = \{L_n\} + \sum_{i=1}^{m} \{E_n^{t_i}(L_n^{t_i} K^{t_i} L_n^{t_i})\} + \{G_n\}$$

from Proposition 5, all operators $W^t\{A_n\}$ with $t \in T \backslash \{t_1, ..., t_m\}$ have a trivial kernel. Defining $k_i := \dim \ker W^{t_i}\{A_n\}$ $(i = 1, ..., m)$ we obtain

$$\alpha(\{A_n\}) = \sum_{i=1}^{m} k_i.$$

For each $i = 1, ..., m$ let $\mathbf{X}^i := \ker W^{t_i}\{A_n\}$ and $(P^i, \hat{P}^i, P_n^i, \hat{P}_n^i, R_n^i)$ be an $\mathbf{X}^i$-corresponding system of projections where, due to the $\mathcal{P}$-dichotomy of $\{A_n\}$, we can choose $P^i \in \mathcal{K}^{t_i}$.

For each i and every $n \geq \max_j n_{t_j}$ let $\{x_{i,l}^n\}_{l=1}^{k_i}$ denote a basis of $\operatorname{im} R_n^i$, respectively, such that for arbitrary scalars $\alpha_{i,j}^n$ the following hold:

$$|\alpha_{i,p}^n| \leq \| \sum_{j=1}^{k_i} \alpha_{i,j}^n x_{i,j}^n \| \quad \text{for all} \quad p = 1, ..., k_i. \tag{7}$$

It is a simple consequence of Auerbach's Lemma (see [13], B.4.8) that such a basis always exists.

In the next proposition we show for each sufficiently large n that, thanks to the separation condition, all of these $x_{i,l}^n$ are linearly independent. In other words: the lifted projections R_n^i are essentially different from each other and span a vectorspace of the dimension $\alpha(\{A_n\})$.

Proposition 8. *There exists a number $N \in \mathbb{N}$, such that*

$$|\alpha_{j,k}| \leq \gamma \| \sum_{i=1}^{m} \sum_{l=1}^{k_i} \alpha_{i,l} x_{i,l}^n \|, \quad \text{where} \quad \gamma = 2 \max_{i=1,...,m} M^{t_i} \|P^i\|,$$

for all $j = 1, ..., m$, $k = 1, ..., k_j$, $n \geq N$ and all scalars $\alpha_{i,l}$.

Proof. Let $i \neq j$. Due to the separation condition II, since $\|P_n^j - P^j\| \to 0$ and $P^j \in \mathcal{K}^{t_j}$, we have

$$\begin{aligned}\|R_n^i R_n^j\| &= \|E_n^{t_i}(P_n^i L_n^{t_i}) E_n^{t_j}(P_n^j L_n^{t_j})\|_{\mathcal{L}(\mathbf{E}_n)} \\ &\leq M^{t_j} \|E_n^{-t_j}(E_n^{t_i}(P_n^i L_n^{t_i})) P_n^j L_n^{t_j}\| \to 0.\end{aligned}$$

Hence, for every $\epsilon \in (0,1)$, there exists a number $N \in \mathbb{N}$ such that

$$\begin{aligned}&\sum_{j=1, j\neq i}^{m} k_j \|R_n^i R_n^j\| \leq \epsilon \text{ and} \\ &\|R_n^i\| \leq M^{t_i}\|P_n^i\| \leq (1+\epsilon) M^{t_i} \|P^i\|\end{aligned} \tag{8}$$

for all $i = 1, ..., m$ and $n \geq N$.

Now let $\alpha_{i,l}$ be arbitrary but fixed scalars and i_0, l_0 be fixed indices such that $|\alpha_{i_0,l_0}| = \max_{i,l} |\alpha_{i,l}|$. Thanks to (7) and (8) we obtain

$$\begin{aligned}\|R_n^{i_0}\| \| \sum_{i=1}^{m} \sum_{l=1}^{k_i} \alpha_{i,l} x_{i,l}^n \| &\geq \| \sum_{l=1}^{k_{i_0}} \alpha_{i_0,l} R_n^{i_0} x_{i_0,l}^n \| - \| \sum_{i=1, i\neq i_0}^{m} \sum_{l=1}^{k_i} \alpha_{i,l} R_n^{i_0} x_{i,l}^n \| \\ &\geq \| \sum_{l=1}^{k_{i_0}} \alpha_{i_0,l} x_{i_0,l}^n \| - \sum_{i=1, i\neq i_0}^{m} \sum_{l=1}^{k_i} |\alpha_{i,l}| \|R_n^{i_0} R_n^i x_{i,l}^n\| \\ &\geq |\alpha_{i_0,l_0}| - |\alpha_{i_0,l_0}| \sum_{i=1, i\neq i_0}^{m} k_i \|R_n^{i_0} R_n^i\| \\ &\geq (1-\epsilon)|\alpha_{i_0,l_0}|.\end{aligned}$$

Thus

$$\begin{aligned}|\alpha_{j,k}| \leq |\alpha_{i_0,l_0}| &\leq \frac{\|R_n^{i_0}\|}{1-\epsilon} \| \sum_{i=1}^{m} \sum_{l=1}^{k_i} \alpha_{i,l} x_{i,l}^n \| \\ &\leq \frac{1+\epsilon}{1-\epsilon} \max_i \left(M^{t_i} \|P^i\| \right) \cdot \| \sum_{i=1}^{m} \sum_{l=1}^{k_i} \alpha_{i,l} x_{i,l}^n \|\end{aligned}$$

for all $n \geq N$, for all $j = 1, ..., m$ and all $k = 1, ..., k_j$. $\epsilon = 1/3$ gives the assertion. □

Now we are in a position to study the $\alpha(\{A_n\})$-th approximation numbers of $\{A_n\}$.

Proposition 9. *Let $\{A_n\} \in \mathcal{F}^T$ be a Fredholm sequence which has the $\mathcal{P}$-dichotomy. Then $s_{\alpha(\{A_n\})}(A_n) \to 0$ as $n \to \infty$.*

Proof. For each $n \geq N$ we introduce functionals $f_{i,j}^n : \operatorname{span}\{x_{1,1}^n, ..., x_{m,k_m}^n\} \to \mathbb{C}$ by the rule

$$f_{i,j}^n \left(\sum_{k=1}^{m} \sum_{l=1}^{k_k} \alpha_{k,l}^n x_{k,l}^n \right) := \alpha_{i,j}^n \quad 1 \leq i \leq m,\ 1 \leq j \leq k_i.$$

By the preceding proposition we have $\|f^n_{i,j}\| \leq \gamma$. These functionals can be extended to the whole Banach spaces $\mathbf{E}_n$ by the Hahn-Banach theorem such that $f^n_{i,j} \in \mathbf{E}^*_n$ and $\|f^n_{i,j}\| \leq \gamma$. Further, we denote by $R_n \in \mathcal{L}(\mathbf{E}_n)$ the linear operators

$$R_n x := \sum_{i=1}^{m} \sum_{j=1}^{k_i} f^n_{i,j}(x) x^n_{i,j}.$$

The operators R_n are projections of the rank $\dim \operatorname{im} R_n = \alpha(\{A_n\})$ and they are uniformly bounded with respect to n. Moreover, for any $x \in \mathbf{E}_n$ we have (since $x^n_{i,j} = R^i_n x^n_{i,j}$)

$$\begin{aligned}\|A_n R_n x\| &= \|A_n \sum_{i=1}^{m} \sum_{j=1}^{k_i} f^n_{i,j}(x) x^n_{i,j}\| \leq \sum_{i=1}^{m} \sum_{j=1}^{k_i} |f^n_{i,j}(x)| \|A_n R^i_n x^n_{i,j}\| \\ &\leq \sum_{i=1}^{m} \sum_{j=1}^{k_i} \gamma \|x\| \|A_n R^i_n\| \|x^n_{i,j}\| = \gamma \|x\| \sum_{i=1}^{m} k_i \|A_n R^i_n\|.\end{aligned}$$

Since, for each i, $\|A_n R^i_n\| \to 0$ (see Proposition 7) it follows that

$$\begin{aligned} s_{\alpha(\{A_n\})}(A_n) &= \inf\{\|A_n + F\|_{\mathcal{L}(\mathbf{E}_n)} : F \in \mathcal{F}_{\dim \mathbf{E}_n - \alpha(\{A_n\})}(\mathbf{E}_n)\} \\ &\leq \|A_n - A_n(L_n - R_n)\|_{\mathcal{L}(\mathbf{E}_n)} \\ &= \|A_n R_n\| \leq \gamma \sum_{i=1}^{m} k_i \|A_n R^i_n\| \to 0. \end{aligned}$$

□

Note that these inequalities and Proposition 7 give us an estimate for the convergence speed of the approximation numbers: $(n \geq N)$

$$\begin{aligned} s_{\alpha(\{A_n\})}(A_n) &\leq 2 \max_{i=1,\ldots,m} \left(M^{t_i} \|P^i\|\right) \cdot \sum_{i=1}^{m} \frac{M^{t_i} k_i}{1 - \|(I^{t_i} - L^{t_i}_n) P^i\|} \|A^{(t_i)}_n P^i\| \\ &\leq \ \text{const} \sum_{i=1}^{m} \|A^{(t_i)}_n P^i\| \to 0. \end{aligned} \tag{9}$$

Starting with equation (4) instead of (3) and considering systems of projections that correspond to the cokernels, that is to complements of the images, one can analogously prove the assertion of this proposition for $s_{\beta(\{A_n\})}(A_n)$. (For this, we extend the functionals $f^n_{i,j}$ by $f^n_{i,j} \circ R^i_n$ and not by Hahn-Banach).

1.2.7 The $\alpha(\{A_n\}) + 1$-st approximation numbers of Fredholm sequences $\{A_n\}$

Proposition 10. *Let $\{A_n\} \in \mathcal{F}^T$ be a Fredholm sequence which has the $\mathcal{P}$-dichotomy. Then* $\liminf_n s_{\alpha(\{A_n\})+1}(A_n) > 0$.

Proof. Due to equation (3)

$$\{B_n\}\{A_n\} = \{L_n\} + \sum_{i=1}^{m} \{E^{t_i}_n (L^{t_i}_n K^{t_i} L^{t_i}_n)\} + \{G_n\}$$

from Proposition 5, all operators $W^t\{A_n\}$ are Fredholm operators and in case $t \in T\backslash\{t_1, ..., t_m\}$ the operator $W^t\{A_n\}$ has a trivial kernel.

Moreover, for every $i = 1, ..., m$, there is an operator $B^i \in \mathcal{L}(\mathbf{E}^{t_i}, \mathcal{P}^{t_i})$ such that $P^i := I^{t_i} - B^i W^{t_i}\{A_n\} \in \mathcal{K}^{t_i}$ is a projection onto the kernel of $W^{t_i}\{A_n\}$ (see Proposition 3). Define $\hat{P}^i := I^{t_i} - P^i$ and furthermore, for every $n \in \mathbb{N}$:

$$D_n := B_n - \sum_{i=1}^{m} E_n^{t_i}(L_n^{t_i} K^{t_i} B^i L_n^{t_i}).$$

It's obvious that $\{D_n\} \in \mathcal{F}^T$. Due to the $\mathcal{P}^{t_i}$-strong convergence of $A_n^{(t_i)}$ we obtain

$$\begin{aligned} &\| \left(E_n^{t_i}(L_n^{t_i} K^{t_i} B^i L_n^{t_i}) A_n - E_n^{t_i}(L_n^{t_i} K^{t_i} \hat{P}^i L_n^{t_i}) \right) \| \\ &= \| E_n^{t_i}(L_n^{t_i} K^{t_i} B^i (A_n^{(t_i)} - W^{t_i}\{A_n\}) L_n^{t_i}) \| \to 0 \end{aligned}$$

for all i, i.e. $\{\sum_{i=1}^m E_n^{t_i}(L_n^{t_i} K^{t_i} B^i L_n^{t_i}) A_n - \sum_{i=1}^m E_n^{t_i}(L_n^{t_i} K^{t_i} \hat{P}^i L_n^{t_i})\} \in \mathcal{G}$.

Thus

$$\begin{aligned} D_n A_n &= B_n A_n - \sum_{i=1}^{m} E_n^{t_i}(L_n^{t_i} K^{t_i} B^i L_n^{t_i}) A_n \\ &= L_n + \sum_{i=1}^{m} E_n^{t_i}(L_n^{t_i} K^{t_i} L_n^{t_i}) - \sum_{i=1}^{m} E_n^{t_i}(L_n^{t_i} K^{t_i} \hat{P}^i L_n^{t_i}) + H_n \\ &= L_n + \sum_{i=1}^{m} E_n^{t_i}(L_n^{t_i} K^{t_i} P^i L_n^{t_i}) + H_n \end{aligned}$$

with $\{H_n\} \in \mathcal{G}$. Since $\dim \operatorname{im} P^i = \dim \ker W^{t_i}\{A_n\}$ for all i, we have

$$\dim \operatorname{im} \left(\sum_{i=1}^{m} E_n^{t_i}(L_n^{t_i} K^{t_i} P^i L_n^{t_i}) \right) \leq \alpha(\{A_n\}).$$

For sufficiently large n we have $\|H_n\| < 1/2$ and, applying (2), it follows

$$\begin{aligned} \frac{1}{2} &\leq (\|(L_n + H_n)^{-1}\|_{\mathcal{L}(\mathbf{E}_n)})^{-1} = s_1(L_n + H_n) \\ &= \inf\{\|L_n + H_n + F\|_{\mathcal{L}(\mathbf{E}_n)} : F \in \mathcal{F}_{\dim \mathbf{E}_n - 1}(\mathbf{E}_n)\} \\ &\leq \inf\{\|L_n + H_n + F + \sum_{i=1}^{l} E_n^{t_i}(L_n^{t_i} K^{t_i} P^i L_n^{t_i})\|_{\mathcal{L}(\mathbf{E}_n)} : \\ &\qquad F \in \mathcal{F}_{\dim \mathbf{E}_n - \alpha(\{A_n\}) - 1}(\mathbf{E}_n)\} \\ &= \inf\{\|D_n A_n + F\|_{\mathcal{L}(\mathbf{E}_n)} : F \in \mathcal{F}_{\dim \mathbf{E}_n - \alpha(\{A_n\}) - 1}(\mathbf{E}_n)\} \\ &\leq \inf\{\|D_n A_n + D_n F\|_{\mathcal{L}(\mathbf{E}_n)} : F \in \mathcal{F}_{\dim \mathbf{E}_n - \alpha(\{A_n\}) - 1}(\mathbf{E}_n)\} \\ &\leq \|D_n\|_{\mathcal{L}(\mathbf{E}_n)} \inf\{\|A_n + F\|_{\mathcal{L}(\mathbf{E}_n)} : F \in \mathcal{F}_{\dim \mathbf{E}_n - \alpha(\{A_n\}) - 1}(\mathbf{E}_n)\} \\ &\leq \|\{D_n\}\| s_{\alpha(\{A_n\})+1}(A_n). \end{aligned}$$

□

Starting with equation (4) and considering the cokernels, one can analogously prove that $\liminf_n s_{\beta(\{A_n\})+1}(A_n) > 0$.

Theorem 10. *The index of a Fredholm sequence having the $\mathcal{P}$-dichotomy is equal to zero.*

Proof. The previous considerations and remarks show that $s_{\alpha(\{A_n\})}(A_n) \to 0$ as well as $\liminf_n s_{\beta(\{A_n\})+1}(A_n) > 0$, hence $\alpha(\{A_n\}) \leq \beta(\{A_n\})$. Similarly, we get $s_{\beta(\{A_n\})}(A_n) \to 0$ and $\liminf_n s_{\alpha(\{A_n\})+1}(A_n) > 0$, hence $\alpha(\{A_n\}) \geq \beta(\{A_n\})$. This proves the assertion. □

1.3 Localization

Let $\mathcal{A}$ be a unital Banach algebra. The center $\mathrm{Cen}\,\mathcal{A}$ of $\mathcal{A}$ is the set of all elements $x \in \mathcal{A}$ with the property that $xa = ax$ for all $a \in \mathcal{A}$. Let $\mathcal{C}$ be a closed subalgebra of $\mathrm{Cen}\,\mathcal{A}$ containing the identity. Further, let $\mathcal{M}_\mathcal{C}$ denote the maximal ideal space of $\mathcal{C}$. If $i \in \mathcal{M}_\mathcal{C}$ is a maximal ideal, then $\mathcal{J}_i$ will denote the smallest closed (two-sided) ideal of $\mathcal{A}$ containing i, and Φ_i the quotient mapping $\mathcal{A} \to \mathcal{A}/\mathcal{J}_i$.

Theorem 11. *(Allan, [17], Theorem 2.3.16)*
$A \in \mathcal{A}$ is invertible if and only if $\Phi_i(A)$ is invertible in $\mathcal{A}/\mathcal{J}_i$ for every $i \in \mathcal{M}_\mathcal{C}$.

2 Finite sections of band-dominated operators

In this section we illustrate the results of this paper by an interesting example.

2.1 Function spaces on $\mathbb{Z}^K$

Let K, N be positive integers and $p \geq 1$. Then we denote by $l^p = l^p(\mathbb{Z}^K, \mathbb{C}^N)$ the Banach space of all functions $f : \mathbb{Z}^K \to \mathbb{C}^N$ such that

$$\|f\|_{l^p}^p := \sum_{x\in\mathbb{Z}^K} \|f(x)\|_p^p < \infty, \text{ where } \|z\|_p^p = \|(z_i)_{i=1}^N\|_p^p := \sum_{i=1}^N |z_i|^p.$$

Further, let $l^\infty = l^\infty(\mathbb{Z}^K, \mathbb{C}^N)$ denote the Banach space of all functions f with

$$\|f\|_{l^\infty} := \sup_{x\in\mathbb{Z}^K} \|f(x)\|_\infty < \infty, \text{ where } \|z\|_\infty = \|(z_i)_{i=1}^N\|_\infty := \max_{i=1..N} |z_i|.$$

$l^0 = l^0(\mathbb{Z}^K, \mathbb{C}^N)$ refers to the closed subspace of all functions $f \in l^\infty$ with

$$\lim_{x\to\infty} \|f(x)\|_\infty = 0.$$

For $1 \leq p < \infty$ the dual space of l^p can be identified with l^q where $1/p + 1/q = 1$ and the dual space of l^0 is isomorphic to l^1.

Finally let $l^{\infty,p}_{N\times N} = l^{\infty,p}(\mathbb{Z}^K, \mathbb{C}^{N\times N})$ (for $p \in \{0\} \cup [1, \infty]$) denote the Banach algebra of all matrix-valued functions $a : \mathbb{Z}^K \to \mathbb{C}^{N\times N}$ with

$$\|a\|_{l^{\infty,p}_{N\times N}} := \sup_{x\in\mathbb{Z}^K} \|a(x)\|_{\mathcal{L}(\mathbb{C}^N, \|\cdot\|_p)} < \infty.$$

Every function $a \in l^{\infty,p}_{N\times N}$ gives rise to an operator $aI \in \mathcal{L}(l^p)$ (a so called multiplication operator) via

$$(af)(x) = a(x)f(x), \quad x \in \mathbb{Z}^K.$$

Evidently, $\|aI\|_{\mathcal{L}(l^p)} = \|a\|_{l^{\infty,p}_{N\times N}}$. By this means, the functions in $l^\infty(\mathbb{Z}^K, \mathbb{C})$ induce multiplication operators as well.

2.2 Band-dominated operators and limit operators

For every subset $F \subset \mathbb{R}^K$ let χ_F denote the characteristic function of F and for a function $f : \mathbb{R}^K \to \mathbb{C}^N$ let $\hat{f}$ denote the restriction of f to $\mathbb{Z}^K$.

Band-dominated operators A band operator A is a finite sum of the form $A = \sum_\alpha a_\alpha V_\alpha$, where $\alpha \in \mathbb{Z}^K$, $a_\alpha \in l^{\infty,p}_{N\times N}$ and V_α denotes the shift operator

$$(V_\alpha f)(x) = f(x-\alpha), \quad x \in \mathbb{Z}^K.$$

An operator is called band-dominated if it is the uniform limit of a sequence of band operators. We denote the class of all band-dominated operators by $\mathcal{A}_{l^p}$.

Let $\Omega \subset \mathbb{R}^K$ be a compact and convex polytope with vertices in $\mathbb{Z}^K$ and suppose that $0 \in \mathbb{Z}^K$ is an inner point of Ω. Further, let

$$\chi_{m\Omega}(x) := \chi_\Omega\left(\frac{x}{m}\right) \text{ and } L_m := \hat{\chi}_{m\Omega} I \quad (m \in \mathbb{N}).$$

Obviously, all $L_m \in \mathcal{L}(l^p)$ are projections with $\|L_m\| = 1$ and $\mathcal{P} := (L_n)_{n\in\mathbb{N}}$ is a uniform approximate identity.

Here is a collection of important properties of band-dominated operators:

Theorem 12. *(see [17], Propositions 2.1.7ff)*

- $\mathcal{A}_{l^p} \subset \mathcal{L}(l^p, \mathcal{P}) \subset \mathcal{L}(l^p)$ *are closed algebras.*
- $\mathcal{A}_{l^p} \subset \mathcal{L}(l^p, \mathcal{P}) \subset \mathcal{L}(l^p)$ *are inverse closed.*
- *The set* $\mathcal{K} := \mathcal{K}(l^p, \mathcal{P})$ *of all* $\mathcal{P}$*-compact operators is a closed ideal in* $\mathcal{A}_{l^p}$*.*
- $\mathcal{A}_{l^p}/\mathcal{K}$ *is inverse closed in the quotient algebra* $\mathcal{L}(l^p, \mathcal{P})/\mathcal{K}$*.*

Theorem 13. *([17], Theorem 2.1.6)*
Let $A \in \mathcal{L}(l^p, \mathcal{P})$*.* A *is band-dominated if and only if, for every* $\epsilon > 0$*, there exists an* M*, such that whenever* F, G *are subsets of* $\mathbb{Z}^K$ *with* $\operatorname{dist}(F,G) > M$ *then*

$$\|\hat{\chi}_F A \hat{\chi}_G I\|_{\mathcal{L}(l^p)} < \epsilon.$$

Proposition 11. *(Duality)*
For $p = 0$ *let* $q = 1$*, and for* $1 \le p < \infty$ *let* $1 < q \le \infty$ *such that* $1/p + 1/p = 1$*, respectively. Then* $(\mathcal{K}(l^p, \mathcal{P}))^* = \mathcal{K}(l^q, \mathcal{P})$ *and* $(\mathcal{A}_{l^p})^* = \mathcal{A}_{l^q}$*.*
A sequence $(A_n) \subset \mathcal{A}_{l^p}$ *converges* $\mathcal{P}$*-strongly to an operator* $A \in \mathcal{A}_{l^p}$ *if and only if* $(A_n^*) \subset \mathcal{A}_{l^q}$ *converges* $\mathcal{P}^*$*-strongly to* $A^* \in \mathcal{A}_{l^q}$*.*

Proof. The relations $(\mathcal{K}(l^p, \mathcal{P}))^* = \mathcal{K}(l^q, \mathcal{P}^*)$ and $(\mathcal{L}(l^p, \mathcal{P}))^* \subset \mathcal{L}(l^q, \mathcal{P}^*)$, as well as the duality of $\mathcal{P}$-strong convergence and $\mathcal{P}^*$-strong convergence were already mentioned in Section 1.1.

Let $a \in l^{\infty,p}_{N\times N}$ and $z \in \mathbb{Z}^K$. Then $(aV_z)^* = V_z^* a^* I^* = \hat{b} V_{-z}$ where b is given by $b(x) = a^*(x+z)$. Thus, the adjoint A^* of a band operator A is a band operator again, and for every band operator B there is a so called preadjoint band operator $(^*B)$ with $(^*B)^* = B$. Since passing to adjoints is an isometry, we obtain this for band-dominated operators as well. Thus $(\mathcal{A}_{l^p})^* = \mathcal{A}_{l^q}$. □

Limit operators Let $A \in \mathcal{L}(l^p, \mathcal{P})$ and $h = (h_n)_{n\in\mathbb{N}} \subset \mathbb{Z}^K$ be a sequence tending to infinity, i.e. $\|h_n\| \to \infty$ as $n \to \infty$. The operator A_h is called limit operator of A with respect to h if

$$V_{-h_n} A V_{h_n} \to A_h \ \mathcal{P}\text{-strongly}.$$

The set $\sigma(A)$ of all limit operators of A is called the operator spectrum of A.

Proposition 12. *$\sigma(K) = \{0\}$ for each $K \in \mathcal{K}$.*

Proof. We fix P_m and obtain for every P_k that

$$\begin{aligned}\|V_{-h_n} K V_{h_n} P_m\| &\leq \|V_{-h_n} K P_k V_{h_n} P_m\| + \|V_{-h_n} K (I - P_k) V_{h_n} P_m\| \\ &\leq \|K\| \|P_k V_{h_n} P_m\| + \|K(I - P_k)\|.\end{aligned}$$

For every given $\epsilon > 0$ there is a k, such that the second summand is smaller than ϵ. Afterwards it is easy to see that the first summand is equal to zero for sufficiently large n.

Analogously we show $\|P_m V_{-h_n} K V_{h_n}\| \to 0$ and the assertion is proved. □

We will see, that limit operators are a very important tool for the examination of band-dominated operators. As a first remarkable result there is the following characterization of Fredholm band-dominated operators:

Theorem 14. *Each band-dominated operator A has the $\mathcal{P}$-dichotomy and the following are equivalent:*

1. *A is Fredholm.*
2. *A is $\mathcal{P}$-Fredholm.*
3. *A is proper $\mathcal{P}$-Fredholm.*
4. *There exists a regularizer $B \in \mathcal{A}_{l^p}$ such that $P := I - BA \in \mathcal{K}$ and $P' := I - AB \in \mathcal{K}$ are projections with $\operatorname{im} P = \ker A$ and $\ker P' = \operatorname{im} A$.*
5. *All limit operators of A are invertible and their inverses are uniformly bounded, i.e. $\sup\{\|(A_h)^{-1}\| : A_h \in \sigma(A)\} < \infty$.*

Proof. At first, let us consider the cases $p \notin \{1, \infty\}$. Then (L_n) as well as (L_n^*) converge strongly to the identity, hence Corollary 1 yields the $\mathcal{P}$-dichotomy and Proposition 3 shows the equivalence of 1, 2, 3 and 4, since $\mathcal{A}_{l^p}/\mathcal{K}$ is inverse closed. The (non-trivial) proof for assertion 5 can be found in [17], Theorem 2.2.1.

Now, let $p = 1$. Since $\mathcal{A}_{l^1} = (\mathcal{A}_{l^0})^*$, each band-dominated operator $A \in \mathcal{A}_{l^1}$ enjoys the $\mathcal{P}$-dichotomy and 1, 2, 3 and 4 are equivalent (see again Proposition 3). For every $A \in \mathcal{A}_{l^1}$ there is a preadjoint operator $({}^*A) \in \mathcal{A}_{l^0}$, i.e. $({}^*A)^* = A$, and A, $({}^*A)$ are Fredholm at the same time. Moreover, it is easy to see that there is a one-to-one correspondence between the limit operators of A and the limit operators of $({}^*A)$. More precisely, the limit operator A_h of A with respect to a sequence h tending to infinity exists, if and only if $({}^*A)_h$ exists. Then $(({}^*A)_h)^* = A_h$ (see Proposition 11). Hence, these limit operators are uniformly invertible at the same time. Therefore A is Fredholm if and only if its limit operators are uniformly invertible.

The case $p = \infty$ can be treated analogously, utilizing the duality between $\mathcal{A}_{l^\infty}$ and $\mathcal{A}_{l^1}$. □

Proposition 13. *Let A be a Fredholm band-dominated operator, B be a regularizer of A, i.e. $AB - I, BA - I \in \mathcal{K}$, and $h = (h_n)$ be a sequence tending to infinity, such that the limit operator A_h of A exists. Then the limit operator of B with respect to h exists too, and equals $(A_h)^{-1}$.*

Proof. A_h is invertible due to Theorem 14, and since $BA - I = K \in \mathcal{K}$ we have

$$\begin{aligned} &V_{-h_n}BV_{h_n} - (A_h)^{-1} \\ &\quad = V_{-h_n}BV_{h_n}(I - V_{-h_n}AV_{h_n}(A_h)^{-1}) + V_{-h_n}KV_{h_n}(A_h)^{-1} \\ &\quad = V_{-h_n}BV_{h_n}(A_h - V_{-h_n}AV_{h_n})(A_h)^{-1} + V_{-h_n}KV_{h_n}(A_h)^{-1} \end{aligned}$$

and hence $\|(V_{-h_n}BV_{h_n} - (A_h)^{-1})J\| \to 0$ for every $J \in \mathcal{K}$. Analogously, we can show that $\|J(V_{-h_n}BV_{h_n} - (A_h)^{-1})\| \to 0$ and obtain the $\mathcal{P}$-strong convergence of $V_{-h_n}BV_{h_n}$ to $(A_h)^{-1}$. □

Proposition 14. *(Strong convergence)*
Let $p \neq \infty$, $A \in \mathcal{A}_{l^p}$ and h be a sequence tending to infinity. The operator sequence $(A_n) := (V_{-h_n}AV_{h_n})$ converges $\mathcal{P}$-strongly to the limit operator A_h, if and only if it converges strongly.

Proof. We note that for $p \notin \{1, \infty\}$ Corollary 1 tells us that strong convergence and $\mathcal{P}$-strong convergence coincide, and that for $p = 1$ every $\mathcal{P}$-strong convergent sequence converges strongly, too.

Finally, we consider strongly convergent sequences (A_n) in case $p = 1$. Obviously, $\|(A_n - A_h)L_m\| \to 0$ for every L_m (which is compact) and it remains to show that $\|L_m(A_n - A_h)\| \to 0$. For this, we fix L_m and $\epsilon > 0$. Then Theorem 13 yields an l with $\|L_m(A_n - A_h)(I - L_l)\| < \epsilon$ for all n. Together with

$$\|L_m(A_n - A_h)\| \leq \|L_m(A_n - A_h)L_l\| + \|L_m(A_n - A_h)(I - L_l)\|$$

and the observation that the first summand tends to zero as $n \to \infty$, due to the compactness of L_l, this proves the $\mathcal{P}$-strong convergence of (A_n). □

Remark 2. Let $\Omega' \subset \mathbb{R}^K$ be another compact and convex polytope with vertices in $\mathbb{Z}^K$ and 0 as its inner point and let $\mathcal{P}' = (L'_n)$ denote the corresponding approximate projection. Then

$$\lim_{n\to\infty} L_m L'_n = \lim_{n\to\infty} L'_n L_m = L_m \text{ and } \lim_{n\to\infty} L_n L'_m = \lim_{n\to\infty} L'_m L_n = L'_m.$$

Therefore $\mathcal{P}$ and $\mathcal{P}'$ are said to be equivalent and [17], Lemma 1.1.10 tells us that

$$\mathcal{K}(l^p, \mathcal{P}) = \mathcal{K}(l^p, \mathcal{P}').$$

Thus, $\mathcal{L}(l^p, \mathcal{P}) = \mathcal{L}(l^p, \mathcal{P}')$ and the notations $\mathcal{P}$-Fredholmness, $\mathcal{P}'$-Fredholmness and $\mathcal{P}$-strong convergence, $\mathcal{P}'$-strong convergence coincide.

This shows that the properties of band-dominated operators or operator sequences do not depend on the concrete choice of Ω.

2.3 Finite sections of band-dominated operators

2.3.1 Definitions

Let $x \in \Omega \backslash \{0\}$. Then we can write x as a convex linear combination of the vertices (extreme points) u_k of Ω (Krein-Milman theorem):

$$x = \sum_i \alpha_i u_i, \quad \alpha_i \geq 0, \quad \sum_i \alpha_i = 1.$$

We choose an i_0 such that $\alpha_{i_0} \neq 0$. Then we denote by $[\alpha]$ the integer part of the real α and for each $n \in \mathbb{N}$ we set

$$x_n := \sum_{i, i \neq i_0} [n\alpha_i] \, u_i + (n - \sum_{i, i \neq i_0} [n\alpha_i]) u_{i_0}.$$

Obviously $x_n \in \mathbb{Z}^K$ for every $n \in \mathbb{N}$ and $\|nx - x_n\|$ is uniformly bounded with respect to n and x, thus nx and x_n are at close quarters. More precisely, the sequences (nx) and (x_n) both tend into the direction of $\frac{x}{\|x\|_2}$ in the sense of Section 2.3.1. in [17]. For $x = 0$ we put $x_n := 0$.

The boundary $\delta\Omega$ of Ω consists of a finite number of $(K-1)$-dimensional polytopes P_j. To each of these polytopes we can assign a uniquely determined K-dimensional half space H_j as follows: $P_j \subset \delta H_j$ and 0 is an inner point of H_j. In what follows, we assume that $H_j \neq H_k$ whenever $j \neq k$. Finally, we set $H_j^0 := H_j - u_j$, where u_j is a vertex of P_j.
According to the considered theory above, we introduce the following notations:

$$\mathbf{E}_n := \operatorname{im} L_n, \; T := \delta\Omega \cup \{0\}, \; K_x := \cap_{j: x \in \delta H_j} H_j^0 \subset \mathbb{Z}^K, \; I^x := \hat{\chi}_{K_x} I \text{ and}$$

$$\begin{array}{ll} \mathbf{E}^0 := l^p(\mathbb{Z}^K, \mathbb{C}^N) & L_n^0 := L_n \\ & E_n^0 : \mathcal{L}(\operatorname{im} L_n^0) \to \mathcal{L}(\mathbf{E}_n), B \mapsto B \\ \mathbf{E}^x := \operatorname{im} I^x & L_n^x := V_{-x_n} L_n V_{x_n} \\ & E_n^x : \mathcal{L}(\operatorname{im} L_n^x) \to \mathcal{L}(\mathbf{E}_n), B \mapsto V_{x_n} B V_{-x_n} \end{array}$$

for every $x \in \delta\Omega$ and every $n \in \mathbb{N}$.

It is easy to check that for each $x \in T$ the following holds:

- $\mathbf{E}^x$ is a Banach space.
- The set $\mathcal{K}^x := \mathcal{K}(\mathbf{E}^x, \mathcal{P}^x)$ equals $\{I^x K I^x : K \in \mathcal{K}(l^p, \mathcal{P})\} \subset \mathcal{K}(\mathbf{E}^x)$. In case $p \notin \{0, \infty\}$ we have $\mathcal{K}^x = \mathcal{K}(\mathbf{E}^x)$.
- $L_m^x \in \mathcal{L}(\mathbf{E}^x)$ are projections of norm 1 and $L_n^x \to I^x$ strongly if $p \neq \infty$.
- $\mathcal{P}^x := (L_n^x)_{n \in \mathbb{N}}$ is a uniform approximate identity.
- E_n^x are surjective isometries, in particular condition I is fulfilled.

We note, that the sets $\mathcal{K}^x$ do not depend on the concrete choice of Ω, which can be seen as in Remark 2, hence the notions of $\mathcal{P}^x$-Fredholmness and $\mathcal{P}^x$-strong convergence are consistent, too.

The general theory now provides the sequence algebras $\mathcal{F}$ and $\mathcal{F}^T$, as well as the ideals $\mathcal{G}$ and $\mathcal{J}^T$. Utilizing the results of Section 1.2 we want to study the finite sections $\{L_n A L_n\}$ of band-dominated operators A. For this, we certainly require the existence of the limit operators $W^x\{L_nAL_n\} = \mathcal{P}\text{-lim}\, V_{x_n}AV_{-x_n}$ that do not exist for general band-dominated operators. Obviously, the shape of the spaces $\mathbf{E}_n$, the transformations E_n^x and the arising limit operators now depend on Ω. Therefore, we will restict our considerations to appropriate subclasses of band-dominated operators, which we will introduce in the next step.

2.3.2 Algebras of band-dominated operators

In what follows we define a subclass of band-dominated operators A which can, roughly spoken, be lifted to sequences $\{E_n^x(L_n^xAL_n^x)\}$ in an appropriate algebra for which the techniques of Section 1.2 are applicable. More precisely, these sequences need to possess all required limit operators. Moreover, these limit operators as well as their inverses or regularizers, if they exist, should be of the same kind.

Definition 8. For every $x \in T$ let $\mathcal{A}_{l^p}^x$ denote the set of all band-dominated operators $A \in \mathcal{A}_{l^p}$ with the property that for every $m \in \mathbb{N}$, $y^0 := x$ and arbitrary $y^1, y^2, ..., y^m \in T$ the following limit operators exist

$$A_{y^i} := \mathcal{P}\text{-}\lim_{n\to\infty} V_{-y_n^i} V_{y_n^{i-1}} A_{y^{i-1}} V_{-y_n^{i-1}} V_{y_n^i} \quad (i = 1, ..., m). \tag{10}$$

Notice, that A_{y^i} is the limit operator of $A_{y^{i-1}}$ with respect to the sequence $(y_n^i - y_n^{i-1})$.

Proposition 15. *Let $x \in T$ and $A \in \mathcal{A}_{l^p}^x$. Then*

- *$\mathcal{A}_{l^p}^x$ is a closed subalgebra of $\mathcal{A}_{l^p}$, containing the ideal $\mathcal{K}$ as well as the operators I and I^x.*
- *$\mathcal{A}_{l^p}^x$ is inverse closed.*
- *$\mathcal{A}_{l^p}^x/\mathcal{K}$ is inverse closed in $\mathcal{A}_{l^p}/\mathcal{K}$ and $\mathcal{L}(l^p, \mathcal{P})/\mathcal{K}$.*
- *Let $y \in T$. Then $A_y \in \mathcal{A}_{l^p}^y$.*

Proof. For $A, B \in \mathcal{A}_{l^p}^x$ and $\alpha, \beta \in \mathbb{C}$ we have

$$(\alpha A + \beta B)_{y^i} = \alpha A_{y^i} + \beta B_{y^i}, \quad (AB)_{y^i} = A_{y^i} B_{y^i},$$

due to Theorem 5. Therefore $\mathcal{A}_{l^p}^x$ is an algebra. Let $(A_k) \subset \mathcal{A}_{l^p}^x$ be a Cauchy sequence and $A \in \mathcal{A}_{l^p}$ its limit. Again from Theorem 5, we obtain that $((A_k)_{y^1})$ is a Cauchy sequence. Thus, there exists a band-dominated operator A_{y^1} such that $(A_k)_{y^1} \to_k A_{y^1}$ and moreover $V_{-y_n^1}V_{x_n}AV_{-x_n}V_{y_n^1} \to A_{y^1}$ $\mathcal{P}$-strongly. Iterating this idea, we obtain operators A_{y^i} for all i and hence $A \in \mathcal{A}_{l^p}^x$. This proves the closedness of $\mathcal{A}_{l^p}^x$. $I, I^x \in \mathcal{A}_{l^p}^x$ is obvious and due to Proposition 12 we have $\mathcal{K} \subset \mathcal{A}_{l^p}^x$. Thus, the first assertion is proved.
Let $A \in \mathcal{A}_{l^p}^x$ be a Fredholm operator and B a regularizer of A. B is band-dominated, since $\mathcal{A}_{l^p}/\mathcal{K}$ is inverse closed in $\mathcal{L}(l^p, \mathcal{P})/\mathcal{K}$. Due to Proposition 13 the $\mathcal{P}$-strong limit of the sequence $V_{-y_n}V_{x_n}BV_{-x_n}V_{y_n}$ exists and equals $(A_y)^{-1}$ for each $y \in T, y \neq x$. This is particularly true if A is invertible with $B = A^{-1}$. Iterating this argument to all limit operators (10) of B leads to $B \in \mathcal{A}_{l^p}^x$ and shows the inverse closedness of $\mathcal{A}_{l^p}^x$ and $\mathcal{A}_{l^p}^x/\mathcal{K}$.
$A_y \in \mathcal{A}_{l^p}^y$ is an immediate consequence from the definition. □

To be able to consider the liftings $\{E_n^x(L_n^x A L_n^x)\}$ of band-dominated operators A, we actually need algebras of operators acting on $\mathbf{E}^x$:

Definition 9. For every $x \in T$, let $\mathcal{B}_{l^p}^x$ denote the set of the restrictions of all operators in $\mathcal{A}_{l^p}^x$ to the space $\mathbf{E}^x$, i.e.

$$\mathcal{B}_{l^p}^x := \{I^x A I^x|_{\mathbf{E}^x} : A \in \mathcal{A}_{l^p}^x\} \subset \mathcal{L}(\mathbf{E}^x, \mathcal{P}^x).$$

Corollary 2. *$\mathcal{B}_{l^p}^x$ is an inverse closed Banach algebra of operators on $\boldsymbol{E}^x$, containing the identity I^x and the ideal $\mathcal{K}^x$.*
$\mathcal{B}_{l^p}^x/\mathcal{K}^x$ is inverse closed in $\mathcal{L}(\boldsymbol{E}^x, \mathcal{P}^x)/\mathcal{K}^x$.

Proof. Obviously, $\mathcal{B}_{l^p}^x$ is a closed algebra with the identity I^x which contains the ideal $\mathcal{K}^x$. Let B be the inverse (or a regulariser) of $A \in \mathcal{B}_{l^p}^x$, respectively. Then $I^x B I^x + (I - I^x) \in \mathcal{A}_{l^p}$ is the inverse (a regulariser) of $I^x A I^x + (I - I^x) \in \mathcal{A}_{l^p}^x$, thus $I^x B I^x + (I - I^x) \in \mathcal{A}_{l^p}^x$, that is $B \in \mathcal{B}_{l^p}^x$. □

Continuous coefficients Here we consider the set $C(\overline{\mathbb{R}}^K, \mathbb{C}^{N\times N})$ of all continuous functions $a : \mathbb{R}^K \to \mathbb{C}^{N\times N}$, such that the following limits exist uniformly with respect to η:

$$a^\infty(\eta) := \lim_{t\to\infty} a(t\eta), \quad \eta \in \mathbb{S}^{K-1},$$

where $\mathbb{S}^{K-1}$ denotes the unit sphere in $\mathbb{R}^K$.
Let $\mathcal{A}_{l^p}(C)$ denote the smallest closed subalgebra of $\mathcal{A}_{l^p}$ containing $\{V_\alpha : \alpha \in \mathbb{Z}^K\}$ and $\{\hat{a}I : a \in C(\overline{\mathbb{R}}^K, \mathbb{C}^{N\times N})\}$.

Proposition 16. *$\mathcal{A}_{l^p}(C)$ is a closed subalgebra of $\mathcal{A}_{l^p}^x$ for every $x \in T$. In particular, $\mathcal{A}_{l^p}(C) \subset \mathcal{B}_{l^p}^0 = \mathcal{A}_{l^p}^0$.*

Proof. For $\alpha \in \mathbb{Z}^K$ and $A = V_\alpha$, it's clear, that $A \in \mathcal{A}_{l^p}^x$.
Let $x, y \in T$ and $x \neq y$, $a \in C(\overline{\mathbb{R}}^K, \mathbb{C}^{N\times N})$ be given and $A = \hat{a}I$. We want to show, that the limit operator $A_y = \mathcal{P}\text{-}\lim V_{-y_n} V_{x_n} A V_{-x_n} V_{y_n}$ exists and equals $a^\infty(\eta)I$, where $\eta := \frac{y-x}{\|y-x\|}$. For this we define a sequence (a_n) of functions a_n by $a_n : z \mapsto a(z + y_n - x_n)$. The sequences $(a_n(z))$ converge to $a^\infty(\eta)$ for every fixed $z \in \mathbb{Z}^K$, due to the continuity of a. Thus,

$$\lim_{n\to\infty} \|(\hat{a}_n I - a^\infty(\eta)I)L_m\| = 0$$

for every L_m. Since the multiplication operators $(\hat{a}_n I - a^\infty(\eta)I)$ and L_m commute, the $\mathcal{P}$-strong convergence of $(\hat{a}_n I)$ to $a^\infty(\eta)I$ is proven. Thanks to $\hat{a}_n I = V_{-y_n} V_{x_n} \hat{a} I V_{-x_n} V_{y_n}$ it is now easy to see, that all generators of $\mathcal{A}_{l^p}(C)$ are contained in $\mathcal{A}_{l^p}^x$. □

For $A = \sum_l \hat{a}_l V_l$ ($a_l \in C(\overline{\mathbb{R}}^K, \mathbb{C}^{N\times N})$) we have $A_y = \sum_l a_l^\infty(\eta) V_l$, and it easily follows, that all limit operators A_y ($y \neq x$) of operators $A \in \mathcal{A}_{l^p}(C) \subset \mathcal{A}_{l^p}^x$ are shift-invariant, that is $A_y = V_{-\alpha} A_y V_\alpha$ for arbitrary $\alpha \in \mathbb{Z}^K$. The set of all shift-invariant band-dominated operators will be denoted by $\mathcal{A}_{l^p}(\mathbb{C})$.

Half spaces and cones Let $\mathcal{I}_{\delta\Omega}$ denote the set of all points in $\delta\Omega$ which are relatively inner points of P_j with respect to H_j for one j (where P_j, H_j are as in 2.3.1). By $\mathcal{H}_\Omega$ we denote the set of all closed half spaces $H \subset \mathbb{R}^K$, such that δH is a $(K-1)$-dimensional linear space and does not contain any point from $\mathcal{I}_{\delta\Omega}$. For $H \in \mathcal{H}_\Omega$ let χ_H denote the characteristic function of H and $P_H := \hat{\chi}_H I$ the related projection.

Proposition 17. *Let $H \in \mathcal{H}_\Omega$. Then $P_H \in \mathcal{A}^x_{l^p}$ whenever $x \in T \cap \delta H$. In particular, $P_H \in \mathcal{B}^0_{l^p} = \mathcal{A}^0_{l^p}$.*

Proof. Let $x = 0$ and $y \in T$. We have to distinguish between different cases: $y \in \operatorname{int} H$, $y \notin H$ or $y \in \delta H$. It is easy to check that the operators $V_{-y_n} P_H V_{y_n}$ converge $\mathcal{P}$-strongly to I or 0 for $y \in \operatorname{int} H$ or $y \notin H$, respectively. The case $y = 0$ is trivial. For $y \in \delta H$, $y \neq 0$ we have, due to the definition of Ω and H, that $y \in P_j \cap P_k \subset \delta H$ with some $j \neq k$ and every representation of y as a convex linear combination of vertices of Ω only contains vertices in $P_j \cap P_k \subset \delta H$. Hence, $y_n \in \delta H$ and consequently $V_{-y_n} P_H V_{y_n} = P_H$ for every n. This proves $(P_H)_y = P_H$.

For arbitrary $x \in T \cap \delta H$ we note that $V_{x_n} P_H V_{-x_n} = P_H$ and obtain again

$$(P_H)_y = \begin{cases} P_H & \text{if } y \in \delta H \\ I & \text{if } y \in \operatorname{int} H \\ 0 & \text{if } y \notin H. \end{cases} \tag{11}$$

Now the assertion easily follows. □

Remark 3. As an explanation for the above definition of the appropriate half spaces $H \in \mathcal{H}_\Omega$ we consider the case $K = 3$: In this situation the definition just means, that δH intersects the boundary of Ω only in its edges and vertices.

Thus, we can consider band-dominated operators on half spaces and, by combination of different half spaces, on cones. Moreover we can compose the restrictions of $\mathcal{A}_{l^p}(C)$-operators to cones. Note that $\mathcal{A}_{l^p}(C)$ is independent of the choice of Ω, but there is a close relation between Ω and the half spaces in $\mathcal{H}_\Omega$.

2.3.3 The sequence algebra $\mathcal{F}_{\mathcal{B}_{l^p}}$

The sequences $E^x_n\{L^x_n A L^x_n\}$ with $A \in \mathcal{B}^x_{l^p}$ are good candidates to be generators of an appropriate sequence algebra which allows us to apply the general theory. In fact, we will need some more sequences.

Definition 10. Let $g \in C(\Omega)$ be a continuous complex valued function on Ω. By the functions $g_n(x) := g(\frac{x}{n})$ we define an operator sequence $\{L_n \hat{g}_n L_n\} \in \mathcal{F}$ and we denote by C_Ω the set of all such sequences arising from continuous complex valued functions.

Proposition 18. *Let $\{L_n \hat{g}_n L_n\} \in C_\Omega$ and $x \in \Omega$. Then $\mathcal{P}$-$\lim V_{-x_n} \hat{g}_n L_n V_{x_n}$ exists for all $x \in \Omega$ and equals $g(x) I^x$, respectively, where $I^x := I$ for $x \in \operatorname{int} \Omega$.*

Proof. We have $V_{-x_n} \hat{g}_n L_n V_{x_n} = \hat{f}_n V_{-x_n} L_n V_{x_n}$ with $f_n(y) := g(\frac{y+x_n}{n})$. Obviously, $f_n(y)$ converges to $g(x)$ as $n \to \infty$ for every fixed $y \in \mathbb{Z}^K$, due to the continuity of g. Thus, $\|(V_{-x_n} \hat{g}_n L_n V_{x_n} - g(x) I^x) L_m\| \to 0$ for every L_m and also $\|L_m (V_{-x_n} \hat{g}_n L_n V_{x_n} - g(x) I^x)\| \to 0$, since these operators commute. This proves the $\mathcal{P}$-strong convergence. □

We recall some notations from Section 1.2: $\mathcal{F}$ denotes the Banach algebra of all bounded sequences $\{A_n\}$ of bounded linear operators $A_n \in \mathcal{L}(\mathbf{E}_n)$, and $\mathcal{F}^T$ denotes the closed subalgebra of all sequences $\{A_n\} \in \mathcal{F}$ for which there exist operators $W^x\{A_n\} \in \mathcal{L}(\mathbf{E}^x, \mathcal{P}^x)$ for all $x \in T$, such that

$$E_n^{-x}(A_n)L_n^x \to W^x\{A_n\} \quad \mathcal{P}^x\text{-strongly.}$$

Condition I and, due to Proposition 12, also condition II are fulfilled. The sets

$$\begin{aligned} \mathcal{G} &:= \{\{G_n\} : \|G_n\| \to 0\}, \\ \mathcal{J} = \mathcal{J}^T &:= \operatorname{clos\,span}\{\{E_n^x(L_n^x K L_n^x)\}, \{G_n\} : x \in T, K \in \mathcal{K}^x, \{G_n\} \in \mathcal{G}\} \end{aligned}$$

are closed ideals in $\mathcal{F}^T$ and a sequence $\{A_n\} \in \mathcal{F}^T$ is called Fredholm sequence, if $\{A_n\} + \mathcal{J}$ is invertible in $\mathcal{F}^T/\mathcal{J}$.

Definition 11. Let $\mathcal{F}_{\mathcal{B}_{lp}}$ denote the smallest closed subalgebra of $\mathcal{F}$, containing all sequences of the form $\{E_n^x(L_n^x A L_n^x)\}$ with $x \in T$ and $A \in \mathcal{B}_{lp}^x$, as well as all sequences from C_Ω and $\mathcal{G}$.

It is clear, that for $\{A_n\} \in \mathcal{F}_{\mathcal{B}_{lp}}$ all limit operators $W^x\{A_n\}$ exist and are contained in $\mathcal{B}_{lp}^x$, respectively, hence $\mathcal{F}_{\mathcal{B}_{lp}} \subset \mathcal{F}^T$. Moreover, the ideals $\mathcal{G}$ and $\mathcal{J}$ are contained in $\mathcal{F}_{\mathcal{B}_{lp}}$ and all sequences in $\mathcal{F}_{\mathcal{B}_{lp}}$ have the $\mathcal{P}$-dichotomy (see Theorem 14). With the help of Proposition 14 it is easy to see that in case $p \neq \infty$ the $\mathcal{P}$-strong convergence to the limit operators is just the strong convergence.

2.3.4 Localization and a finite section algebra $\mathcal{F}_{\mathcal{A}_{lp}(C)}$

For every Fredholm sequence we obtain the Fredholm property of all its limit operators from Theorem 6. The aim of this section is to determine an appropriate subalgebra of $\mathcal{F}_{\mathcal{B}_{lp}}$, for which the converse is true, namely that a sequence $\{A_n\}$ is Fredholm, if all $W^x\{A_n\}$ are Fredholm.

Proposition 19. *The set $C_\Omega^{\mathcal{J}}$ of all $\{L_n \hat{g}_n L_n\} + \mathcal{J}$ with $\{L_n \hat{g}_n L_n\} \in C_\Omega$ is a C^*-subalgebra of the center of $\mathcal{F}_{\mathcal{B}_{lp}}^{\mathcal{J}} := \mathcal{F}_{\mathcal{B}_{lp}}/\mathcal{J}$, and $C_\Omega^{\mathcal{J}}$ is isometrically *-isomorphic to $C(\Omega)$.*

Proof. At first, we have to show, that the elements of $C_\Omega^{\mathcal{J}}$ commute with all elements of $\mathcal{F}_{\mathcal{B}_{lp}}^{\mathcal{J}}$. Since

$$\begin{aligned} &\{L_n \hat{g}_n L_n\}\{E_n^x(L_n^x A L_n^x)\} - \{E_n^x(L_n^x A L_n^x)\}\{L_n \hat{g}_n L_n\} \\ &\quad = \{L_n \hat{g}_n L_n V_{x_n} A V_{-x_n} L_n - L_n V_{x_n} A V_{-x_n} L_n \hat{g}_n L_n\} \\ &\quad = \{L_n(\hat{g}_n L_n V_{x_n} A V_{-x_n} - V_{x_n} A V_{-x_n} \hat{g}_n) L_n\} \end{aligned}$$

for $x \in T$, $A \in \mathcal{B}_{lp}^x$ and $\{L_n \hat{g}_n L_n\} \in C_\Omega$, and since $\mathcal{B}_{lp}^x$ is generated by shifts and multiplication operators, it suffices to consider the cases $A = aI$ and $A = V_y$, where $a \in l_{N\times N}^{\infty,p}$ and $y \in \mathbb{Z}^K$. Obviously, we have $\hat{g}_n V_{x_n} a V_{-x_n} = V_{x_n} a V_{-x_n} \hat{g}_n$ and furthermore

$$\begin{aligned} L_n(\hat{g}_n L_n V_{x_n} V_y V_{-x_n} - V_{x_n} V_y V_{-x_n} \hat{g}_n) L_n &= L_n(\hat{g}_n L_n V_y - V_y \hat{g}_n) L_n \\ &= L_n(\hat{g}_n L_n - V_y \hat{g}_n L_n V_{-y}) V_y L_n \\ &= L_n \hat{f}_n V_y L_n \end{aligned}$$

with $f_n(x) := \chi_{n\Omega+y}\chi_{n\Omega}(g(\frac{x}{n}) - g(\frac{x-y}{n}))$. The continuous function g on the compact set Ω is uniformly countinuous and therefore $\|f_n\| \to 0$. Thus, the considered commutators form sequences in $\mathcal{G}$ and $C_\Omega^{\mathcal{J}}$ is a central subalgebra of $\mathcal{F}_{\mathcal{B}_{l^p}}^{\mathcal{J}}$.

By $^*: \{L_n\hat{g}_nL_n\} + \mathcal{J} \mapsto \{L_n\hat{\bar{g}}_nL_n\} + \mathcal{J}$ an involution in $C_\Omega^{\mathcal{J}}$ is given and the mapping $C(\Omega) \to C_\Omega^{\mathcal{J}}, g \mapsto \{L_n\hat{g}_nL_n\} + \mathcal{J}$ is a surjective $*$-homomorphism. The assertion can be easily proved, if this mapping is shown to be isometric. For this we fix $g \in C(\Omega)$ and $\{J_n\} \in \mathcal{J}$. Then, for every $\epsilon > 0$, there exist a sequence $\{G_n\} \in \mathcal{G}$, a $k \in \mathbb{N}$, certain $x_1, ..., x_k \in T$ and $K^i \in \mathcal{K}^{x_i}$, such that

$$\|\{\sum_{i=1}^{k} E_n^{x_i}(L_n^{x_i}K^iL_n^{x_i}) + G_n\} - \{J_n\}\| < \epsilon/4.$$

Moreover, it is easy to see that there are a number $m \in \mathbb{N}$ and a function $f \in l^p$, $\|f\| = 1$, such that

$$\|G_m\| < \epsilon/4,$$

$$\|\sum_{i=1}^{k} E_m^{x}(L_m^{x_i}K^iL_m^{x_i})f\| < \epsilon/4 \text{ and}$$

$$\|(L_m\hat{g}_mL_m)f\| > \|g\| - \epsilon/4,$$

hence

$$\begin{aligned}
\|\{L_n\hat{g}_nL_n + J_n\}\| &\geq \|(L_m\hat{g}_mL_m + J_m)L_mf\| \\
&\geq \|(L_m\hat{g}_mL_m)f\| - \|J_mL_mf\| \\
&\geq \|(L_m\hat{g}_mL_m)f\| - \|J_m - \sum_{i=1}^{k} E_m^{x_i}(L_m^{x_i}K^iL_m^{x_i}) + G_m\|\|f\| \\
&\quad - \|\sum_{i=1}^{k} E_m^{x}(L_m^{x_i}K^iL_m^{x_i})L_mf\| - \|G_m\|\|f\| \\
&\geq \|g\| - \epsilon.
\end{aligned}$$

Since $\{J_n\}$ and ϵ were chosen arbitrarily, we have

$$\|g\| \leq \inf_{\{J_n\}\in\mathcal{J}} \|\{L_n\hat{g}_nL_n + J_n\}\| \leq \|\{L_n\hat{g}_nL_n\}\| = \|g\|,$$

hence $\|\{L_n\hat{g}_nL_n\} + \mathcal{J}\| = \|g\|$. □

Consequently, the maximal ideal space $\mathcal{M}(C_\Omega^{\mathcal{J}})$ is homeomorphic to Ω, that is the maximal ideals are exactly the sets

$$i_z := \{\{L_n\hat{g}_nL_n\} + \mathcal{J} : g(z) = 0\}, \quad z \in \Omega.$$

For $z \in \Omega$ let $\mathcal{J}_z$ denote the smallest closed ideal in $\mathcal{F}_{\mathcal{B}_{l^p}}^{\mathcal{J}}$, which contains i_z and let Φ_z denote the canonical homomorphism $\Phi_z : \mathcal{F}_{\mathcal{B}_{l^p}}^{\mathcal{J}} \to \mathcal{F}_{\mathcal{B}_{l^p}}^{\mathcal{J}}/\mathcal{J}_z$.
Let $\{A_n\} \in \mathcal{F}_{\mathcal{B}_{l^p}}$. The coset $\{A_n\} + \mathcal{J}$ is invertible (that is $\{A_n\}$ is a Fredholm sequence), if all $\Phi_z(\{A_n\} + \mathcal{J})$ $(z \in \Omega)$ are invertible, due to Theorem 11 (Allan localisation).

Applying this observation, we will tighten the relations between Fredholmness of sequences and Fredholmess of their limit operators. For this, we define

$$\mathcal{F}_{\mathcal{A}_{l^p}(C)} := \text{alg}\{\{L_n A L_n\}, \{L_n P_H L_n\}, \{E_n^x(L_n^x K^x L_n^x)\}, \{G_n\} : \\ A \in \mathcal{A}_{l^p}(C), H \in \mathcal{H}_\Omega, K^x \in \mathcal{K}^x, x \in \delta\Omega, \{G_n\} \in \mathcal{G}\}$$

and obtain

Proposition 20. *A sequence* $\{A_n\} \in \mathcal{F}_{\mathcal{A}_{l^p}(C)}$ *is Fredholm, if and only if all operators* $W^x\{A_n\}$ *are Fredholm.*

Proof. 1. For $z \in \delta\Omega$ we put $x := z$ and if $z \in \text{int}\,\Omega$ we put $x := 0$. If $\{A_n\}$ is a Fredholm sequence, then all $W^x\{A_n\}$ are Fredholm, due to Theorem 6 and for the reverse implication it suffices to show that the invertibility of $W^x\{A_n\} + \mathcal{K}^x$ implies the invertibility of the corresponding cosets $\Phi_z(\{A_n\} + \mathcal{J})$.

2. At first we consider $z \in \Omega$ and show for $A, B \in \mathcal{B}_{l^p}^x$ that

$$\Phi_z(\{E_n^x(L_n^x A B L_n^x)\} + \mathcal{J}) = \Phi_z(\{E_n^x(L_n^x A L_n^x)\}\{E_n^x(L_n^x B L_n^x)\} + \mathcal{J}). \tag{12}$$

For every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ and an open bounded neighborhood U of z, such that for all $n > n_0$

$$\|V_{-x_n}\hat{\chi}_{nU}V_{x_n}A(I^x - L_n^x)B\| < \epsilon, \tag{13}$$

as Theorem 13 shows. Let $g \in C(\Omega)$ with $\|g\| = 1$, $g(z) = 1$ and $g(1 - \chi_U) \equiv 0$. Then $\{L_n\hat{g}_n L_n\} \in C_\Omega$ and $\Phi_z(\{L_n - L_n\hat{g}_n L_n\} + \mathcal{J}) = 0$. Moreover, we have $L_n\hat{g}_n L_n = L_n\hat{g}_n L_n\hat{\chi}_{nU}$. Thus

$$\begin{aligned}
&\{E_n^x(L_n^x A B L_n^x) - E_n^x(L_n^x A L_n^x)E_n^x(L_n^x B L_n^x)\} + \mathcal{J} \\
&\quad = \{E_n^x(L_n^x A(I^x - L_n^x)B L_n^x)\} + \mathcal{J} \\
&\quad = \{L_n - L_n\hat{g}_n L_n\}\{E_n^x(L_n^x A(I^x - L_n^x)B L_n^x)\} + \mathcal{J} \\
&\qquad + \{L_n\hat{g}_n L_n\}\{E_n^x(L_n^x A(I^x - L_n^x)B L_n^x)\} + \mathcal{J} \\
&\quad = \{L_n - L_n\hat{g}_n L_n\}\{E_n^x(L_n^x A(I^x - L_n^x)B L_n^x)\} + \mathcal{J} \\
&\qquad + \{L_n\hat{g}_n L_n\}\{E_n^x(L_n^x V_{-x_n}\hat{\chi}_{nU}V_{x_n}A(I^x - L_n^x)B L_n^x)\} + \mathcal{J}.
\end{aligned}$$

The first summand belongs to the kernel of Φ_z and the operators of the latter sequence are smaller than ϵ in the norm for sufficiently large n, due to (13). This proves the assertion of step 2.

3. Let $z \in \Omega$. We show: If $W^x\{A_n\}$ is Fredholm (i.e. $W^x\{A_n\} + \mathcal{K}$ is invertible in $\mathcal{B}_{l^p}^x/\mathcal{K}^x$), then the cosets $\Phi_z(\{E_n^x(L_n^x W^x\{A_n\}L_n^x)\} + \mathcal{J})$ are invertible. For this we choose $B \in (W^x\{A_n\} + \mathcal{K}^x)^{-1}$ and obtain from (12) that

$$\begin{aligned}
&\Phi_z(\{E_n^x(L_n^x W^x\{A_n\}L_n^x)\}\{E_n^x(L_n^x B L_n^x)\} + \mathcal{J}) \\
&\quad = \Phi_z(\{E_n^x(L_n^x W^x\{A_n\}B L_n^x)\} + \mathcal{J}) = \Phi_z(\{L_n\} + \mathcal{J})
\end{aligned}$$

as well as $\Phi_z(\{E_n^x(L_n^x B L_n^x)\}\{E_n^x(L_n^x W^x\{A_n\}L_n^x)\} + \mathcal{J}) = \Phi_z(\{L_n\} + \mathcal{J})$.

4. Now it remains to prove that

$$\Phi_z(\{A_n\} + \mathcal{J}) = \Phi_z(\{E_n^x(L_n^x W^x\{A_n\}L_n^x)\} + \mathcal{J}) \tag{14}$$

for all generating sequences $\{A_n\}$ of $\mathcal{F}_{\mathcal{A}_{l^p}(C)}$, since (12) yields that this equality then holds true for all sequences in $\mathcal{F}_{\mathcal{A}_{l^p}(C)}$.

At first we consider the case $\{L_nAL_n\}$ with $A \in \mathcal{A}_{l^p}(C)$. If $z \in \operatorname{int}\Omega$ then (14) is obvious, since $W^0\{L_nAL_n\} = A$. In the case $z \in \delta\Omega$ let, at first, $A = \hat{a}V_y$ with $a \in C(\overline{\mathbb{R}}^K, \mathbb{C}^{N\times N})$, $y \in \mathbb{Z}^K$. We put $\eta := \frac{x}{\|x\|} = \frac{z}{\|z\|}$ and obtain $W^x\{L_nAL_n\} = I^x a^\infty(\eta)V_yI^x$. Moreover, for every $\epsilon > 0$, there exist numbers $n_0, \delta_0 > 0$ such that $\|\hat{a}(nu) - a^\infty(\eta)\| < \epsilon$ for all $n > n_0$ and all $u \in U_{\delta_0}(x)$.
Let $g \in C(\Omega)$ with $\|g\| = 1$, $g(z) = 1$ and $g(\Omega \setminus U_{\delta_0}(x)) = \{0\}$. Then

$$\begin{aligned}&\{L_nAL_n - E_n^x(L_n^xW^x\{L_nAL_n\}L_n^x)\} + \mathcal{J}\\ &\quad = \{L_n\hat{a}V_yL_n - L_na^\infty(\eta)V_yL_n\} + \mathcal{J}\\ &\quad = \{\hat{g}_nL_n(\hat{a} - a^\infty(\eta))V_yL_n\} + \mathcal{J} + \{(1-\hat{g}_n)L_n(\hat{a} - a^\infty(\eta))V_yL_n\} + \mathcal{J}\end{aligned}$$

where the first summand is smaller than ϵ and the second one is contained in the kernel of Φ_z. Since ϵ was chosen arbitrarily, and the kernel of Φ_z is closed, we obtain (14) for $A = \hat{a}V_y$, as well as for sums of such operators, that is for band operators and, due to the continuity of Φ_z, for all band-dominated operators in $\mathcal{A}_{l^p}(C)$.

Now let $H \in \mathcal{H}_\Omega$. Due to equation (11) we have $W^x\{L_nP_HL_n\} = I^xP_HI^x$ if $x \in \delta H$, and equation (14) obviously holds for all $z \in \operatorname{int}\Omega$ and $z \in \delta\Omega \cap \delta H$. For $z \in \operatorname{int} H \cap \delta\Omega$ or $z \in H^C \cap \delta\Omega$ the operator $W^x\{L_nP_HL_n\}$ equals I^x or 0, respectively, and we can choose a continuous function $g \in C(\Omega)$ with $\|g\| = 1$ and $g(z) = 1$, whose support is contained in an open ball, which does not intersect δH, i.e. its support is located completely on one side of δH. The observation that $\{\hat{g}_nL_nP_HL_n\}$ then equals $\{\hat{g}_nL_n\}$ or $\{0L_n\}$ for $z \in \operatorname{int} H$ or $z \in H^C$, respectively, yields (14) again. □

2.3.5 Main theorem

Since all sequences $\{A_n\} \in \mathcal{F}_{\mathcal{B}_{l^p}}$ enjoy the $\mathcal{P}$-dichotomy, Theorems 7, 9 and 10 and Proposition 20 yield the following picture for the finite sections of band-dominated operators.

Theorem 15. *Let $\{A_n\} \in \mathcal{F}_{\mathcal{B}_{l^p}}$.*

- *If $\{A_n\}$ is Fredholm, then all $W^x\{A_n\}$ are Fredholm, the number of non-invertible operators among them is finite and the approximation numbers of $\{A_n\}$ have the k-splitting-property with $k = \sum_{x\in T} \dim\ker W^x\{A_n\}$. Moreover,*

$$\sum_{x\in T} \operatorname{ind} W^x\{A_n\} = 0.$$

- *If one $W^x\{A_n\}$ is not Fredholm, then $\lim_{n\to\infty} s_l(A_n) = 0$ for each $l \in \mathbb{N}$.*
- *$\{A_n\}$ is stable, iff $\{A_n\}$ is Fredholm and all $W^x\{A_n\}$ are invertible.*
- *If $\{A_n\} \in \mathcal{F}_{\mathcal{A}_{l^p}(C)}$, then $\{A_n\}$ is Fredholm iff all $W^x\{A_n\}$ are Fredholm.*

Remark 4. Let $\{A_n\}$ be a Fredholm sequence, let $k^x := \dim\ker W^x\{A_n\}$ denote the kernel dimension of $W^x\{A_n\}$ and let $P^x \in \mathcal{L}(\mathbf{E}^x, \mathcal{P}^x)$ denote a projection onto the kernel of $W^x\{A_n\}$

for each $x \in T$, respectively. This definition is correct, since all limit operators are proper $\mathcal{P}^x$-Fredholm. In view of (9) the following estimate for the convergence speed of the $\alpha(\{A_n\})$-th approximation numbers holds:

$$\begin{aligned} s_{\alpha(\{A_n\})}(A_n) &\leq 2 \max_{x \in T} \|P^x\| \cdot \sum_{x \in T} \frac{k^x}{1 - \|(I^x - L_n^x)P^x\|} \|E_n^{-x}(A_n) L_n^x P^x\| \\ &\leq \text{ const } \sum_{x \in T} \|E_n^{-x}(A_n) L_n^x P^x\| \to 0. \end{aligned} \tag{15}$$

Note, that just a finite number of the projections P^x are non-trivial.

Remark 5. This result particularly treats finite sections $\{L_n A L_n\}$ of operators $A \in \mathcal{A}_{l^p}(\mathbb{C})$, which are usually called convolution operators. Since these operators are shift-invariant, it is clear, that the limit operators $W^x\{L_n A L_n\}$ coincide for all points $x \in \delta\Omega$ that are relatively inner points of the same polytope P_j (i.e. lateral surfaces), or the same intersection of two or more P_{j_i} (i.e. edges and vertices), and so on. Hence there is just a finite number of different limit operators, one for each lateral surface, edge, vertex, ...

Thanks to the fact that the sequences $\{L_n P_H L_n\}$ for $H \in \mathcal{H}_\Omega$ are contained in $\mathcal{F}_{\mathcal{A}_{l^p}(C)}$, we can consider convolution operators on half spaces and cones as well.

Such stability results for finite sections were obtained for the first time in [10] and [11] for $p \neq \infty$.

2.4 Band-dominated operators on $l^p(\mathbb{Z}, \mathbb{C}^N)$

2.4.1 Sequences and subsequences

In the preceding section we were able to apply the general theory on stability and approximation numbers to the finite sections of a certain class of band-dominated operators acting on $l^p(\mathbb{Z}^K, \mathbb{C}^N)$. In this section we consider general band-dominated operators in the case $K = 1$ and we will obtain a fairly complete theory concerning the asymptotical behavior of the finite sections of general band-dominated operators. As the subsequent example shows, this situation is more sophisticated.

First of all, we put $\Omega = [-1, 1]$ and denote by $\mathcal{P} = (L_n)$ the uniform approximate identity, which we can deduce from Ω.

Example We consider the band-dominated operator A on $l^p(\mathbb{Z}, \mathbb{C})$, whose matrix representation with respect to the canonical basis is as follows:

$$A = \text{diag}(..., B, B, B, ...), \text{ where } B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Obviously, A is invertible as well as all of its limit operators, and its finite section $L_n A L_n$ is invertible if and only if n is even, thus $\{L_n A L_n\}$ is not stable. Moreover, we have $V_{-n} A V_n = A$ if n is even and $V_{-n} A V_n = V_{-1} A V_1$ if n is odd, thus the operators $W^{\pm 1}\{L_n A L_n\}$ does not exist at all.

But, if we consider $\tilde{\Omega} = [-2, 2]$ and $\tilde{\mathcal{P}} = (\tilde{L}_n) = (L_{2n})$, the corresponding finite section sequence becomes $\{\tilde{L}_n A \tilde{L}_n\} = \{L_{2n} A L_{2n}\}$, that is a stable sequence, all desired operators $W^{\pm 1}\{\tilde{L}_n A \tilde{L}_n\}$ exist and they are invertible on $\mathbb{Z}_\pm$, respectively.

Let $\mathcal{H}_+$ ($\mathcal{H}_-$) denote the set of all sequences $h : \mathbb{N} \to \mathbb{N}$ ($h : \mathbb{N} \to \mathbb{Z} \setminus \mathbb{N}$) tending to $+\infty$ ($-\infty$). Moreover, let $\mathcal{H} := \mathcal{H}_+ \cup \mathcal{H}_-$. In the example the convenient choice of $\tilde{\Omega}$ leads to a subsequence of $\mathcal{P}$, such that the required limit operators of A actually exist and the known results are applicable. In general the following holds (see [17], Corollary 2.1.17).

Proposition 21. *Let $A \in \mathcal{A}_{l^p}$. Then every sequence $h \in \mathcal{H}$ contains a subsequence $g \in \mathcal{H}$ such that the limit operator A_g of A with respect to g exists.*

For a band-dominated operator A and a sequence $h \in \mathcal{H}$ there is a subsequence j of h such that A_j exists. By the same argument, j itself contains a subsequence g such that the limit operator A_{-g} exists, too.

Therefore, it seens to be feasible and valuable to pass to appropriate subsequences for which the limit operators exist and hence the ideas of the general theory apply:

For a given sequence $h = (h_n)_{n\in\mathbb{N}} \in \mathcal{H}_+$ we define operators $L_{h_n} := \hat{\chi}_{[-h_n,h_n]}I$ and obtain a uniform approximate identity $\mathcal{P}_h := (L_{h_n})$ of projections L_{h_n} of finite rank. Obviously, $\mathcal{P}_h$ and $\mathcal{P}$ are equivalent, and thus all "$\mathcal{P}_h$-notations" coincide with their corresponding "$\mathcal{P}$-notations".

Let $\mathbf{E}_{h_n} := \operatorname{im} L_{h_n}$, $T := \{-1, 0, 1\}$, $I^0 := I$, $I^{\pm 1} := \hat{\chi}_{\mathbb{Z}_\mp} I$ and

$$
\begin{aligned}
&\mathbf{E}^0 := l^p(\mathbb{Z}, \mathbb{C}^N) &\quad& L^0_{h_n} := L_{h_n}\\
& && E^0_{h_n} : \mathcal{L}(\operatorname{im} L^0_{h_n}) \to \mathcal{L}(\mathbf{E}_{h_n}), B \mapsto B\\
&\mathbf{E}^{\pm 1} := \operatorname{im} I^{\pm 1} && L^{\pm 1}_{h_n} := V_{\mp h_n} L_{h_n} V_{\pm h_n}\\
& && E^{\pm 1}_{h_n} : \mathcal{L}(\operatorname{im} L^{\pm 1}_{h_n}) \to \mathcal{L}(\mathbf{E}_{h_n}), B \mapsto V_{\pm h_n} B V_{\mp h_n}
\end{aligned}
$$

for every n. I^x are the identities in $\mathbf{E}^x$, respectively, and the uniform approximate identities $\mathcal{P}^x_h := (L^x_{h_n})$ are equivalent to $\mathcal{P}^x := (L^x_n)$. Thus $\mathcal{K}(\mathbf{E}^x, \mathcal{P}^x_h) = \mathcal{K}(\mathbf{E}^x, \mathcal{P}^x) =: \mathcal{K}^x$ for $x \in T$. Again, we let $\mathcal{F}^T_h$ denote the set of all bounded sequences $\{A_{h_n}\}$ of bounded linear operators $A_{h_n} \in \mathcal{L}(\mathbf{E}_{h_n})$ for which there exist operators $W^x\{A_{h_n}\} \in \mathcal{L}(\mathbf{E}^x, \mathcal{P}^x)$ such that for $n \to \infty$

$$E^{-x}_{h_n}(A_{h_n}) L^x_{h_n} \to W^x\{A_{h_n}\} \quad \mathcal{P}^x - \text{strongly}$$

The sets

$$
\begin{aligned}
&\mathcal{G}_h := \{\{G_{h_n}\} : \|G_{h_n}\| \to 0\},\\
&\mathcal{J}_h := \operatorname{span}\{\{E^x_{h_n}(L^x_{h_n} K L^x_{h_n})\}, \{G_{h_n}\} : x \in T, K \in \mathcal{K}^x, \{G_{h_n}\} \in \mathcal{G}_h\}
\end{aligned}
$$

are closed ideals in $\mathcal{F}^T_h$ and a sequence $\{A_{h_n}\} \in \mathcal{F}^T_h$ is said to be Fredholm, if $\{A_{h_n}\} + \mathcal{J}_h$ is invertible in $\mathcal{F}^T_h / \mathcal{J}_h$.

The finite section algebra $\mathcal{F}_{\mathcal{A}_{l^p}}$ Let $\mathcal{F}$ be the algebra of all bounded sequences $\{A_n\}$ of bounded linear operators $A_n \in \mathcal{L}(\mathbf{E}_n)$ and let $\mathcal{F}_{\mathcal{A}_{l^p}}$ denote the smallest closed subalgebra of $\mathcal{F}$ containing all sequences $\{L_n A L_n\}$ with $A \in \mathcal{A}_{l^p}$. For $\mathbb{A} = \{A_n\} \in \mathcal{F}_{\mathcal{A}_{l^p}}$ and a sequence $h = (h_n)_{n\in\mathbb{N}} \in \mathcal{H}_+$ let $\mathbb{A}_h$ denote the subsequence $\{A_{h_n}\}$. It is obvious, that for each $\mathbb{A} = (A_n) \in \mathcal{F}_{\mathcal{A}_{l^p}}$ the operator

$$W(\mathbb{A}) := \mathcal{P}\text{-}\lim_{n\to\infty} A_n L_n = W^0\{A_{h_n}\} = \mathcal{P}\text{-}\lim_{n\to\infty} A_{h_n} L_{h_n}$$

exists for every $h \in \mathcal{H}_+$ and is independent from the choice of h.

Appropriate subsequences Let $\mathbb{A} \in \mathcal{F}_{\mathcal{A}_{l^p}}$ and $h \in \mathcal{H}_+$. By $\mathcal{H}_{\mathbb{A}_h}$ we denote the collection of all subsequences g of h such that the following holds

1. $\mathbb{A}_g \in \mathcal{F}_g^T$ (which means that the operators $W(\mathbb{A})$ and $W^{\pm 1}(\mathbb{A}_g)$ exist).
2. $\mathbb{A}_g - \{L_{g_n} W(\mathbb{A}) L_{g_n}\} \in \mathcal{J}_g$.

Furthermore, we put $\mathbb{Q}^l := \{L_{n-l}\}$ and $\mathbb{P}^l := \{L_n\} - \mathbb{Q}^l$.

Proposition 22. *Let $\mathbb{A} \in \mathcal{F}_{\mathcal{A}_{l^p}}$ and let $h \in \mathcal{H}_+$. Then there exists a subsequence $g \in \mathcal{H}_{\mathbb{A}_h}$ of h. If $W(\mathbb{A})$ is Fredholm then $\mathbb{A}_g$ is a Fredholm sequence in $\mathcal{F}_g^T$ and if B is a regularizer of $W(\mathbb{A})$ then $\{L_{g_n} B L_{g_n}\} \in \mathcal{F}_g^T$, too. Moreover*

$$\lim_{l\to\infty} \|(\mathbb{A} - \{L_n W(\mathbb{A}) L_n\})\mathbb{Q}^l\| = \lim_{l\to\infty} \|\mathbb{Q}^l(\mathbb{A} - \{L_n W(\mathbb{A}) L_n\})\| = 0.$$

Proof. First of all, we prove that for every sequence $g \in \mathcal{H}_+$ and for each pair of operators $B, C \in \mathcal{A}_{l^p}$ with $\{L_{g_n} B L_{g_n}\}, \{L_{g_n} C L_{g_n}\} \in \mathcal{F}_g^T$ there are operators $K_{\pm 1} \in \mathcal{K}^{\pm 1}$ and a sequence $\{G_{g_n}\} \in \mathcal{G}_g$, such that

$$\begin{aligned} \{D_{g_n}\} &:= \{L_{g_n} BC L_{g_n}\} - \{L_{g_n} B L_{g_n}\}\{L_{g_n} C L_{g_n}\} \\ &= \{E^1_{g_n}(L^1_{g_n} K_1 L^1_{g_n})\} + \{E^{-1}_{g_n}(L^{-1}_{g_n} K_{-1} L^{-1}_{g_n})\} + \{G_{g_n}\} \in \mathcal{J}_g. \end{aligned} \tag{16}$$

Since the limit operators $B_{\pm g}, C_{\pm g}$ exist, we also have $\{L_{g_n} BC L_{g_n}\} \in \mathcal{F}_g^T$, hence $\{D_{g_n}\} \in \mathcal{F}_g^T$. For every m the sequence $\{D_{g_n}\}\mathbb{P}_g^m$ is contained in $\mathcal{J}_g$ and

$$\|\{D_{g_n}\}\mathbb{Q}_g^m\| \leq \|B\| \sup_n \|(I - L_{g_n}) C L_{g_n - m}\|$$

which can be chosen arbitrarily small, due to Theorem 13. Thus, the closedness of $\mathcal{J}_g$ implies (16).

Let $\mathbb{A}$ and h be given. By definition there is a sequence $(\mathbb{A}^{(m)}) \subset \mathcal{F}_{\mathcal{A}_{l^p}}$ such that $\mathbb{A}^{(m)} \to \mathbb{A}$ in the norm (of $\mathcal{F}_{\mathcal{A}_{l^p}}$) as $n \to \infty$, and such that all $\mathbb{A}^{(m)}$ are of the form

$$\mathbb{A}^{(m)} = \sum_{i=1}^{k_m} \prod_{j=1}^{l_m} \{L_n A_{ij}^{(m)} L_n\}, \quad A_{ij}^{(m)} \in \mathcal{A}_{l^p}.$$

Applying Proposition 21 several times, it is easy to see that there exists a subsequence g^1 of h such that $\{L_{g_n^1} A_{ij}^{(1)} L_{g_n^1}\} \in \mathcal{F}_{g^1}^T$ for all $i = 1, ..., k_1$ and $j = 1, ..., l_1$. Relation (16) yields $g^1 \in \mathcal{H}_{\mathbb{A}^{(1)}}$. Repeating this argument we obtain a subsequence g^2 of g^1 such that $g^2 \in \mathcal{H}_{\mathbb{A}^{(2)}}$, and so on. Hence, there are sequences $h \supset g^1 \supset g^2 \supset ... \supset g^m \supset ...$ such that $g^m \in \mathcal{H}_{\mathbb{A}^{(1)}}$, ..., $g^m \in \mathcal{H}_{\mathbb{A}^{(m)}}$ for every m. Now let the sequence $g = (g_n)$ be defined by $g_n := g_n^n$. Then it follows that $g \in \mathcal{H}_{\mathbb{A}^{(m)}}$ for all $m \in \mathbb{N}$. Since $\mathbb{A}_g$ is the norm limit of the sequence $(\mathbb{A}_g^{(m)})_{m\in\mathbb{N}}$, we easily obtain $\mathbb{A}_g \in \mathcal{F}_g^T$ and $\mathbb{A}_g - \{L_{g_n} W(\mathbb{A}) L_{g_n}\} \in \mathcal{J}_g$, that is $g \in \mathcal{H}_{\mathbb{A}_h}$.

Now let $W(\mathbb{A})$ be Fredholm, B be a regularizer and $g \in \mathcal{H}_{\mathbb{A}_h}$. Then Proposition 13 immediately shows that $\{L_{g_n} B L_{g_n}\} \in \mathcal{F}_g^T$. Applying equation (16) to the operators $W(\mathbb{A})$ and B proves the Fredholmness of $\{L_{g_n} W(\mathbb{A}) L_{g_n}\}$, that is the Fredholmness of $\mathbb{A}_g$ since $\mathbb{A}_g - \{L_{g_n} W(\mathbb{A}) L_{g_n}\} \in \mathcal{J}_g$.

Finally, we assume that $\limsup_{l\to\infty} \|(\mathbb{A} - \{L_n W(\mathbb{A}) L_n\})\mathbb{Q}^l\| \geq \epsilon > 0$, i.e.

$$\|(\mathbb{A} - \{L_n W(\mathbb{A}) L_n\})\mathbb{Q}^{l_k}\| \geq \frac{\epsilon}{2} \tag{17}$$

for a strongly increasing sequence $(l_k)_{k\in\mathbb{N}}$ of positive integers. If we pass to a subsequence $g \in \mathcal{H}_{\mathbb{A}}$ then (17) is of the form

$$\|(\{E^1_{g_n}(L^1_{g_n} K_1 L^1_{g_n})\} + \{E^{-1}_{g_n}(L^{-1}_{g_n} K_{-1} L^{-1}_{g_n})\} + \{G_{g_n}\})\mathbb{Q}^{l_k}\| \geq \frac{\epsilon}{2}$$

with operators $K_{\pm 1} \in \mathcal{K}^{\pm 1}$ and $\{G_{g_n}\} \in \mathcal{G}_g$ for every k. This yields a contradiction since for every $\mathcal{P}^x$-compact operator K and every l we have

$$\|\{E^{-x}_{g_n}(L^x_{g_n} K L^x_{g_n})\}\mathbb{Q}^l\| = \|\{E^{-x}_{g_n}(L^x_{g_n} K(I - L_{l-1}) L^x_{g_n})\}\| \leq \|K(I - L_{l-1})\| \text{ and}$$
$$\|\{G_{g_n}\}\mathbb{Q}^l\| = \sup_{n\in\mathbb{N}} \|G_{g_n} L_{g_n - l}\| \leq \sup_{n\in\mathbb{N}: g_n \geq l} \|G_{g_n}\|$$

which tend to 0 as l goes to ∞. The second part of the last assertion can be proved analogously. □

Now Theorems 7, 9, 10 and 14 provide the following result.

Theorem 16. *Let $\mathbb{A} = \{A_n\} \in \mathcal{F}_{\mathcal{A}_{l^p}}$ and $g \in \mathcal{H}_{\mathbb{A}}$.*

- *If $W(\mathbb{A})$ is Fredholm, then $W^{\pm 1}(\mathbb{A}_g)$ are Fredholm. Moreover*

$$\operatorname{ind} W(\mathbb{A}) = -\operatorname{ind} W^1(\mathbb{A}_g) - \operatorname{ind} W^{-1}(\mathbb{A}_g)$$

  and the approximation numbers of $\mathbb{A}_g$ have the k-splitting-property with

$$k = \dim\ker W(\mathbb{A}) + \dim\ker W^1(\mathbb{A}_g) + \dim\ker W^{-1}(\mathbb{A}_g).$$

- *If $W(\mathbb{A})$ is not Fredholm, then $\lim_{n\to\infty} s_l(A_{g_n}) = 0$ for each $l \in \mathbb{N}$.*
- *$\mathbb{A}_g$ is stable if and only if $W(\mathbb{A})$ and $W^{\pm 1}(\mathbb{A}_g)$ are invertible.*

Proposition 14 reveals that in case $p \neq \infty$ the $\mathcal{P}^x$-strong convergence to the limit operators is just the strong convergence.

2.4.2 Finite sections and the α-number

The previous theorem provides some information on the behavior of subsequences of a sequence $\mathbb{A} \in \mathcal{F}_{\mathcal{A}_{l^p}}$. We now want to state a similar result for $\mathbb{A}$ itself.

Definition 12. Let $\mathbb{A} = \{A_n\} \in \mathcal{F}_{\mathcal{A}_{l^p}}$. If there is a finite number $\alpha \in \mathbb{Z}_+$ with

$$\liminf_{n\to\infty} s_\alpha(A_n) = 0 \text{ and } \liminf_{n\to\infty} s_{\alpha+1}(A_n) > 0,$$

then this number is called the α-number of $\mathbb{A}$ and it is denoted by $\alpha(\mathbb{A})$.

Theorem 17. *Let $\mathbb{A} = \{A_n\} \in \mathcal{F}_{\mathcal{A}_{l^p}}$.*

- $\mathbb{A}$ *has an α-number if and only if $W(\mathbb{A})$ is a Fredholm operator. In this case the operators $W^{\pm 1}(\mathbb{A}_h)$ are Fredholm for each $h \in \mathcal{H}_{\mathbb{A}}$, the formula* $\operatorname{ind} W(\mathbb{A}) = -\operatorname{ind} W^1(\mathbb{A}_h) - \operatorname{ind} W^{-1}(\mathbb{A}_h)$ *holds and*

$$\alpha(\mathbb{A}) = \dim\ker W(\mathbb{A}) + \max_{h\in\mathcal{H}_{\mathbb{A}}}(\dim\ker W^1(\mathbb{A}_h) + \dim\ker W^{-1}(\mathbb{A}_h)). \tag{18}$$

- $\mathbb{A}$ *is a stable sequence if and only if $W(\mathbb{A})$ and all operators $W^{\pm 1}(\mathbb{A}_h)$ with $h \in \mathcal{H}_{\mathbb{A}}$ are invertible.*

Proof. Suppose $W(\mathbb{A})$ is Fredholm, but for each $n \in \mathbb{N}$ there is a number h_n such that $h_n > h_{n-1}$ and $s_n(A_{h_n}) < \frac{1}{n}$. Due to Proposition 22, there is a subsequence $g \in \mathcal{H}_{\mathbb{A}_h}$ and since $W(\mathbb{A})$ is Fredholm, Theorem 16 provides a splitting number k for the sequence $\mathbb{A}_g$ and a constant $c > 0$ such that

$$0 < c \le s_{k+1}(A_{g_m}) \le s_m(A_{g_m}) < \frac{1}{m}$$

for all sufficiently large m. This is a contradiction.

On the other hand, let $\alpha(\mathbb{A}) < \infty$ be given. Then there exists a sequence $h \in \mathcal{H}_+$ such that $\lim_{n\to\infty} s_{\alpha(\mathbb{A})}(A_{h_n}) = 0$ and there is again a subsequence $g \in \mathcal{H}_{\mathbb{A}_h}$ of h. Hence, Theorem 16 applies to $\mathbb{A}_g$ and in view of the relation $\liminf_{n\to\infty} s_{\alpha(\mathbb{A})+1}(A_{g_n}) > 0$ it implies that $W(\mathbb{A})$ is Fredholm. Moreover, this proves the relation "$\le$" in equation (18). To prove the equality in (18), we assume that there is a sequence $h \in \mathcal{H}_{\mathbb{A}}$ such that

$$\alpha(\mathbb{A}) + 1 \le \dim\ker W(\mathbb{A}) + \dim\ker W^1(\mathbb{A}_h) + \dim\ker W^{-1}(\mathbb{A}_h).$$

Then we obtain again from Theorem 16 that $\liminf_{n\to\infty} s_{\alpha(\mathbb{A})+1}(A_{h_n}) = 0$, a contradiction.

If $\mathbb{A}$ is a stable sequence, then every subsequence $\mathbb{A}_h$ of $\mathbb{A}$ is stable, hence the operator $W(\mathbb{A})$ and all operators $W^{\pm 1}(\mathbb{A}_h)$ with $h \in \mathcal{H}_{\mathbb{A}}$ are invertible, due to Theorem 16. Suppose $\mathbb{A}$ is not stable, then there is a sequence $h \in \mathcal{H}_+$ such that $\|A_{h_n}^{-1}\| \to \infty$ (where $\|B^{-1}\| := \infty$ whenever B is not invertible). Choosing $g \in \mathcal{H}_{\mathbb{A}_h}$, we obtain a non-stable sequence $\mathbb{A}_g \in \mathcal{F}_g^T$ and Theorem 16 yields that at least one of its limit operators is not invertible. □

In some sense, the α-number provides a measure for the almost stability of an operator sequence. Later on, we will present an idea for modifications of the finite section method, that admits to handle such almost stable sequences.

2.4.3 Stability and the index formula

The stability of finite section sequences for band-dominated operators has already been considered in [16] for continuous coefficients, in [21] for general band-dominated operators in case $p = 2$ and in chapter 6 of [17] also for the cases $p \in (1,\infty)$, where the results read as follows: The sequence $\{L_n A L_n\}$ is stable if and only if A and certain limit operators of A are invertible and if the norms of their inverses are uniformly bounded. In [18] the uniform boundedness condition could be shown to be redundant for band-dominated operators on $l^p(\mathbb{Z},\mathbb{C})$ with $p \in (1,\infty)$.

To be consistent with these results, we should introduce some further notations.

Let $P := \chi_{\mathbb{Z}_+} I$ and $Q := I - P$. For $A \in \mathcal{A}_{l^p}$ we easily check that PAQ and QAP are $\mathcal{P}$-compact (see Theorem 13), and we put $A_+ := PAP + Q$ as well as $A_- := P + QAQ$. The equality $PAP + QAQ = A_+A_- = A_-A_+$ shows that the operators A_+ and A_- are Fredholm, if and only if A is Fredholm and

$$\operatorname{ind} A = \operatorname{ind}^+ A + \operatorname{ind}^- A, \tag{19}$$

where $\operatorname{ind}^\pm A := \operatorname{ind} A_\pm$ are called the plus- and the minus-index of A, respectively. Let $\sigma_\pm(A)$ denote the set of all limit operators of $A \in \mathcal{A}_{l^p}$ with respect to sequences in $\mathcal{H}_\pm$.

Corollary 3. *Let $A \in \mathcal{A}_{l^p}$. The sequence $\{L_nAL_n\}$ is stable if and only if A and all operators B_+, C_-, with $B \in \sigma_-(A)$ and $C \in \sigma_+(A)$, respectively, are invertible.*

Proof. In view of Theorem 17 it suffices to annotate that for each $B \in \sigma_-(A)$ there is a sequence $h \in \mathcal{H}_+$ such that $B = A_{-h}$. Passing to $g \in \mathcal{H}_{\{L_{h_n}AL_{h_n}\}}$ we get that B_+ and $W^{-1}\{L_{g_n}AL_{g_n}\}$ are simultaneously invertible. The operators $C \in \sigma_+(A)$ can be treated in the same way, because $C \in \sigma_+(A)$ if and only if $V_{-1}1CV_1 \in \sigma_+(A)$, and for $h \in \mathcal{H}_{\{L_nAL_n\}}$ with $C = A_h$ we get the invertibility of $(V_{-1}CV_1)_-$ and $W^1\{L_{h_n}AL_{h_n}\}$ at the same time. □

Corollary 4. *Let $A \in \mathcal{A}_{l^p}$ be Fredholm.*

- *All operators in $\sigma_+(A)$ have the same plus-index and this number coincides with the plus-index of A. Similarly, A and all operators in $\sigma_-(A)$ have the same minus-index.*
- *For arbitrarily taken $B \in \sigma_-(A)$, $C \in \sigma_+(A)$ the following always holds*

$$\operatorname{ind} A = \operatorname{ind}^- B + \operatorname{ind}^+ C.$$

Proof. It needs only to prove $\operatorname{ind}^- A = \operatorname{ind}^- B$ and $\operatorname{ind}^+ A = \operatorname{ind}^+ C$ for each $B \in \sigma_-(A)$ and $C \in \sigma_+(A)$. For a given $B \in \sigma_-(A)$ there is a sequence $g \in \mathcal{H}_{\{L_nA_-L_n\}}$ such that the limit operator $W^{-1}\{L_{g_n}A_-L_{g_n}\}$ coincides with PBP. Since B is invertible, we obtain from theorem 16 and equation (19) that

$$\operatorname{ind}^- A = \operatorname{ind} A_- = W\{L_nA_-L_n\} = -W^{-1}\{L_{g_n}A_-L_{g_n}\} = -\operatorname{ind}^+ B = \operatorname{ind}^- B.$$

C can be treated analogously. □

Remark 6. This index formula was proved in [15] in case $p = 2$ via K-theory. A generalization to $l^p(\mathbb{Z}, \mathbb{C})$, $p \in (1, \infty)$ was done in [20] and recently, by a different approach, in [19]. Notice that it can be used to get new insight into the stability problem for the finite section method concerning band-dominated operators with slowly oscillating coefficients which belong to $l^p(\mathbb{Z}, \mathbb{C})$, $1 \le p \le \infty$. The particular case $1 < p < \infty$ was already treated in [20].

2.4.4 Modified finite sections

In what follows we want to present a method to handle sequences $\mathbb{A} \in \mathcal{F}_{\mathcal{A}_{l^p}}$, which have an invertible limit operator $W(\mathbb{A})$, but which are unfortunately not stable. The main idea is to study modified sequences and the convergence of their generalized inverses. Modified finite sections for Toeplitz operators were already considered in [9] and [25]. The results here are a straightforward extension of Section 5 in [24].

Definition 13. For a given sequence $\mathbb{A} = \{A_n\} \in \mathcal{F}_{\mathcal{A}_{l^p}}$ let $\mathbb{A}^{\underline{m}}$ denote the modified sequence

$$\mathbb{A}^{\underline{m}} = \{A_n^{\underline{m}}\} := \mathbb{A}\mathbb{Q}^m.$$

If we interpret A_n as a matrix with respect to the standard basis, then we obtain the matrix $A_n^{\underline{m}}$ from A_n by putting m columns at the left and at the right of A_n to 0.

Theorem 18. *Let $\mathbb{A} = \{A_n\} \in \mathcal{F}_{\mathcal{A}_{l^p}}$ and $W(\mathbb{A})$ be Fredholm. Then there is a number $m \in \mathbb{Z}_+$ such that*

$$\lim_{n\to\infty} s_{\dim\ker W(\mathbb{A})+2m}(A_n^{\underline{m}}) = 0 \quad \textit{and} \quad \liminf_{n\to\infty} s_{\dim\ker W(\mathbb{A})+2m+1}(A_n^{\underline{m}}) > 0.$$

Proof. We firstly consider the sequence $\mathbb{B} = \{B_n\} := \{L_n W(\mathbb{A}) L_n\}$. Then there is a number $k \geq 0$ such that

$$\ker W^x(\mathbb{B}_h^{\underline{k}}) = \ker(W(\mathbb{A}))_{xh} I^x (I - L_{m-1}) I^x = \ker I^x (I - L_{m-1}) I^x$$

for all $h \in \mathcal{H}_{\mathbb{B}}$, $x \in \{-1, 1\}$ and $m \geq k$ (see [24], Proposition 5.1). Since $W(\mathbb{B}^{\underline{m}})$ equals $W(\mathbb{A})$, Theorem 17 now implies for all $m \geq k$ that

$$\alpha(\mathbb{B}^{\underline{m}}) = \dim\ker W(\mathbb{A}) + 2m.$$

Furthermore, for every $m \geq k$ we have

$$\begin{aligned}
&s_j(A_n^{\underline{m}})\\
&\geq \inf\{\|B_n^{\underline{m}} + F\| : \dim \operatorname{im} F \leq 2n+1-j\} - \|B_n^{\underline{m}} - A_n^{\underline{m}}\|\\
&\geq \inf\{\|B_n^{\underline{k}} + (B_n^{\underline{m}} - B_n^{\underline{k}}) + F\| : \dim \operatorname{im} F \leq 2n+1-j\} - \|\mathbb{B}^{\underline{m}} - \mathbb{A}^{\underline{m}}\|\\
&\geq \inf\{\|B_n^{\underline{k}} + G\| : \dim \operatorname{im} G \leq 2n+1-j+2(m-k)\} - \|\mathbb{B}^{\underline{m}} - \mathbb{A}^{\underline{m}}\|\\
&= s_{j-2m+2k}(B_n^{\underline{k}}) - \|(\mathbb{B} - \mathbb{A})\mathbb{Q}^m\|
\end{aligned}$$

for all $n > m$ and all $j \leq 2n+1$. Thus,

$$\liminf_{n\to\infty} s_{\dim\ker W(\mathbb{A})+2m+1}(A_n^{\underline{m}}) \geq d - \|(\mathbb{B} - \mathbb{A})\mathbb{Q}^m\|$$

for all $m \geq k$, where $d := \liminf_{n\to\infty} s_{\dim\ker W(\mathbb{A})+2k+1}(B_n^{\underline{k}}) > 0$. Assume that $\inf_{l\geq k} \|(\mathbb{B} - \mathbb{A})\mathbb{Q}^l\| \geq d$. Since the first $2l$ entries of $\mathbb{Q}^l = \{Q_n^l\}$ are equal to 0, it is easy to show that there is a sequence $h \in \mathcal{H}_+$ such that

$$\|(B_{h_l} - A_{h_l})Q_{h_l}^l\| = \|(B_{h_l} - A_{h_l})(I - L_{l-1})\| \geq \frac{d}{2} \quad \text{for every} \quad l \in \mathbb{N}. \tag{20}$$

Passing to a subsequence $g \in \mathcal{H}_{\mathbb{B}_h - \mathbb{A}_h}$, we obtain that

$$\begin{aligned}
(\mathbb{B}_g - \mathbb{A}_g)\mathbb{Q}^l &= (\{E_{g_n}^1(L_{g_n}^1 K_1 L_{g_n}^1)\} + \{E_{g_n}^{-1}(L_{g_n}^{-1} K_{-1} L_{g_n}^{-1})\} + \{G_{g_n}\})\mathbb{Q}^l\\
&= \{E_{g_n}^1(L_{g_n}^1 K_1 (I - L_{l-1}) L_{g_n}^1)\}\\
&\quad + \{E_{g_n}^{-1}(L_{g_n}^{-1} K_{-1} (I - L_{l-1}) L_{g_n}^{-1})\} + \{G_{g_n}(I - L_{l-1})\},
\end{aligned}$$

which tends to 0 in the norm as $l \to \infty$. This contradicts (20). Thus, we have proved that there is a number $m \in \mathbb{Z}_+$ such that $\alpha(\mathbb{A}^{\underline{m}}) = \dim\ker W(\mathbb{A}) + 2m$, that is

$$\liminf_{n\to\infty} s_{\dim\ker W(\mathbb{A})+2m}(A_n^{\underline{m}}) = 0 \quad \text{and} \quad \liminf_{n\to\infty} s_{\dim\ker W(\mathbb{A})+2m+1}(A_n^{\underline{m}}) > 0.$$

Assume that the first lim inf cannot be replaced by lim. Then there is a sequence $h \in \mathcal{H}_+$ such that the limit $\lim_{n\to\infty} s_{\dim\ker W(\mathbb{A})+2m}(A_{h_n}^{\underline{m}}) > 0$ exists and there is again a subsequence $g \in \mathcal{H}_{\mathbb{A}_h}$ of h. Applying Theorem 16 to the sequence $\mathbb{A}_g^{\underline{m}}$ leads to $\lim_{n\to\infty} s_{\dim\ker W(\mathbb{A})+2m}(A_{g_n}^{\underline{m}}) = 0$, which is a contradiction. □

Theorem 19. *Let $\mathbb{A} = \{A_n\} \in \mathcal{F}_{\mathcal{A}_{l^p}}$ and $W(\mathbb{A})$ be invertible. Then there is a number m such that $\alpha(\mathbb{A}^{\underline{m}}) = 2m$. Moreover, there is a uniformly bounded sequence $\{C_n\}$ of generalized inverses C_n for $A_n^{\underline{m}}$, that is*

$$C_n A_n^{\underline{m}} C_n = C_n, \quad A_n^{\underline{m}} C_n A_n^{\underline{m}} = A_n^{\underline{m}} \quad \text{and} \quad \sup_n \|C_n\| < \infty,$$

such that $\|(C_n L_n - (W(\mathbb{A}))^{-1})L_m\| \to 0$ for every m. In case $p \neq \infty$ this is strong convergence.

Proof. The previous theorem provides the number m. For $n > m$ let $z \in \operatorname{im} L_{n-m}$ with $\|z\| = 1$ be arbitrarily chosen. Applying the Hahn-Banach Theorem, it is easy to prove that there is always a projection P_z onto $\operatorname{span}\{z\}$ of norm 1. Therefore there are positive numbers d and N such that for all $n \geq N$

$$\begin{aligned} 0 < d \leq s_{2m+1}(A_n^{\underline{m}}) &= \inf\{\|A_n^{\underline{m}} + F\| : \dim\operatorname{im} F \leq 2n + 1 - (2m+1)\} \\ &\leq \inf\{\|A_n^{\underline{m}} + A_n^{\underline{m}}(P_z - I)L_{n-m}\| : z \in \operatorname{im} L_{n-m},\ \|z\| = 1\} \\ &= \inf\{\|A_n^{\underline{m}} P_z L_{n-m}\| : z \in \operatorname{im} L_{n-m},\ \|z\| = 1\} \\ &\leq \inf\{\|A_n^{\underline{m}} z\| : z \in \operatorname{im} L_{n-m},\ \|z\| = 1\}. \end{aligned}$$

Thus $A_n^{\underline{m}} : \operatorname{im} L_{n-m} \to \operatorname{im} A_n^{\underline{m}}$ is invertible and $\|(A_n^{\underline{m}})^{-1}|_{\operatorname{im} A_n^{\underline{m}}}\| \leq \frac{1}{d}$ whenever $n \geq N$. Since all these operators $A_n^{\underline{m}}$ have the codimension of $2m$, there are a constant C and a sequence of projections $P_{\operatorname{im} A_n^{\underline{m}}}$, each onto the range of $A_n^{\underline{m}}$, such that $\|P_{imA_n^{\underline{m}}}\| \leq C$ for all $n \geq N$ (see [13], B.4.10). Now we put

$$C_n := L_{n-m}(A_n^{\underline{m}})^{-1} P_{\operatorname{im} A_n^{\underline{m}}} \text{ if } n \geq N \quad \text{and} \quad C_n := 0 \text{ otherwise.}$$

Then $\{C_n\}$ is uniformly bounded (by $\frac{C}{d}$) and for $n \geq N$ we obviously get

$$C_n A_n^{\underline{m}} C_n = C_n P_{\operatorname{im} A_n^{\underline{m}}} = C_n \quad \text{and} \quad A_n^{\underline{m}} C_n A_n^{\underline{m}} = A_n^{\underline{m}} (A_n^{\underline{m}})^{-1} P_{\operatorname{im} A_n^{\underline{m}}} A_n^{\underline{m}} = A_n^{\underline{m}},$$

that is C_n is a generalized inverse for $A_n^{\underline{m}}$. Finally, with $\hat{A} := W(\mathbb{A})$, we have

$$\begin{aligned} \|(C_n L_n - \hat{A}^{-1})L_m\| &= \|(C_n L_n \hat{A} - I)\hat{A}^{-1} L_m\| \\ &\leq \|(C_n L_n \hat{A} - L_{n-m})\hat{A}^{-1} L_m\| + \|(I - L_{n-m})\hat{A}^{-1} L_m\| \\ &\leq \|C_n\| \|(\hat{A} - A_n^{\underline{m}})\hat{A}^{-1} L_m\| + \|(I - L_{n-m})\hat{A}^{-1} L_m\|, \end{aligned}$$

for each m, where the first term tends to 0, since $(A_n^{\underline{m}} L_n)$ tends $\mathcal{P}$-strongly to $\hat{A}$, and the second one tends to 0 due to Theorem 13. □

Remark 7. In the case $p = 2$ we obtain the Moore-Penrose inverses (for a definition see [8], for instance)

$$(A_n^{\underline{m}})^+ = C_n = Q_n^m (A_n^{\underline{m}})^{-1} P_{\operatorname{im} A_n^{\underline{m}}}$$

if we choose the orthogonal projections $P_{\operatorname{im} A_n^{\underline{m}}}$.

2.4.5 Toeplitz operators

Here is a nice application of the results of the preceding sections.

Let $W := \{a \in L^\infty(\mathbb{T}) : \|a\|_W := \sum_{n\in\mathbb{Z}} |a_n| < \infty\}$ (where $(a_n)_{n\in\mathbb{Z}}$ are the Fourier coefficients of a) denote the Wiener algebra, and $W_{N\times N}$ the set of $N \times N$-matrices with entries in W.

For $a \in W_{N\times N}$ let $L(a) := \sum_{n\in\mathbb{Z}} a_n V_n$ be the Laurent operator and let $T(a) := PL(a)P$ be the Toeplitz operator with the symbol a. In what follows, we will consider the compressions of Toeplitz operators $T(a)$ to $l^p_N := l^p(\mathbb{Z}_+, \mathbb{C}^N)$. Let P_n denote the compression of $L_n := \chi_{[-n,n]}I$ to l^p_N and W_n be the operator on l^p_N which sends $(x_0, x_1 ..., x_n, x_{n+1}, ...)$ to $(x_n, x_{n-1}, ..., x_0, 0, ...)$. Obviously, $\hat{\mathcal{P}} = (P_n)$ is a uniform approximate identity on l^p_N.

Now we consider the finite section sequence

$$\{A_n\} := \{P_nT(a)P_n + P_nKP_n + W_nLW_n + G_n\},$$

where $K, L \in \mathcal{K}(l^p_N, \hat{\mathcal{P}})$ and $\|G_n\| \to 0$. This sequence can be embedded into $\mathcal{F}^T$, which was introduced in the sections above, as follows:

$$\{B_n\} := \{L_n(P(T(a) + K)P + PE^1_n(L^1_nJPLPJL^1_n)P + PG_nP + JPJ)L_n\},$$

where J is the operator $J : (x_n) \mapsto (x_{-n})$.

It is clear that the sequence $\{A_n\}$ is stable if and only if $\{B_n\}$ is stable, and that their approximation numbers behave in the same vein. Obviously the preceding theory applies to $\{B_n\}$ and its limit operators are

$$W\{B_n\} = P(T(a) + K)P + JPJ, \quad W^{-1}\{B_n\} = P \quad \text{and}$$
$$W^1\{B_n\} = JPJ(L(a) + JLJ)JPJ.$$

The first one can be identified with the operator $T(a) + K \in \mathcal{L}(l^p_N)$ and the last one with $T(\tilde{a}) + L \in \mathcal{L}(l^p_N)$ (where $\tilde{a}(t) := a(\frac{1}{t})$). Since the Toeplitz operator $T(a)$ is Fredholm if and only if its symbol a is invertible (see [14], 4.108), we arrive at

Theorem 20. *Let $p \in \{0\} \cup [1, \infty]$, $a \in W_{N\times N}$, $K, L \in \mathcal{K}(l^p_N, \hat{\mathcal{P}})$ and $\|G_n\| \to 0$ as $n \to \infty$. Then we obtain for the sequence*

$$\{A_n\} := \{P_nT(a)P_n + P_nKP_n + W_nLW_n + G_n\} :$$

- *If the symbol function a is invertible, then the operators $T(a) + K$ and $T(\tilde{a}) + L$ are Fredholm on l^p_N and the approximation numbers of A_n have the k-splitting property with $k = \dim\ker(T(a) + K) + \dim\ker(T(\tilde{a}) + L)$.*
- *If the symbol function a is not invertible then $s_l(A_n) \to 0$ for each $l \in \mathbb{N}$.*
- *$\{A_n\}$ is stable if and only if $T(a) + K$ and $T(\tilde{a}) + L$ are invertible.*
- *If the symbol function a is invertible and the operator $T(a) + K$ is invertible, then there is a number m such that for the sequence $\{A_nW_nP_mW_n\}$ there is a bounded sequence of generalized inverses C_n with*

$$\|(C_nP_n - (T(a) + K)^{-1})P_l\| \to 0 \quad \textit{for all} \quad l \in \mathbb{N}.$$

Applying the general theory of Section 1.2 directly to the case of Toeplitz operators on l^p_N, one can actually prove this result for a larger class of symbol functions, namely $(C_p+\overline{H^\infty_p})_{N\times N}$, which coincides with $W_{N\times N}$ in the cases $p \in \{0, 1, \infty\}$ (see [5] and [23]).

The earliest results on the stability of finite section sequences $\{P_nT(a)P_n\}$ were obtained in [1] for $p = 1$ and in [6] for $1 < p < \infty$. The case $p = \infty$ already appeared in [22].

Appendix

Proof of Proposition 3:

Proof. Since rank P_n is finite for every n, it is obvious that $\mathcal{K}(\mathbf{X}, \mathcal{P}) \subset \mathcal{K}(\mathbf{X})$ and hence every $\mathcal{P}$-compact projection is of finite rank. Therefore $A \in \mathcal{D}(\mathbf{X}, \mathcal{P})$ is either proper $\mathcal{P}$-Fredholm or proper deficient by the definition, that is the relation $1 \Leftrightarrow 4$ is proved.
$5 \Rightarrow 2$ is obvious from the definition.
$2 \Rightarrow 3$: If A is a $\mathcal{P}$-Fredholm operator, then there exists a $B \in \mathcal{L}(\mathbf{X}, \mathcal{P}) \subset \mathcal{L}(\mathbf{X})$, such that $I - AB, I - BA \in \mathcal{K}(\mathbf{X}, \mathcal{P}) \subset \mathcal{K}(\mathbf{X})$ and hence A is Fredholm.
$3 \Rightarrow 4$ immediately follows from Theorem 3.
$1 \Rightarrow 5$ is the delicate part of the proof. For this let A be proper $\mathcal{P}$-Fredholm and $P, P' \in \mathcal{K}(\mathbf{X}, \mathcal{P})$ be the appropriate projections from the definition. As a consequence of the Banach inverse mapping theorem, $A|_{\ker P} : \ker P \to \ker P'$ is invertible. Let $A^{(-1)}$ be its inverse and $B := (I - P)A^{(-1)}(I - P')$. Then $B \in \mathcal{L}(\mathbf{X})$ and $AB = I - P'$ as well as $BA = I - P$. We have to show that $B \in \mathcal{L}(\mathbf{X}, \mathcal{P})$, but this can be done by a slight modification of the proof of Theorem 1.1.9 in [17], which was taken from Kozak and Simonenko [11]:
Let (Q_n) stand for the sequence $(I - P_n)$. Further, we will write $m \ll n$ if $P_lQ_n = Q_nP_l = 0$ for all $l \le m$. By Theorem 4, what we have to show is that

$$\|P_kBQ_n\| \to 0 \text{ and } \|Q_nBP_k\| \to 0 \text{ as } n \to \infty.$$

Given $\epsilon > 0$ and $k \in \mathbb{N}$, we can choose a number $N \in \mathbb{N}$ such that $k \ll N$ and the norms $\|PQ_l\|, \|Q_lP\|, \|P'Q_l\|$ and $\|Q_lP'\|$ are less than $\frac{\epsilon}{C_\mathcal{P}(C_\mathcal{P}+1)\|B\|}$ for all $l \ge N$ (note that P, P' are $\mathcal{P}$-compact). We fix a positive number m with $\|B\|^2\|A\|/m < \epsilon$, and choose integers

$$0 = r_1^{(1)} < r_2^{(1)} < r_3^{(1)} < r_4^{(1)} < r_1^{(2)} < ... < r_4^{(m-1)} < r_1^{(m)} < r_2^{(m)} < r_3^{(m)} < r_4^{(m)}$$

such that $N \le k + r_2^{(1)}$, $k + r_l^{(i)} \ll k_{l+1}^{(i)}$ and $k + r_4(j) \ll k_1^{(j+1)}$ for all $1 \le l \le 3$, $1 \le i \le m$ and $1 \le j \le m - 1$ and such that (by Theorem 4)

$$\|P_{k+r_1^{(i)}}AQ_{k+r_2^{(i)}}\| < \epsilon/\|B\|^2, \tag{21}$$

$$\|Q_{k+r_3^{(i)}}AP_{k+r_2^{(i)}}\| < \epsilon/\|B\|^2, \tag{22}$$

$$\|P_{k+r_3^{(i)}}AQ_{k+r_4^{(i)}}\| < \epsilon/\|B\|^2, \tag{23}$$

$$\|Q_{k+r_1^{(i+1)}}AP_{k+r_4^{(i)}}\| < \epsilon/\|B\|^2. \tag{24}$$

That is, given $r_1^{(i)}$ we choose $r_2^{(i)} > r_1^{(i)}$ such that $k + r_2^{(i)} \gg k + r_1^{(i)}$ and that (21) holds, then $r_3^{(i)} > r_2^{(i)}$ such that $k + r_3^{(i)} \gg k + r_2^{(i)}$ and that (22) is satisfied, then $r_4^{(i)} > r_3^{(i)}$ which fulfills $k + r_4^{(i)} \gg k + r_3^{(i)}$ and (23), and finally $r_1^{(i+1)} > r_4^{(i)}$ such that $k + r_1^{(i+1)} \gg k + r_4^{(i)}$ and that (24) is valid.

Let $n \gg k + r_4^{(m)}$. We set

$$U_i := (k + r_1^{(i)}, k + r_3^{(i)}], \qquad V_i := (k + r_2^{(i)}, k + r_4^{(i)}],$$
$$U_i' := [0, k + r_3^{(i)}] \qquad V_i' := [0, k + r_2^{(i)}].$$

Then, since $n \gg k + r_4^{(m)} \gg k + r_2^{(m)}$ for all i,

$$\begin{aligned} P_k B Q_n &= P_k P_{V_i'} B Q_n = P_k P_{V_i'} (I - P) B Q_n = P_k P_{V_i'} B Q_n - P_k P_{V_i'} P B Q_n \\ &= P_k (BA + P) P_{V_i'} B Q_n - P_k P B Q_n = P_k B A P_{V_i'} B Q_n - P_k P Q_{V_i'} B Q_n \\ &= P_k B P_{U_i'} A P_{V_i'} B Q_n + P_k B Q_{U_i'} A P_{V_i'} B Q_n - P_k P Q_{V_i'} B Q_n. \end{aligned}$$

Obviously, the third item is smaller than ϵ because of the choice of N, and for the middle term we get

$$\begin{aligned} \| P_k B Q_{U_i'} A P_{V_i'} B Q_n \| &\le C_{\mathcal{P}} (C_{\mathcal{P}} + 1) \|B\|^2 \| Q_{U_i'} A P_{V_i'} \| \\ &= C_{\mathcal{P}} (C_{\mathcal{P}} + 1) \|B\|^2 \| Q_{k + r_3^{(i)}} A P_{k + r_2^{(i)}} \| \\ &\le C_{\mathcal{P}} (C_{\mathcal{P}} + 1) \epsilon \end{aligned}$$

due to (22). Further, since $n \gg k + r_3^{(i)}$,

$$\begin{aligned} P_k B P_{U_i'} A P_{V_i'} B Q_n &= P_k B P_{U_i'} A B Q_n - P_k B P_{U_i'} A Q_{V_i'} B Q_n \\ &= P_k B P_{U_i'} Q_n - P_k B P_{U_i'} P' Q_n - P_k B P_{U_i'} A Q_{V_i'} B Q_n \end{aligned}$$

where again $\| P_k B P_{U_i'} Q_n - P_k B P_{U_i'} P' Q_n \| < \epsilon$ and

$$\begin{aligned} & P_k B P_{U_i'} A Q_{V_i'} B Q_n \\ &\quad = P_k B P_{U_i'} A Q_{k + r_4^{(i)}} B Q_n + P_k B P_{U_i'} A P_{V_i} B Q_n \\ &\quad = P_k B P_{U_i'} A Q_{k + r_4^{(i)}} B Q_n + P_k B P_{k + r_1^{(i)}} A P_{V_i} B Q_n + P_k B P_{U_i} A P_{V_i} B Q_n. \end{aligned}$$

For the first term we have by (23),

$$\| P_k B P_{U_i'} A Q_{k + r_4^{(i)}} B Q_n \| \le C_{\mathcal{P}} (C_{\mathcal{P}} + 1) \|B\|^2 \| P_{k + r_3^{(i)}} A Q_{k + r_4^{(i)}} \| \le C_{\mathcal{P}} (C_{\mathcal{P}} + 1) \epsilon$$

and, for the middle term, due to (21),

$$\begin{aligned} \| P_k B P_{k + r_1^{(i)}} A P_{V_i} B Q_n \| &\le \| P_k B P_{k + r_1^{(i)}} A Q_{k + r_2^{(i)}} P_{k + r_4^{(i)}} B Q_n \| \\ &\le C_{\mathcal{P}}^2 (C_{\mathcal{P}} + 1) \|B\|^2 \| P_{k + r_1^{(i)}} A Q_{k + r_2^{(i)}} \| \\ &\le C_{\mathcal{P}}^2 (C_{\mathcal{P}} + 1) \epsilon. \end{aligned}$$

Hence, we conclude that

$$P_k B Q_n = -P_k B P_{U_i} A P_{V_i} B Q_n + D_i \tag{25}$$

where D_i is an operator with norm less than $c\epsilon$ with c being a constant independent of i and ϵ. Summarizing the identities (25) we get

$$\begin{aligned} mP_kBQ_n &= -\sum_{i=1}^m P_kBP_{U_i}AP_{V_i}BQ_n + \sum_{i=1}^m D_i \\ &= -P_kBP_{U_1\cup\ldots\cup U_m}AP_{V_1\cup\ldots\cup V_m}BQ_n + \sum_{i=1}^m D_i \\ &\quad + \sum_{i=1}^m P_kBP_{U_i}AP_{(V_1\cup\ldots\cup V_m)\setminus V_i}BQ_n, \end{aligned} \tag{26}$$

where we used that $U_i \cap U_j = V_i \cap V_j = \emptyset$ for $i \neq j$. For the first item we find

$$\|P_kBP_{U_1\cup\ldots\cup U_m}AP_{V_1\cup\ldots\cup V_m}BQ_n\| \leq C_{\mathcal{P}}^3(C_{\mathcal{P}}+1)\|B\|^2\|A\|.$$

Since $P_{U_i} = Q_{k+r_1^{(i)}}P_{k+r_3^{(i)}} = P_{k+r_3^{(i)}}Q_{k+r_1^{(i)}}$ as well as

$$\begin{aligned} P_{(V_1\cup\ldots\cup V_m)\setminus V_i)} &= P_{(V_1\cup\ldots\cup V_{i-2})} + P_{V_{i-1}} + P_{V_{i+1}} + P_{(V_{i+2}\cup\ldots\cup V_m)} \\ &= P_{k+r_4^{(i-1)}}P_{(V_1\cup\ldots\cup V_{i-2})} + P_{k+r_4^{(i-1)}}Q_{k+r_2^{(i-1)}} \\ &\quad + Q_{k+r_2^{(i+1)}}P_{k+r_4^{(i+1)}} + Q_{k+r_2^{(i+1)}}P_{(V_{i+2}\cup\ldots\cup V_m)}, \end{aligned}$$

we can estimate every term in the last sum of (26) by

$$\begin{aligned} &\|P_kBP_{U_i}AP_{(V_1\cup\ldots\cup V_m)\setminus V_i}BQ_n\| \\ &\quad \leq C_{\mathcal{P}}(C_{\mathcal{P}}+1)\|B\|^2\|P_{U_i}AP_{(V_1\cup\ldots\cup V_m)\setminus V_i}\| \\ &\quad \leq C_{\mathcal{P}}(C_{\mathcal{P}}+1)\|B\|^2[\|P_{U_i}AP_{k+r_4^{(i-1)}}\|\|P_{(V_1\cup\ldots\cup V_{i-2})} + Q_{k+r_2^{(i-1)}}\| \\ &\qquad\qquad + \|P_{U_i}AQ_{k+r_2^{(i+1)}}\|\|P_{k+r_4^{(i+1)}} + P_{(V_{i+2}\cup\ldots\cup V_m)}\|] \\ &\quad \leq C_{\mathcal{P}}(C_{\mathcal{P}}+1)\|B\|^2[C_{\mathcal{P}}\|Q_{k+r_1^{(i)}}AP_{k+r_4^{(i-1)}}\|(2C_{\mathcal{P}}+1) \\ &\qquad\qquad + (C_{\mathcal{P}}+1)\|P_{k+r_3^{(i+1)}}AQ_{k+r_2^{(i+1)}}\|2C_{\mathcal{P}}] \\ &\quad \leq d\epsilon \end{aligned}$$

with a constant d independent of i and ϵ due to (23) and (24). Inserting these estimates into (26), and dividing by m, we arrive at

$$\|P_kBQ_n\| \leq (C_{\mathcal{P}}^3(C_{\mathcal{P}}+1) + c + d)\epsilon$$

for all $n \gg k + r_4^{(m)}$. Therefore, $\|P_kBQ_n\| \to 0$ as $n \to \infty$, and the dual assertion $\|Q_nBP_k\| \to 0$ can be checked analogously. This completes the proof of $1 \Rightarrow 5$.

Finally, let A be proper $\mathcal{P}$-Fredholm and $P, P' \in \mathcal{K}(\mathbf{X}, \mathcal{P})$ be the appropriate projections from the definition. Since $P'A = 0$, we have $A^*(P')^* = 0$ and consequently $\operatorname{im}(P')^* \subset \ker A^*$. On the other hand

$$\begin{aligned} \dim\ker A^* &= \dim\operatorname{coker} A = \dim\operatorname{im} P' = \dim\ker(I - P') \\ &= \dim\operatorname{coker}(I - (P')^*) = \dim\operatorname{im}(P')^*, \end{aligned}$$

hence $\operatorname{im}(P')^* = \ker A^*$. In this vein we also show $\ker P^* = \operatorname{im} A^*$. Thus A^* is proper $\mathcal{P}^*$-Fredholm, since $P^*, (P')^* \in \mathcal{K}(\mathbf{X}^*, \mathcal{P}^*)$. If A is proper deficient, then it is obvious from the definition that A^* is proper deficient, too. $\square$

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