Sequential tests for Weibull distributed observations

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Abstract
This paper deals with two kinds of sequential tests for grouped observations based on Weibull distributed random variables. Based on the Weibull distribution we construct two sequential ratio test for testing the form parameter β and determine the appropriate OC- and ASN-functions. The problem of discrimination between more than two hypotheses is here solved by means of a Sobel-Wald test where we consider three hypotheses. The computation of corresponding characteristics of this test is presented.

Keywords
Hypotheses testing; grouped observations; sequential analysis; Sobel-Wald test; OC-function; ASN-function; Weibull distribution

Subject Classification
62F03 [Hypotheses testing], 62L10 [Sequential analysis]

Contents

1 Introduction

This Preprint deals with two kinds of sequential tests for grouped observations based on Weibull distributed random variables. At first we introduce the sequential ratio test where we are interested in the computation of the characteristics, namely the OC-function and the average sample number. Calculations of these functions are difficult because we have to solve Fredholm integral equations of type 2. Solutions of these equations are known so for only for the exponential and the Erlang distribution (see [1] and [9]). On the other hand, in practice we cannot observe continuous random variables exactly, so it is sensible to transform this test into a test based on grouped observations. Due to applications one has a fixed grouping, a so-called passive grouping scheme, or a classification where the group bounds are freely chosen, so-called active, which we will consider here. In the sequel corresponding sequential ratio tests are observed when Weibull distributed random variables are given.
The Weibull distribution has its application in the analysis of fatigues of material or failures of electronic components. Based on this distribution we construct two tests for testing the form parameter β, followed by robustness considerations. The test for early failures discriminates between early failures, i.e. $\beta < 1$, and random failures whereas the test for late failures decides between random failures and late failures, i.e. $\beta > 1$. Both tests can be interpreted as tests of the form of the Weibull distribution or testing for exponential distribution because we have an exponential distribution if $\beta = 1$ holds. As mentioned above we will investigate how the characteristics of the tests considered above vary if the parameter α changes to an unknown parameter.
It is often necessary to discriminate not only between two hypotheses. This problem is here solved by means of a Sobel-Wald test where we consider three hypotheses. This test consists of two sequential ratio tests which influences the OC-function and the average sample number function of the Sobel-Wald test. Computation of these characteristics is the main subject of the second part. The OC-functions often can easily calculated from the OC-functions of the underlying sequential tests. The determination of the average sample number is more difficult, so we present a special algorithm solving this problem.

2 Sequential test

In this section we introduce the WALD sequential ratio test for grouped observations and calculate the characteristics, namely the OC-function and the average sample number function. We give a general method for computing the characteristics. A discrete approach is considered in [3]. Further on we are going to construct two tests based on the Weibull distribution to test for early and late failures and present some numerical results.

2.1 General description

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with a given density function $f_\theta(x), \theta \in \Theta$, with respect to a certain measure μ. Our aim is to discriminate between the hypotheses

$$H_0 : \theta = \theta_0 \quad \text{and} \quad H_1 : \theta = \theta_1 \qquad \theta_0,\ \theta_1 \in \Theta, \qquad \theta_0 \neq \theta_1. \tag{1}$$

For $n = 1, 2, \ldots$, the likelihood ratio

$$L_{n,\theta_0,\theta_1} = \prod_{i=1}^{n} \frac{f_{\theta_1}(X_i)}{f_{\theta_0}(X_i)}$$

is used as a test statistic and for given stopping bounds $0 < B < 1 < A < \infty$ the sample size N and the decision rule δ are determined by

$$N = \min\{n \geq 1 : L_{n,\theta_0,\theta_1} \notin (B, A)\} \text{ and } \delta = \mathbf{1}_{\{L_{N,\theta_0,\theta_1} \leq B\}}.$$

As long as the critical inequality $B < L_{n,\theta_0,\theta_1} < A$ holds, we continue the observations for $n = 1, 2, \ldots$ If on observation stage $n = 1, 2, \ldots$ $L_{n,\theta_0,\theta_1} \leq B$ or $L_{n,\theta_0,\theta_1} \geq A$ holds for the first time we will stop the test and accept the null hypothesis H_0 or the alternative hypothesis H_1, respectively. This test, defined by (N, δ), is called *WALD sequential probability ratio test (SPRT)*. We denote this test by $S(B, A)$. For further investigations it will be helpful to consider the logarithmic likelihood ratio as follows

$$Z_{n,\theta_0,\theta_1} = \ln L_{n,\theta_0,\theta_1} = \sum_{i=1}^{n} \ln \frac{f_{\theta_1}(X_i)}{f_{\theta_0}(X_i)} = \sum_{i=1}^{n} Y_i$$

with

$$Y_i = \ln \frac{f_{\theta_1}(X_i)}{f_{\theta_0}(X_i)} \qquad \text{for } i = 1, 2, \ldots$$

In accordance to this, we get a modified sample size and a modified decision rule

$$N = \min\{n \geq 1 : Z_{n,\theta_0,\theta_1} \notin (b, a)\} \qquad \text{and} \qquad \delta = \mathbf{1}_{\{Z_{N,\theta_0,\theta_1} \leq b\}},$$

respectively, with $a = \ln A$ and $b = \ln B$.
If $P_\theta(L_{1,\theta_0,\theta_1} = 1) < 1$ holds, the WALD SPRT has the following properties:

- The test $S(B, A)$ is closed, that means $P_\theta(N < \infty) = 1$.
- Moments of the sample size are finite, that means $E_\theta N^k < \infty$ for $k = 1, 2, \ldots$
- The theorem of WALD and WOLFOWITZ holds which ensures a pointwise optimality of SPRT at the points θ_0 and θ_1.

In general, the calculation of the characteristics of a WALD SPRT is difficult. In the case of continuous random variables, Fredholm integral equations of type 2 has to be solved to get the OC- and the ASN-function. Exact solutions are given for the exponential and the Erlang distribution (see [1] and [9], respectively).
In this context so-called generalized SPRT's are considered. A generalized WALD SPRT is a SPRT which starts on stage 0 in point x,$-\infty < x < \infty$. Again we discriminate between two simple hypotheses

$$H_0 : \theta = \theta_0 \quad \text{and} \quad H_1 : \theta = \theta_1 \qquad \theta_0,\ \theta_1 \in \Theta, \qquad \theta_0 \neq \theta_1 \tag{2}$$

and apply the logarithmic likelihood ratio Z_{n,θ_0,θ_1} for $n = 1, 2, \ldots$ So we get a modified sample size as well as a modified decision rule as follows

$$N(x) = \inf\{n \geq 1 : x + Z_{n,\theta_0,\theta_1} \notin (b, a)\} \quad \text{and} \quad \delta(x) = \mathbf{1}_{\{x + Z_{N,\theta_0,\theta_1} \leq b\}}.$$

We denote this general SPRT by $(N(x), \delta(x))$. If the test starts on stage 0 in point $x = 0$ we will get the previous test $(N(0), \delta(0)) = (N, \delta)$.

Theorem 2.1. *Let $(N(x), \delta(x))$, $x \in \mathbb{R}$, be a general WALD SPRT for the hypotheses (2) with the operating characteristic function $q_\theta(x)$ and the average sample number function $e_\theta(x)$. Under the boundary conditions*

$$q_\theta(x) = 1 \ \text{ for } \ x \leq b \ \text{ and } \ q_\theta(x) = 0 \quad \text{for} \quad x \geq a$$

as well as

$$e_\theta(x) = 0 \ \text{ for } \ x \notin (b, a),$$

the OC-function and the ASN-function can be determined by

$$q_\theta(x) = P_\theta(Y_1 \leq b - x) + \int_b^a q_\theta(z) g_\theta(z - x) dz, \tag{3}$$

$$e_\theta(x) = 1 + \int_b^a e_\theta(z) g_\theta(z - x) dz \tag{4}$$

for $b \leq x \leq a$. $g_\theta(y)$ is the density function of the random variable $Y_1 = \ln \frac{f_{\theta_1}(X_1)}{f_{\theta_0}(X_1)}$.

Proof. see [3] □

As mentioned before we can solve these integral equations only for exponential and Erlang distributed random variables. Approximate solutions can be obtained by two ways. On the one hand we can get an approximately solution by discretization of the range of X. On the other hand a solution can be found by dividing the range of Y into intervals of equal length. Both methods transform continuous random variables into discrete variables. In the following we apply the first method.

We divide the range $\mathcal{X}$ into disjunct groups or classes $\mathcal{X}_1, \ldots, \mathcal{X}_m$ with

$$\mathcal{X}_i = [x_{i-1}, x_i),\ X_i \cap \mathcal{X}_j = \emptyset,\ i \neq j \quad \text{and} \quad \bigcup_{i=1}^{m} \mathcal{X}_i = \mathcal{X}.$$

We denote this partition by $G = (\mathcal{X}_1, \ldots, \mathcal{X}_m)$. A detailed description of such a classification is given in [3]. Now we assume that instead of the random variable X a random variable X^G is observed where

$$X^G = k \Leftrightarrow X \in \mathcal{X}_k \qquad k = 1, \ldots, m$$

holds. The probability function of X^G is determined by

$$p_\theta^G(k) = P_\theta(X^G = k) = P_\theta(X \in \mathcal{X}_k) \qquad k = 1, \ldots, m.$$

Now we consider a WALD SPRT for the hypotheses (2) based on classification G which starts on stage 0 in point $x \in \mathbb{R}$. Let

$$Y_i^G = \ln \frac{p_{\theta_1}^G(X_i^G)}{p_{\theta_0}^G(X_i^G)}, \quad i = 1, 2, \ldots$$

be on observation stage i the corresponding test statistic to X_i^G. Then we obtain on stage n a modified test statistic

$$Z_{n,\theta_0,\theta_1}^G = \sum_{i=1}^{n} Y_i^G.$$

With stopping bounds a and b, $-\infty < b < 0 < a < \infty$, the sample size and the decision rule for a discretized generalized SPRT is then given by

$$N^G(x) = \inf\{n \geq 1 : x + Z_{n,\theta_0,\theta_1}^G \notin (b, a)\} \quad \text{and} \quad \delta^G(x) = \mathbf{1}_{\{x + Z_{N^G(x),\theta_0,\theta_1}\}}.$$

This test is denoted by $(N^G(x), \delta^G(x))$. Because of the random variables $X_1, X_2, \ldots$ are assumed to be i.i.d. random variables we only have to consider X_1^G and Y_1^G, respectively. Let be $\mathcal{Y}^G = (y_1^G, \ldots, y_m^G)$ the range of Y_1^G where y_k^G defined by $y_k^G = \ln \frac{p_{\theta_1}^G(k)}{p_{\theta_0}^G(k)}, \quad k = 1, \ldots, m$. In order to transform the integral equations into a linear system of equations, the range of the test statistic $\{x + Z_{n,\theta_0,\theta_1}^G\}_{n=1}^{\infty}$ has to be adapted to the discretization. Let $s > 0$ be a discretization parameter and h an interval length according to

$$h = \frac{a - b}{s}.$$

Then the critical inequality $b < \sum_{i=1}^{n} Y_i^G < a$ of test $(N^G(x), \delta^G(x))$ can be written as

$$0 < -\frac{b}{h} + \sum_{i=1}^{n} \frac{Y_i^G}{h} < \frac{a-b}{h} = s, \qquad n = 1, 2, \ldots$$

By rounding we get a whole-numbered random variable

$$\tilde{Y}_i^G = round\left(\frac{Y_i^G}{h}\right) \qquad i = 1, 2, \ldots$$

and a whole-numbered starting point

$$\tilde{c} = round\left(-\frac{b}{h}\right).$$

The probability function $\tilde{p}_\theta^G(k)$ of the i.i.d. random variables $\tilde{Y}_1^G, \tilde{Y}_2^G, \ldots$ can be determined by

$$\tilde{p}_\theta^G(k) = P_\theta(\tilde{Y}_1^G = k) = \sum_{i=1}^{m} p_\theta^G(i) \mathbf{1}_{\{\tilde{y}_i^G = k\}} \qquad k \in \Gamma \tag{5}$$

with $\tilde{y}_i^G = round(\frac{y_i^G}{h})$ for $i = 1, \ldots, m$ and $p_\theta^G(i) = P_\theta(Y_1^G = y_i^G) = P_\theta(X_1^G = i) = P_\theta(X_1 \in \mathcal{X}_i)$. For sufficiently large values of s the variables $\tilde{y}_i^G$, $i = 1, \ldots, m$, are different from each other and we obtain instead of (5)

$$\tilde{p}_\theta^G(k) = \begin{cases} p_\theta^G(i) & \text{if } k \in \{\tilde{y}_1^G, \ldots, \tilde{y}_m^G\} \text{ and } k = \tilde{y}_i^G \\ 0 & \text{otherwise.} \end{cases}$$

The sample size and the decision rule of test $(\tilde{N}^G, \tilde{\delta}^G)$ are now given by

$$\tilde{N}^G = \inf\{n \geq 1 : \tilde{c} + \sum_{i=1}^{n} \tilde{Y}_i^G \notin (0, s)\} \quad \text{and} \quad \tilde{\delta}^G = \mathbf{1}_{\{\tilde{c} + Z_{\tilde{N}^G, \theta_0, \theta_1}\}}$$

with $Z_{\tilde{N}^G, \theta_0, \theta_1} = \sum\limits_{i=1}^{\tilde{N}^G} \tilde{Y}_i^G$.

For the computation of the characteristics of test $(\tilde{N}^G, \tilde{\delta}^G)$ it is necessary again to consider corresponding generalized SPRT's $(\tilde{N}_k^G, \tilde{\delta}_k^G)$, which start on stage 0 in point $k \in \Gamma$. The sample size $\tilde{N}_k^G$ and the decision rule $\tilde{\delta}_k^G$ for $k = 1, \ldots, s-1$ are defined by

$$\tilde{N}_k^G = \inf\{n \geq 1 : k + \sum_{i=1}^{n} \tilde{Y}_i^G \notin (0, s)\} \quad \text{and} \quad \tilde{\delta}_k^G = \mathbf{1}_{\{k + Z_{\tilde{N}_k^G, \theta_0, \theta_1}\}}$$

with $Z_{\tilde{N}_k^G, \theta_0, \theta_1} = \sum\limits_{i=1}^{\tilde{N}_k^G} \tilde{Y}_i^G$.

Computation of the OC-function

Let $\tilde{q}_\theta^G(k) = E_\theta \tilde{\delta}_k^G$ be the OC-function of test $(\tilde{N}_k^G, \tilde{\delta}_k^G)$ for $k \in \Gamma$ and $\theta \in \Theta$. Under boundary conditions

$$\tilde{q}_\theta^G(k) = 1 \quad \text{for} \quad k \leq 0 \qquad \text{and} \qquad \tilde{q}_\theta^G(k) = 0 \quad \text{for} \quad k \geq s \tag{6}$$

the integral equation (3) becomes a system of simultaneous linear equations

$$\tilde{q}_\theta^G(k) = \sum_{j=-\infty}^{\infty} \tilde{c}_{kj}^\theta \tilde{q}_\theta^G(j) = \tilde{a}_k^G + \sum_{j=1}^{s-1} \tilde{c}_{kj}^\theta \tilde{q}_\theta^G(j), \qquad k = 1, \ldots, s-1, \tag{7}$$

with unknowns $\tilde{q}_\theta^G(1), \ldots, \tilde{q}_\theta^G(s-1)$. The coefficients $\tilde{c}_{kj}^\theta$ and $\tilde{a}_k^\theta$ are determined by

$$\begin{aligned} \tilde{c}_{kj}^\theta &= P_\theta(k + \tilde{Y}_1^G = j) = P_\theta(\tilde{Y}_1^G = j - k) = \tilde{p}_\theta^G(j-k), \\ \tilde{a}_k^\theta &= P_\theta(k + \tilde{Y}_1^G \leq 0) = \sum_{j=-\infty}^{0} 1 \cdot \tilde{c}_{kj}^\theta = \sum_{j=-\infty}^{0} \tilde{p}_\theta^G(j-k) = P_\theta(\tilde{Y}_1^G \leq -k). \end{aligned} \tag{8}$$

We obtain the OC-function $Q^G(\theta)$ of test (N^G, δ^G) approximately from

$$Q^G(\theta) = E_\theta \delta^G \approx E_\theta \tilde{\delta}_{\tilde{c}}^G = \tilde{q}_\theta^G(\tilde{c}) \tag{9}$$

with $\tilde{c} = round\left(-\frac{b}{h}\right)$.

Computation of the ASN-function

The ASN-function can be determined analogously to the OC-function. Let $\tilde{e}_\theta^G(k)$ be the average sample number of test $(\tilde{N}_k^G, \tilde{\delta}_k^G)$, $k \in \Gamma$. With the boundary condition

$$\tilde{e}_\theta^G(k) = 0 \qquad \text{for} \quad k \notin (0, s)$$

the integral equation (4) can be modified to a linear system of equations

$$\tilde{e}_\theta^G(k) = 1 + \sum_{j=1}^{s-1} \tilde{c}_{kj}^\theta \tilde{e}_\theta^G(j), \qquad k = 1, \ldots, s-1, \tag{10}$$

with unknowns $\tilde{e}_\theta^G(1), \ldots, \tilde{e}_\theta^G(s-1)$. The probabilities $\tilde{c}_{kj}^\theta$ can be calculated in accordance to (8). Finally, we get an approximation of the average sample number of test (N^G, δ^G) through

$$E_\theta N^G \approx E_\theta \tilde{N}^G = \tilde{e}_\theta^G(\tilde{c}) \tag{11}$$

with $\tilde{c} = round\left(-\frac{b}{h}\right)$.

2.2 Sequential test for Weibull distributed observations

In this section we consider the Weibull distribution itself and design two WALD SPRTs for testing the parameter β of the Weibull distribution while the parameter α remains constant 1.

2.2.1 Testing for early failure

The Weibull distribution, which is named after the Swedish engineer and mathematician Waloddi Weibull, plays an important role in analyzing lifetime and reliability. Fatigue of material, failure of electronic components or statistical determination of wind velocities are typical applications of the Weibull distribution.

Definition 2.1. *A random variable X will be called* **Weibull distributed** *with parameters $\alpha > 0$ and $\beta > 0$, if it has the density function*

$$f(x) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} & x > 0 \\ 0 & x \leq 0. \end{cases} \tag{12}$$

The corresponding distribution function, expectation value and variance are

$$F(x) = \begin{cases} 1 - e^{-\alpha x^{\beta}} & x > 0 \\ 0 & x \leq 0, \end{cases}$$

$$EX = \left(\frac{1}{\alpha}\right)^{\left(\frac{1}{\beta}\right)} \Gamma\left(\frac{1}{\beta}+1\right),$$
$$D^2X = (\frac{1}{\alpha})^{\left(\frac{2}{\beta}\right)} \left[\Gamma\left(\frac{2}{\beta}+1\right) - \Gamma\left(\frac{1}{\beta}+1\right)^2\right].$$

with $\alpha > 0, \beta > 0$. Figure 1 shows the effects of parameter variation with respect to the density function (12).

In the following we assume $\alpha = 1$ and denote the parameter β by θ. Under the assumption of a Weibull distributed population X with density function

$$f_\theta(x) = \begin{cases} \theta x^{\theta-1} e^{-x^{\theta}} & x > 0 \\ 0 & x \leq 0 \end{cases} \qquad \text{with } \theta > 0$$

and range $\mathcal{X} = [0, \infty)$ we consider a sequence of i.i.d. random variables $X_1, X_2, \ldots$ for discriminating between hypotheses

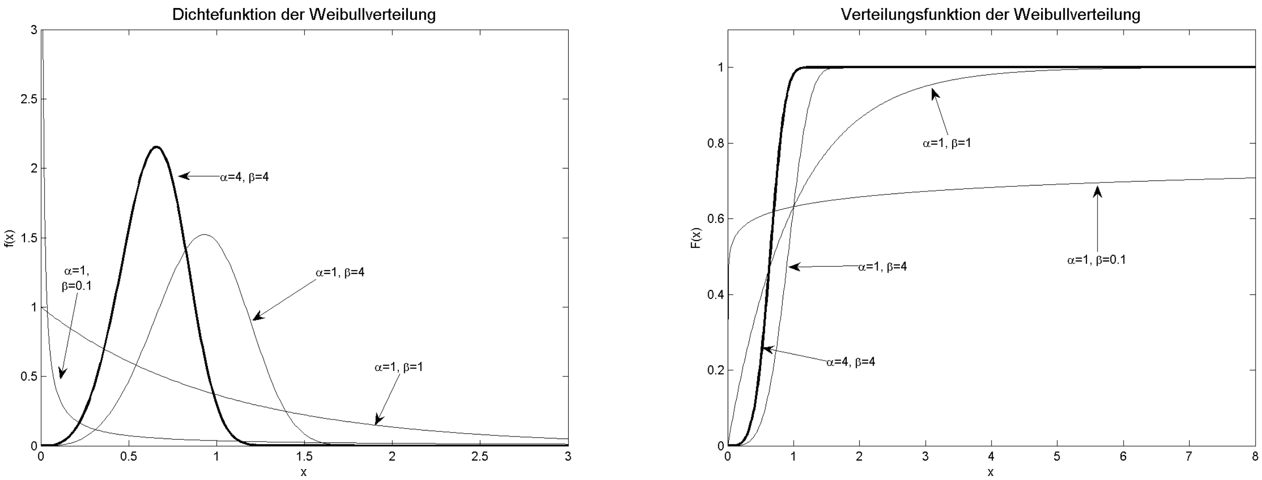


Figure 1: Density function and distribution function

$$H_0^{(1)}:\ \theta = \theta_0 = 1 - \varepsilon_1 \quad \text{and} \quad H_1^{(1)}:\ \theta = \theta_1 = 1 \qquad 0 < \varepsilon_1 < 1.$$

Then we get for the likelihood ratio for $n = 1, 2, \ldots$

$$L_{n,\theta_0,\theta_1}^{(1)} \;=\; \prod_{i=1}^{n} \frac{f_{\theta_1}(X_i)}{f_{\theta_0}(X_i)} = \prod_{i=1}^{n} \frac{e^{-X_i}}{(1-\varepsilon_1)\,\exp(-X_i^{1-\varepsilon_1})\,X_i^{-\varepsilon_1}} = \prod_{i=1}^{n} \frac{\exp(-X_i + X_i^{1-\varepsilon_1})\,X_i^{\varepsilon_1}}{1-\varepsilon_1}$$

as well as for the logarithmic likelihood ratio

$$Z_{n,\theta_0,\theta_1}^{(1)} \;=\; \ln\, L_{n,\theta_0,\theta_1}^{(1)} = \sum_{i=1}^{n} \left[-X_i + X_i^{1-\varepsilon_1} + \ln X_i^{\varepsilon_1} - \ln(1-\varepsilon_1)\right] = \sum_{i=1}^{n} Y_i^{(1)}.$$

We continue our observations as long as

$$\ln B_1 < Z_{n,\theta_0,\theta_1}^{(1)} < \ln A_1 \tag{13}$$

holds. The sample size and the decision rule are then

$$N^{(1)} = \min\{n \geq 1 : Z_{n,\theta_0,\theta_1}^{(1)} \notin (b_1, a_1)\} \quad \text{and} \quad \delta^{(1)} = \mathbf{1}_{\{Z_{N,\theta_0,\theta_1}^{(1)} \leq b_1\}}$$

with $b_1 = \ln B_1$ and $a_1 = \ln A_1$ and denote this test by $(N^{(1)}, \delta^{(1)})$. Characteristics of this test are considered in subsection 2.2.3.

2.2.2 Testing for late failure

We consider again the Weibull distributed population as in the previous subsection. But now we discriminate between the hypotheses

$$H_0^{(2)} : \theta = \theta_0 = 1 \quad \text{and} \quad H_1^{(2)} : \theta = \theta_1 = 1 + \varepsilon_2 \qquad 0 < \varepsilon_2.$$

For $n = 1, 2, \ldots$ we obtain the likelihood ratios as follows:

$$\begin{aligned} L_{n,\theta_0,\theta_1}^{(2)} &= \prod_{i=1}^{n} \frac{f_{\theta_1}(X_i)}{f_{\theta_0}(X_i)} = \prod_{i=1}^{n} \frac{(1+\varepsilon_2)\,\exp(-X_i^{1+\varepsilon_2})\,X_i^{\varepsilon_2}}{e^{-X_i}} = \prod_{i=1}^{n}(1+\varepsilon_2)\,\exp(X_i - X_i^{1+\varepsilon_2})\,X_i^{\varepsilon_2}, \\ Z_{n,\theta_0,\theta_1}^{(2)} &= \ln L_{n,\theta_0,\theta_1}^{(2)} = \sum_{i=1}^{n} \left[X_i - X_i^{1+\varepsilon_2} + \ln X_i^{\varepsilon_2} + \ln(1+\varepsilon_2)\right] = \sum_{i=1}^{n} Y_i^{(2)}. \end{aligned}$$

As long as

$$\ln B_2 < Z_{n,\theta_0,\theta_1}^{(2)} < \ln A_2 \tag{14}$$

holds and the decision rule is

$$\delta^{(2)} = \mathbf{1}_{\{Z_{N,\theta_0,\theta_1}^{(2)} \le b_2\}}$$

we continue our observations and get the sample size from

$$N^{(2)} = \min\{n \ge 1 : Z_{n,\theta_0,\theta_1}^{(2)} \notin (b_2, a_2)\}$$

with $b_2 = \ln B_2$ and $a_2 = \ln A_2$. Then we obtain this test $(N^{(2)}, \delta^{(2)})$.

2.2.3 Determination of the characteristics

Analogous to section 2.1 we determine the characteristics of both tests by a discretization scheme. Let the group bounds $x_{(k)}$, $k = 0, \ldots, m$, are chosen in accordance to

$$0 = x_0 < x_1 < \ldots < x_{m-1} < x_m = \infty.$$

For $k = 1, \ldots, m$ the probabilities $p_\theta(k) = P_\theta(X^G = k)$ are identical for both tests, because of the same underlying population X. We get

$$\begin{aligned} p_\theta(k) &= P_\theta(X^G = k) = P_\theta(X \in \mathcal{X}_k) = P_\theta(X \in [x_{k-1}, x_k)) \\ &= F_\theta(x_k) - F_\theta(x_{k-1}) \\ &= e^{-x_{k-1}^\theta} - e^{-x_k^\theta}. \end{aligned}$$

Due to this grouping we consider instead of random variables $Y_1^{(1)}$ and $Y_1^{(2)}$ the grouped random variables $Y_1^{G,1}$ and $Y_1^{G,2}$. The ranges of the random variables $\mathcal{Y}^{G,i} = \{y_1^{(i)}, \ldots, y_m^{(i)}\}$, where

$$y_k^{(1)} = \ln\left[\frac{e^{-x_{k-1}} - e^{-x_k}}{e^{-x_{k-1}^{1-\varepsilon}} - e^{-x_k^{1-\varepsilon}}}\right] \quad \text{and} \quad y_k^{(2)} = \ln\left[\frac{e^{-x_{k-1}^{1+\varepsilon}} - e^{-x_k^{1+\varepsilon}}}{e^{-x_{k-1}} - e^{-x_k}}\right],$$

$k = 1, \dots, m$. Analogously to the previous section we have to adapt the ranges of $\{x + \sum_{i=1}^n Y_i^{G,j}\}_{n=1}^\infty$, $j = 1,2$, to the discretization scheme. Let to a given integer s interval lengths h_1 and h_2 be defined by $h_j = \frac{a_j - b_j}{s}$, $j = 1,2$. Then the critical inequalities (13) and (14) transform into

$$0 < -\frac{b_j}{h_j} + \sum_{i=1}^n \frac{Y_i^{G,j}}{h_j} < \frac{a_j - b_j}{h_j} =: s_j \qquad j = 1,2.$$

After rounding we obtain integer-valued random variables $\tilde{Y}_i^{G,1}$ and $\tilde{Y}_i^{G,2}$ and integer-valued starting points $\tilde{c}_1$ and $\tilde{c}_1$ according to

$$\tilde{Y}_i^{G,j} = round\left(\frac{Y_i^{G,j}}{h_j}\right) \quad j = 1,2,\ i = 1,2,\dots \tag{15}$$

and

$$\tilde{c}_j = round\left(-\frac{b_j}{h_j}\right) \quad j = 1,2.$$

The corresponding group probabilities are then

$$\tilde{p}_\theta^{G,j}(k) = P_\theta(\tilde{Y}_1^{(j)} = k) = \sum_{r=1}^m p_\theta(r) \mathbf{1}_{\{\tilde{y}_r^{G,j} = k\}} \qquad k = 0, \pm 1, \pm 2, \dots,\ j = 1,2$$

with $\tilde{y}_k^{G,j} = round\left(\frac{y_k^{G,j}}{h_j}\right)$, $j = 1,2$ and $k = 1, \dots, m$. If s is large enough the values $\tilde{y}_1^{G,j}, \dots, \tilde{y}_m^{G,j}$ will be different and can be calculated by

$$\tilde{p}_\theta^{(j)}(k) = \begin{cases} p_\theta(r) & \text{for } k \in \{\tilde{y}_1^{G,j}, \dots, \tilde{y}_m^{G,j}\} \quad \text{and } k = \tilde{y}_r^{G,j}, \\ 0 & \text{otherwise.} \end{cases}$$

Using these corresponding group probabilities we can solve the linear systems of equations (7) and (10) and obtain characteristics $Q^{G,j}(\theta)$ and $E_\theta N^{G,j}$, $j = 1,2$, for both tests.

2.2.4 Example and robustness analysis

In this section we will illustrate the tests for early and for late failures considered above. With respect to a test for random failures which is described in section 3.3 we use F-optimal group bounds according to [4]. Table 1 shows these bounds for $m = 2, \dots, 10$.
In the following we denote by α and β the error probabilities of the corresponding tests and use the approximations of WALD as stopping bounds in accordance to [2], i.e.

m	2	3	4	5	6	7	8	9	10
x_0	0	0	0	0	0	0	0	0	0
x_1	1.5936	1.0176	0.7540	0.6004	0.4993	0.4276	0.3739	0.3323	0.2991
x_2		2.6112	1.7716	1.3545	1.0997	0.9269	0.8015	0.7062	0.6314
x_3			3.3652	2.3720	1.8538	1.5273	1.3008	1.1338	1.0053
x_4				3.9657	2.8714	2.2813	1.9012	1.6331	1.4329
x_5					4.4650	3.2989	2.6553	2.2336	1.9322
x_6						4.8925	3.6729	2.9876	2.5326
x_7							5.2665	4.0052	3.2867
x_8								5.5988	4.3042
x_9									5.8979

Table 1: F-optimal group bounds for $\theta = 1$

$$B = \frac{\beta}{1-\alpha} = B^* \quad \text{and} \quad A = \frac{1-\beta}{\alpha} = A^*. \tag{16}$$

With these stopping bounds

$$Q(\theta_0) \approx 1 - \alpha \qquad\qquad Q(\theta_1) \approx \beta,$$

holds for a WALD SPRT.

Example 1:

We choose the error probabilities $\alpha = \beta = 0.05$ and the discretization parameter $s = 500$. The parameter α of the Weibull distribution is chosen 1.

Test for early failures

For testing for early failures let be $\varepsilon_1 = 0.2$. Then we obtain

$$H_0^{(1)}: \ \theta = \theta_0 = 0.8 \quad \text{and} \quad H_1^{(1)}: \ \theta = \theta_1 = 1.$$

By relation (16) we get stopping bounds $a_1 = \ln A_1^* = 2.94444$ and $b_1 = \ln B_1^* = -2.94444$ and according to (15) a starting point $\tilde{c}_1 = 250$ with $h_1 = \frac{a_1 - b_1}{s}$.
We only consider partitions with 3 till 10 groups because dividing the range $\mathcal{X}$ of random variable X into two groups do not give us enough information about the hypotheses. The

reason is that the group bound lies between two intersections of the density function of the Weibull distribution for $\beta = 0.8$ and $\beta = 1$ and $\beta = 1$ and $\beta = 1.24$, respectively. This is shown in figure 2.

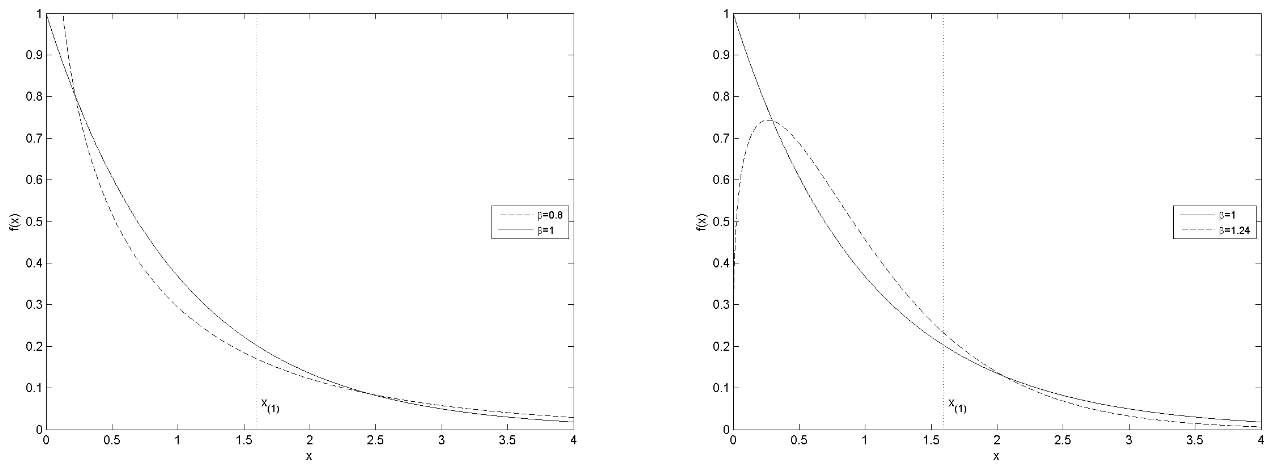


Figure 2: Comparison of the density function for both tests

The computed OC-function for groups $3, \ldots, 10$ and corresponding the ASN-function is shown in figure 3.

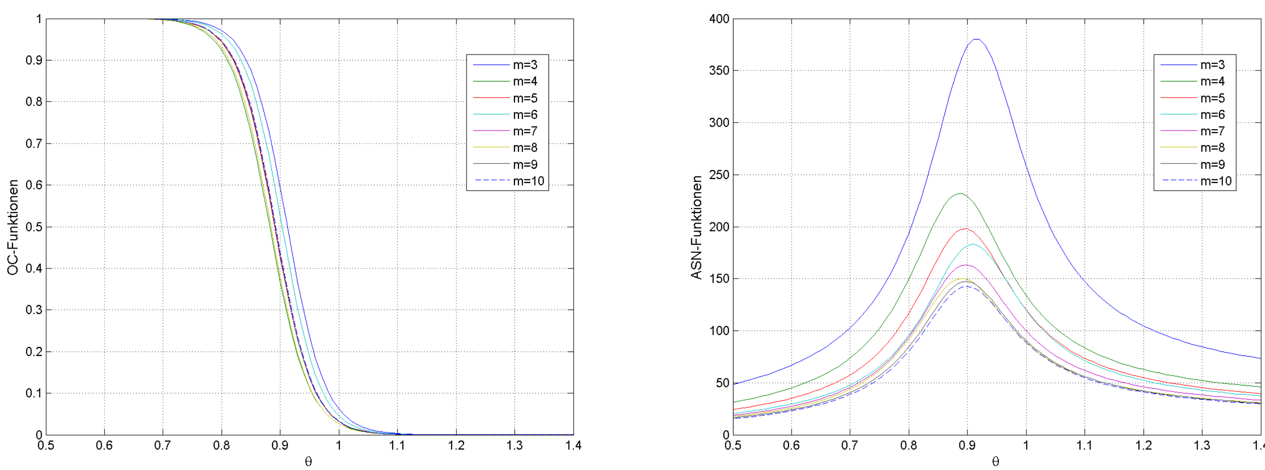


Figure 3: Characteristics for groups $3, \ldots, 10$ and $\varepsilon_1 = 0.2$

The OC-functions of different groupings are quite similar. Differences between the OC-functions of different groups can be explained by the "small" discretization parameter. We can see, the average sample number decreases when the number of groups increases. Thus, we can choose a higher number of groups and get a smaller sample number whereas the selectivity remains approximately unaltered.

Robustness considerations for testing for early failures

In practice the parameter value α may change to a new parameter value $\tilde{\alpha} \neq \alpha$. To determine the influence of the true value of α on the characteristics, it is necessary to include this parameter in our calculations. Random variable $\tilde{Y}_1^{G,1}$ remains unchanged, whereas the group probabilities alter to

$$\tilde{p}^G_{\theta,\tilde{\alpha}}(k) = \begin{cases} p^G_{\theta,\tilde{\alpha}}(i) & if\ k \in \{\tilde{y}^G_1, \ldots, \tilde{y}^G_m\}\ and\ k = \tilde{y}^G_i \\ 0 & otherwise. \end{cases}$$

where $p^G_{\theta,\tilde{\alpha}}(i) = P_{\theta,\tilde{\alpha}}(X^G = k) = P_{\theta,\tilde{\alpha}}(X = \mathcal{X}_k)$.
By solving the linear systems of equations (7) and (10) we obtain a generalized OC-function $Q^{G,1}(\theta, \tilde{\alpha})$ and a generalized ASN-function $E_{\theta,\tilde{\alpha}}N^{G,1}$. Figure 4 presents the influence of parameter value α for $m = 10$ groups on the OC-function.

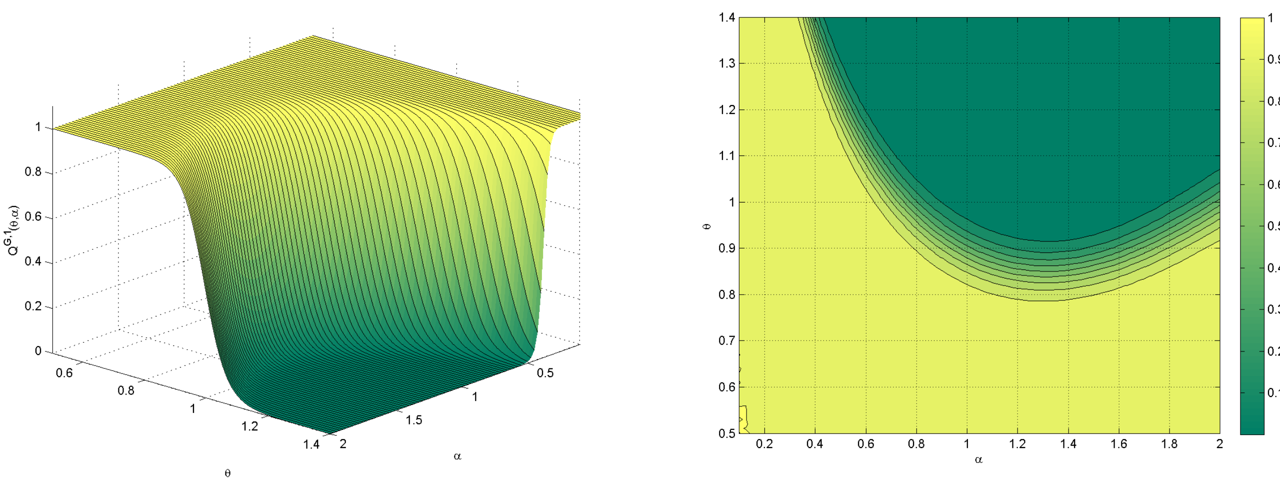


Figure 4: Two-dimensional OC-function and contour lines for $m = 10$ and $\varepsilon_1 = 0.2$

These considerations show that for $\alpha \leq 0.3$ the test opts for H_0 without reference to parameter $\theta \in [0.5, 1.4]$. Consequently the test accepts early failures permanently. If $\alpha > 0.3$ holds, the OC-function will have its typical shape but with different probabilities for accepting the null or the alternative hypothesis depending on α. Analogously, figure 5 shows the ASN-function. The test needs a small average sample number for small values of α to decide between the hypotheses. Some values of the characteristics are displayed in table 2.

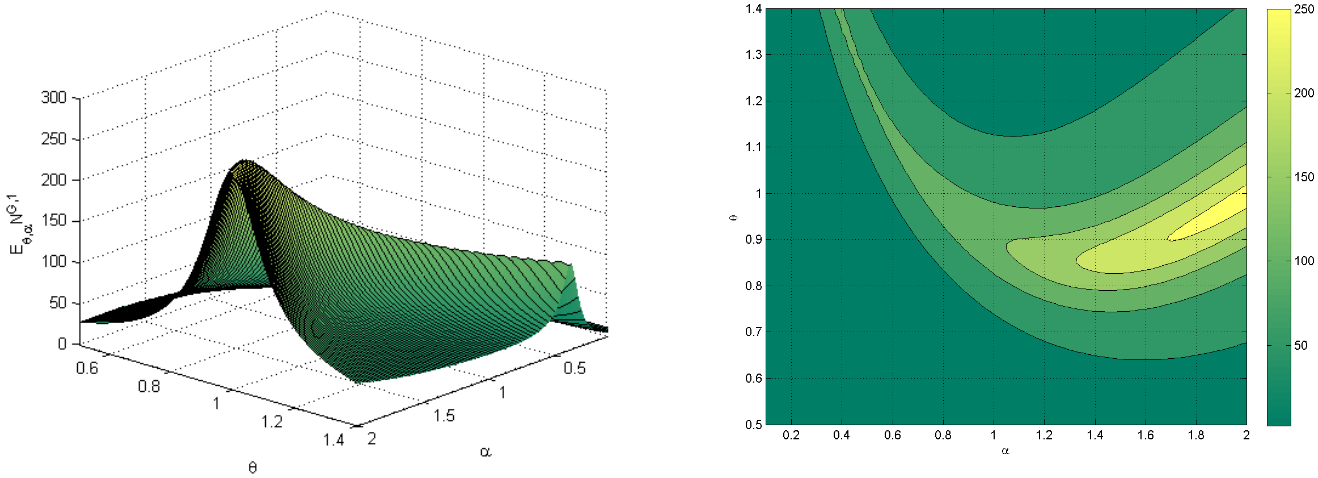


Figure 5: Two-dimensional ASN-function and contour lines for $m = 10$ and $\varepsilon_1 = 0.2$

$\tilde{\alpha}$	0.25	0.5	0.75	**1**	1.25	1.5	1.75	2
$Q^{G,1}(\theta_0, \tilde{\alpha})$	1	1	0.996	**0.9464**	0.8528	0.8967	0.977	0.9973
$Q^{G,1}(\theta_1, \tilde{\alpha})$	1	0.9952	0.5019	**0.0328**	0.0069	0.0111	0.0716	0.4526
$E_{\theta_0,\tilde{\alpha}} N^{G,1}$	5.174	12.8659	32.9387	**80.5858**	142.4465	162.2592	125.4194	86.6043
$E_{\theta_1,\tilde{\alpha}} N^{G,1}$	7.3245	29.0338	112.9595	**88.6749**	84.597	108.3077	173.4256	260.8086

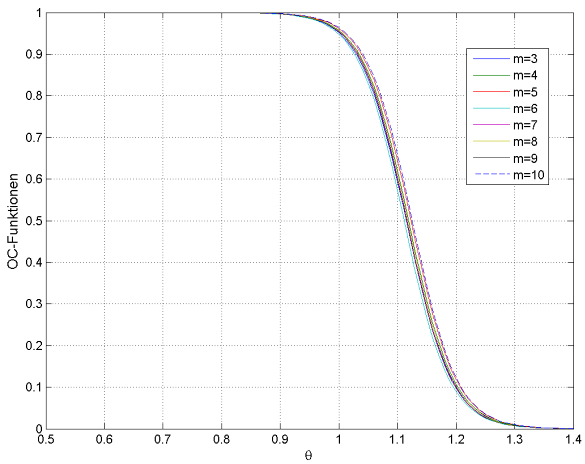
Table 2: Probabilities for accepting hypotheses $H_0^{(1)}$ and $H_1^{(1)}$ for different values of $\tilde{\alpha}$

Testing for late failures

Let be $\varepsilon_2 = 0.24$. Then we get

$$H_0^{(2)} : \ \theta = \theta_0 = 1 \quad \text{and} \quad H_1^{(2)} : \ \theta = \theta_1 = 1.24.$$

By relation (16) the stopping bounds are $a_2 = \ln A_2^* = 2.94444$ and $b_2 = \ln B_2^* = -2.94444$. According to (15) the test starts in $\tilde{c}_2 = 250$. The characteristics are shown in figure 6 and we get a similar result as in the previous test. But the characteristics are shifted to the right because of different hypotheses.



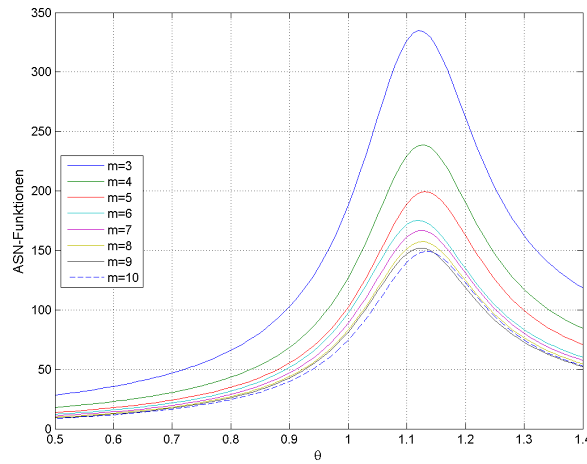
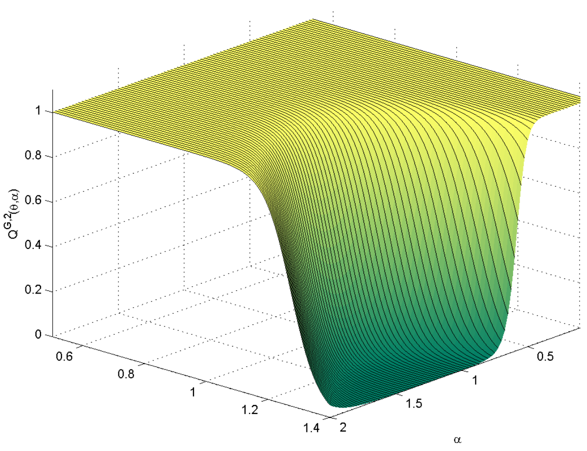


Figure 6: Characteristics for groups $3, \ldots, 10$ and $\varepsilon_2 = 0.24$

Robustness consideration for testing for late failures

Again we consider the influences of parameter α if α changes to a new parameter $\tilde{\alpha} \neq \alpha$. The procedure to compute the characteristics is the same like in the test for early failures. Hence we only present the corresponding results in figure 7 and 8.



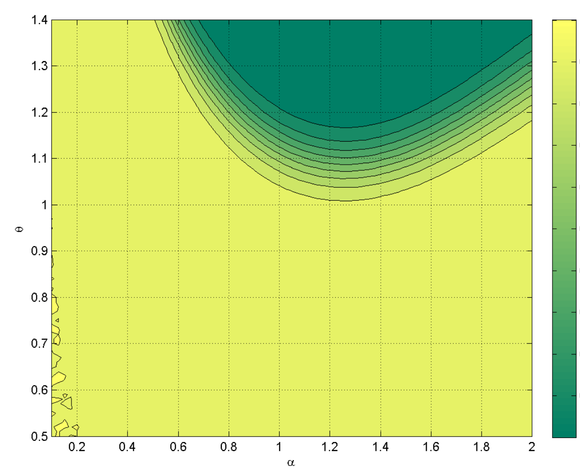


Figure 7: Two-dimensional OC-function and contour linesfor $m = 10$ and $\varepsilon_2 = 0.24$

The pictures of the OC-function correspond to the pictures of the previous test, but one can see the differences between both tests because of different hypotheses. Especially the projection of the OC-function in the $(\alpha\theta)-$ level shows these differences. The ASN-function reveals the same effect. Some values of the characteristics are presented in table 3.

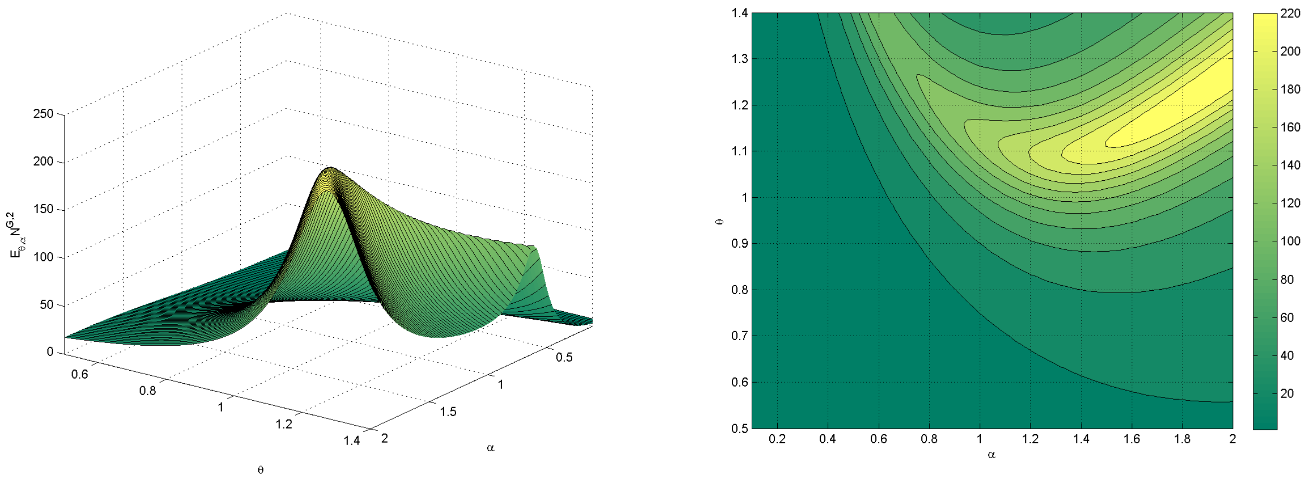


Figure 8: Two-dimensional ASN-function and contour linesfor $m = 10$ and $\varepsilon_2 = 0.24$

$\tilde{\alpha}$	0.25	0.5	0.75	**1**	1.25	1.5	1.75	2
$Q^{G,2}(\theta_0, \tilde{\alpha})$	1	1	0.9970	**0.9636**	0.9201	0.9581	0.9919	0.9990
$Q^{G,2}(\theta_1, \tilde{\alpha})$	1	0.9965	0.5551	**0.0458**	0.0146	0.0342	0.1957	0.6920
$E_{\theta_0,\tilde{\alpha}} N^{G,2}$	3.8712	11.4525	31.1848	**74.6657**	121.612	122.5196	91.8208	66.3865
$E_{\theta_1,\tilde{\alpha}} N^{G,2}$	76.2761	28.1285	117.6429	**99.6438**	97.4744	129.6633	203.9023	225.1246

Table 3: Probabilities for accepting hypotheses $H_0^{(2)}$ and $H_1^{(2)}$ for different values of $\tilde{\alpha}$

3 Sobel-Wald test

In practice there are many situations where we have to decide between more than two hypotheses. That is why this chapter deals with the Sobel-Wald test for discriminating between three hypotheses based on grouped observations. Again our aim is to calculate the OC- and the ASN-function. This can be done for the OC-function, easily by means of the OC-functions of the SPRTs of the previous section, see [6]. Unfortunately for the average sample number function such formulas do not exist. In the sequel we introduce a special algorithm to calculate these characteristics.

3.1 General description

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with a given density function $f_\theta(x), \theta \in \Theta$, with respect to a measure μ and range $\mathcal{X}$. Our aim is to discriminate between the three simple hypotheses

$$H_0 : \theta = \theta_0, \quad H_1 : \theta = \theta_1 \quad \text{and} \quad H_2 : \theta = \theta_2 \tag{17}$$

with $\theta_0,\ \theta_1,\ \theta_2 \in \Theta, \quad \theta_0 < \theta_1 < \theta_2,$. According to the Sobel-Wald test two WALD SPRTs $S(b_i, a_i),\ i = 1, 2$, according to section 2.1 are considered for discriminating between the hypotheses

$$H_{i-1} : \theta = \theta_{i-1} \quad \text{and} \quad H_i : \theta = \theta_i \qquad \theta_{i-1},\ \theta_i, \in \Theta, \quad \theta_{i-1} < \theta_i \qquad i = 1, 2$$

with $b_i < a_i,\ i = 1, 2$. The appropriate likelihood ratios for test $S(b_i, a_i),\ i = 1, 2$, are given by

$$L_{n,\theta_{i-1},\theta_i} = \prod_{j=1}^{n} \frac{f_{\theta_i}(X_j)}{f_{\theta_{i-1}}(X_j)}, \qquad i = 1, 2,$$

or

$$Z_{n,\theta_{i-1},\theta_i} = \ln L_{n,\theta_{i-1},\theta_i} = \sum_{j=1}^{n} \ln \frac{f_{\theta_i}(X_j)}{f_{\theta_{i-1}}(X_j)} = \sum_{j=1}^{n} Y_j^{(i)},$$

respectively, with the critical inequalities

$$B_i < L_{n,\theta_{i-1},\theta_i} < A_i \Leftrightarrow b_i < Z_{n,\theta_{i-1},\theta_i} < a_i, \qquad i = 1, 2,$$

where $a_i = \ln A_i$ and $b_i = \ln B_i$ for $i = 1, 2,$. The sample sizes of these tests $S(b_i, a_i),\ i = 1, 2$, as well as the decision rules are given by

$$N_i = \inf\{n \geq 1 : Z_{n,\theta_{i-1},\theta_i} \notin (b_i, a_i)\} \qquad \text{and} \qquad \delta_i = \mathbf{1}_{\{Z_{N,\theta_{i-1},\theta_i} \leq b_i\}} \qquad \text{for } i = 1, 2.$$

Now we define the Sobel-Wald test as follows.

Definition 3.1. (Sobel-Wald test)
Tests $S(b_1, a_1)$ and $S(b_2, a_2)$ are performed simultaneously by observing $X_1, X_2, \ldots$ On each observation stage $n = 1, 2, \ldots$ we decide between the following options:

(i) Acceptance of hypothesis H_0 if and only if $S(b_1, a_1)$ accepts H_0.

(ii) The test will accept hypothesis H_1 if and only if test $S(b_1, a_1)$ accepts hypothesis H_1 after acceptance of this hypothesis by test $S(b_2, a_2)$ or if test $S(b_2, a_2)$ accepts H_1 after acceptance of H_1 by test $S(b_1, a_1)$.

(iii) Acceptance of hypothesis H_2 if and only if $S(b_2, a_2)$ accepts H_2.

(iv) Continue testing by observing X_{n+1} if none of (i)-(iii) is true.

This test is called a Sobel-Wald test $S(b_1, b_2, a_1, a_2)$.

Test $S(b_1, b_2, a_1, a_2)$ for instance may accept hypothesis H_1 if the test statistic exceeds the boundary b_2 and later our test statistic exceeds the boundary a_1, see figure 9. Once the test statistic drops below b_2 test $S(b_2, a_2)$ is finished and only test $S(b_1, a_1)$ has to discriminate between hypotheses H_0 and H_1. In this case hypothesis H_2 cannot be accepted by the Sobel-Wald test $S(b_1, b_2, a_1, a_2)$ anymore. If the test statistic exceeds a_1, hypothesis H_1 will be accepted. This holds also vice versa.
Figure 9 shows the regions of acceptance of test $S(b_1, b_2, a_1, a_2)$ for the standard situation where we can transform the likelihood ratios into a sum of random variables X_i. If we apply the Weibull distribution to the Sobel-Wald test, then we cannot transform the likelihood ratios in this manner. So we do not have a standard situation here.

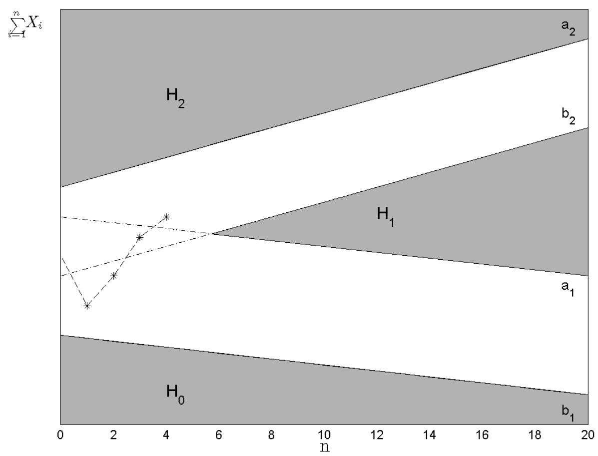


Figure 9: Regions of acceptance of a Sobel-Wald test in the standard situation

Let N denote the sample size of Sobel-Wald tests $S(b_1, b_2, a_1, a_2)$ and N_1 and N_2 the sample sizes of test $S(b_1, a_1)$ and $S(b_2, a_2)$, respectively. Then

$$N = \max\{N_1, N_2\}$$

is valid. From this equation it follows for every observation stage $n \geq 1$ and $\theta_0 < \theta_1 < \theta_2$ that

$$P_\theta(N > n) = P_\theta(N_1 > n) + P_\theta(N_2 > n) - P_\theta(\{N_1 > n\} \cap \{N_2 > n\})$$

holds. If both SPRTs $S(b_1, a_1)$ and $S(b_2, a_2)$ are closed, test $S(b_1, b_2, a_1, a_2)$ will be closed as well and all moments of the sample size N will be finite (see [5], p.259).

Theorem 3.1. *We assume, that the tests $S(b_1, a_1)$ and $S(b_2, a_2)$ are closed and some compatibility conditions are fulfilled, so that we have our standard situation. Then the OC-functions of the Sobel-Wald test $S(b_1, b_2, a_1, a_2)$ satisfy for $\theta_0 < \theta_1 < \theta_2$ the relations*

$$\begin{aligned} Q_0(\theta) &= Q^{(1)}(\theta), \\ Q_1(\theta) &= Q^{(2)}(\theta) - Q^{(1)}(\theta), \\ Q_2(\theta) &= 1 - Q^{(2)}(\theta), \end{aligned}$$

and

$$Q_0(\theta_0) + Q_1(\theta_1) + Q_2(\theta_2) = 1,$$

where $Q^{(i)}(\theta)$ *denotes the OC-function of test* $S(b_i, a_i)$, $i = 1, 2$.

Proof. See [5] on p. 259f. □

Hence, the calculation of the OC-functions of the Sobel-Wald test $S(b_1, b_2, a_1, a_2)$ according to (18) is quite simple, if one knows the OC-functions of the SPRTs $S(b_1, a_1)$ and $S(b_2, a_2)$ and if some compatibility conditions are fulfilled. For the ASN-function such identities do not exist. So we present now a special algorithm for computing the ASN-function.

3.2 Determination of the characteristics by discretization

As mentioned above we will now present an algorithm to compute the average sample number. This algorithm can also compute the OC-function of the Sobel-Wald test. Another method is mentioned in [7] for Erlang distributed random variables.
The algorithm in this section is based on the random variables $Y_1^{G,i}$, $i = 1, 2$, which are transformed according to the discretization method shown in section 2 in integer-valued random variables $\tilde{Y}_1^{G,i}$, $i = 1, 2$. The interval $(0, s_i)$ corresponds to interval (b_i, a_i), $i = 1, 2$. By composing these two tests, one gets a grid, which is shown in figure 10.
Denote by

$$G = \{(i, j) \in \Gamma^2 : 0 \le i \le s_1, 0 \le j \le s_2\}.$$

the set of the corresponding grid points. Because both random variables $\tilde{Y}_1^{G,1}$ and $\tilde{Y}_1^{G,2}$ depend on the same population X, we can determine the group probabilities $p_\theta(k)$ by

$$p_\theta(k) := \tilde{p}_\theta^{(1)}(k) = \tilde{p}_\theta^{(2)}(k) \quad k = 0, \pm 1, \pm 2, \dots$$

and the random variables $\tilde{Y}_1^{G,1}$ and $\tilde{Y}_1^{G,2}$ have the form

$$\tilde{Y}_1^{G,1} = \begin{cases} \tilde{y}_1^{G,1} \text{ with } \tilde{p}_\theta(1) \\ \tilde{y}_2^{G,1} \text{ with } \tilde{p}_\theta(2) \\ \dots \\ \tilde{y}_m^{G,1} \text{ with } \tilde{p}_\theta(m) \end{cases} \qquad \tilde{Y}_1^{G,2} = \begin{cases} \tilde{y}_1^{G,2} \text{ with } \tilde{p}_\theta(1) \\ \tilde{y}_2^{G,2} \text{ with } \tilde{p}_\theta(2) \\ \dots \\ \tilde{y}_m^{G,2} \text{ with } \tilde{p}_\theta(m). \end{cases}$$

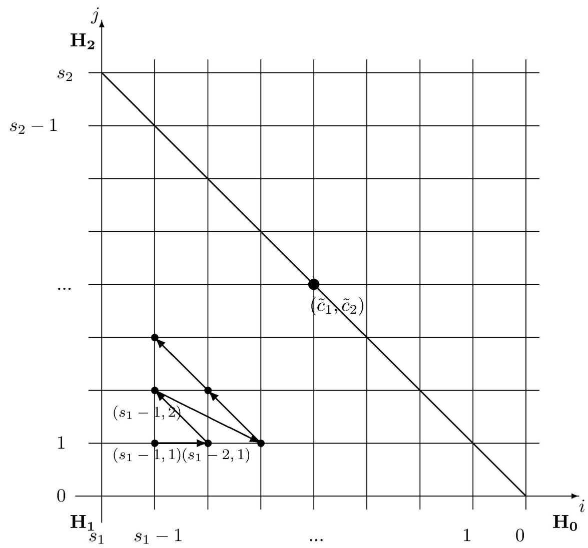


Figure 10: Discretization

We assume

$$\tilde{y}_i^{G,1} \geq \tilde{y}_i^{G,2} \qquad \text{for } i = 1, \ldots, m. \tag{19}$$

If $\tilde{y}_i^{G,1} = \tilde{y}_i^{G,2}$ holds we suppose

$$\tilde{y}_i^{G,o} < 0 \qquad \text{for } i = 1, \ldots, m \;\text{ and }\; o = 1, 2.$$

Let be $S(b_1, a_1)$ and $S(b_2, a_2)$ two generalized SPRTs, introduced in section 2, which starts on stage 0 in point $x \in (-\infty, \infty)$ and $y \in (-\infty, \infty)$, respectively. Analogous to the theory of SPRTs we define a generalized Sobel-Wald test that starts in point (x, y). After discretization both SPRTs, they start on stage 0 in point $\tilde{c}_1 \in (0, s_1)$ and $\tilde{c}_2 \in (0, s_2)$, respectively. Then the generalized Sobel-Wald test is transformed into a discretized and generalized Sobel-Wald test with the integer-valued starting point $(\tilde{c}_1, \tilde{c}_2)$. We assume that this point lies on or below the diagonal, respectively, in our grid (see figure 10). Analog to section 2.1 we consider a discretized and generalized Sobel-Wald test which starts on stage 0 in point (i, j) with $i \in (0, s_1)$ and $j \in (0, s_2)$.

Definition 3.2. *The Sobel-Wald test* $S(b_1, b_2, a_1, a_2)$ *for discriminating between the hypotheses* (17) *which starts on observation stage* 0 *in point* (x, y), $x, y \in (-\infty, \infty)$, *is called* **generalized Sobel-Wald test**

$$\mathbf{S_{xy}(b_1, b_2, a_1, a_2)}.$$

Let denote $q^{(l)}_{\theta,xy}$, $l = 0, 1, 2$, the probabilities of accepting hypothesis H_l by Sobel-Wald test $S_{xy}(b_1, b_2, a_1, a_2)$ and $e_{\theta,xy} = E_\theta N_{xy}$ the average sample number with the sample size N_{xy} of the Sobel-Wald test $S_{xy}(b_1, b_2, a_1, a_2)$. Because of discretization, shown in section 2.1, we get a **discretized and generalized Sobel-Wald test $\mathbf{S_{ij}(b_1, b_2, a_1, a_2)}$** *which starts on stage 0 in point (i, j), $i \in (0, s_1)$, $j \in (0, s_2)$.*
The decision rules for accepting hypothesis H_l, $l = 0, 1, 2$, by Sobel-Wald test $S_{ij}(b_1, b_2, a_1, a_2)$ is denoted by $\delta^{(l)}_{ij}$ and the corresponding OC-functions by $q^{(l)}_{\theta,ij} = E_\theta \delta^{(l)}_{ij}$. Additionally let N_{ij} be the sample size and $e_{\theta,ij} = E_\theta N_{ij}$ the average sample number of test $S_{ij}(b_1, b_2, a_1, a_2)$.

The Sobel-Wald test $S(b_1, b_2, a_1, a_2)$, which we are interested in, is a specialized, generalized Sobel-Wald test and corresponds to test $S_{00}(b_1, b_2, a_1, a_2)$. Hence we only consider discretized and generalized Sobel-Wald tests and donate them with S_{ij}. This test can be interpreted as a two-dimensional random walk in the grid, shown in figure 10, which starts in point $(\tilde{c}_1, \tilde{c}_2)$ and is dependent on the values $\tilde{y}^{G,i}_r$, $i = 1, 2$, $r = 1, \ldots, m$. The test accepts the hypotheses (17) depending on where the random walk leaves the region G. Further we refer to the values $\tilde{y}^{G,i}_r$ as $\tilde{y}^{(i)}_r$ for $i = 1, 2$. We will see that we can start a corresponding algorithm in point $(s_1 - 1, 1)$ and that we can calculate then the characteristics stepwise till we reach our starting point $(\tilde{c}_1, \tilde{c}_2)$.

Computation of the OC-functions

For computing the OC-functions $q^{(o)}_{\theta,ij}, o = 0, 1, 2$, we need the values on the x- and the y-axis and suitable boundary conditions for both underlying SPRTs (6). The values on the x-axis and the y-axis are solutions of the linear systems of equations (7) of test $S(b_1, a_1)$ and $S(b_2, a_2)$, respectively. We denote them by the vectors

$$\tilde{q}^{(1)}_\theta = \begin{bmatrix} \tilde{q}^{(1)}_\theta(1) \\ \ldots \\ \tilde{q}^{(1)}_\theta(s_1 - 1) \end{bmatrix} \quad \text{and} \quad \tilde{q}^{(2)}_\theta = \begin{bmatrix} \tilde{q}^{(2)}_\theta(1) \\ \ldots \\ \tilde{q}^{(2)}_\theta(s_2 - 1) \end{bmatrix}.$$

Now we can formulate a theorem for computing the OC-functions.

Theorem 3.2. *Let be S_{ij} a discretized and generalized Sobel-Wald test with starting point (i, j) according to definition 3.2. Then we can compute the OC-functions $q^{(l)}_{\theta,ij}$, $l = 0, 1, 2$ under the constraints*

$$\begin{array}{llll} q^{(0)}_{\theta,ij} = \tilde{q}^{(1)}_\theta(i) & \textit{for} & 0 < i < s_1, & j \le 0, \\ q^{(0)}_{\theta,ij} = 0 & \textit{for} & i \ge s_1, & -\infty < j < \infty, \\ q^{(0)}_{\theta,ij} = 1 & \textit{for} & i \le 0, & -\infty < j < \infty, \end{array}$$

$$
\begin{array}{llll}
q^{(1)}_{\theta,ij} = \tilde{q}^{(2)}_{\theta}(j) & \textit{for} & i \geq s_1, & 0 < j < s_2, \\
q^{(1)}_{\theta,ij} = 1 - \tilde{q}^{(1)}_{\theta}(i) & \textit{for} & 0 < i < s_1, & j \leq 0, \\
q^{(1)}_{\theta,ij} = 0 & \textit{for} & i \leq 0, & -\infty < j < \infty \\
 & \textit{or} & -\infty < i < \infty, & j \geq s_2, \\
q^{(1)}_{\theta,ij} = 1 & \textit{for} & i \geq s_1, & j \leq 0
\end{array}
$$

$$
\textit{and} \quad
\begin{array}{llll}
q^{(2)}_{\theta,ij} = 1 - \tilde{q}^{(2)}_{\theta}(j) & \textit{for} & i \geq s_1, & 0 < j < s_2, \\
q^{(2)}_{\theta,ij} = 0 & \textit{for} & -\infty < i < \infty, & j \leq 0, \\
q^{(2)}_{\theta,ij} = 1 & \textit{for} & -\infty < i < \infty, & j \geq s_2
\end{array}
$$

by

$$
q^{(0)}_{\theta,ij} = \sum_{r=1}^{m} q^{(0)}_{\theta,i+\tilde{y}^{(1)}_r,j+\tilde{y}^{(2)}_r} \tilde{c}^{\theta}_{ii+\tilde{y}^{(1)}_r,jj+\tilde{y}^{(2)}_r}, \tag{20}
$$

$$
q^{(1)}_{\theta,ij} = \sum_{r=1}^{m} q^{(1)}_{\theta,i+\tilde{y}^{(1)}_r,j+\tilde{y}^{(2)}_r} \tilde{c}^{\theta}_{ii+\tilde{y}^{(1)}_r,jj+\tilde{y}^{(2)}_r}, \tag{21}
$$

$$
q^{(2)}_{\theta,ij} = \sum_{r=1}^{m} q^{(2)}_{\theta,i+\tilde{y}^{(1)}_r,j+\tilde{y}^{(2)}_r} \tilde{c}^{\theta}_{ii+\tilde{y}^{(1)}_r,jj+\tilde{y}^{(2)}_r}. \tag{22}
$$

Here $\tilde{c}^{\theta}_{ii+\tilde{y}^{(1)}_r,jj+\tilde{y}^{(2)}_r}$ *denotes the "transition probability" from point* (i,j) *to point* $(i+\tilde{y}^{(1)}_r, j+\tilde{y}^{(2)}_r)$*. With group probabilities* $p_\theta(r)$ *it holds*

$$
\tilde{c}^{\theta}_{ii+\tilde{y}^{(1)}_r,jj+\tilde{y}^{(2)}_r} = p_\theta(r).
$$

Proof. First we show equation (20). This proof is analogue to the proof of theorem 2.1 which one can see in [3]. Let δ^l_{ij}, $l = 0,1,2$ be the decision rules. The recursion equation for determining the OC-function can be shown by means of the law of total probability

$$
\begin{aligned}
q^{(0)}_{\theta,ij} &= E_\theta \delta^{(0)}_{ij} = E_\theta \left[E_\theta \left(\delta^{(0)}_{ij} | X^G_1 \right) \right] = \sum_{r=1}^{m} E_\theta \left(\delta^{(0)}_{ij} | X^G_1 = r \right) P_\theta(X^G_1 = r) \\
&= \sum_{r=1}^{m} q_{\theta,i+\tilde{y}^{(1)}_r,j+\tilde{y}^{(2)}_r} P_\theta(X^G_1 = r).
\end{aligned}
$$

A transition from point (i,j) to point $(i+\tilde{y}^{(1)}_r, j+\tilde{y}^{(2)}_r)$ follows from condition $\{X^G_1 = r\}$ and the probability for accepting hypothesis H_0 in point $(i+\tilde{y}^{(1)}_r, j+\tilde{y}^{(2)}_r)$ is

$$
q^{(0)}_{\theta,i+\tilde{y}^{(1)}_r,j+\tilde{y}^{(2)}_r} = E_\theta \delta^{(0)}_{i+\tilde{y}^{(1)}_r,j+\tilde{y}^{(2)}_r}.
$$

The transition probabilities are $\tilde{c}^{\theta}_{ii+\tilde{y}_r^{(1)},jj+\tilde{y}_r^{(2)}} = P_\theta(X_1^G = r) = p_\theta(r)$.
Equations (21) and (22) can be shown in an analogous way. □

Now we consider condition (19) again. If for one $uin\{1,\ldots,m\}$ equation $\tilde{y}_u^{(1)} = \tilde{y}_u^{(2)} = 0$ holds, we have to compute the OC-functions as follows

$$q^{(0)}_{\theta,ij} = \frac{1}{1-\tilde{c}^{\theta}_{ii+\tilde{y}_u^{(1)},jj+\tilde{y}_u^{(2)}}} \left[\sum_{r=1}^{u-1} q^{(0)}_{\theta,i+\tilde{y}_r^{(1)},j+\tilde{y}_r^{(2)}} \tilde{c}^{\theta}_{ii+\tilde{y}_r^{(1)},jj+\tilde{y}_r^{(2)}} + \sum_{r=u+1}^{m} q^{(0)}_{\theta,i+\tilde{y}_r^{(1)},j+\tilde{y}_r^{(2)}} \tilde{c}^{\theta}_{ii+\tilde{y}_r^{(1)},jj+\tilde{y}_r^{(2)}}\right]$$
$$q^{(1)}_{\theta,ij} = \frac{1}{1-\tilde{c}^{\theta}_{ii+\tilde{y}_u^{(1)},jj+\tilde{y}_u^{(2)}}} \left[\sum_{r=1}^{u-1} q^{(1)}_{\theta,i+\tilde{y}_r^{(1)},j+\tilde{y}_r^{(2)}} \tilde{c}^{\theta}_{ii+\tilde{y}_r^{(1)},jj+\tilde{y}_r^{(2)}} + \sum_{r=u+1}^{m} q^{(1)}_{\theta,i+\tilde{y}_r^{(1)},j+\tilde{y}_r^{(2)}} \tilde{c}^{\theta}_{ii+\tilde{y}_r^{(1)},jj+\tilde{y}_r^{(2)}}\right]$$
$$q^{(2)}_{\theta,ij} = \frac{1}{1-\tilde{c}^{\theta}_{ii+\tilde{y}_u^{(1)},jj+\tilde{y}_u^{(2)}}} \left[\sum_{r=1}^{u-1} q^{(2)}_{\theta,i+\tilde{y}_r^{(1)},j+\tilde{y}_r^{(2)}} \tilde{c}^{\theta}_{ii+\tilde{y}_r^{(1)},jj+\tilde{y}_r^{(2)}} + \sum_{r=u+1}^{m} q^{(2)}_{\theta,i+\tilde{y}_r^{(1)},j+\tilde{y}_r^{(2)}} \tilde{c}^{\theta}_{ii+\tilde{y}_r^{(1)},jj+\tilde{y}_r^{(2)}}\right].$$

If $\tilde{y}_u^{(1)} = \tilde{y}_u^{(2)} = 0$, $u \in \{1,\ldots,m\}$ holds for more than one u, we must modify the equations for computing the OC-function again.
According to equation (9) we can determine the OC-functions of the Sobel-Wald test $S(b_1,b_2,a_1,a_2)$ for the hypotheses H_0, H_1 and H_2 by

$$Q_0(\theta) \approx q^{(0)}_{\theta,\tilde{c}_1,\tilde{c}_2}, \qquad Q_1(\theta) \approx q^{(1)}_{\theta,\tilde{c}_1,\tilde{c}_2} \qquad \text{and} \qquad Q_2(\theta) \approx q^{(2)}_{\theta,\tilde{c}_1,\tilde{c}_2}$$

with $\tilde{c}_i = \left(-\frac{b_i}{h_i}\right)$, $i = 1,2$.

Computation of the ASN-function

For computing the average sample number function of our Sobel-Wald test, we have to know the ASN-functions of tests $S(b_1,a_1)$ and $S(b_2,a_2)$ on the x- and y-axis, respectively. Let the solutions of the linear system of equations (10) be

$$\tilde{e}^{(1)}_\theta = \begin{bmatrix} \tilde{e}^{(1)}_\theta(1) \\ \ldots \\ \tilde{e}^{(1)}_\theta(s_1-1) \end{bmatrix} \qquad \text{and} \qquad \tilde{e}^{(2)}_\theta = \begin{bmatrix} \tilde{e}^{(2)}_\theta(1) \\ \ldots \\ \tilde{e}^{(2)}_\theta(s_2-1) \end{bmatrix}.$$

Then we can establish a corresponding theorem for the average sample number function.

Theorem 3.3. *Let S_{ij} be a discretized and generalized Sobel-Wald test with starting point (i,j) and sample number N_{ij} according to definition 3.2. Then we obtain the ASN-function under the boundary conditions*

$$
\begin{array}{llll}
e_{\theta,ij} = 0 & \mathit{for} & i \le 0, & -\infty < j < \infty, \\
 & & \mathit{or} & -\infty < i < \infty, j \ge s_2, \\
 & & \mathit{or} & i \ge s_1, j \le 0, \\
e_{\theta,ij} = \tilde{e}_\theta^{(1)}(i) & \mathit{for} & 0 < i < s_1, & j \le 0, \\
e_{\theta,ij} = \tilde{e}_\theta^{(2)}(j) & \mathit{for} & i \ge s_2, & 0 < j < s_2,
\end{array}
$$

by

$$e_{\theta,ij} = 1 + \sum_{r=1}^{m} e_{\theta,i+\tilde{y}_r^{(1)} j+\tilde{y}_r^{(2)}} \tilde{c}^\theta_{ii+\tilde{y}_r^{(1)},jj+\tilde{y}_r^{(2)}}$$

with transition probabilities $\tilde{c}^\theta_{ii+\tilde{y}_r^{(1)},jj+\tilde{y}_r^{(2)}} = p_\theta(r)$ *from point* (i,j) *to point* $(i+\tilde{y}_r^{(1)}, j+\tilde{y}_r^{(2)})$.

Proof. Because of the law of total probability we can compute the average sample number as follows:

$$
\begin{aligned}
e_{\theta,ij} &= E_\theta N_{ij} = E_\theta \left[E_\theta \left(N_{ij} | X_1^G \right) \right] = \sum_{r=1}^{m} E_\theta \left(N_{ij} | X_1^G = r \right) P_\theta (X_1^G = r) \\
&= \sum_{r=1}^{m} E_\theta \left(1 + N_{i+\tilde{y}_r^{(1)},j+\tilde{y}_r^{(2)}} \right) P_\theta (X_1^G = r) = 1 + \sum_{r=1}^{m} E_\theta \left(N_{i+\tilde{y}_r^{(1)},j+\tilde{y}_r^{(2)}} \right) P_\theta (X_1^G = r) \\
&= 1 + \sum_{r=1}^{m} e_{\theta,i+\tilde{y}_r^{(1)},j+\tilde{y}_r^{(2)}} P_\theta (X_1^G = r).
\end{aligned}
$$

By condition $\{X_1^G = r\}$ we have a transition from point (i,j) to point $(i+\tilde{y}_r^{(1)}, j+\tilde{y}_r^{(2)})$ and the average sample number in point $(i+\tilde{y}_r^{(1)}, j+\tilde{y}_r^{(2)})$ is

$$e_{\theta,i+\tilde{y}_r^{(1)},j+\tilde{y}_r^{(2)}} = E_\theta N_{i+\tilde{y}_r^{(1)},j+\tilde{y}_r^{(2)}}.$$

The transition probabilities are again $\tilde{c}^\theta_{ii+\tilde{y}_r^{(1)},jj+\tilde{y}_r^{(2)}} = P_\theta(X_1^G = r) = p_\theta(r)$. □

If equation $\tilde{y}_u^{(1)} = \tilde{y}_u^{(2)} = 0$, $u \in \{1, \dots, m\}$ holds for one u, we have to modify our formulas and get

$$e_{\theta,ij} = \frac{1}{1 - \tilde{c}^\theta_{ii+\tilde{y}_u^{(1)},jj+\tilde{y}_u^{(2)}}} \left[1 + \sum_{r=1}^{u-1} e_{\theta,i+\tilde{y}_r^{(1)},j+\tilde{y}_r^{(2)}} \tilde{c}^\theta_{ii+\tilde{y}_r^{(1)},jj+\tilde{y}_r^{(2)}} + \sum_{r=u+1}^{m} e_{\theta,i+\tilde{y}_r^{(1)},j+\tilde{y}_r^{(2)}} \tilde{c}^\theta_{ii+\tilde{y}_r^{(1)},jj+\tilde{y}_r^{(2)}} \right].$$

The average sample number can be determined in line with (11) by

$$E_\theta N = e_{\theta,\tilde{c}_1,\tilde{c}_2}.$$

Algorithm for computing the OC-functions and the ASN-function

Under the conditions for the random variables and for the point $(\tilde{c}_1, \tilde{c}_2)$, as mentioned above, we start the algorithm in point $(s_1-1, 1)$ and calculate the OC-functions $q^{(l)}_{\theta,s_1-1,1}$, $l = 0, 1, 2$, as well as the average sample number $e_{\theta,s_1-1,1}$ in this point. After that we compute the characteristics in point $(s_1-2, 1)$ and so on. The order of computation is shown in figure 10. Because of equation (9) holds for both SPRTs, we have to finish our algorithm in point $(\tilde{c}_1, \tilde{c}_2)$. For each point $(i + \tilde{y}^{(1)}_r, j + \tilde{y}^{(2)}_r)$ we have to distinguish between the following cases.

(1) $\boxed{s_1 - k + \tilde{y}^{(1)}_r < s_1 \text{ and } l + \tilde{y}^{(2)}_r > 0}$

Because (19) holds, only points below the diagonal (see figure 10) the test of $0 < s_1 - k + \tilde{y}^{(1)}_r$ and $l + \tilde{y}^{(2)}_r > s_2$ can be neglected. If above condition holds, the random walk will reach an inner point of the grid and the values of the unknowns $q^{(o)}_{i+\tilde{y}^{(1)}_r, j+\tilde{y}^{(2)}_r}$, $o = 0, 1, 2$ and $e_{\theta,i+\tilde{y}^{(1)}_r j+\tilde{y}^{(2)}_r}$ are

$$\begin{aligned}
q^{(0)}_{i+\tilde{y}^{(1)}_r, j+\tilde{y}^{(2)}_r} &= q^{(0)}_{\theta,s_1-(l+\tilde{y}^{(2)}_r),k-\tilde{y}^{(1)}_r}, \\
q^{(1)}_{i+\tilde{y}^{(1)}_r, j+\tilde{y}^{(2)}_r} &= q^{(1)}_{\theta,s_1-(l+\tilde{y}^{(2)}_r),k-\tilde{y}^{(1)}_r}, \\
q^{(2)}_{i+\tilde{y}^{(1)}_r, j+\tilde{y}^{(2)}_r} &= q^{(2)}_{\theta,s_1-(l+\tilde{y}^{(2)}_r),k-\tilde{y}^{(1)}_r}, \\
e_{\theta,i+\tilde{y}^{(1)}_r j+\tilde{y}^{(2)}_r} &= e_{\theta,s_1-(l+\tilde{y}^{(2)}_r),k-\tilde{y}^{(1)}_r}.
\end{aligned}$$

(2) $\boxed{s_1 - k + \tilde{y}^{(1)}_r \geq s_1}$

The random walk crosses the y-axis which terminates test $S(b_1, a_1)$ and hypothesis H_0 cannot be accepted anymore. That means $q^{(0)}_{i+\tilde{y}^{(1)}_r, j+\tilde{y}^{(2)}_r} = 0$. For the other OC-functions only test $S(b_2, a_2)$ is decisive where three cases has are possible:

(a) $l + \tilde{y}^{(2)}_r \geq s_2$:

$$q^{(1)}_{i+\tilde{y}^{(1)}_r, j+\tilde{y}^{(2)}_r} = 0, \quad q^{(2)}_{i+\tilde{y}^{(1)}_r, j+\tilde{y}^{(2)}_r} = 1, \quad e_{\theta,i+\tilde{y}^{(1)}_r j+\tilde{y}^{(2)}_r} = 0,$$

(b) $l + \tilde{y}^{(2)}_r \leq 0$:

$$q^{(1)}_{i+\tilde{y}^{(1)}_r, j+\tilde{y}^{(2)}_r} = 1, \quad q^{(2)}_{i+\tilde{y}^{(1)}_r, j+\tilde{y}^{(2)}_r} = 0, \quad e_{\theta,i+\tilde{y}^{(1)}_r j+\tilde{y}^{(2)}_r} = 0,$$

(c) $0 < l + \tilde{y}^{(2)}_r < s_2$:

$$\begin{aligned}
q^{(1)}_{i+\tilde{y}^{(1)}_r, j+\tilde{y}^{(2)}_r} &= \tilde{q}^{(2)}_\theta(l + \tilde{y}^{(2)}_r), \quad q^{(2)}_{i+\tilde{y}^{(1)}_r, j+\tilde{y}^{(2)}_r} = 1 - \tilde{q}^{(2)}_\theta(l + \tilde{y}^{(2)}_r), \\
e_{\theta,i+\tilde{y}^{(1)}_r j+\tilde{y}^{(2)}_r} &= \tilde{e}^{(2)}_\theta(l + \tilde{y}_r).
\end{aligned}$$

(3) $\boxed{l+\tilde{y}_r^{(2)} \le 0}$

In this case test $S(b_2, a_2)$ is stopped because the random walk crosses the x-axis. Then hypothesis H_2 cannot be accepted anymore. So we obtain $q^{(2)}_{i+\tilde{y}_r^{(1)}, j+\tilde{y}_r^{(2)}} = 0$. Again we have to distinguish between three cases where only test $S(b_1, a_1)$ is decisive.

(a) $s_1 - k + \tilde{y}_r^{(1)} \ge s_1$:

$$q^{(0)}_{i+\tilde{y}_r^{(1)}, j+\tilde{y}_r^{(2)}} = 0, \quad q^{(1)}_{i+\tilde{y}_r^{(1)}, j+\tilde{y}_r^{(2)}} = 1, \quad e_{\theta, i+\tilde{y}_r^{(1)} j+\tilde{y}_r^{(2)}} = 0,$$

(b) $s_1 - k + \tilde{y}_r^{(1)} \le 0$:

$$q^{(0)}_{i+\tilde{y}_r^{(1)}, j+\tilde{y}_r^{(2)}} = 1, \quad q^{(1)}_{i+\tilde{y}_r^{(1)}, j+\tilde{y}_r^{(2)}} = 0, \quad e_{\theta, i+\tilde{y}_r^{(1)} j+\tilde{y}_r^{(2)}} = 0,$$

(c) $0 < s_1 - k + \tilde{y}_r^{(1)} < s_1$:

$$\begin{aligned} q^{(0)}_{i+\tilde{y}_r^{(1)}, j+\tilde{y}_r^{(2)}} &= \tilde{q}_\theta^{(2)}(l+\tilde{y}_r^{(1)}), \quad q^{(1)}_{i+\tilde{y}_r^{(1)}, j+\tilde{y}_r^{(2)}} = 1 - \tilde{q}_\theta^{(2)}(l+\tilde{y}_r^{(1)}), \\ e_{\theta, i+\tilde{y}_r^{(1)} j+\tilde{y}_r^{(2)}} &= \tilde{e}_\theta^{(1)}(l+\tilde{y}_r). \end{aligned}$$

3.3 Testing for random failures for Weibull distributed observations

Let $X_1, X_2, \ldots$ be a sequence of Weibull distributed random variables with density function according to definition 2.1. For parameter α of the Weibull distribution we suppose $\alpha = 1$. We denote the parameter β of the Weibull distribution as θ. We consider a sequence of i.i.d. random variables $X_1, X_2, \ldots$ for discriminating between the hypotheses

$$H_0 : \theta = \theta_0 = 1 - \varepsilon_1, \quad H_1 : \theta = \theta_1 = 1 \quad \text{and} \quad H_2 : \theta = \theta_2 = 1 + \varepsilon_2$$

with $0 < \theta_0 < \theta_1 < \theta_2$ and $0 < \varepsilon_1 < 1$, $\varepsilon_2 > 0$. This test can be interpreted in two ways. On the one hand this test proves the parameter θ and decides between early failures, H_0, random failures, H_1, and late failures, H_2, and on the other hand we will be able to decide if an exponential distributed population is present, i.e. $\theta = 1$. Let $S(b_1, a_1)$ and $S(b_2, a_2)$ be tests according to section 2.2.1 and 2.2.2, respectively, with likelihood ratios Z_{n,θ_0,θ_1} and Z_{n,θ_1,θ_2} as follows

$$S(b_1, a_1): \; H_0: \theta = \theta_0 = 1 - \varepsilon_1, \qquad H_1: \theta = \theta_1 = 1 \qquad 0 < \varepsilon_1 < 1$$

$$Z_{n,\theta_0,\theta_1} = \sum_{i=1}^{n} Y_i^{(1)} = \sum_{i=1}^{n} -X_i + X_i^{1-\varepsilon_1} + \varepsilon_1 \ln X_i - \ln(1 - \varepsilon_1),$$

$$S(b_2, a_2): \; H_1: \theta = \theta_1 = 1, \qquad H_2: \theta = \theta_2 = 1 - \varepsilon_2 \qquad 0 < \varepsilon_2$$

$$Z_{n,\theta_1,\theta_2} = \sum_{i=1}^{n} Y_i^{(2)} = \sum_{i=1}^{n} X_i - X_i^{1+\varepsilon_2} + \varepsilon_2 \ln X_i + \ln(1 + \varepsilon_1).$$

Both tests are based on the same sequence of random variables X. Both tests are executed as seen in section 2 so that all conditions for the algorithm are fulfilled. Then we can determine the OC-functions $Q_0(\theta)$, $Q_1(\theta)$, $Q_2(\theta)$ and the ASN-function $E_\theta N$ of the Sobel-Wald test as described in section 3.2.

3.4 Example and robustness analysis

We want to illustrate the test for random failures on the basis of one example. Again we do not consider the case $m = 2$, because we cannot compute the OC-functions of the single SPRTs due to multiple intersections of the density function (see figure 2).

Example 2 (Continuation of example 1):

As mentioned before the Sobel-Wald test consists of two SPRTs. For this example we compose the test for early failures and the test for late failures (see example 1), i.e. $\varepsilon_1 = 0.2$ and $\varepsilon_2 = 0.24$. So the Sobel-Wald test based on the Weibull distribution discriminates between the hypotheses

$$H_0: \; \theta = \theta_0 = 0.8, \qquad H_1: \; \theta = \theta_1 = 1 \quad \text{and} \quad H_1: \theta = \theta_2 = 1.24.$$

With the error probabilities $\alpha = \beta = 0.05$ we construct an admissible test and choose the stopping bounds according to the WALDs approximations by $a_1 = a_2 = 2.94444$, and $b_1 = b_2 = -2.94444$. The discretization parameters are $s_1 = 500$ and $s_2 = 500$ and group bounds from table 1 are used. The condition (19) is fulfilled for the random variables $\tilde{Y}_1^{G,1}$ and $\tilde{Y}_1^{G,2}$. The Sobel-Wald test starts in point $(\tilde{c}_1, \tilde{c}_2) = (250, 250)$ because both single tests are starting in point $\tilde{c}_1 = 250$ and $\tilde{c}_2 = 250$, respectively. Table 4 shows some values of the OC-functions on points $\theta_0 = 0.8$, $\theta_1 = 1$ and $\theta_2 = 1.24$ as well as the average sample number on these points for $3 - 10$ groups. The characteristics are illustrated in picture 11.
The OC-functions for hypothesis H_2 are close together as we could see in figure 11 as well as in table 4 whereas the OC-functions for the hypotheses H_0 and H_1 show clear differences depending on the grouping. In contrast to this the average sample number is

m	3	4	5	6	7	8	9	10
$Q_0(\theta_0)$	0.9714	0.9240	0.9428	0.9624	0.9446	0.9307	0.9442	0.9464
$Q_1(\theta_1)$	0.8889	0.9302	0.9290	0.9024	0.9229	0.9297	0.9221	0.9308
$Q_2(\theta_2)$	0.9612	0.9594	0.9552	0.9670	0.9636	0.9618	0.9647	0.9542
$E_{\theta_0}N$	195.0462	151.0912	117.8287	96.3040	94.9525	92.6933	85.2092	81.1812
$E_{\theta_1}N$	334.1613	192.8083	164.6219	161.5785	139.7476	127.0308	127.2405	120.7312
$E_{\theta_2}N$	215.7308	155.9937	133.3472	110.4768	106.9654	102.0171	97.0260	100.4760

Table 4: Characteristics for $3, \ldots, 10$ groups on the points $\theta_0 = 0.8$, $\theta_1 = 1$ and $\theta_2 = 1.24$.

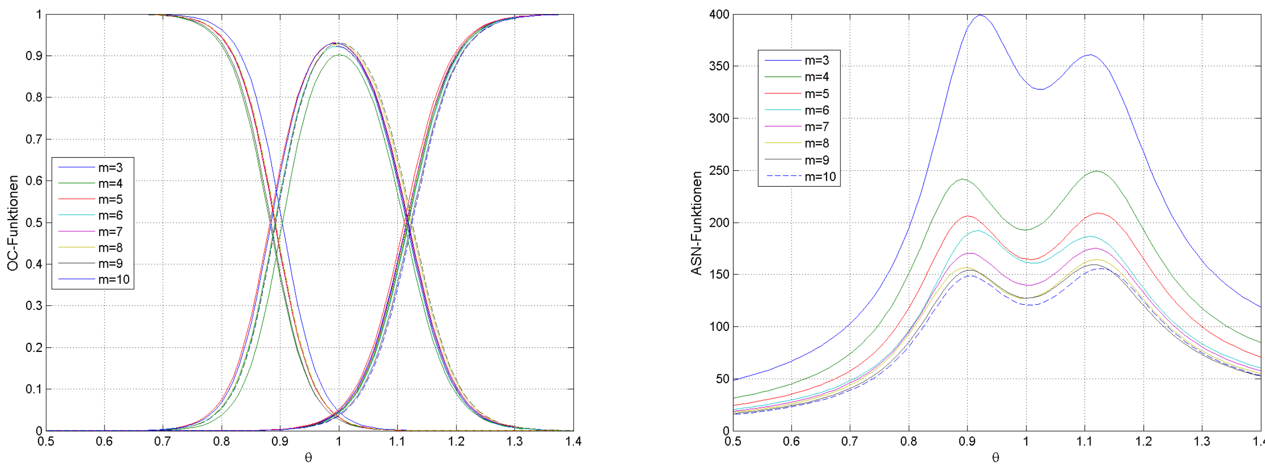


Figure 11: Characteristics for groups $3, \ldots, 10$ and $\varepsilon_1 = 0.2$ as well as $\varepsilon_2 = 0.24$

showing great differences between. If the number of groups increases, the average sample number will decrease drastically. This difference will be extremely large if one choose 4 classes instead of 3 groups whereas the differences will be not so large anymore for $8 - 10$ classes.

Robustness consideration for testing for random failures

Again we want to consider the robustness of the Sobel-Wald test against the impact of changes of the parameter value α to an unknown parameter value $\tilde{\alpha} \neq \alpha$ as in example 1. The group probabilities depending on $\tilde{\alpha} \neq \alpha$ are given from example 1. The random variables $\tilde{Y}_1^{G,1}$ and $\tilde{Y}_1^{G,2}$ remain unchanged. We execute the algorithm above for $\tilde{\alpha} = 0.025(0.025)2$ in order to compute the two-dimensional OC-functions $Q_l(\theta, \tilde{\alpha}), l = 0, 1, 2$ and the two-dimensional ASN-function $E_{\theta,\tilde{\alpha}}N$. The results are shown in figure 12-15 for ten classes.

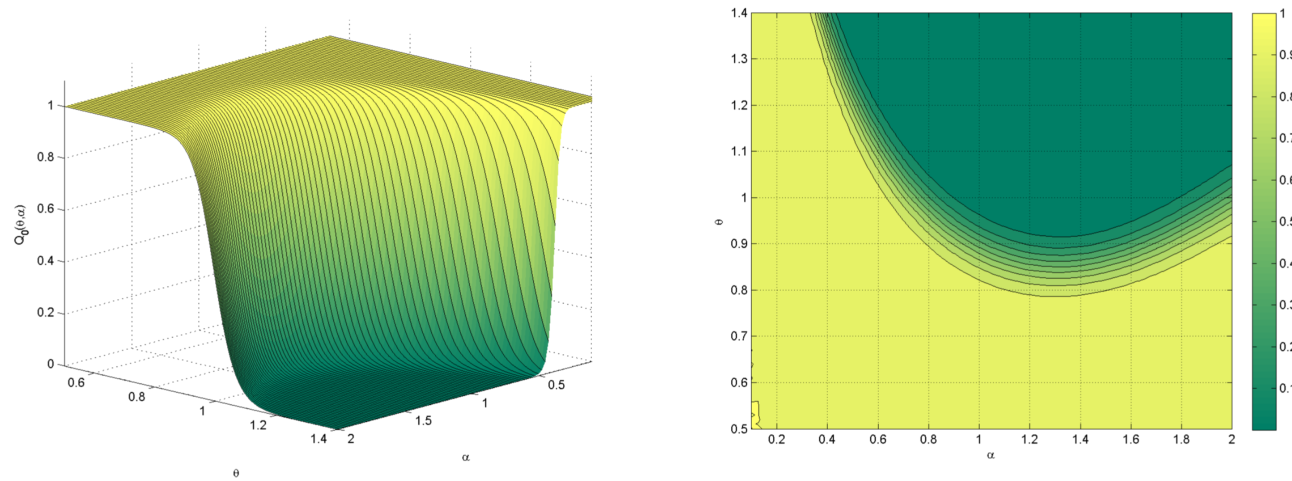


Figure 12: Two-dimensional OC-function for hypothesis H_0 and corresponding contour lines for $m = 10$

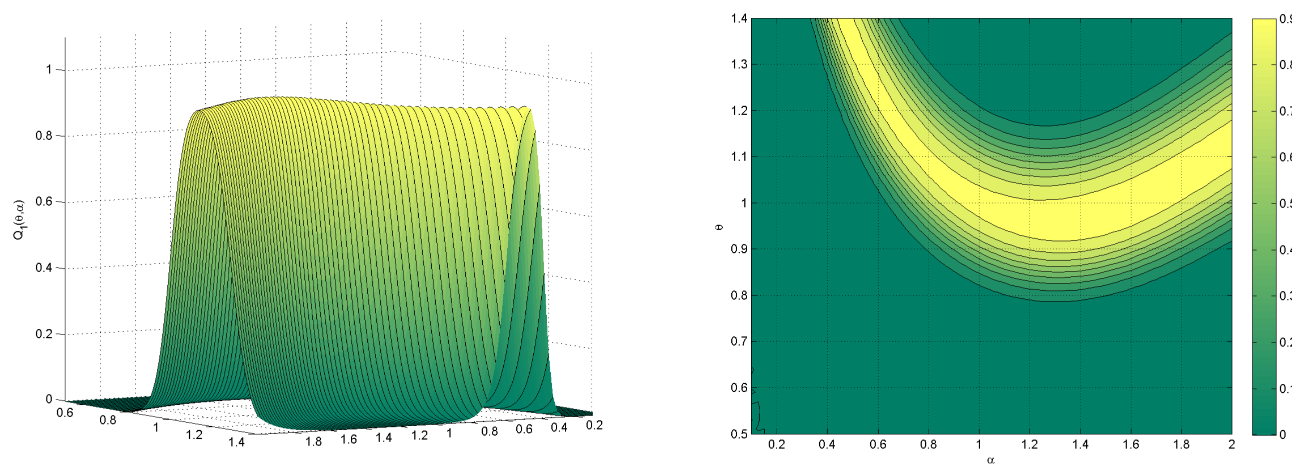


Figure 13: Two-dimensional OC-function for hypothesis H_1 and corresponding contour lines for $m = 10$

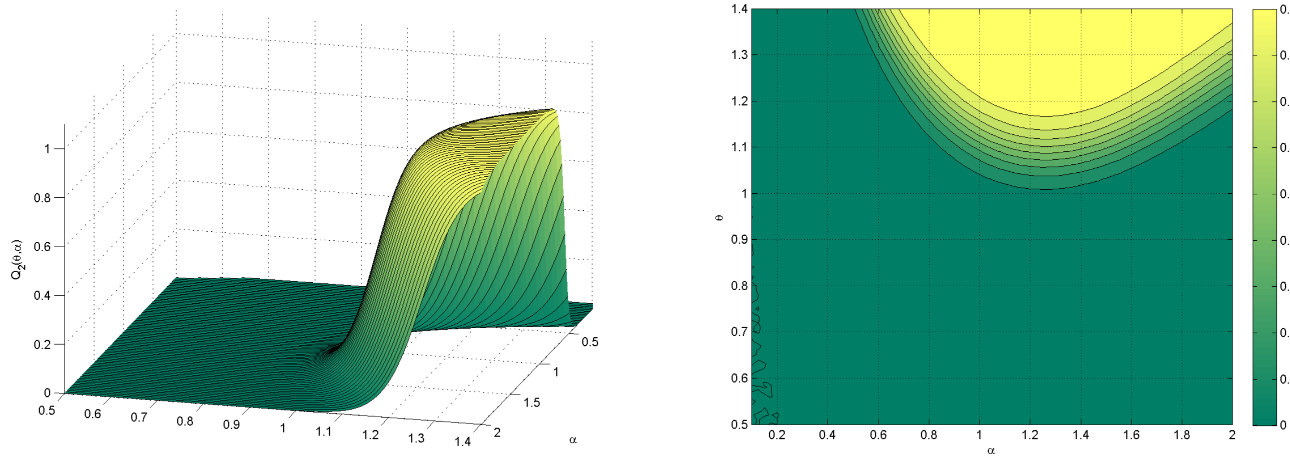


Figure 14: Two-dimensional OC-function for hypothesis H_2 and corresponding contour lines for $m = 10$

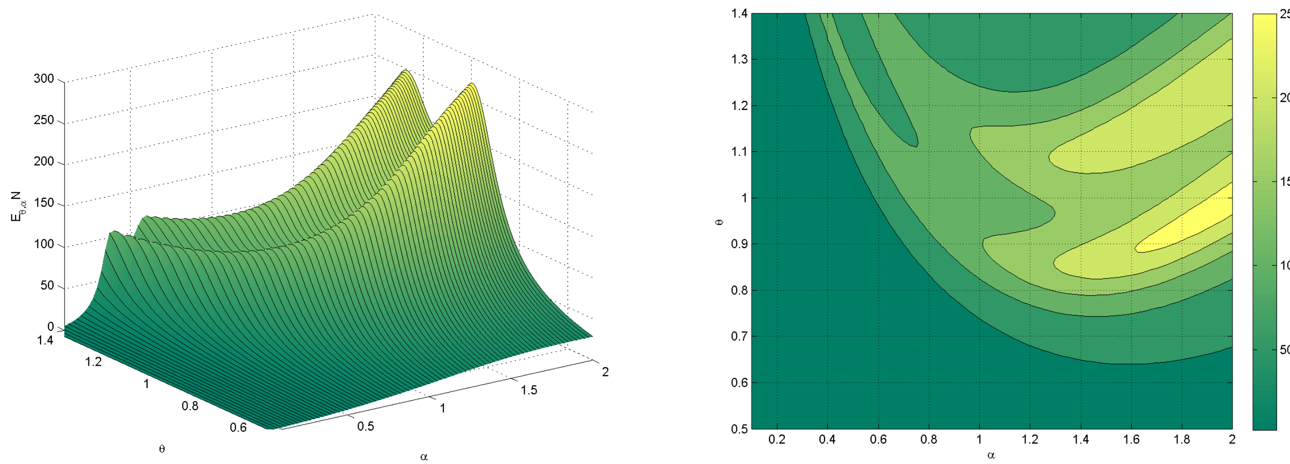


Figure 15: Two-dimensional ASN-function for the Sobel-Wald test and corresponding contour lines for $m = 10$

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