Weak regularity conditions for maximal monotonicity in separable Asplund spaces

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Abstract. In this paper we give weak regularity conditions that ensure maximal monotonicity of the operator $S + A^*TA$, where $S : X \rightrightarrows X^*$ and $T : Y \rightrightarrows Y^*$ are two maximal monotone operators, $A : X \to Y$ is a linear and continuous mapping and X, Y are separable Asplund spaces. In particular, it follows that Rockafellar's conjecture is true in these spaces.

Key Words. maximal monotone operator, Fitzpatrick function, representative function

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1 Introduction

Rockafellar proved in [21] that, whenever X is a reflexive Banach space and $S, T : X \rightrightarrows X^*$ are two maximal monotone operators such that $\text{int}(D(S)) \cap D(T) \neq \emptyset$, then $S + T$ is maximal monotone and conjectured that this result holds also in general Banach spaces. Since than, an intensive research was made in the theory of maximal monotone operators aiming, among other things, to prove this conjecture. A comprehensive study on this topic may be found in the monographs of Simons [22, 23] and in the lecture notes [19] due to Phelps, which are important references for the theory of maximal monotone operators.

In the last decade the convex analysis played a determinant role in this field. The link between the theory of maximal monotone operators and convex analysis is mainly made via the *Fitzpatrick function* associated to a monotone operator. It was introduced in [11], where it is also proved that every maximal monotone operator is representable by a proper, convex and lower semicontinuous function. Rediscovered after some years in [7, 15], this function proved to be crucial in the theory of maximal monotone operators. Motivated by the properties of the Fitzpatrick function, the notion of *representative function* associated to a monotone operator was also introduced in the literature.

Different regularity conditions, weaker than the one in [21], have been given in the past for guaranteeing the maximality of $S + T$, but in reflexive Banach spaces (see, for instance, [1, 2, 4–6, 9, 13, 16–18, 24, 29]). Let us consider X a reflexive Banach space and

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$S, T : X \rightrightarrows X^*$ two maximal monotone operators with representative functions f_S and f_T, respectively, such that $\text{pr}_X(\text{dom}\, h_S) \cap \text{pr}_X(\text{dom}\, h_T) \neq \emptyset$. Each of the conditions below (starting with the so-called *generalized interior point conditions* and coming to the *closedness type conditions*) guarantees the maximality of the monotone operator $S + T$:

(i_r) (cf. [21]): $\text{int}(D(S)) \cap D(T) \neq \emptyset$;

(ii_r) (cf. [2]): $0 \in \text{core}\left[\text{co}(D(S)) - \text{co}(D(T))\right]$;

(iii_r) (cf. [16]): $0 \in \text{ri}(D(S) - D(T))$;

(iv_r) (cf. [9]): $0 \in \text{ri}(\text{co}(D(S)) - \text{co}(D(T)))$;

(v_r) (cf. [24]): $0 \in {}^{ic}(\text{pr}_X(\text{dom}\, \varphi_S) - \text{pr}_X(\text{dom}\, \varphi_T))$;

(vi_r) (cf. [18]): $0 \in {}^{ic}(D(S) - D(T))$;

(vii_r) (cf. [18]): $0 \in {}^{ic}(\text{pr}_X(\text{dom}\, h_S) - \text{pr}_X(\text{dom}\, h_T))$;

(viii_r) (cf. [13]): $\{(x^* + y^*, x, y, r) : \varphi_S^*(x^*, x) + \varphi_T^*(y^*, y) \leq r\}$ is closed;

(ix_r) (cf. [5, 6]): $\{(x^* + y^*, x, y, r) : \varphi_S^*(x^*, x) + \varphi_T^*(y^*, y) \leq r\}$ is closed regarding the subspace $X^* \times \Delta_X \times \mathbb{R}$, where $\Delta_X = \{(x, x) : x \in X\}$;

(x_r) (cf. [4]): $\{(x^*+y^*, x, y, r) : h_S^*(x^*, x)+h_T^*(y^*, y) \leq r\}$ is closed regarding the subspace $X^* \times \Delta_X \times \mathbb{R}$.

On the other hand, in the last years, an increasing number of characterizations of the maximality of monotone operators as well as different sufficient conditions for the maximality of the sum of two maximal monotone operators in general Banach spaces have been given (see, for instance, [3, 14, 23, 26–28]). In case X is a Banach space, the following conditions ensure the maximality of the operator $S + T$:

(i) (cf. [3]): $\text{int}(D(S)) \cap \text{int}\, D(T) \neq \emptyset$;

(ii) (cf. [3]): $D(S) \cap D(T)$ is closed and convex and $\text{int}(D(S)) \cap D(T) \neq \emptyset$;

(iii) (cf. [3, 27]): both $D(S)$ and $D(T)$ are closed and convex and $0 \in \text{core}(D(S) - D(T))$;

(iv) (cf. [28]): both $D(S)$ and $D(T)$ are closed and convex and $0 \in {}^{ic}(D(S) - D(T))$.

We give in this paper weak sufficient generalized interior point conditions for the maximal monotonicity of the operator $S + A^*TA$, where $S : X \rightrightarrows X^*$ and $T : Y \rightrightarrows Y^*$ are two maximal monotone operators, $A : X \to Y$ is a linear and continuous mapping and X, Y are separable Asplund spaces. The approach we use is based on two important results recently given in the literature in [10, 14] combined with some techniques of convex analysis. Particularizing the main result of the paper to the case $S+T$, we prove that the hypotheses both $D(S)$ and $D(T)$ are closed and convex in conditions (iii) and (iv) above are not necessary in the framework of separable Asplund spaces. Moreover, we obtain that Rockafellar's conjecture holds in these spaces.

2 Preliminaries

In order to make the paper self-contained we introduce some preliminary notions and results. Let X be a nonzero real Banach space, X^* its topological dual space and X^{**} its bidual. By $\langle\cdot,\cdot\rangle$ we denote the duality products in both $X \times X^*$ and $X^* \times X^{**}$, i.e. for $x \in X, x^* \in X^*$ and $x^{**} \in X^{**}$ we have $\langle x, x^*\rangle := x^*(x)$ and $\langle x^*, x^{**}\rangle := x^{**}(x^*)$, respectively. The canonical embedding of X into X^{**} is defined by $\widehat{\ }: X \to X^{**}$, $\langle x^*, \widehat{x}\rangle := \langle x, x^*\rangle$ for all $x \in X$ and $x^* \in X^*$.

For a subset C of X we denote by $\operatorname{int}(C)$, $\overline{C}$, $\operatorname{co}(C)$, $\operatorname{lin}(C)$, $\operatorname{core}(C)$ and ${}^{ic}C$ its *interior*, *closure*, *convex hull*, *linear hull*, *algebraic interior*, and *intrinsic relative algebraic interior*, respectively. Let us note that if C is a convex set, then (cf. [29]):

(i) $x \in \operatorname{core}(C)$ if and only if $\bigcup_{\lambda>0} \lambda(C - x) = X$;

(ii) $x \in {}^{ic}C$ if and only if $\bigcup_{\lambda>0} \lambda(C - x)$ is a closed linear subspace of X.

We also consider the *indicator function* of the set C, denoted by δ_C, which is zero for $x \in C$ and $+\infty$ otherwise.

For a function $f : X \to \overline{\mathbb{R}}$ we denote by $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ its *domain* and we call f *proper* if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. The *Fenchel-Moreau conjugate* of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by $f^*(x^*) = \sup_{x\in X}\{\langle x, x^*\rangle - f(x)\}$ for all $x^* \in X^*$. Having a function $h : X \times X^* \to \overline{\mathbb{R}}$ we denote by $\widehat{h^*} : X^* \times X^{**} \to \overline{\mathbb{R}}$ its conjugate function and by $h^* : X^* \times X \to \overline{\mathbb{R}}$, $h^*(x^*, x) = \widehat{h^*}(x^*, \hat{x})$ its canonical embedding to $X^* \times X$.

Having $f, g : X \to \overline{\mathbb{R}}$ two proper functions we consider their *infimal convolution*, namely the function denoted by $f\Box g : X \to \overline{\mathbb{R}}$, $f\Box g(x) = \inf_{u\in X}\{f(u) + g(x - u)\}$ for all $x \in X$. For a function $f : A \times B \to \overline{\mathbb{R}}$, where A and B are nonempty sets, we denote by $f^\top$ the *transpose* of f, namely the function $f^\top : B \times A \to \overline{\mathbb{R}}, f^\top(b, a) = f(a, b)$ for all $(b, a) \in B \times A$. We consider also the *projection operator* $\operatorname{pr}_A : A \times B \to A$, $\operatorname{pr}_A(a, b) = a$ for all $(a, b) \in A\times B$ and the *identity function* on A, $\operatorname{id}_A : A \to A, \operatorname{id}_A(a) = a$ for all $a \in A$. When an infimum or a supremum is attained we write min, respectively max instead of inf, respectively sup.

Given a linear and continuous mapping $A : X \to Y$, where Y is another nonzero real Banach space, we denote by $\operatorname{Im} A$ its *image-set*, $\operatorname{Im} A = \{Ax : x \in X\}$, by A^* its *adjoint operator*, $A^* : Y^* \to X^*$ given by $\langle x, A^*y^*\rangle = \langle Ax, y^*\rangle$ for all $(y^*, x) \in Y^* \times X$ and by A^{**} its *bi-adjoint operator*, $A^{**} : X^{**} \to Y^{**}$ given by $\langle y^*, A^{**}x^{**}\rangle = \langle A^*y^*, x^{**}\rangle$ for all $(x^{**}, y^*) \in X^{**} \times Y^*$. For F a subspace of X, we consider the *annihilator* of F, defined by $F^\perp = \big\{x^* \in X^* : \langle F, x^*\rangle = \{0\}\big\}$.

We introduce now further notions and results concerning monotone operators. A set-valued operator $S : X \rightrightarrows X^*$ is said to be *monotone* if $\langle y - x, y^* - x^*\rangle \geq 0,$ whenever $x^* \in S(x)$ and $y^* \in S(y)$. We denote by $G(S) = \{(x, x^*) : x^* \in S(x)\} \subseteq X \times X^*$ the *graph* and by $D(S) = \{x \in X : S(x) \neq \emptyset\}$ the *domain* of S, respectively.

The monotone operator S is called *maximal monotone* if $G(S)$ is not properly contained in the graph of any other monotone operator $S' : X \rightrightarrows X^*$. The classical example of a maximal monotone operator is the subdifferential of a proper, convex and lower semicontinuous function defined on a Banach space (see [20]). However, in case the dimension of X is strictly greater than 1, there exist maximal monotone operators which are not subdifferentials (see [22, 23]).

Having a monotone operator $S : X \rightrightarrows X^*$ one can associate to it the so-called *Fitzpatrick function* $\varphi_S : X \times X^* \to \overline{\mathbb{R}}$, defined by

$$\varphi_S(x, x^*) = \sup\{\langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle : y^* \in S(y)\},$$

which is obviously convex and norm-weak* lower semicontinuous. Introduced by Fitzpatrick (see [11]), it proved to be very important in the theory of maximal monotone operators, revealing some connections between convex analysis and monotone operators (see [2, 4–7, 17, 18, 23, 24, 28] and the references therein). Considering the function $c : X \times X^* \to \mathbb{R}$, $c(x, x^*) = \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$, we have $\varphi_S = (c + \delta_{G(S)})^{*\top}$.

If S is a maximal monotone operator, then (cf. [11]) $\varphi_S(x, x^*) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$ and $G(S) = \{(x, x^*) \in X \times X^* : \varphi_S(x, x^*) = \langle x, x^* \rangle\}$. Motivated by these properties of the Fitzpatrick function, the notion of *representative function* of a monotone operator was introduced and studied in the literature. For $S : X \rightrightarrows X^*$ a monotone operator, we call *representative function* of S a convex and norms-weak* lower semicontinuous function $h_S : X \times X^* \to \overline{\mathbb{R}}$ fulfilling

$$h_S \geq c \text{ and } G(S) \subseteq \{(x, x^*) \in X \times X^* : h_S(x, x^*) = \langle x, x^* \rangle\}.$$

Remark 1 *The above definition is in the sense considered by J.M. Borwein in [3]. In the case of maximal monotone operators it coincides with the usual definition for representative functions considered, for instance, in [4, 18] (see Proposition 1 below).*

We observe that if $G(S) \neq \emptyset$ (in particular if S is maximal monotone), then every representative function of S is proper. It follows immediately that the Fitzpatrick function associated to a maximal monotone operator is a representative function of the operator. The next result is a direct consequence of [3, Proposition 2 and Corollary 4].

Proposition 1 *Let $S : X \rightrightarrows X^*$ be a maximal monotone operator and h_S be a representative function of S. Then:*

(i) $\varphi_S \leq h_S \leq \varphi_S^{*\top}$;

(ii) $h_S^{*\top}$ *is also a representative function of S;*

(iii) $\{(x, x^*) \in X \times X^* : h_S(x, x^*) = \langle x, x^* \rangle\} = \{(x, x^*) \in X \times X^* : h_S^{*\top}(x, x^*) = \langle x, x^* \rangle\} = G(S)$.

Remark 2 *These properties of representative functions are well-known in the framework of reflexive Banach spaces (see [18]). It is shown in [3] that these characterizations hold also in a general Banach space. For more on the properties of representative functions we refer to [2–4, 18] and the references therein.*

The main theorem of the paper is based on two important results recently introduced in the literature. The next theorem is a part of a result given by M. Marques Alves and B.F. Svaiter in [14], which generalizes to general Banach spaces some results given in [8, 18].

Theorem 2 *(cf. [14, Theorem 4.2]) Suppose that $h : X \times X^* \to \overline{\mathbb{R}}$ is a proper, convex and norm-norm lower semicontinuous function such that $h(x, x^*) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$ and $\widehat{h^*}(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle$ for all $(x^*, x^{**}) \in X^* \times X^{**}$. Define $S : X \rightrightarrows X^*$ by $G(S) = \{(x, x^*) \in X \times X^* : h(x, x^*) = \langle x, x^* \rangle\}$. Then $G(S) = \{(x, x^*) \in X \times X^* : h^{*\top}(x, x^*) = \langle x, x^* \rangle\}$ and S is maximal monotone.*

The following result is proved by A.C. Eberhard and J.M. Borwein in [10].

Theorem 3 *(cf. [10, Theorem 15]) Let $S : X \rightrightarrows X^*$ be a maximal monotone operator defined on a Banach space X such that X^* is separable and h_S be a representative function of S. Then it holds $\widehat{h_S^*}(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle$ for all $(x^*, x^{**}) \in X^* \times X^{**}$.*

3 Regularity conditions for maximal monotonicity

We start with the following lemma, the proof of which uses techniques taken from [22, p. 57–62 and p. 87–88]. A similar result is given in [24, Lemma 5.3] in case of reflexive Banach spaces. As it can be seen, this holds also in general Banach spaces.

Lemma 4 *Let $S : X \rightrightarrows X^*$ and $T : Y \rightrightarrows Y^*$ be two maximal monotone operators with representative functions h_S and h_T, respectively, and $A : X \to Y$ a linear and continuous mapping. The following statements are true:*

(a) If F is a closed subspace of Y, $w \in Y$ and $A\big(D(S)\big) \subseteq F + w$ then $A\big(\operatorname{pr}_X(\operatorname{dom} h_S)\big) \subseteq F + w$;

(b) If F is a closed subspace of X, $w \in X$ and $D(S) \subseteq F + w$ then $\operatorname{pr}_X(\operatorname{dom} h_S) \subseteq F + w$;

(c) $\bigcup_{\lambda > 0} \lambda\big[A\big(\operatorname{pr}_X(\operatorname{dom} h_S)\big) - \operatorname{pr}_Y(\operatorname{dom} h_T)\big] \subseteq \overline{\operatorname{lin}\big(A(D(S)) - D(T)\big)}$.

Proof. In order to prove (a), let us take an arbitrary $x \in \operatorname{pr}_X(\operatorname{dom} h_S)$ and $u \in D(S)$ (the existence of u is guaranteed by the maximality of the monotone operator S). Then there exist $u^* \in Su$ and $x^* \in X^*$ such that $h_S(x, x^*) < +\infty$. Take an arbitrary $y^* \in F^\perp$. We claim that

$$u^* + A^* y^* \in Su. \tag{1}$$

Let (s, s^*) be an arbitrary element of $G(S)$. Then, since $A(u - s) = Au - As \in A(D(S)) - A(D(S)) \subseteq (F + w) - (F + w) = F$, we have $\langle A(u - s), y^* \rangle = 0$. Combining this with the monotonicity of S we get

$$\langle u - s, (u^* + A^* y^*) - s^* \rangle = \langle u - s, u^* - s^* \rangle \geq 0.$$

The maximality of S ensures (1). From Proposition 1 and the definition of φ_S we obtain

$$+\infty > h_S(x, x^*) \geq \varphi_S(x, x^*) \geq \langle u, x^* \rangle + \langle x, u^* + A^* y^* \rangle - \langle u, u^* + A^* y^* \rangle$$

from which

$$+\infty > h_S(x, x^*) - \langle u, x^* \rangle - \langle x, u^* \rangle + \langle u, u^* \rangle \geq \langle Ax - Au, y^* \rangle.$$

As $F^\perp$ is a subspace of Y^* we get $\langle Ax - Au, F^\perp\rangle = \{0\}$, which implies, in view of the bipolar theorem, that $Ax - Au \in F$. Hence $Ax = (Ax - Au) + Au \in F + A(D(S)) \subseteq F + F + w = F + w$ and the proof of (a) is complete.

The assertion in (b) follows by taking $Y := X$ and $A := \mathrm{id}_X$.

For (c) we make the notation $F := \overline{\mathrm{lin}\big(A(D(S)) - D(T)\big)}$. Let x be an arbitrary element of $\mathrm{pr}_X(\mathrm{dom}\, h_S)$ and y an arbitrary element of $\mathrm{pr}_Y(\mathrm{dom}\, h_T)$. Let t be an arbitrary element of $D(T)$. Then $A(D(S)) - t \subseteq A(D(S)) - D(T) \subseteq F$, thus $A(D(S)) \subseteq F + t$. Part (a) guarantees that $Ax \in F + t$, that is $t \in F + Ax$. Since t is arbitrary in $D(T)$, we get $D(T) \subseteq F + Ax$. By using now part (b), applied for the operator T, we obtain $y \in F + Ax$, hence $Ax - y \in F$. This holds for all $x \in \mathrm{pr}_X(\mathrm{dom}\, h_S)$ and all $y \in \mathrm{pr}_Y(\mathrm{dom}\, h_T)$, implying $A\big(\mathrm{pr}_X(\mathrm{dom}\, h_S)\big) - \mathrm{pr}_Y(\mathrm{dom}\, h_T) \subseteq \overline{\mathrm{lin}\big(A(D(S)) - D(T)\big)}$. ■

Remark 3 *It follows easily from Proposition 1 and Lemma 4 that for $S : X \rightrightarrows X^*$, $T : Y \rightrightarrows Y^*$ maximal monotone operators and $A : X \to Y$ a linear and continuous mapping the following inclusions hold*

$$\bigcup_{\lambda>0} \lambda\big(A(D(S)) - D(T)\big) \subseteq \bigcup_{\lambda>0} \lambda\big(\mathrm{co}\, A(D(S)) - \mathrm{co}\, D(T)\big)$$

$$\subseteq \bigcup_{\lambda>0} \lambda\big[A\big(\mathrm{pr}_X(\mathrm{dom}\,\varphi_S^*)\big) - \mathrm{pr}_Y(\mathrm{dom}\,\varphi_T^*)\big] \subseteq \bigcup_{\lambda>0} \lambda\big[A\big(\mathrm{pr}_X(\mathrm{dom}\,\varphi_S)\big) - \mathrm{pr}_Y(\mathrm{dom}\,\varphi_T)\big]$$

$$\subseteq \overline{\mathrm{lin}\big(A(D(S)) - D(T)\big)} \subseteq \overline{\mathrm{lin}\Big(A\big(\mathrm{pr}_X(\mathrm{dom}\,\varphi_S^*)\big) - \mathrm{pr}_Y(\mathrm{dom}\,\varphi_T^*)\Big)}$$

$$\subseteq \overline{\mathrm{lin}\Big(A\big(\mathrm{pr}_X(\mathrm{dom}\,\varphi_S)\big) - \mathrm{pr}_Y(\mathrm{dom}\,\varphi_T)\Big)} \subseteq \overline{\mathrm{lin}\big(A(D(S)) - D(T)\big)},$$

thus

$$\overline{\mathrm{lin}\big(A(D(S)) - D(T)\big)} = \overline{\mathrm{lin}\Big(A\big(\mathrm{pr}_X(\mathrm{dom}\,\varphi_S^*)\big) - \mathrm{pr}_Y(\mathrm{dom}\,\varphi_T^*)\Big)}$$

$$= \overline{\mathrm{lin}\Big(A\big(\mathrm{pr}_X(\mathrm{dom}\,\varphi_S)\big) - \mathrm{pr}_Y(\mathrm{dom}\,\varphi_T)\Big)}.$$

The next result will be important in deriving the main result of the paper.

Theorem 5 *Suppose that $S : X \rightrightarrows X^*$, $T : Y \rightrightarrows Y^*$ are two maximal monotone operators with representative functions h_S and h_T, respectively, and $A : X \to Y$ is a linear and continuous mapping fulfilling*

$$0 \in {}^{ic}\big(A(\mathrm{pr}_X(\mathrm{dom}\, h_S^*)) - \mathrm{pr}_Y(\mathrm{dom}\, h_T^*)\big).$$

Then the function $h : X \times X^* \to \overline{\mathbb{R}}$ defined by $h(x, x^*) := \inf\{h_S(x, u^*) + h_T(Ax, v^*) : u^* \in X^*, v^* \in Y^*, u^* + A^*v^* = x^*\}$ is convex and norm-weak* lower semicontinuous. Further, for all $(x, x^*) \in X \times X^*$ we have $h(x, x^*) \geq \langle x, x^*\rangle$ and the infimum in the definition of h is attained. The function h is proper if and only if $A(\mathrm{pr}_X(\mathrm{dom}\, h_S)) \cap \mathrm{pr}_Y(\mathrm{dom}\, h_T) \neq \emptyset$. Moreover, $G(S + A^*TA) = \{(x, x^*) \in X \times X^* : h(x, x^*) = \langle x, x^*\rangle\}$.

Proof. The convexity of h follows immediately.

In order to show that h is norm-weak* lower semicontinuous we define the functions $f_S : X \times X^* \to \overline{\mathbb{R}}$, $f_S := h_S^{*\top}$ and $f_T : Y \times Y^* \to \overline{\mathbb{R}}$, $f_T := h_T^{*\top}$. Further, consider the functions $F_S : X \times X^* \times Y^* \to \overline{\mathbb{R}}$, defined by $F_S(x, x^*, y^*) = f_S(x, x^*)$ for all $(x, x^*, y^*) \in X \times X^* \times Y^*$ and $F_T : Y \times X^* \times Y^* \to \overline{\mathbb{R}}$, defined by $F_T(y, x^*, y^*) = f_T(y, y^*)$ for all $(y, x^*, y^*) \in Y \times X^* \times Y^*$. Let us define also the linear and continuous mapping $B : X \times X^* \times Y^* \to Y \times X^* \times Y^*$ by $B(x, x^*, y^*) = (Ax, x^*, y^*)$ for all $(x, x^*, y^*) \in X \times X^* \times Y^*$. One can easily deduce that

$$B(\operatorname{dom} F_S) - \operatorname{dom} F_T = \big[A(\operatorname{pr}_X(\operatorname{dom} h_S^*)) - \operatorname{pr}_Y(\operatorname{dom} h_T^*)\big] \times X^* \times Y^*.$$

The hypotheses imply $0 \in {}^{ic}\big(B(\operatorname{dom} F_S) - \operatorname{dom} F_T\big)$. By using [29, Theorem 2.8.3] we obtain $(\widehat{F_S + F_T} \circ B)^*(x^*, x^{**}, y^{**}) = \min\{\widehat{F_S^*}(z^*, a^{**}, b^{**}) + \widehat{F_T^*}(y^*, \alpha^{**}, \beta^{**}) : (z^*, a^{**}, b^{**}) + B^*(y^*, \alpha^{**}, \beta^{**}) = (x^*, x^{**}, y^{**})\}$ for all $(x^*, x^{**}, y^{**}) \in X^* \times X^{**} \times Y^{**}$. One can show that for all (z^*, a^{**}, b^{**}) and $(y^*, \alpha^{**}, \beta^{**})$ we have

$$\widehat{F_S^*}(z^*, a^{**}, b^{**}) = \begin{cases} \widehat{f_S^*}(z^*, a^{**}), & \text{if } b^{**} = 0, \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$\widehat{F_T^*}(y^*, \alpha^{**}, \beta^{**}) = \begin{cases} \widehat{f_T^*}(y^*, \beta^{**}), & \text{if } \alpha^{**} = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

respectively. The adjoint operator $B^* : Y^* \times X^{**} \times Y^{**} \to X^* \times X^{**} \times Y^{**}$ has the form $B^*(y^*, x^{**}, y^{**}) = (A^*y^*, x^{**}, y^{**})$ for all $(y^*, x^{**}, y^{**}) \in Y^* \times X^{**} \times Y^{**}$. After some calculations we get $(\widehat{F_S + F_T} \circ B)^*(x^*, x^{**}, y^{**}) = \min\{\widehat{f_S^*}(z^*, x^{**}) + \widehat{f_T^*}(y^*, y^{**}) : z^* + A^*y^* = x^*\}$ for all $(x^*, x^{**}, y^{**}) \in X^* \times X^{**} \times Y^{**}$. Restricting the last equality to (x^*, x, Ax) we obtain $(F_S + F_T \circ B)^*(x^*, x, Ax) = \min\{f_S^*(z^*, x) + f_T^*(y^*, Ax) : z^* + A^*y^* = x^*\}$ for all $(x^*, x, Ax) \in X^* \times X \times Y$. The functions h_S and h_T being proper, convex and norm-weak* lower semicontinuous we have $f_S^*(z^*, x) = h_S(x, z^*)$ and $f_T^*(y^*, Ax) = h_T(Ax, y^*)$, which implies $(F_S + F_T \circ B)^*(x^*, x, Ax) = \min\{h_S(x, z^*) + h_T(Ax, y^*) : z^* + A^*y^* = x^*\}$ for all $(x^*, x, Ax) \in X^* \times X \times Y$. The last relation shows that the function h is norm-weak* lower semicontinuous and the infimum in the definition of h is attained.

The inequality $h(x, x^*) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$ follows from the definition of the function h and thus the statement regarding the properness of this function is obvious.

Finally, employing the properties of the functions h, h_S and h_T we get

$$\begin{aligned}
&\{(x, x^*) : h(x, x^*) = \langle x, x^* \rangle\} \\
&= \{(x, x^*) : \exists v^* \in Y^* \text{ such that } h_S(x, x^* - A^*v^*) + h_T(Ax, v^*) = \langle x, x^* \rangle\} \\
&= \{(x, x^*) : \exists v^* \in Y^* \text{ such that } h_S(x, x^* - A^*v^*) - \langle x, x^* - A^*v^* \rangle + h_T(Ax, v^*) - \langle Ax, v^* \rangle = 0\} \\
&= \{(x, x^*) : \exists v^* \in Y^* \text{ such that } \begin{array}{l} h_S(x, x^* - A^*v^*) = \langle x, x^* - A^*v^* \rangle \\ \text{and } h_T(Ax, v^*) = \langle Ax, v^* \rangle\} \end{array} \\
&= \{(x, x^*) : \exists v^* \in Y^* \text{ such that } x^* - A^*v^* \in S(x) \text{ and } v^* \in T(Ax)\} \\
&= \{(x, x^*) : x^* \in (S + A^*TA)(x)\} = G(S + A^*TA),
\end{aligned}$$

hence the desired conclusion follows. ■

We give in the following the main result of the paper, which provides a weak sufficient condition for the maximal monotonicity of the operator $S + A^*TA$ in case of separable Asplund spaces. A similar result was proved in [18] in the framework of reflexive Banach spaces.

Theorem 6 *Let $S : X \rightrightarrows X^*$, $T : Y \rightrightarrows Y^*$ be two maximal monotone operators defined on separable Asplund spaces, $A : X \to Y$ a linear and continuous mapping fulfilling*

$$0 \in {}^{ic}\big(A(\mathrm{pr}_X(\mathrm{dom}\,\varphi_S^*)) - \mathrm{pr}_Y(\mathrm{dom}\,\varphi_T^*)\big).$$

Then $S + A^*TA$ is a maximal monotone operator.

Proof. From Proposition 1, we have $A(\mathrm{pr}_X(\mathrm{dom}\,\varphi_S^*)) \cap \mathrm{pr}_Y(\mathrm{dom}\,\varphi_T^*) \subseteq A(\mathrm{pr}_X(\mathrm{dom}\,\varphi_S)) \cap \mathrm{pr}_Y(\mathrm{dom}\,\varphi_T)$. Applying now Theorem 5 we obtain that the function $h : X \times X^* \to \overline{\mathbb{R}}$ defined by $h(x, x^*) := \inf\{\varphi_S(x, u^*) + \varphi_T(Ax, v^*) : u^* \in X^*, v^* \in Y^*, u^* + A^*v^* = x^*\}$ is proper, convex and norm-weak* lower semicontinuous, $h(x, x^*) \geq \langle x, x^*\rangle$ for all $(x, x^*) \in X \times X^*$ and

$$G(S + A^*TA) = \{(x, x^*) : h(x, x^*) = \langle x, x^*\rangle\}.$$

In view of Theorem 2, it remains to prove that $\widehat{h^*}(x^*, x^{**}) \geq \langle x^*, x^{**}\rangle$ for all $(x^*, x^{**}) \in X^* \times X^{**}$.

Take an arbitrary $(x^*, x^{**}) \in X^* \times X^{**}$. It follows by the definition of the function h that

$$\widehat{h^*}(x^*, x^{**}) = \sup_{z\in X, u^*\in X^*, v^*\in Y^*} \{\langle z, x^*\rangle + \langle u^*, x^{**}\rangle + \langle v^*, A^{**}x^{**}\rangle - \varphi_S(z, u^*) - \varphi_T(Az, v^*)\}.$$

Let us define the functions $H_S : X \times X^* \times Y^* \to \overline{\mathbb{R}}$, by $H_S(z, u^*, v^*) = \varphi_S(z, u^*)$ for all $(z, u^*, v^*) \in X \times X^* \times Y^*$ and $H_T : Y \times X^* \times Y^* \to \overline{\mathbb{R}}$, by $H_T(y, u^*, v^*) = \varphi_T(y, v^*)$ for all $(y, u^*, v^*) \in Y \times X^* \times Y^*$. Let us consider again the linear and continuous mapping $B : X \times X^* \times Y^* \to Y \times X^* \times Y^*$ defined as in the proof of Theorem 5. Then

$$\widehat{h^*}(x^*, x^{**}) = (\widehat{H_S + H_T \circ B})^*(x^*, x^{**}, A^{**}x^{**}).$$

One can deduce that $B(\mathrm{dom}\,H_S) - \mathrm{dom}\,H_T = \big[A(\mathrm{pr}_X(\mathrm{dom}\,\varphi_S)) - \mathrm{pr}_Y(\mathrm{dom}\,\varphi_T)\big] \times X^* \times Y^*$. Combining the condition from the hypotheses with the sequence of inclusions in Remark 3 it follows $0 \in {}^{ic}\big(B(\mathrm{dom}\,H_S) - \mathrm{dom}\,H_T\big)$. By using [29, Theorem 2.8.3] we obtain $(\widehat{H_S + H_T \circ B})^*(x^*, x^{**}, A^{**}x^{**}) = \min\{\widehat{H_S^*}(z^*, a^{**}, b^{**}) + \widehat{H_T^*}(y^*, \alpha^{**}, \beta^{**}) : (z^*, a^{**}, b^{**}) + B^*(y^*, \alpha^{**}, \beta^{**}) = (x^*, x^{**}, A^{**}x^{**})\}$. After some calculations we obtain that for all (z^*, a^{**}, b^{**}) and $(y^*, \alpha^{**}, \beta^{**})$ we have

$$\widehat{H_S^*}(z^*, a^{**}, b^{**}) = \begin{cases} \widehat{\varphi_S^*}(z^*, a^{**}), & \text{if } b^{**} = 0, \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$\widehat{H_T^*}(y^*, \alpha^{**}, \beta^{**}) = \begin{cases} \widehat{\varphi_T^*}(y^*, \beta^{**}), & \text{if } \alpha^{**} = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

respectively. Further we get

$$(\widehat{H_S + H_T \circ B})^*(x^*, x^{**}, A^{**}x^{**}) = \min\{\widehat{\varphi_S^*}(z^*, x^{**}) + \widehat{\varphi_T^*}(y^*, A^{**}x^{**}) : z^* + A^*y^* = x^*\}.$$

Hence, employing Theorem 3 we obtain $\widehat{h^*}(x^*, x^{**}) = \min\{\widehat{\varphi_S^*}(z^*, x^{**}) + \widehat{\varphi_T^*}(y^*, A^{**}x^{**}) : z^* + A^*y^* = x^*\} \geq \min\{\langle z^*, x^{**}\rangle + \langle y^*, A^{**}x^{**}\rangle : z^* + A^*y^* = x^*\} = \langle x^*, x^{**}\rangle$, so the proof is complete. ■

Theorem 6 and Remark 3 imply the following result.

Corollary 7 *Let $S : X \rightrightarrows X^*$, $T : Y \rightrightarrows Y^*$ be two maximal monotone operators defined on separable Asplund spaces and $A : X \to Y$ a linear and continuous mapping fulfilling*

$$0 \in {}^{ic}\big(\operatorname{co} A(D(S)) - \operatorname{co} D(T)\big).$$

Then $S + A^*TA$ is a maximal monotone operator.

In case $S : X \rightrightarrows X^*$, $Sx = 0$ for all $x \in X$, we have $S + A^*TA = A^*TA$ and $\varphi_S = \delta_{X\times\{0\}} = \varphi_S^{*\top}$. From Theorem 6 and Corollary 7 we obtain the following conditions for the maximality of the operator A^*TA.

Corollary 8 *Let $T : Y \rightrightarrows Y^*$ be a maximal monotone operator defined on a separable Asplund space and $A : X \to Y$ a linear and continuous mapping fulfilling*

$$0 \in {}^{ic}\big(\operatorname{Im} A - \operatorname{pr}_Y(\operatorname{dom} \varphi_T^*)\big).$$

Then A^*TA is a maximal monotone operator.

Corollary 9 *Let $T : Y \rightrightarrows Y^*$ be a maximal monotone operator defined on a separable Asplund space and $A : X \to Y$ a linear and continuous mapping fulfilling*

$$0 \in {}^{ic}\big(\operatorname{Im} A - \operatorname{co} D(T)\big).$$

Then A^*TA is a maximal monotone operator.

Finally, the following corollaries are easy consequences of Theorem 6 and Corollary 7 by taking $X = Y$ and $A = \mathrm{id}_X$.

Corollary 10 *Let $S, T : X \rightrightarrows X^*$ be two maximal monotone operators defined on separable Asplund spaces such that*

$$0 \in {}^{ic}\big(\operatorname{pr}_X(\operatorname{dom} \varphi_S^*) - \operatorname{pr}_X(\operatorname{dom} \varphi_T^*)\big).$$

Then $S + T$ is a maximal monotone operator.

Corollary 11 *Let $S, T : X \rightrightarrows X^*$ be two maximal monotone operators defined on separable Asplund spaces fulfilling*

$$0 \in {}^{ic}\big(\operatorname{co} D(S) - \operatorname{co} D(T)\big).$$

Then $S + T$ is a maximal monotone operator.

Remark 4 *(a) Voisei obtained in [28] a similar result for the maximality of the operator $S+T$ in case of general Banach spaces. It follows that the conditions $D(S)$ and $D(T)$ are convex and closed are not needed anymore in the framework of separable Asplund spaces.*

(b) Let us notice that Corollary 10 and Corollary 11 can be derived also from Corollary 8 and Corollary 9, respectively. Indeed, take $Y = X \times X$, $A : X \to X \times X$, $Ax = (x, x)$ and $(S, T) : X \times X \rightrightarrows X^* \times X^*$, $(S, T)(x, y) = (S(x), T(y))$. In case S and T are maximal monotone operators, (S, T) is also a maximal monotone operator and it holds $A^*(S, T)A(x) = S(x) + T(x)$ for all $x \in X$. The details are left for the reader.

(c) As the condition $\operatorname{int}(D(S)) \cap D(T) \neq \emptyset$ implies $\bigcup_{\lambda>0} \lambda\big(\operatorname{co} D(S) - \operatorname{co} D(T)\big) = X$, we obtain that Rockafellar's conjecture concerning the maximal monotonicity of the operator $S + T$ under the condition $\operatorname{int}(D(S)) \cap D(T) \neq \emptyset$ holds in separable Asplund spaces.

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