On an open problem regarding totally Fenchel unstable functions

Radu Ioan Boţ * Ernö Robert Csetnek †

Abstract. We give an answer to the Problem 11.5 posed by Stephen Simons in his book "From Hahn-Banach to Monotonicity".

Key Words. conjugate function, Fenchel duality, minimax theorem, weak* - extreme point

AMS subject classification. 90C25, 90C46, 42A50, 90C47, 46B20

1 Introduction and problem formulation

Before introducing the problem proposed by Stephen Simons, we recall some preliminary notions and results. Throughout this note, E denotes a nontrivial real Banach space, E^* its topological dual space and E^{**} its bidual space. The canonical embedding of E into E^{**} is defined by $\widehat{\ }: E \to E^{**}$, $\langle x^*, \widehat{x}\rangle := \langle x, x^*\rangle$, for all $x \in E$ and $x^* \in E^*$, where $\langle x, x^*\rangle$ denotes the value of the linear continuous functional x^* at x. For $D \subseteq E$, we denote by $\widehat{D}$ the image of the set D through the canonical embedding, that is $\widehat{D} = \{\widehat{x} : x \in D\}$.

The *indicator function* of $D \subseteq E$, denoted by δ_D, is defined as $\delta_D : E \to \overline{\mathbb{R}}$,

$$\delta_D(x) = \begin{cases} 0, & \text{if } x \in D, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. For a function $f : E \to \overline{\mathbb{R}}$ we denote by $\text{dom}(f) = \{x \in E : f(x) < +\infty\}$ its *domain* and by $\text{epi}(f) = \{(x, r) \in E \times \mathbb{R} : f(x) \le r\}$ its *epigraph.* We call f *proper* if $\text{dom}(f) \neq \emptyset$ and $f(x) > -\infty$ for all $x \in E$. By $\text{cl}(f)$ we denote the *lower semicontinuous hull* of f, namely the function of which epigraph is the closure of $\text{epi}(f)$ in $E \times \mathbb{R}$, that is $\text{epi}(\text{cl}(f)) = \text{cl}(\text{epi}(f))$. The *Fenchel-Moreau conjugate* of f is the function $f^* : E^* \to \overline{\mathbb{R}}$ defined by $f^*(x^*) = \sup_{x \in E}\{\langle x, x^*\rangle - f(x)\}$ for all $x^* \in E^*$.

Consider $f, g : E \to \overline{\mathbb{R}}$ two arbitrary convex functions. We say that f and g satisfy *stable Fenchel duality* if for all $x^* \in E^*$, there exists $z^* \in E^*$ such that

$$(f + g)^*(x^*) = f^*(x^* - z^*) + g^*(z^*).$$

*Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: radu.bot@mathematik.tu-chemnitz.de

†Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: robert.csetnek@mathematik.tu-chemnitz.de

If this property holds for $x^* = 0$, then f and g satisfy the classical *Fenchel duality*. The pair f, g is *totally Fenchel unstable* (see [14]) if f and g satisfy Fenchel duality but

$$y^*, z^* \in E^* \text{ and } (f+g)^*(y^*+z^*) = f^*(y^*) + g^*(z^*) \Longrightarrow y^* + z^* = 0.$$

A geometric characterization of these notions, in terms of the epigraphs of the conjugates of the functions involved can be given, as we point out below.

It is known (see Proposition 2.2 in [1]) that if f and g are proper functions such that $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$, the stable Fenchel duality is equivalent to the relation

$$\mathrm{epi}(f+g)^* = \mathrm{epi}(f^*) + \mathrm{epi}(g^*).$$

Moreover, if f and g are proper, convex and lower semicontinuous functions such that $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$, then f and g satisfy stable Fenchel duality if and only if $\mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ is closed in the product topology of $(E^*, \omega(E^*, E)) \times \mathbb{R}$, where $\omega(E^*, E)$ is the weak* topology on E^* (see [3] for the Banach setting and [1] for the more general case when E is a separated locally convex space).

In case f and g are proper functions such that $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$, one can prove that Fenchel duality is equivalent to the relation

$$\mathrm{epi}(f+g)^* \cap (\{0\} \times \mathbb{R}) = (\mathrm{epi}(f^*) + \mathrm{epi}(g^*)) \cap (\{0\} \times \mathbb{R}). \tag{1}$$

Furthermore, various sufficient conditions were given in the literature in order to guarantee Fenchel duality, starting with the so-called interior-point conditions (see [8] for an overview on these conditions) and coming to the recently introduced closedness-type conditions (see [1]).

Finally, it is not difficult to show that a pair f, g of proper functions such that $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$ is totally Fenchel unstable if and only if (1) holds and if for $x^* \in E^*$ we have

$$\mathrm{epi}(f+g)^* \cap (\{x^*\} \times \mathbb{R}) = (\mathrm{epi}(f^*) + \mathrm{epi}(g^*)) \cap (\{x^*\} \times \mathbb{R}),$$

then $x^* = 0$.

Obviously, stable Fenchel duality implies Fenchel duality, but the converse is not true (see the example in [1], pp. 2798-2799 and Example 11.1 in [14]). Nevertheless, each of these examples (which are given in $\mathbb{R}^2$) fails when one tries to verify total Fenchel unstability. Surprisingly, in the finite dimensional case, it is still an open question if there exists a pair of functions which is totally Fenchel unstable (see Problem 11.6 in [14]). In the infinite dimensional setting this problem receives an answer, due to the existence of extreme points which are not support points of certain convex sets. Recall that if C is a convex subset of E, then $x \in C$ is a *support point* of C if there exists $x^* \in E^*$, $x^* \neq 0$ such that $\langle x, x^* \rangle = \sup \langle C, x^* \rangle$. We give below an example, proposed in [14], of a pair f, g which is totally Fenchel unstable.

Example 1. Let C be a nonempty, bounded, closed and convex subset of E such that there exists an extreme point x_0 of C which is not a support point of C (an example of a set C and a point x_0 with the above mentioned properties was given in the space l_2, following an idea due to Jonathan Borwein, see [14]). Take $A := x_0 - C$, $B := C - x_0$, $f := \delta_A$ and $g := \delta_B$. One can prove that the pair f, g is totally Fenchel unstable (see Example 11.3 in [14]).

Regarding the functions defined in the above example, Stephen Simons asks whether, denoting $E^* \setminus \{0\}$ with $\{0\}^c$, the following representation of the Minkowski sum of the sets $\text{epi}(f^*)$ and $\text{epi}(g^*)$ is true:

$$\text{epi}(f^*) + \text{epi}(g^*) = (\{0\} \times [0, \infty)) \cup (\{0\}^c \times (0, \infty)). \tag{2}$$

The justification of this question comes from a similar representation of the set $\text{epi}(f_0^*) + \text{epi}(g_0^*)$, proved in [14] for a pair of functions f_0, g_0 defined on the space $\mathbb{R}^2$ in a similar way as in Example 1 above (see Example 11.1 and Example 11.2 in [14]).

We give in the following a reformulation of this problem (as in [14]). The conjugates of the functions f and g are

$$f^*(y^*) = \langle x_0, y^* \rangle - \inf\langle C, y^* \rangle \geq 0 \text{ for all } y^* \in E^* \text{ and}$$

$$g^*(y^*) = \sup\langle C, y^* \rangle - \langle x_0, y^* \rangle \geq 0 \text{ for all } y^* \in E^*.$$

One can use the boundedness of the set C to conclude that $\text{dom}(f^*) = \text{dom}(g^*) = E^*$, thus f^* and g^* are continuous functions (see Theorem 2.2.9 in [15]). The inclusion " $\subseteq$ " in (2) holds and, since $(0, 0) = (0, 0) + (0, 0) \in \text{epi}(f^*) + \text{epi}(g^*)$, relation (2) is equivalent to

$$\text{epi}(f^*) + \text{epi}(g^*) \supset E^* \times (0, \infty). \tag{3}$$

Let us mention that for the implication (3)$\Rightarrow$(2) the assumption that x_0 is not a support point of C is decisive.

In case E is reflexive, this question gets a positive answer. Although the proof is given in [14] (Example 11.3), we give the details for the reader's convenience. Let $y^* \in E^*$ be arbitrary. Consider the functions $h : E^* \to \mathbb{R}$ and $k : E^* \to \mathbb{R}$ defined by $h(z^*) := f^*(z^*)$ and $k(z^*) := g^*(y^* - z^*)$ for all $z^* \in E^*$. Since h and k are continuous, it follows that h and k satisfy Fenchel duality (see Theorem 2.8.7 in [15]). This and the reflexivity of the space E gives

$$-\inf_{E^*}[h + k] = (h + k)^*(0) = \min_{z \in E}[h^*(z) + k^*(-z)].$$

A simple computation shows that $h^*(z) = f(z)$ and $k^*(-z) = g(z) - \langle z, y^* \rangle$, for all $z \in E$. Hence

$$-\inf_{E^*}[h + k] = \min_{E}[f + g - y^*] = \min_{E}[\delta_{\{0\}} - y^*] = 0,$$

so, for all $\varepsilon > 0$, there exists $z^* \in E^*$ such that $h(z^*) + k(z^*) \leq \varepsilon$, that is $f^*(z^*) + g^*(y^* - z^*) \leq \varepsilon$. This means exactly that $(y^*, \varepsilon) \in \text{epi}(f^*) + \text{epi}(g^*)$, hence the proof of (3) is complete.

Remark 1. Regarding the proof given above, one can easily notice that relation (2) is fulfilled if and only if for all $y^* \in E^*$ and for all $\varepsilon > 0$ there exists

$z^* \in E^*$ such that $f^*(z^*) + g^*(y^* - z^*) \leq \varepsilon$. This is equivalent to the statement that there exists $z^* \in E^*$ such that for all $x, y \in E$, $f(x) + g(y) - \langle x - y, z^* \rangle \geq \langle y, y^* \rangle - \varepsilon$. Using the Hahn-Banach-Lagrange theorem (see Theorem 1.11 in [14]), this is equivalent to the following: there exists $M \geq 0$ such that for all $x, y \in E$, $f(x) + g(y) + M\|x - y\| \geq \langle y, y^* \rangle - \varepsilon$, that is to say there exists $M \geq 0$ such that for all $u, v \in C$, $M\|u + v - 2x_0\| \geq \langle v - x_0, y^* \rangle - \varepsilon$.

Following this observation, Stephen Simons proposed the following problem (Problem 11.5 in [14]):

Problem 1. Let C be a nonempty, bounded, closed and convex subset of a nonreflexive Banach space E, x_0 be an extreme point of C, $y^* \in E^*$ and $\varepsilon > 0$. Then does there always exist $M \geq 0$ such that, for all $u, v \in C$, $M\|u + v - 2x_0\| \geq \langle v - x_0, y^* \rangle - \varepsilon$? If the answer to this question is positive, then $\operatorname{epi}(f^*) + \operatorname{epi}(g^*) \supset E^* \times (0, \infty)$.

2 The solution to Problem 1

We give in this section an answer to the problem raised by Stephen Simons. We show that in the nonreflexive case the answer depends on whether x_0 is a weak*-extreme point of C or not. We recall that x_0 is a *weak*-extreme* point of the nonempty, bounded, closed and convex set $C \subseteq E$ if $\widehat{x_0}$ is an extreme point of $\operatorname{cl}\widehat{C}$, where the closure is taken with respect to the weak* topology $\omega(E^{**}, E^*)$ (see [10]). One can show that if x_0 is a weak*-extreme point of C, then x_0 is an extreme point of C. The history of this notion goes back to the paper of Phelps (see [12]), where the author asked the following: must the image $\widehat{x}$ of an extreme point of $x \in B_E$ (the unit ball of E) be an extreme point of $B_{E^{**}}$ (the unit ball of the bidual)? We recall that by the Goldstine theorem, the closure of $\widehat{B_E}$ in the weak* topology $\omega(E^{**}, E^*)$ is $B_{E^{**}}$ (hence the generalization to a nonempty, bounded, closed and convex set is natural). Several papers from the literature deal with this notion, see [2], [5], [7], [10], [11], [12]. In the spaces $C(X)$ and $L^p (1 \leq p \leq \infty)$ all the extreme points of the corresponding unit balls are weak*-extreme points (see [11]). The first example of a Banach space of which unit ball contains elements which are not weak*-extreme was suggested by K. de Leeuw and proved by Y. Katznelson (see the note added at the end of [12]). If E is a separable Banach space containing an isomorphic copy of c_0, then E is isomorphic to a strictly convex space F such that B_F has no weak*-extreme points (see [11]). For the general case when C is a bounded, closed and convex set, we refer to [2] and [10] for more on this subject. We recall from [2] the following result: a Banach space E has the Radon-Nikodým property if and only if every bounded, closed and convex subset C of E has a weak*-extreme point. Of course, in a Radon-Nikodým space it is possible that some of the extreme points are not weak*-extreme points (see [9] for other equivalent formulations of the Radon-Nikodým property).

2.1 First solution

Before we give the solution to Problem 1, we prove in this subsection some results regarding functions with certain properties and then we particularize these results to the functions considered in Problem 1.

For $f : E \to \overline{\mathbb{R}}$ we define $\widehat{f} : E^{**} \to \overline{\mathbb{R}}$ by $\widehat{f}(x^{**}) = f(x)$, if $x^{**} = \widehat{x} \in \widehat{X}$ and $\widehat{f}(x^{**}) = +\infty$, otherwise. Let us start with the following result.

Lemma 1. *We assume that f is convex with* $\mathrm{dom}(f) \neq \emptyset$ *and that* $\mathrm{cl}(\widehat{f})$ *is proper, where the lower semicontinuous hull is considered with respect to the topology $\omega(E^{**}, E^*)$. Then* $f^{**} = \mathrm{cl}(\widehat{f})$.

Proof. Let $x^{**} \in E^{**}$ be fixed. We have:

$$f^{**}(x^{**}) = \sup_{x^* \in E^*} \{\langle x^*, x^{**}\rangle - f^*(x^*)\} = \sup_{x^* \in E^*, r \in \mathbb{R}} \{\langle x^*, x^{**}\rangle - r : r \geq f^*(x^*)\}$$

$$= \sup_{x^* \in E^*, r \in \mathbb{R}} \{\langle x^*, x^{**}\rangle - r : f(y) \geq \langle y, x^*\rangle - r \quad \forall y \in E\}$$

$$= \sup_{x^* \in E^*, r \in \mathbb{R}} \{\langle x^*, x^{**}\rangle - r : \widehat{f}(y^{**}) \geq \langle x^*, y^{**}\rangle - r \quad \forall y^{**} \in E^{**}\}$$

$$= \sup_{x^* \in E^*, r \in \mathbb{R}} \{\langle x^*, x^{**}\rangle - r : \mathrm{cl}(\widehat{f})(y^{**}) \geq \langle x^*, y^{**}\rangle - r \quad \forall y^{**} \in E^{**}\}.$$

Since $\mathrm{cl}(\widehat{f})$ is proper, convex and $\omega(E^{**}, E^*)$-lower semicontinuous, it is equal to the pointwise supremum of the set of its affine minorants (see [6]) and so the conclusion follows. □

Let us consider in the following the proper convex functions $f, g : E \to \overline{\mathbb{R}}$ with the properties: $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$, $\mathrm{cl}(\widehat{f})$ and $\mathrm{cl}(\widehat{g})$ are proper, $f^{**}(0) + g^{**}(0) \geq 0$ and $\mathrm{dom}(f^*) + \mathrm{dom}(g^*) = E^*$. Define the function $P : E^* \to \overline{\mathbb{R}}$, $P(z^*) = (f^{**} + g^{**})^*(z^*)$, for all $z^* \in E^*$.

Let $y^* \in E^*$ be fixed. Consider also the functions $h : E^* \to \overline{\mathbb{R}}$ and $k : E^* \to \overline{\mathbb{R}}$ defined by $h(z^*) := f^*(z^*)$ and $k(z^*) := g^*(y^* - z^*)$ for all $z^* \in E^*$.

Lemma 2. *We have:*

(a) $\inf_{E^*}[h + k] = P(y^*)$;

(b) If $\lambda \in \mathbb{R}$ *then*

$(y^*, \lambda) \in \mathrm{epi}(f^*) + \mathrm{epi}(g^*) \Leftrightarrow$ *there exists* $z^* \in E^*$ *such that* $(h+k)(z^*) \leq \lambda$.

Proof. Since $\mathrm{dom}(h) = \mathrm{dom}(f^*)$ and $\mathrm{dom}(k) = y^* - \mathrm{dom}(g^*)$ we get $\mathrm{dom}(h) - \mathrm{dom}(k) = -y^* + \mathrm{dom}(f^*) + \mathrm{dom}(g^*) = E^*$. It follows that h and k satisfy Fenchel duality (see Theorem 2.8.7 in [15]). We obtain

$$\inf_{E^*}[h + k] = \sup_{z^{**} \in E^{**}} [-h^*(z^{**}) - k^*(-z^{**})].$$

As $h^*(z^{**}) = f^{**}(z^{**})$ and $k^*(-z^{**}) = g^{**}(z^{**}) - \langle y^*, z^{**}\rangle$, for all $z^{**} \in E^{**}$, the conclusion follows easily.

(b) This is immediate from the definitions of the functions h and k. $\square$

Lemma 3. *Let* $(y^*, \lambda) \in E^* \times \mathbb{R}$. *Then:*

$$\lambda > P(y^*) \Rightarrow (y^*, \lambda) \in \operatorname{epi}(f^*) + \operatorname{epi}(g^*) \Rightarrow \lambda \geq P(y^*).$$

Proof. If $\lambda > P(y^*)$, then Lemma 2(a) gives $z^* \in E^*$ such that $(h+k)(z^*) < \lambda$ and so Lemma 2(b) implies that $(y^*, \lambda) \in \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$.

On the other hand, if $(y^*, \lambda) \in \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$, from Lemma 2(b), there exists $z^* \in E^*$ such that $(h+k)^*(z^*) \leq \lambda$. Hence, $\inf_{E^*}[h+k] \leq \lambda$ and so, from Lemma 2(a), we obtain $\lambda \geq P(y^*)$. $\square$

Corollary 1. *We have* $E^* \times (0, \infty) \subset \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$ *if and only if* $\operatorname{dom}(\operatorname{cl}(\widehat{f})) \cap \operatorname{dom}(\operatorname{cl}(\widehat{g})) = \{0\}$.

Proof. Applying Lemma 1 we have that for all $y^* \in E^*$:

$$P(y^*) = \sup_{z^{**} \in \operatorname{dom}(\operatorname{cl}(\widehat{f})) \cap \operatorname{dom}(\operatorname{cl}(\widehat{g}))} \{\langle y^*, z^{**}\rangle - \operatorname{cl}(\widehat{f})(z^{**}) - \operatorname{cl}(\widehat{g})(z^{**})\}.$$

Let us suppose first that $E^* \times (0, \infty) \subset \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$. For all $y^* \in E^*$ and for all $\lambda > 0$, $(y^*, \lambda) \in \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$ and so, from Lemma 3, we get $P(y^*) \leq \lambda$, for all $y^* \in E^*$ and for all $\lambda > 0$. We obtain $P(y^*) \leq 0$, for all $y^* \in E^*$, that is $\langle y^*, z^{**}\rangle - \operatorname{cl}(\widehat{f})(z^{**}) - \operatorname{cl}(\widehat{g})(z^{**}) \leq 0$, for all $y^* \in E^*$ and for all $z^{**} \in \operatorname{dom}(\operatorname{cl}(\widehat{f})) \cap \operatorname{dom}(\operatorname{cl}(\widehat{g}))$, from which it follows that $\operatorname{dom}(\operatorname{cl}(\widehat{f})) \cap \operatorname{dom}(\operatorname{cl}(\widehat{g})) = \{0\}$.

On the other hand, when $\operatorname{dom}(\operatorname{cl}(\widehat{f})) \cap \operatorname{dom}(\operatorname{cl}(\widehat{g})) = \{0\}$, then $P(y^*) = -\operatorname{cl}(\widehat{f})(0) - \operatorname{cl}(\widehat{g})(0) = -f^{**}(0) - g^{**}(0) \leq 0$, for all $y^* \in E^*$. From Lemma 3, for all $y^* \in E^*$ and for all $\lambda > 0$, $(y^*, \lambda) \in \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$, hence $E^* \times (0, \infty) \subset \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$. $\square$

Let us consider now the following particular functions: $f := \delta_A$, $g := \delta_B$, where $A := x_0 - C$, $B := C - x_0$, $x_0 \in C$ and C is a nonempty, bounded and convex subset of the Banach space E. In this case we have $f^* = \sup\langle A, \cdot\rangle$, $g^* = \sup\langle B, \cdot\rangle$, $\operatorname{dom}(f^*) = \operatorname{dom}(g^*) = E^*$, $\widehat{f} = \delta_{\widehat{A}}$, $\operatorname{cl}(\widehat{f}) = \delta_{\operatorname{cl}(\widehat{A})}$, thus, in view of Lemma 1, $f^{**} = \delta_{\operatorname{cl}(\widehat{A})}$, where the closure is considered in the topology $\omega(E^{**}, E^*)$. We mention that the formula $\delta_A^{**} = \delta_{\operatorname{cl}(\widehat{A})}$ was obtained also in section 4 of [4] and thus Lemma 1 is a generalization of this result. Further, $g^{**} = \delta_{\operatorname{cl}(\widehat{B})}$ and $P(y^*) = \sup\langle y^*, D\rangle$, for all $y^* \in E^*$, where $D = \operatorname{cl}(\widehat{A}) \cap \operatorname{cl}(\widehat{B})$. Applying Corollary 1 to this particular case (the hypotheses regarding the functions f and g are obviously fulfilled) we obtain the following result.

Corollary 2. *We have* $E^* \times (0, \infty) \subset \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$ *if and only if* x_0 *is a weak***-extreme point of* C.

Remark 2. The above result gives the solution to Problem 1 (see Remark 1), namely the answer is positive if and only if x_0 is a weak*-extreme point of C. Let us mention that the closedness of the set C, requested in [14], is not needed anymore for this result.

2.2 An alternative solution

By means of a minimax theorem we give in this subsection an alternative proof of Corollary 2, hence an alternative solution to Problem 1 (see Remark 2).

Proof. Let $y^* \in E^*$ and $\varepsilon > 0$ be arbitrary. In view of Remark 1, the condition $(y^*, \varepsilon) \in \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$ is equivalent to the statement that there exists $z^* \in E^*$ such that for all $x, y \in E$, $f(x) + g(y) - \langle x - y, z^* \rangle \geq \langle y, y^* \rangle - \varepsilon$, which is nothing else than there exists $z^* \in E^*$ such that for all $u, v \in C$, $\langle u + v - 2x_0, z^* \rangle + \langle x_0 - v, y^* \rangle \geq -\varepsilon$. Hence the inclusion $E^* \times (0, \infty) \subset \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$ is fulfilled if and only if:

$$\sup_{z^* \in E^*} \inf_{(u,v) \in C \times C} [\langle u + v - 2x_0, z^* \rangle + \langle x_0 - v, y^* \rangle] \geq 0 \text{ for all } y^* \in E^*. \quad (4)$$

Let us suppose first that x_0 is a weak*-extreme point of C. Take $y^* \in E^*$. For $z^* \in E^*$, we have

$$\inf_{(u,v) \in C \times C} [\langle u+v-2x_0, z^* \rangle + \langle x_0 - v, y^* \rangle] = \inf_{(u,v) \in \widehat{C} \times \widehat{C}} [\langle z^*, u+v-2\widehat{x_0} \rangle + \langle y^*, \widehat{x_0} - v \rangle]$$

$$= \inf_{(u,v) \in \operatorname{cl} \widehat{C} \times \operatorname{cl} \widehat{C}} [\langle z^*, u + v - 2\widehat{x_0} \rangle + \langle y^*, \widehat{x_0} - v \rangle],$$

where the first equality follows by the definition of the canonical embedding and the second one is a consequence of the continuity (in the weak* topology $\omega(E^{**}, E^*)$) of the functions $\langle x^*, \cdot \rangle : E^{**} \to \mathbb{R}$, for all $x^* \in E^*$. The set C being bounded, we use the celebrated Banach-Alaoglu theorem to conclude that the set $\operatorname{cl} \widehat{C}$ is weak*-compact. We apply a minimax theorem (see for example Theorem 3.1 in [13]) and obtain that

$$\sup_{z^* \in E^*} \inf_{(u,v) \in C \times C} [\langle u + v - 2x_0, z^* \rangle + \langle x_0 - v, y^* \rangle] =$$

$$\sup_{z^* \in E^*} \inf_{(u,v) \in \operatorname{cl} \widehat{C} \times \operatorname{cl} \widehat{C}} [\langle z^*, u + v - 2\widehat{x_0} \rangle + \langle y^*, \widehat{x_0} - v \rangle] =$$

$$\inf_{(u,v) \in \operatorname{cl} \widehat{C} \times \operatorname{cl} \widehat{C}} \sup_{z^* \in E^*} [\langle z^*, u + v - 2\widehat{x_0} \rangle + \langle y^*, \widehat{x_0} - v \rangle] = \inf_{\substack{(u,v) \in \operatorname{cl} \widehat{C} \times \operatorname{cl} \widehat{C} \\ u+v=2\widehat{x_0}}} \langle y^*, \widehat{x_0} - v \rangle.$$

Since x_0 is a weak*-extreme point of C we get that $\{(u, v) \in \operatorname{cl} \widehat{C} \times \operatorname{cl} \widehat{C} : u + v = 2\widehat{x_0}\} = \{(\widehat{x_0}, \widehat{x_0})\}$, hence $\inf_{\substack{(u,v) \in \operatorname{cl} \widehat{C} \times \operatorname{cl} \widehat{C} \\ u+v=2\widehat{x_0}}} \langle y^*, \widehat{x_0} - v \rangle = 0$. Thus relation (4) is fulfilled, implying $E^* \times (0, \infty) \subset \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$.

On the other hand, consider the case $E^* \times (0, \infty) \subset \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$ and suppose that x_0 is not a weak*-extreme point of C. Then there exist $u_0, v_0 \in \operatorname{cl} \widehat{C} \times \operatorname{cl} \widehat{C}, u_0 + v_0 = 2\widehat{x_0}$ such that $u_0 \neq \widehat{x_0}$ and $v_0 \neq \widehat{x_0}$. We can choose

$y_0^* \in E^*$ such that $\langle y_0^*, \widehat{x_0} - v_0 \rangle < 0$. Thus there exists $\varepsilon_0 > 0$ such that $\langle y_0^*, \widehat{x_0} - v_0 \rangle < -\varepsilon_0$, hence $\inf_{\substack{(u,v) \in \operatorname{cl} \widehat{C} \times \operatorname{cl} \widehat{C} \\ u+v=2\widehat{x_0}}} \langle y_0^*, \widehat{x_0} - v \rangle < -\varepsilon_0$. As above, we get

$$\sup_{z^* \in E^*} \inf_{(u,v) \in C \times C} [\langle u + v - 2x_0, z^* \rangle + \langle x_0 - v, y_0^* \rangle] < -\varepsilon_0 < 0,$$

which contradicts (4), hence the proof is complete. □

Acknowledgements. The authors would like to express their gratitude to Stephen Simons for carefully reading a preliminary version of the paper and for providing some ideas in a personal communication which are generalized in subsection 2.1.

References

[1] Boţ, R.I., Wanka G. (2006): *A weaker regularity condition for subdifferential calculus and Fenchel duality in infinite dimensional spaces*, Nonlinear Analysis: Theory, Methods & Applications 64 (12), 2787–2804.

[2] Bourgain, J. (1978): *A geometric characterization of the Radon-Nikodým property in Banach spaces*, Compositio Mathematica 36 (1), 3–6.

[3] Burachik, R.S., Jeyakumar, V., Wu, Z.-Y. (2006): *Necessary and sufficient conditions for stable conjugate duality*, Nonlinear Analysis 64 (9), 1998–2006.

[4] Chakrabarty, A.K., Shunmugaraj, P., Zălinescu, C. (2007): *Continuity properties for the subdifferential and ε-subdifferential of a convex function and its conjugate*, Journal of Convex Analysis 14 (3), 479–514.

[5] Dutta, S., Rao, T.S.S.R.K. (2003): *On weak*-extreme points in Banach spaces*, Journal of Convex Analysis 10 (2), 531–539.

[6] Ekeland, I., Temam, R. (1976): *Convex analysis and variational problems*, North-Holland Publishing Company, Amsterdam.

[7] Godun, B.V., Lin, B.-L, Troyanski, S.L. (1992): *On the strongly extreme points of convex bodies in separable Banach spaces*, Proceedings of the American Mathematical Society 114 (3), 673–675.

[8] Gowda, M.S., Teboulle, M. (1990): *A comparison of constraint qualifications in infinite-dimensional convex programming*, SIAM Journal on Control and Optimization 28, 925–935.

[9] Huff, R.E., Morris, P.D. (1975): *Dual spaces with the Krein-Milman property have the Radon-Nikodým property*, Proceedings of the American Mathematical Society 49 (1), 104–108.

[10] Kunen, K., Rosenthal, H. (1982): *Martingale proofs of some geometrical results in Banach space theory*, Pacific Journal of Mathematics 100 (1), 153–175.

[11] Morris, P. (1983): *Disappearance of extreme points*, Proceedings of the American Mathematical Society 88 (2), 244–246.

[12] Phelps, R.R. (1961): *Extreme points of polar convex sets*, Proceedings of the American Mathematical Society 12 (2), 291–296.

[13] Simons, S. (1998): *Minimax and Monotonicity*, Springer-Verlag, Berlin.

[14] Simons, S.: *From Hahn-Banach to Monotonicity*, Springer-Verlag, Berlin, in press.

[15] Zălinescu, C. (2002): *Convex analysis in general vector spaces*, World Scientific, Singapore.