Fredholmness and index of operators in the Wiener algebra are independent of the underlying space.

Marko Lindner

February 7, 2008

Abstract. The purpose of this paper is to demonstrate the so-called Fredholm-inverse closedness of the Wiener algebra $\mathcal{W}$ and to deduce independence of the Fredholm property and index of the underlying space. More precisely, we look at operators $A \in \mathcal{W}$ as acting on a family of vector valued ℓ^p-spaces and show that the Fredholm regularizer of A for one of these spaces can always be chosen in $\mathcal{W}$ as well and therefore regularizes A (modulo compact operators) on all of the ℓ^p-spaces under consideration. We conclude that both Fredholmness and the index of A do not depend on the ℓ^p-space that A is considered as acting on.

Mathematics subject classification (2000): 47A53, 47B37, 47L10, 47B36, 46E40.
Keywords and phrases: Fredholm operator, Fredholm index, Wiener algebra, operators on ℓ^p spaces.

1 Introduction and Preliminaries

We study bounded linear operators A on a family $\{E^p\}_{p\in P}$ of sequence spaces. Our operators can be identified with infinite matrices which have an absolutely summable off-diagonal decay, and we are interested in their Fredholm property and index if considered as acting on one of the spaces E^p. Our main result is that neither Fredholmness nor the index of A depend on the parameter p of the underlying space.

Let $N \in \mathbb{N}$ and let X be a complex Banach space. The spaces E^p that we have in mind are spaces of functions $u : \mathbb{Z}^N \to X$. In particular, for $p \in [1, \infty]$, we put $E^p := \ell^p(\mathbb{Z}^N, X)$, equipped with the usual norm $\|u\|_{E^p} := \| (\|u(k)\|_X)_k \|_{\ell^p}$. In addition, we let $E^0 := c_0(\mathbb{Z}^N, X)$ refer to the closure in E^∞ of the space of all sequences with finite support. If we simply write of E then the corresponding statement is meant to hold with any of the spaces E^p, $p \in P := \{0\} \cup [1, \infty]$, in place of E.

Let $L(X)$ denote the set of all bounded linear operators on X. Given a matrix $M = [m_{ij}]_{i,j\in\mathbb{Z}^N}$ with entries $m_{ij} \in L(X)$, we say that M *induces* a bounded operator A on E if

$$(Au)(i) \;=\; \sum_{j\in\mathbb{Z}^N} m_{ij}\, u(j), \qquad i \in \mathbb{Z}^N, \tag{1}$$

the sum converges in X for every $i \in \mathbb{Z}^N$ and every $u \in E$ and if the resulting operator A is a bounded mapping $E \to E$. An operator $A \in L(E)$ is called a *band operator* if it is induced by a banded matrix M, that means $m_{ij} = 0$ if $|i - j| > w$ for some $w \geq 0$. Clearly, if A is bounded on one space E^p then every diagonal d_k of the inducing matrix M is a bounded sequence of elements in $L(X)$ and therefore A is bounded on all spaces E^p. We now put

$$\|A\|_{\mathcal{W}} \;:=\; \sum_{k\in\mathbb{Z}^N} \|d_k\|_\infty \;=\; \sum_{k\in\mathbb{Z}^N} \sup_{j\in\mathbb{Z}^N} \|m_{j+k,j}\|_{L(X)}$$

and denote by $\mathcal{W}$ the closure of the set of all band operators in the norm $\|.\|_{\mathcal{W}}$. The set $\mathcal{W}$, equipped with addition, multiplication by scalars, operator composition and with the norm $\|.\|_{\mathcal{W}}$, turns out to be a Banach algebra (with unit $I : u \mapsto u$) and is called *the Wiener algebra*. Note that this is a natural (non-stationary) extension of the classical algebra of all operators with constant diagonals and $\|A\|_W < \infty$

(which is isomorphic, via Fourier transform, to the algebra of all periodic functions with absolutely summable sequence of Fourier coefficients). Like band operators, operators in the Wiener algebra act boundedly on all spaces E^p. A deep and remarkable result about $\mathcal{W}$ is its inverse closedness; that is, if $A \in \mathcal{W}$ is invertible on one of the spaces E^p, its inverse A^{-1} is automatically in $\mathcal{W}$ again [18, Theorem 2.5.2] (see [1] for the classic stationary case) and therefore acts as the inverse of A on all spaces E^p.

What's new? One of the main aims of this paper is to show that a very similar result holds for Fredholmness in place of invertibility. Recall that, by Calkin's theorem, $A \in L(E)$ is a Fredholm operator iff there is a so-called regularizer $B \in L(E)$ and two compact operators K and L on E such that $AB = I + K$ and $BA = I + L$ hold. We will show that $\mathcal{W}$ is Fredholm-inverse closed, meaning that, if $A \in \mathcal{W}$ is Fredholm on one of the spaces E^p in our family then its regularizer $B \in L(E^p)$ can always be chosen in the Wiener algebra $\mathcal{W}$ as well and the operators K and L are not only compact on this particular space E^p but on all the spaces under consideration, which clearly shows that A is Fredholm on all of them. We then carry on showing that also the index of A does not depend on the space that A is considered as acting on.

The general result about Fredholm-inverse closedness of $\mathcal{W}$ appears to be new. Results of the type

$$\begin{array}{c}\text{If } A \in \mathcal{W} \text{ is Fredholm on one space } E^p \text{ with } p \in P \\ \text{then } A \text{ is Fredholm on all spaces } E^q \text{ with } q \in Q.\end{array} \tag{2}$$

and

$$\text{Moreover, the index of } A \text{ on } E^q \text{ is the same for all } q \in Q. \tag{3}$$

are known but in less general settings: In [10, 17], statement (2) was shown with $P = Q = \{0\} \cup [1, \infty]$ in the particular case when $X = \mathbb{C}$. In [18], (2) with $P = Q = \{0\} \cup (1, \infty)$ was extended to arbitrary reflexive Banach spaces X. Later, in [19], statement (3) was shown for $Q = (1, \infty)$ and $X = \mathbb{C}$. Note that, by a general result [22] about Fredholm operators on interpolational families of Banach spaces, statement (2) automatically implies that the index of A is the same on all spaces E^q with $q \in Q'$ for every open interval $Q' \subset Q$. Also note that, for particular operator classes, statements of the form (2) and (3) are well-studied in the literature (e.g. [8, 20, 21] for Schrödinger operators and [2, 6] for Toeplitz operators with continuous symbol).

In [15, 3] the step to an arbitrary Banach space X was done but for the price that $A \in \mathcal{W}$ has to be of the form $I + C$ with C being induced by a matrix with (collectively) compact entries. In this setting, it was shown in [3] that (2) holds with $P = \{0\} \cup [1, \infty)$, $Q = \{0\} \cup [1, \infty]$ and, under the additional assumption that X is the dual of another space, denoted by $X^{\triangleleft}$, and A, if considered as acting on $E^\infty = \ell^\infty(\mathbb{Z}^N, X)$, is the adjoint of another operator, say $A^{\triangleleft}$ on $\ell^1(\mathbb{Z}^N, X^{\triangleleft})$, (2) was also shown for $P = \{\infty\}$ and $Q = \{0\} \cup [1, \infty]$. Moreover, in the same paper, statement (3) was shown for arbitrary $A \in \mathcal{W}$ in the case of a finite-dimensional space X and $Q = \{0\} \cup [1, \infty]$.

Now, in the current paper, we show that, for an arbitrary $A \in \mathcal{W}$ and a vast selection of Banach spaces X (namely those of finite dimension plus those possessing a subspace of codimension 1 that is isomorphic to X), statement (2) holds with $P = \{0\} \cup [1, \infty)$, $Q = \{0\} \cup [1, \infty]$ and, under the additional condition that $X^{\triangleleft}$ and $A^{\triangleleft}$ exist, with $P = \{\infty\}$ and $Q = \{0\} \cup [1, \infty]$. We moreover show that both of these statements are complemented by (3) with $Q = \{0\} \cup [1, \infty]$. These results follow almost immediately from the observation that $A \in \mathcal{W}$ has a Fredholm regularizer in $\mathcal{W}$ if Fredholm on one of the spaces E^p.

Our proof takes the idea of that of [19, Lemma 2.1] a bit further and combines it with duality results from [3, §6]. Note that, unlike in most of the papers cited above, our arguments are based on Fredholm properties only and do not make a detour, via so-called invertibility at infinity (alias $\mathcal{P}$-Fredholmness), to the invertibility (and this is where the p-invariance usually comes in) of all so-called limit operators of A. The benefit of not using this heavy machinery is that we can even extend our results to larger families of spaces E, for which the limit operator approach has not been developed (yet). We will say a bit more about these possibilities in §4 at the end of the paper. Concerning limit operators [18, 11], it should be added that they have proven an effective tool to actually check Fredholmness [10, 17, 18, 11] and in some cases even calculate the index [16, 19, 15] of $A \in \mathcal{W}$. But this shall not be the subject of this paper.

Contents of the paper. In §2 we make a short intermezzo on Stefan Banach's famous hyperplane problem and its connection with the existence of Fredholm operators of a certain index. In §3 we state and prove the main theorem of this paper before we discuss some possible extensions in §4.

Acknowledgements. The author would like to thank his friend and colleague Leslie J Bunce from Reading, UK, for the exciting conversations on the geometry of Banach spaces and the astonishing surprises they are keeping for us. Moreover, I would like to acknowledge the financial support by the European Union (Marie Curie Fellowship, MEIF-CT-2005-009758).

2 Fredholm operators and the hyperplane problem

Let X be an infinite-dimensional complex Banach space and $A \in L(X)$. As usual, denote by

$$\ker A := \{x \in X : Ax = 0\} \qquad \text{and} \qquad \operatorname{im} A := \{Ax : x \in X\}$$

the *kernel* (or *null-space*) and *image* (or *range*) of A. If

$$\alpha(A) := \dim \ker A < \infty \qquad \text{and} \qquad \beta(A) := \operatorname{codim}_X \operatorname{im} A < \infty$$

(in which case $\operatorname{im} A$ is automatically closed), then we say that A is a *Fredholm operator* on X. In this case we refer to the integer

$$\operatorname{ind} A := \alpha(A) - \beta(A)$$

as its *index* and let $\operatorname{coim} A$ and $\operatorname{coker} A$ denote a complement space of $\ker A$ and $\operatorname{im} A$, respectively, in X. (Note that $\ker A$ and $\operatorname{im} A$ are complementable since $\alpha(A), \beta(A) < \infty$.) Clearly, it holds that

$$\operatorname{coim} A \;\cong\; X \,/\, \ker A \;\cong\; \operatorname{im} A.$$

Lemma 2.1 *$A \in L(X)$ is Fredholm of index zero iff there exist an invertible operator $B \in L(X)$ and a compact operator $K \in L(X)$ such that $A = B + K$.*

Proof. If $A = B + K$ with B invertible (hence Fredholm of index 0) and K compact then A is Fredholm and $\operatorname{ind} A = \operatorname{ind}(B + K) = \operatorname{ind} B = 0$.

Conversely, let $A \in L(X)$ be Fredholm of index zero. Then $\ker A$ and $\operatorname{coker} A$ are isomorphic since they have both dimension $\alpha(A) = \beta(A) < \infty$. Let $T : \ker A \to \operatorname{coker} A$ be an isomorphism (i.e. an invertible linear operator) and put $B := A + TP_{\ker A}$ with $P_{\ker A}(k+c) := k$ for all $k \in \ker A$ and $c \in \operatorname{coim} A$. Then, with $x \in X$ decomposed as $x = k + c$, we get

$$Bx = (A + TP_{\ker A})(k + c) = Ak + Ac + TP_{\ker A}(k + c) = Ac + Tk$$

for all $k \in \ker A$ and $c \in \operatorname{coim} A$. Consequently, $\operatorname{im} B = \operatorname{im} A + \operatorname{im} T = \operatorname{im} A + \operatorname{coker} A = X$ and $\ker B = \{0\}$ since $0 = Bx = Ac + Tk$ implies $Ac = Tk = 0$ and hence $c = 0$, $k = 0$ and $x = 0$. So we have $A = B + K$ with $B \in L(X)$ invertible and $K = -TP_{\ker A}$ of finite rank and therefore compact. ∎

Lemma 2.2 *The following are equivalent for an infinite-dimensional complex Banach space X.*

- *(i)* *X is isomorphic to a subspace $Y \subset X$ of codimension* 1.
- *(ii)* *X is isomorphic to $X \times \mathbb{C} = \{(x, \lambda) : x \in X, \lambda \in \mathbb{C}\}$.*
- *(iii)* *There exists a Fredholm operator $A \in L(X)$ with $\operatorname{ind} A = 1$.*
- *(iv)* *There exists a Fredholm operator $B \in L(X)$ with $\operatorname{ind} B = -1$.*

Proof. *(i)* ⇒ *(ii)* Take $Y \subset X$ with $\operatorname{codim}_X Y = 1$ and $X \cong Y$. Pick $z \in X \setminus Y$ and let $Z = \{\lambda z : \lambda \in \mathbb{C}\}$. Then $X = \{y + \lambda z : y \in Y, \lambda \in \mathbb{C}\} \cong Y \times \mathbb{C} \cong X \times \mathbb{C}$ since $Y \cong X$.

(ii) ⇒ *(iii)* Take $A : X \cong X \times \mathbb{C} \to X$ with $A : (x, \lambda) \mapsto x$ for all $x \in X, \lambda \in \mathbb{C}$.

(iii) ⇒ *(iv)* Let $A \in L(X)$ be Fredholm with index 1. Then there are $B, K, L \in L(X)$ with $AB = I+K$, $BA = I + L$ and K, L compact. But this shows that B is Fredholm with index -1 since $\operatorname{ind} A + \operatorname{ind} B = \operatorname{ind} AB = \operatorname{ind}(I + K) = 0$.

(iv) ⇒ *(i)* Let $B \in L(X)$ be Fredholm with index -1. Choose a subspace Z of $\operatorname{coker} B$ with $\dim Z = \dim \operatorname{coker} B - 1 = \beta(B) - 1 = \alpha(B) = \dim \ker B$ and let $T : \ker B \to Z$ be an isomorphism. Then $A : X \to Z + \operatorname{im} B$ with $A(k+c) := Tk + Bc$ for all $k \in \ker B$ and $c \in \operatorname{coim} B$ is an isomorphism between X and its 1-codimensional subspace $Z + \operatorname{im} B$. ■

Definition 2.3 *An infinite-dimensional complex Banach space X is said to have the hyperplane property if it is subject to property (i) (and therefore any of (i) – (iv)) of Lemma 2.2. We write $\mathcal{H}_\infty$ for the set of all infinite-dimensional complex Banach spaces with the hyperplane property, and we let $\mathcal{H}$ denote the union of $\mathcal{H}_\infty$ with the set of all finite-dimensional complex spaces.*

Remark 2.4 It has been an open problem, the so-called *hyperplane problem*, posed by Stefan Banach in his famous "Scottish Book", whether or not there are any complex Banach spaces outside of $\mathcal{H}$. In 1993, more than 50 years later, it was William Timothy Gowers who solved this and two more of Banach's classical problems by constructing a Banach space that is not in $\mathcal{H}$ [7]. Gowers was subsequently awarded the Fields Medal in 1998 for his important contributions to functional analysis by combining it with combinatorial ideas. Note that Gowers constructed a Banach space X which is not isomorphic to any of its finite-codimensional subspaces. As a consequence, in $L(X)$ there are no Fredholm operators with a non-zero index! □

In the current paper we will prove a theorem about Fredholm operators on spaces of functions $\mathbb{Z}^N \to X$ with $X \in \mathcal{H}$. Judging by the fact that the discovery of Banach spaces $X \notin \mathcal{H}$ took a long time (and was worth a Fields Medal) it seems pretty safe to assume that your given Banach space X at hand is contained in $\mathcal{H}$ and is therefore covered by the main result of this paper. We will however give some sufficient criteria here for membership in $\mathcal{H}$. The following lemma is the result of personal communication with Les Bunce from Reading, UK.

Lemma 2.5 *Let X be an infinite-dimensional complex Banach space. Then the following hold.*

(i) *$X \in \mathcal{H}_\infty$ implies that $X^* \in \mathcal{H}_\infty$. The converse is in general not true.*

(ii) *The direct sum $X = Y \dotplus Z$ is in $\mathcal{H}_\infty$ if one of Y and Z is in $\mathcal{H}_\infty$.*

(iii) *The spaces $c_0 := c_0(\mathbb{N}, \mathbb{C})$ and $\ell^p := \ell^p(\mathbb{N}, \mathbb{C})$ with $1 \le p \le \infty$ are in $\mathcal{H}_\infty$. Consequently, all spaces $c_0(\Omega, Y)$ and $\ell^p(\Omega, Y)$ with Ω at most countable, Y a finite-dimensional complex space, and $1 \le p \le \infty$ are in $\mathcal{H}$.*

(iv) *If $c_0 \precapprox X$ (meaning that X contains an isomorphic copy of c_0) and X is separable, then $X \in \mathcal{H}_\infty$.*

(v) *If $c_0 \precapprox X^*$, then $X \in \mathcal{H}_\infty$.*

(vi) *If $\ell^\infty \precapprox X$, then $X \in \mathcal{H}_\infty$.*

(vii) *If μ is a σ-finite nonatomic measure over an infinite set Ω and $1 \le p \le \infty$, then $L^p(\Omega, \mu) \in \mathcal{H}_\infty$.*

(viii) *If K is an infinite compact metric space, then $C(K) \in \mathcal{H}_\infty$.*

(ix) *If X is a separable C^*-algebra, then $X \in \mathcal{H}_\infty$. There are (non-separable) C^*-algebras $X \notin \mathcal{H}$.*

(x) *If X is a C^*-algebra then $X^* \in \mathcal{H}_\infty$.*

(xi) *If X is a von Neumann algebra, then both X and its (unique) predual $X^\triangleleft$ are in $\mathcal{H}_\infty$.*

Remark 2.6 In connection with *(viii)*, we would like to remark that already an infinite compact (not necessarily metrisable) space K is enough if it has a nontrivial convergent sequence as this sequence can be used to construct a complementable copy of c_0 in $C(K)$ (see e.g. [14]).

We would also like to mention that there exist (non-separable) examples of $C(K) \notin \mathcal{H}$. For an example of a non-metrisable compact Hausdorff space K with this property see [9, 14]. □

Proof.

(i) Recall Lemma 2.2 and that if $A \in L(X)$ is Fredholm of index 1, then $B = A^* \in L(X^*)$ is Fredholm of index -1 on X^*. For an example of $X \notin \mathcal{H}_\infty$ and $X^* \in \mathcal{H}_\infty$ see (ix) and (x).

(ii) Suppose $Y \in \mathcal{H}_\infty$ and $A \in L(Y)$ is Fredholm of index 1. Then $A' \in L(X)$ with $A'(y+z) = Ay + z$ for all $y \in Y$ and $z \in Z$ is Fredholm of index 1 on X.

(iii) The backward shift $(Au)(k) = u(k+1)$, $k \in \mathbb{N}$, on c_0 and ℓ^p, $1 \le p \le \infty$, is Fredholm of index 1. Now let Ω be at most countable and Y be a finite-dimensional complex space. If Ω is finite, then $c_0(\Omega, Y)$ and all $\ell^p(\Omega, Y)$ are finite-dimensional and therefore in $\mathcal{H}$. If Ω is countable, then $c_0(\Omega, Y) \cong c_0 \in \mathcal{H}_\infty$ and $\ell^p(\Omega, Y) \cong \ell^p \in \mathcal{H}_\infty$ for all $p \in [1, \infty]$.

(iv) By Theorem 5 of [23] (or see [4, 24]) we have that the isomorphic copy of c_0 is complementable in X, i.e. $X \cong c_0 \dotplus Z$ for some space Z. From (iii) and (ii) we get that $X \in \mathcal{H}_\infty$.

(v) By $c_0 \precsim X^*$ we know that a copy of ℓ^1 is complementable in X (see e.g. [4, 24]). By (iii) and (ii) we get $X \in \mathcal{H}_\infty$.

(vi) By [12] (or see [4, 24]) we know that the copy of ℓ^∞ is automatically complementable in X, and hence $X \in \mathcal{H}_\infty$ by (iii) and (ii).

(vii) Let μ be a σ-finite nonatomic measure on Ω. For every $p \in [1, \infty]$, there is a complementable copy of ℓ^p in $L^p(\Omega, \mu)$. To see this, let $\Omega_1, \Omega_2, ...$ be disjoint subsets of Ω each having positive measure and take $S : L^p(\Omega, \mu) \to \ell^p$ with $(Sf)(k) = 1/\mu(\Omega_k) \int_{\Omega_k} f \, d\mu$ and $R : \ell^p \to L^p(\Omega, \mu)$ with $(Ru)(x) = u(k) \,/\, \mu(\Omega_k)^{1/p}$ (putting $1/\infty = 0$) for $x \in \Omega_k$ and 0 otherwise. Then S is bounded (Hölder inequality), R is an isometry, $SR = I$ and ℓ^p can be identified with the image of the projector RS in $L^p(\Omega, \mu)$; so it has a complement (e.g. $\ker RS$). Hence, by (iii) and (ii), $L^p(\Omega, \mu) \in \mathcal{H}_\infty$.

(viii) If K is an infinite compact metric space, then $C(K)$ is separable and $c_0 \precsim C(K)$ (take an infinite sequence $K_1, K_2, ...$ of pairwise disjoint open subsets of K and look at functions $f \in C(K)$ which are constant on each K_n to see the latter). Together with (iv) we get $C(K) \in \mathcal{H}_\infty$.

(ix) Let X be a separable C^*-algebra. We may assume that X has a unit e and that there exists an $a \in X$ with $a = a^*$ and an infinite spectrum $\sigma(a)$. Let $\mathcal{A}$ denote the C^*-subalgebra of X that is generated by e and a. Then $\mathcal{A} \cong C(\sigma(a))$ contains an isomorphic copy of c_0 (as seen in the proof of (viii)) so that $c_0 \precsim \mathcal{A} \subset X$ and hence $X \in \mathcal{H}_\infty$ by (iv).

(x) If X is a C^*-algebra, then (as seen before) $c_0 \precsim X \precsim X^{**} = (X^*)^*$. Hence, by (v), $X^* \in \mathcal{H}_\infty$.

(xi) If X is a von Neumann algebra, then $\ell^\infty \precsim X$ so that $X \in \mathcal{H}_\infty$ by (vi). But also $X^\triangleleft \in \mathcal{H}_\infty$ by $c_0 \subset \ell^\infty \precsim X = (X^\triangleleft)^*$ and (v).

■

3 Main result

Theorem 3.1 *Let $X \in \mathcal{H}$ and $A \in \mathcal{W}$. Then the following hold.*

a) *If A is Fredholm on one of the spaces E^p with $p \in \{0\} \cup [1, \infty)$ then A is Fredholm on all the spaces E^q with $q \in \{0\} \cup [1, \infty]$.*

b) *If X has a predual $X^\triangleleft$ and A, considered as acting on $E^\infty = \ell^\infty(\mathbb{Z}^N, X)$, has a preadjoint $A^\triangleleft$ on $\ell^1(\mathbb{Z}^N, X^\triangleleft)$ and if A is Fredholm on E^∞ then A is Fredholm on all the spaces E^q with $q \in \{0\} \cup [1, \infty]$.*

In both cases, the index of A is the same on all these spaces E^q with $q \in \{0\} \cup [1, \infty]$.

Remark 3.2 a) We conjecture that the condition $X \in \mathcal{H}$ is not necessary in this statement (see §4).

b) In the particularly simple case of a finite-dimensional space X we know that $X \in \mathcal{H}$, the predual of X exists (it can be identified with X^* and therefore with X itself) and the preadjoint operator of $A \in L(E^\infty)$ always exists and is induced by $[m_{ji}^*]$ on $\ell^1(\mathbb{Z}^N, X^*)$. So in this case we can drop these two conditions from statement **b)** of Theorem 3.1 and merge **a)** and **b)** into one statement with $p \in \{0\} \cup [1, \infty]$.

c) Note that, for $A \in L(Y)$ with a Banach space Y that is the dual of another space Z, the statements

(i) A is the adjoint of an operator $B \in L(Z)$.
(ii) The adjoint A^* maps Z, understood as a subspace of its second dual $Z^{**} = Y^*$, into itself.
(iii) A is continuous in the weak-$*$ topology on Y.

are equivalent. □

The rest of this section is devoted to the proof of Theorem 3.1. We start with two lemmas. But first we define the truncation operator $P_m : E \to E$ by

$$(P_m u)(k) \;:=\; \begin{cases} u(k), & k \in \{-m, ..., m\}^N, \\ 0, & \text{otherwise} \end{cases}$$

for every $u \in E$, $m \in \mathbb{N}$ and $k \in \mathbb{Z}^N$, and we put $Q_m := I - P_m$.

Lemma 3.3 *If $p \in \{0\} \cup (1, \infty)$ and K is compact on E^p then $\|K - P_m K P_m\| \to 0$ as $m \to \infty$.*

Proof. The claim follows from the bound

$$\begin{aligned} \|K - P_m K P_m\| &= \|P_m K Q_m + Q_m K\| \;\le\; \|(P_m K Q_m)^*\| + \|Q_m K\| \\ &\le \|Q_m^* K^*\| \cdot \|P_m^*\| + \|Q_m K\| \;\to\; 0 \end{aligned}$$

as $m \to \infty$ since

$$\|P_m\| \text{ remains bounded}, \quad Q_m \to 0 \text{ and } Q_m^* \to 0 \text{ pointwise} \quad \text{as } m \to \infty \tag{4}$$

on E^p and $(E^p)^*$, respectively, and since K and K^* are compact on E^p and $(E^p)^*$, respectively. ■

Lemma 3.4 *Let $m \in \mathbb{N}$ and $p \in \{0\} \cup [1, \infty]$. If $P_m K P_m$ is compact on E^p then it is compact on all spaces E^q with $q \in \{0\} \cup [1, \infty]$.*

Proof. Let $P_m K P_m$ be compact on E^p. Now let $q \in \{0\} \cup [1, \infty]$ and take an arbitrary bounded sequence $(u_k) \subset E^q$. We have to show that $(P_m K P_m u_k)_k$ has an E^q-convergent subsequence. W.l.o.g. we can restrict ourselves to elements $u_k \in \operatorname{im} P_m$. Now note that on $\operatorname{im} P_m$ all the E^q-norms are equivalent. So (u_k) is also bounded in E^p and, by our assumption, $(P_m K P_m u_k)_k$ has an E^p-convergent subsequence. But since $P_m K P_m u_k \in \operatorname{im} P_m$ for every k and again since the norms are equivalent on $\operatorname{im} P_m$, the same subsequence also converges in the norm of E^q. ■

Remark 3.5 From the proof of Lemma 3.4 we see that this statement generalizes to any family of spaces E of functions $u : \mathbb{Z}^N \to X$ the different norms of which are equivalent on $\operatorname{im} P_m$. This is, for example, the case when $\|u\|_E$ is defined in terms of the scalar sequence $(\|u(k)\|_X)_{k \in \mathbb{Z}^N}$ for every E. □

In the following, we will write $\operatorname{ind}_p A$ for the index of $A \in \mathcal{W}$ on E^p. An essential ingredient to the proof of Theorem 3.1 is a family $\{S_\kappa\}_{\kappa \in \mathbb{Z}}$ of operators in $\mathcal{W}$ with $\operatorname{ind}_p S_\kappa = \kappa$ for all $p \in \{0\} \cup [1, \infty]$ and all $\kappa \in \mathbb{Z}$. Here is one way to choose this family.

If $n := \dim X < \infty$, let $e_1, ..., e_n$ be a basis in X, write $u \in E$ as

$$u(k_1, k_2, ..., k_N) \;=\; \sum_{i=1}^{n} u_i(k_1, k_2, ..., k_N)\, e_i$$

with $u_i(k_1, k_2, ..., k_N) \in \mathbb{C}$ for all $k_1, ..., k_N \in \mathbb{Z}$ and $i = 1, ..., n$, and put

$$(S_{-1}u)(k_1, k_2, ..., k_N) := \begin{cases} 0e_1 + \sum_{i=2}^n u_{i-1}(k_1, k_2, ..., k_N)e_i, & k_1 = ... = k_N = 0, \\ u_n(k_1 - 1, k_2, ..., k_N)e_1 + \sum_{i=2}^n u_{i-1}(k_1, k_2, ..., k_N)e_i, & k_1 > 0, k_2 = ... = k_N = 0, \\ u(k_1, k_2, ..., k_N), & \text{otherwise} \end{cases}$$

and

$$(S_1 u)(k_1, k_2, ..., k_N) := \begin{cases} \sum_{i=1}^{n-1} u_{i+1}(k_1, k_2, ..., k_N)e_i + u_1(k_1 + 1, k_2, ..., k_N)e_n, & k_1 \geq 0, k_2 = ... = k_N = 0, \\ u(k_1, k_2, ..., k_N), & \text{otherwise.} \end{cases}$$

If $\dim X = \infty$ choose $T_{-1}, T_1 \in L(X)$ with $\operatorname{ind} T_{\pm 1} = \pm 1$, respectively, which is possible by $X \in \mathcal{H}_\infty$ and Lemma 2.2. Now, for every $u \in E$, put

$$(S_{\pm 1}u)(k) = \begin{cases} T_{\pm 1}(u(0)), & k = 0, \\ u(k), & k \neq 0, \end{cases}$$

respectively, for all $k \in \mathbb{Z}^N$, i.e. $S_{\pm 1} = \operatorname{diag}(..., I_X, I_X, T_{\pm 1}, I_X, I_X, ...)$ with $T_{\pm 1}$ at position zero.

In either case, $\dim X$ finite or infinite, now put

$$S_\kappa := \begin{cases} S_1^\kappa, & \kappa > 0, \\ I, & \kappa = 0, \\ S_{-1}^{-\kappa}, & \kappa < 0 \end{cases}$$

for all $\kappa \in \mathbb{Z}$, and it follows from $\operatorname{ind}_p S_{\pm 1} = \pm 1$ for all $p \in \{0\} \cup [1, \infty]$ that $\operatorname{ind}_p S_\kappa = \kappa$ for all $\kappa \in \mathbb{Z}$ and all p. Also note that, by our construction, $S_\kappa \in \mathcal{W}$ for all $\kappa \in \mathbb{Z}$.

We are now ready for the proof of Theorem 3.1.

Proof. Suppose $X \in \mathcal{H}$, $A \in \mathcal{W}$, $p \in \{0\} \cup [1, \infty]$, and A is Fredholm on E^p with index $\kappa := \operatorname{ind}_p A$.

Case 1. $p \in \{0\} \cup (1, \infty)$.
Since $AS_{-\kappa}$ is Fredholm of index zero on E^p, we know from Lemma 2.1 that there exists a compact operator K on E^p such that $AS_{-\kappa} + K$ is invertible on E^p. By Lemma 3.3 and a simple perturbation argument, we know that, for a sufficiently large $m \in \mathbb{N}$, also $A' := AS_{-\kappa} + P_m K P_m$ is invertible on E^p. Moreover, $A' \in \mathcal{W}$ since $A, S_{-\kappa}, P_m K P_m \in \mathcal{W}$. From the inverse closedness of $\mathcal{W}$ [18, Theorem 2.5.2] we know that $B' := (A')^{-1} \in \mathcal{W}$. Summarizing,

$$I = A'B' = AS_{-\kappa}B' + P_m K P_m B', \tag{5}$$

i.e. $AB = I - K'$ with $B = S_{-\kappa}B' \in \mathcal{W}$ and $K' = P_m K P_m B' \in \mathcal{W}$ being compact on all spaces E^q with $q \in \{0\} \cup [1, \infty]$ by Lemma 3.4. By a completely symmetric argument for $A'' := S_{-\kappa}A + P_m L P_m$ with L and m accordingly chosen, one gets that $CA = I - L'$ for some $C \in \mathcal{W}$ and $L' \in \mathcal{W}$ compact on all E^q with $q \in \{0\} \cup [1, \infty]$. Looking at $C - CK' = C(AB) = (CA)B = B - L'B$, we see that the left and right regularizers B and C only differ by an operator $L'B - CK' \in \mathcal{W}$ that is compact on all spaces E^q so that we can use one of them as regularizer for both sides. This shows that A is Fredholm on all spaces E^q. The q-independence of the index now follows by looking at (5) as an equality on E^q and taking the index on both sides, i.e.

$$0 = \operatorname{ind}_q I = \operatorname{ind}_q A + \operatorname{ind}_q S_{-\kappa} + \operatorname{ind}_q B' = \operatorname{ind}_q A + (-\kappa) + 0,$$

showing that $\operatorname{ind}_q A = \kappa = \operatorname{ind}_p A$ for all $q \in \{0\} \cup [1, \infty]$.

Case 2. $p = \infty$ with existence of $X^\triangleleft$ and $A^\triangleleft$.
We get from Proposition 6.18 in [3](which is applicable since $A \in \mathcal{W}$ and since $X^\triangleleft$ and $A^\triangleleft$ exist) that A is also Fredholm, with the same index κ, if restricted to $E^0 \subset E^\infty$. Now the claim follows from Case 1 with $p = 0$.

Case 3. $p = 1$.
If A is Fredholm with index κ on $E^1 = \ell^1(\mathbb{Z}^N, X)$ then A^* is Fredholm of index $-\kappa$ on $\ell^\infty(\mathbb{Z}^N, X^*)$. By Case 2 (note that X^* and A^* clearly have a predual and preadjoint) we get that A^* is Fredholm on $\ell^2(\mathbb{Z}^N, X^*)$ with the same index $-\kappa$. But consequently, A is Fredholm on $E^2 = \ell^2(\mathbb{Z}^N, X)$ with index κ, so that the claim follows from Case 1 with $p = 2$. ■

Note that, as an important interim result of this proof, we get the following so-called Fredholm-inverse closedness of $\mathcal{W}$.

Proposition 3.6 *If $X \in \mathcal{H}$ and $A \in \mathcal{W}$ is Fredholm on one of the spaces E^p (existence of $X^\triangleleft$ and $A^\triangleleft$ assumed if $p = \infty$) then its Fredholm regularizer $B \in L(E^p)$ can be chosen in $\mathcal{W}$ as well, and the remainders $AB - I$ and $BA - I$ are compact on all spaces of the $\{E^p\}$ family.*

4 Outlook: Generalizations and Improvements?

We end this paper with an outlook to some possible future work on this subject. There are two or three things about Theorem 3.1 that look like they could possibly be extended or improved.

Firstly, it would be rather surprising if the condition $X \in \mathcal{H}$ really turned out to be necessary. The reason why we need this condition here is to be able to define a family $\{S_\kappa\}_{\kappa\in\mathbb{Z}}$ of Fredholm operators in $\mathcal{W}$ containing an operator with Fredholm index κ on E for each integer κ. If $X \notin \mathcal{H}$ then there exist no Fredholm operators of index 1 or -1 on X. Instead there is either a smallest positive integer $\varphi(X)$ for which a Fredholm operator $T \in L(X)$ of that index (or equivalently: an isomorphic subspace of X of that codimension) exists or there is no Fredholm operator on X with a nonzero index (i.e. no isomorphic subspace of X with a finite codimension, see Remark 2.4 or the spaces constructed in [9, 14]) in which case we put $\varphi(X) := 0$. It is clear that all Fredholm operators on X then have an index that is an integer multiple of $\varphi(X)$. We conjecture that, for every space E under consideration here (maybe even for every Banach space E of functions $\mathbb{Z}^N \to X$), it is also true that all Fredholm operators on E have an index that is a multiple of $\varphi(X)$; in other words, it holds that $\varphi(E) = \varphi(X)$. If that conjecture was true then it would be sufficient (and of course possible) to define the family $\{S_\kappa\}$ only for all $\kappa \in \varphi(X)\mathbb{Z}$. The condition $X \in \mathcal{H}$ could then be erased from Theorem 3.1 without any changes to the proof since every Fredholm operator A on E would also have an index in $\varphi(X)\mathbb{Z}$.

Secondly, is the existence of the preadjoint $A^\triangleleft$ really necessary?

Finally, the proof of Theorem 3.1 suggests that we can extend/exchange our family of spaces E^p to/by other families of Banach spaces E of functions $\mathbb{Z}^N \to X$ for which the following holds:

- The Wiener algebra $\mathcal{W}$ is contained and inverse closed in every $L(E)$;
- Property (4) in the proof of Lemma 3.3 holds for all E;
- The statement of Lemma 3.4 generalizes to the new family of spaces E, i.e. if $m \in \mathbb{N}$ and $P_m K P_m$ is compact on one space E then it is compact on all spaces E.

As likely candidates for such extensions, we suggest looking at the following spaces.

Example 4.1 One generalization of our family of X-valued ℓ^p spaces is the family of so-called *weak ℓ^p spaces* [5] with values in X; that is the set of all functions $u : \mathbb{Z}^N \to X$ with

$$\|u\| \ := \sup_{f\in X^*, \|f\|=1} \left\| \Big(f\big(x(k)\big) \Big)_{k\in\mathbb{Z}^N} \right\|_{\ell^p} \ < \ \infty,$$

usually denoted by $\ell^p_{\mathsf{weak}}(\mathbb{Z}^N, X)$. □

Example 4.2 Another direction of generalization – in particular a generalization of $\ell^1(\mathbb{Z}^N, X)$ – is the so-called *Rademacher sequence space* [5, 13], denoted by $\mathrm{Rad}(\mathbb{Z}^N, X)$. This is the set of all functions $u : \mathbb{Z}^N \to X$ with

$$\|u\| \;:=\; \sup_{n\in\mathbb{N}} \frac{1}{|S_n|} \sum_{\sigma\in S_n} \left\| \sum_{k\in\{-n,\dots,n\}^N} \sigma(k)\, u(k) \right\|_X \;<\; \infty,$$

where S_n denotes the set of all functions $\sigma : \{-n, ..., n\}^N \to \{-1, 1\}$. □

References

[1] S. Bochner and R. S. Phillips: Absolutely convergent Fourier expansions for non-commutative normed rings, *Ann. of Math.* **43** (1942), 409–418.

[2] A. Böttcher and B. Silbermann: *Analysis of Toeplitz Operators*, 1st ed. Akademie-Verlag 1989 and Springer 1990; 2nd ed. Springer 2006.

[3] S. N. Chandler-Wilde and M. Lindner: *Limit Operators, Collective Compactness, and the Spectral Theory of Infinite Matrices*, submitted for publication.

[4] J. Diestel: *Sequences and Series in Banach Spaces*, Springer-Verlag 1984.

[5] J. Diestel, H. Jarchow and A. Tonge: *Absolutely Summing Operators*, Cambridge University Press 1995.

[6] I. Gohberg and N. Krupnik: *One-Dimensional Linear Singular Integral Equations*, Volume II, Birkhäuser 1992.

[7] W. T. Gowers: A Solution to Banach's Hyperplane Problem, *Bulletin of the LMS* **26** (1994), no. 6, 523–530.

[8] R. Hempel and J. Voigt: The spectrum of a Schrödinger operator in L^p is p-independent, *Commun. Math. Phys.* **104** (1986), 243–250.

[9] P. Koszmider: A Banach space of continuous functions with few operators, *Math. Ann.* **330** (2004), 151–183.

[10] B. V. Lange and V. S. Rabinovich: On the Noether property of multidimensional discrete convolutions, *Mat. Zametki* **37** (1985), no. 3, 407–421 (Russian).

[11] M. Lindner: *Infinite Matrices and their Finite Sections: An Introduction to the Limit Operator Method*, Frontiers in Mathematics, Birkhäuser 2006.

[12] R. S. Phillips: On linear transformations, *Trans. Amer. Math. Soc.*, **48** (1940), 516–541.

[13] A. Pietsch and J. Wenzel: *Orthonormal Systems and Banach Space Geometry*, Cambridge University Press 1998.

[14] G. Plebanek: A construction of a Banach space C(K) with few operators, *Topology and its Applications* **143** (2004), 217–239.

[15] V. S. Rabinovich and S. Roch: The Fredholm index of locally compact band-dominated operators on $L^p(\mathbb{R})$, *Integral Equations and Operator Theory*, **57** (2007), no. 2, 263–281.

[16] V. S. Rabinovich, S. Roch and J. Roe: Fredholm indices of band-dominated operators, *Integral Equations Operator Theory* **49** (2004), no. 2, 221–238.

[17] V. S. Rabinovich, S. Roch and B. Silbermann: Fredholm Theory and Finite Section Method for Band-dominated operators, *Integral Equations Operator Theory* **30** (1998), no. 4, 452–495.

[18] V. S. Rabinovich, S. Roch and B. Silbermann: *Limit Operators and Their Applications in Operator Theory*, Birkhäuser 2004.

[19] S. Roch: Band-dominated operators on ℓ^p-spaces: Fredholm indices and finite sections, *Acta Sci. Math.* **70** (2004), no. 3–4, 783–797.

[20] I. M. Sigal: A generalized Weyl theorem and L^p-spectra of Schrödinger operators. *J. Oper. Theory* **13** (1985), 119–129.

[21] B. Simon: Brownian motion, L^p properties of Schrödinger operators and the localization of binding. *J. Funct. Anal.* **35** (1980), 215–225.

[22] I. Ja. Sneiberg: The solvability of linear equations in interpolational families of Banach spaces. (Russian) *Dokl. Akad. Nauk SSSR* **212** (1973), 57-59.

[23] A. Sobczyk: Projection of the space m on its subspace c_0, *Bull. Amer. Math. Soc.* **47** (1941), 938–947.

[24] P. Wojtaszczyk: *Banach Spaces for Analysts*, Cambridge University Press 1991.